AN INVERSE PROBLEM IN PÓLYA–SCHUR THEORY. II. EXACTLY SOLVABLE OPERATORS AND COMPLEX DYNAMICS

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Abstract. This paper being the sequel of [ABS] studies a class of linear ordinary differential operators with polynomial coefficients called *exactly solvable*; such an operator sends every polynomial of sufficiently large degree to a polynomial of the same degree. We focus on invariant subsets of the complex plane for such operators when their action is restricted to polynomials of a fixed degree and discover a connection of this topic with the classical complex dynamics. As a very special case, we encounter as invariant sets the class of Julia sets of rational functions.

If a new result is to have any value, it must unite elements long since known, but till then scattered and seemingly foreign to each other, and suddenly introduce order where the appearance of disorder reigned. Then it enables us to see at a glance each of these elements at a place it occupies in the whole.

H. Poincaré, Science and method

Contents

1. Introduction

In 1914, generalizing some earlier results of E. Laguerre, G. Pólya and I. Schur created a new branch of mathematics now referred to as the Pólya–Schur theory, see [PS14]. The main question considered in the Pólya–Schur theory can be formulated as follows, [CC04].

prob1 **Problem 1.** *Given a subset $S \subset \mathbb{C}$ of the complex plane, describe the semigroup of all linear operators $T : \mathbb{C}[z] \to \mathbb{C}[z]$ sending any polynomial with roots in S to a polynomial with roots in S (or to 0).*

def0 **Definition 2.** Given a subset $S \subset \mathbb{C}$ of the complex plane, if an operator T solves Problem 1, then we say that S *is a T-invariant set*, or that T *preserves* S.

In [ABS], we initiated the study of the following natural inverse problem in the set-up of Pólya–Schur theory.

prob:main **Problem 3.** *Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, find a sufficiently large class of T-invariant sets in the complex plane. Ultimately, characterize all non-trivial T-invariant subsets.*

For example, if $T = \frac{d}{dx}$, then a subset $S \subseteq \mathbb{C}$ is T-invariant if and only if it is convex. Although it is too optimistic to hope for a complete solution of Problem 3, we obtained a number of relevant results valid for arbitrary linear differential operators of finite order, see [ABS]. In particular, among other results, we were able to prove the existence and uniqueness of the closed and minimal by inclusion T-invariant set for linear differential operator T whose leading coefficient is not a constant. In what follows we need the following important notion.

defFuchs **Definition 4.** Given a linear ordinary differential operator

$$T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j} \tag{1.1}$$ {eq:main}

of order $k \geq 1$ with polynomial coefficients, define its *Fuchs index* as

$$\rho = \max_{0 \leq j \leq k} (\deg Q_j - j).$$

T is called *non-degenerate* if $\deg Q_k - k = \rho$, and *degenerate* otherwise. In other words, T is non-degenerate if ρ is realized by the leading coefficient of T.

We say that T is *exactly solvable* if its Fuchs index is zero.

This paper is mainly devoted to the study of exactly solvable operators and certain types of invariant sets different from those in Definition 2. The main reason for our choice of this class of operators is that exactly solvable operators can be alternatively characterized as "degree preserving", see (3.1) and that the invariant sets we consider below resemble those appearing in complex dynamics. In particular, they have very different properties than those studied in [ABS].

1.1. Two additional types of invariant sets.

def:degreen **Definition 5.** Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, we denote by $\mathcal{I}_n^T$ the collection of all closed subsets $S \subset \mathbb{C}$ such that for every polynomial of degree n with roots in S, its image $T(p)$ is either 0 or has all roots in S. In this situation, we say that S *belongs to the class* $\mathcal{I}_n^T$ or that S is *T-invariant in degree n*.

We say that a set $S \in \mathcal{I}_n^T$ is *minimal* if there is no closed proper nonempty subset of S belonging to $\mathcal{I}_n^T$. Observe that families of sets belonging to $\mathcal{I}_n^T$ are closed under taking the intersection. Denote by M_n^T the intersection of all sets in $\mathcal{I}_n^T$.

Notice the difference of this definition with Definition 2 above. We will study this type of invariant and minimal sets in Section 3. In particular, we show the existence and uniqueness of M_n^T under certain weak assumptions on T, give its alternative characterization, and study the asymptotic behavior of the sequence $\{\mathrm{M}_n^T\}$ when $n \to \infty$. To describe the asymptotic behavior we need the following notion.

Definition 6. Given an operator (1.1) with $Q_k(x) \neq const$, denote by $Conv(Q_k) \subset \mathbb{C}$ the convex hull of the zero locus of $Q_k(x)$. We will refer to $Conv(Q_k)$ as the *fundamental polygon* of T.

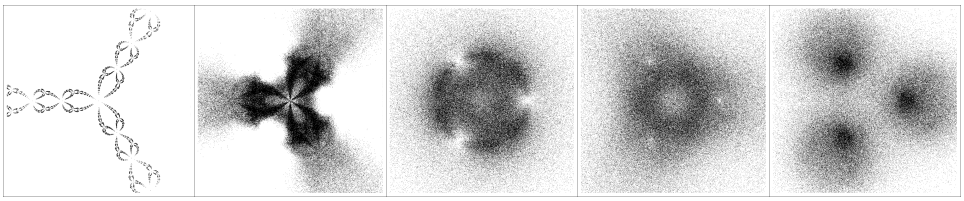

fig:newton1 (A) $n = 1$ fig:newton2 (B) $n = 2$ fig:newton3 (C) $n = 3$ fig:newton4 (D) $n = 4$ fig:newton5 (E) $n = 5$

FIGURE 1. The set M_n^T for $T(p) = (1 - 3x^3)p' + (3x^2)p$. For $n = 1$ this gives the classical Newton fractal, [Bea00, p. 23], derived from the iterative Newton–Raphson method for finding roots of a polynomial. As seen in the leftmost figure this fractal is an unbounded Julia set. As n grows, the sequence M_n^T approaches the convex hull of the roots of $1 - 3x^3 = 0$.

fig1

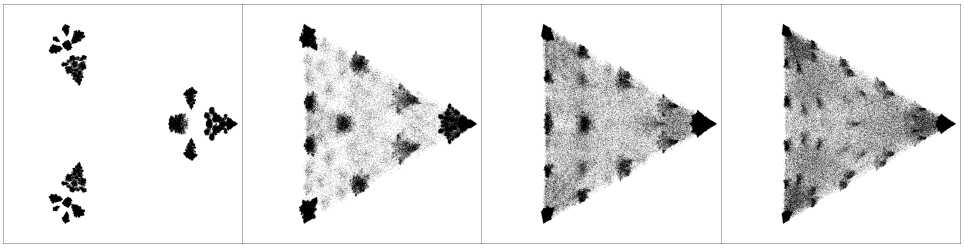

fig:deg3n2 (A) $n = 2$ fig:deg3n3 (B) $n = 3$ fig:deg3n4 (C) $n = 4$ fig:deg3n5 (D) $n = 5$

FIGURE 2. The set M_n^T for $T(p) = (x^3 - 1)p'' + x^2p' + p$. For $n = 2$, the last term has no effect, and is excluded. As in previous example, the sequence $\{\mathrm{M}_n^T\}$ approaches the convex hull of the roots of $x^3 - 1 = 0$ as n grows.

fig2

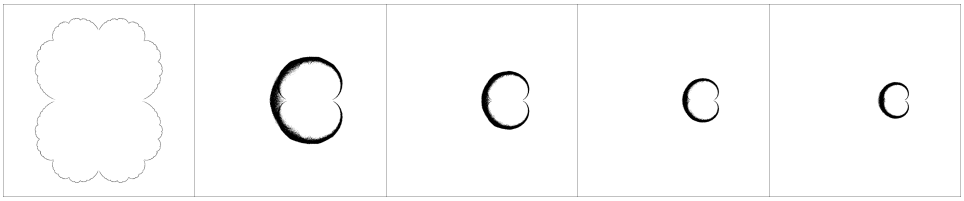

fig:julia_c025n1 (A) $n = 1$ fig:julia_c025n2 (B) $n = 2$ fig:julia_c025n3 (C) $n = 3$ fig:julia_c025n4 (D) $n = 4$ fig:julia_c025n5 (E) $n = 5$

FIGURE 3. The set M_n^T for $T(p) = (x^2 - x + \frac{1}{4})p' + p$. For $n = 1$ this gives the Julia fractal for $f(x) = x^2 + \frac{1}{4}$. When $n \to \infty$, the set M_n^T seems to approach the root of $(x - \frac{1}{2})^2 = 0$; so in this case, we can argue that M_∞^T could be defined as the single point $\frac{1}{2}$.

fig3

In Section 3 we will prove that under some asymptions on T, the sequence $\{\mathrm{M}_n^T\}$ approaches $Conv(Q_k)$, see Figures 1– 3.

In the context of exactly solvable operators (and even for more general classes of operators (1.1)) it is also natural to introduce still another concept of invariant sets.

def7 **Definition 7.** Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, we say that S is *Hutchinson-invariant in degree* n if whenever $z \in S$, $T[(x - z)^n]$ either is 0 or has all roots in S.

The set of all Hutchinson-invariant sets will be denoted by $\mathcal{HI}_n^T$. The intersection of all closed Hutchinson-invariant sets is denoted H_n^T.

The difference of Definition 7 from Definition 5 is that we now only require that the set S is invariant with respect to the single type of polynomials of degree n, namely, those with an n-tuple root, rather than all polynomials of degree n with roots in S. It is then straightforward to see that $H_n^T \subseteq M_n^T$. In Section 4, we show that H_n^T is closely related to so called Hutchinson-operators and they exists under certain assumptions on T. We observe that the H_n^T are, in general, fractal in nature and, in particular, we will show that the classical fractals such as the Sierpinski triangle, the Koch curve and the Lévy curve can be realized as such minimal Hutchinson-invariant sets.

In Section 5, we focus on the case $n = 1$. In this case one has that $M_1^T = H_1^T$. (We impose no restriction on T being exactly solvable or non-degenerate.) We can then without loss of generality assume that $T : \mathbb{C}[x] \to \mathbb{C}[x]$ acts as

$$T[(x - z)] = P_1(x) - zP_0(x).$$

In this case, we show that M_1^T is equal to the Julia set associated with the rational function $R(x) = \frac{P_1(x)}{P_0(x)}$. We also show that M_n^T for larger values of n must contain a family of Julia sets.

Remark 8. Observe that being a Julia set, M_1^T looks extremely different from the "nice" convex $\mathrm{M}_{\geq 1}^T$ studied in [ABS], see examples in Section 6. Illustrations of M_n^T for different T and n are given in Figures 1–3.

The structure of the paper is as follows. In Section 3, we present and prove some general results about $\mathcal{I}_n^T$ and M_n^T. Section 4 discusses Hutchnson-invariant sets. Section 5 contains our results about the class of exactly solvable operators and their relation to the Julia sets of rational functions. Section 6 presents a number of illustrating examples. Finally, Section 7 contains a number of open problems connected to this topic.

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2. Preliminary results: large invariant disks

Define the symbol of the operator $T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j}$ as

$$F_T(x, y) := \sum_{j=0}^{k} Q_j(x) y^j.$$

Define the modified symbol as

$$G_T(x, y) := \sum_{j=0}^{\infty} \frac{(-1)^j T(z^j)}{n!} y^n.$$

Observe that $G_T(x, y)e^{xy} = F_T(x, -y)$. CHECK THAT THIS FITS FUTHER MATERIAL!

Proposition 9. *An operator $T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j}$ is exactly solvable and non-degenerate if and only if*

$$T[(x - z)^k] = \lambda_k(x^k + R_1(z)x^{k-1} + \cdots + R_k(z)),$$

where $\deg R_i(z) \leq i,\ i = 1, \ldots, k$ *and* $\lambda_k \neq 0$.

Proof. BLA □

Now we will formulate and prove a number of results about the existence of large invariant disks which are valid for a general class of operators of the form (1.1). We will later apply them in the case of exactly solvable operators. (These results are also presented in [ABS].) Let us first provide an analog of Fuchs index in Definition 4 for the case of finite degree.

Define the nth *Fuchs index* of a linear operator $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$ to be

$$\rho = \rho_n = \max_{0\le j\le n} (\deg T(x^j) - j),$$

and call T *non-degenerate* if $\deg T(x^n) - n = \rho_n$. Note that we may write

$$G_T(x, y) = T[(1 + xy)^n] := \sum_{-n\le \ell\le \rho} x^\ell P^n_\ell(xy),$$

for polynomials $P^n_\ell(t)$ uniquely determined by T. Thus T is non-degenerate if and only if the degree of P^n_ρ is n.

genethm **Theorem 10.** *Suppose $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$ is a non-degenerate linear operator with nth Fuchs index ρ. Let $g(x)$ be the greatest common divisor of $\{P^n_\ell(x)\}_\ell$. Then the closed disk $D(0, R)$ is invariant in degree n for all sufficiently large $R > 0$ if and only if all zeros of $g(x)$ lie in $\{z : |z| \le 1\}$ and all zeros of $P^n_\rho(x)/g(x)$ lie in $\{z : |z| < 1\}$.*

Proof. Suppose all zeros of $P^n_\rho(x)$ lie in $\{z : |z| < 1\}$ and all zeros of $g(x)$ lie in $\{z : |z| \le 1\}$. The disk $D(0, R)$ is invariant in degree n if and only if $G_T(z, w) \ne 0$ for all $(z, w) \in \Omega_R = \{(z, w) \in \mathbb{C}^2 : |z| > R \text{ and } |w| > 1/R\}$. Since P^n_ρ has degree n and all the zeros of $P_\rho(x)/g(x)$ lie in the open disk $\{z : |z| < 1\}$, there is a positive constant C such that $|P^n_\ell(z)/P^n_\rho(z)| < C$ for all $|z| \ge 1$. Hence if $(z, w) \in \Omega_R$, then

$$\left| \frac{G_T(z, w)}{z^\rho P^n_\rho(zw)} - 1 \right| \le \sum_{\ell=-n}^{\rho-1} R^{-(\rho-\ell)} C < 1, \tag{2.1}$$ {tratt}

for all sufficiently large R.

If $g(x)$ has a zero in $\{z : |z| \le 1\}$, then $G_T(z, w) = 0$ for some $(z, w) \in \Omega_R = \{(z, w) \in \mathbb{C}^2 : |z| > R \text{ and } |w| > 1/R\}$, and then $D(0, R)$ is not invariant. Suppose $(P^n_\rho/g)(\xi) = 0$, where $|\xi| \ge 1$, and that $g(z)$ has no zeros in $\{z : |z| \le 1\}$. Consider a sequence $\{\xi_k\}_{k=1}^\infty$, where $P^n_\rho(\xi_k) \ne 0$, $|\xi_k| > 1$, and $\lim_{k\to\infty} \xi_k = \xi$. Let now

$$Q_k(z) = \sum_{\ell\le\rho} z^\ell (P^n_\ell/g)(\xi_k) = G_T(z, \xi_k/z)/g(\xi_k).$$

Since $(P^n_\rho/g)(\xi) = 0$, we see that at least one zero, say z_k, of $Q_k(z)$ tends to ∞ as $k \to \infty$. Hence for $R_k = |z_k|(1 + |\xi_k|)/2$ we have $(z_k, \xi_k/z_k) \in \Omega_{R_k}$, while $G_T(z_k, \xi_k/z_k) = 0$. □

Suppose $T = \sum_{j=0}^k Q_j(x) \frac{d^j}{dx^j}$, $Q_k(x) \ne 0$, is a differential operator. Note that

$$T[(1 + xy)^n] = \sum_{j=0}^k j! x^{-j} Q_j(x) \binom{n}{j} (xy)^j (1 + xy)^{n-j}$$

Hence

$$P^n_\ell(x) = \sum_{j=0}^k j! a_{\ell,j} \binom{n}{j} x^j (1 + x)^{n-j}, \tag{2.2}$$ {prn}

where $a_{\ell,j}$ is the coefficient of $x^{j+\ell}$ in $Q_j(x)$.

Theorem 11. *Suppose that $T = \sum_{j=0}^{k} Q_j(x)\frac{d^j}{dx^j}$, $Q_k(x) \neq 0$ is a non-degenerate differential operator. If $n \geq k$, $\sum_{j=0}^{k} j! a_{\rho,j}\binom{n}{j} \neq 0$, and all zeros of $\sum_{j=0}^{k} j! a_{\rho,j}\binom{n}{j} x^j$ have real part greater than $-1/2$, then $D(0,R)$ is invariant in degree n for all sufficiently large R.*

Proof. The condition $\sum_{j=0}^{k} j! a_{\rho,j}\binom{n}{j} \neq 0$ guarantees that $P_\rho^n(x)$ has degree n, by (2.2). By the assumptions we see that $(1+x)^{n-k}$ divides $P_\ell^n(x)$ for all ℓ. Now $P_\rho^n(x)/(1+x)^{n-k}$ has all its zeros in $\{z : |z| < 1\}$ if and only if all zeros of $\sum_{j=0}^{k} j! a_{\rho,j}\binom{n}{j} x^j$ have real part greater than $-1/2$. The proof now follows from Theorem 10. □

Corollary 12. *If T is a non-degenerate differential operator, then there is an integer N_0 and a positive number R_0 such that $D(0,R)$ is invariant in degree n whenever $n \geq N_0$ and $R \geq R_0$.*

Proof. From (2.2) it follows that the estimate in (2.1) can be made uniform in n, for all $n \geq k$ for which $\sum_{j=0}^{k} j! a_{\rho,j}\binom{n}{j} \neq 0$. □

inrem **Remark 13.** Note that if T is a linear operator and $\Omega \subseteq \mathbb{C}$ is closed and unbounded, then Ω is invariant in degree n if and only it is invariant in degree j for all $j \leq n$. Indeed if $f(z)$ has degree $j \leq n$ we may take a sequence $\{w_i\}_{i=1}^{\infty}$ for which $|w_i| \to \infty$ as $i \to \infty$. Then the zeros of

$$T(f) = \lim_{i\to\infty} T[(1 - x/w_i)^{n-j} f(z)]$$

is in Ω by Hurwitz' theorem.

Proposition 14. *Let H be a closed half-plane given by $H = \{az + b : \mathrm{Im}(z) \leq 0\}$, and let $T = \sum_{j=0}^{k} Q_j(x)\frac{d^j}{dx^j}$ be a differential operator. The following are equivalent:*

(1) The set of positive integers for which H is invariant in degree n is unbounded,
(2) H is invariant in degree n for all $n \geq 0$,
(3) The polynomial $\sum_{j=0}^{k} Q_j(ax+b)(-y/a)^j$ is stable.

Proof. By Remark 13 we see that (1) and (2) are equivalent. Now, (2) is equivalent to that the operator $S : \mathbb{C}[x] \to \mathbb{C}[x]$ defined by

$$S(f)(x) = T(f(\phi^{-1}(x)))(\phi(x)),$$

where $\phi(x) = ax + b$, preserves stability. The operator S is again a differential operator, so the equivalence of (2) and (3) now follows from

ADD REF? Petter, which reference should it be?

. □

3. Properties of minimal invariant sets M_n^T

sec:general

3.1. Existence of M_n^T.

We start with the next statement which can be found in Theorem 1 of [BR02].

lm:trivial **Theorem A.** (i) For any exactly solvable operator T, observe that for each non-negative integer i,

$$T(x^i) = \lambda_i^T x^i + \text{ lower order terms.} \tag{3.1}$$

{eq:spectrum}

Additionally one can show that for i large, the λ_i have monotone increasing absolute values.
(ii) For any exactly solvable operator T and any sufficiently large positive integer n, there exists a unique (up to a constant factor) eigenpolynomial $p_n^T(x)$ of T of degree n. Additionally, the eigenvalue of p_n^T equals λ_n^T, where λ_n^T is given by (3.1).

Remark 15. In addition to Theorem A, observe that for any exactly solvable operator T as in (1.1) and any non-negative integer n, T has a basis of eigenpolynomials in the linear space $\mathbb{C}_n[x]$ consisting of all univariate polynomials of degree at most n. This follows immediately from e.g., the fact that T is triangular in the monomial basis $\{1, x, \dots, x^n\}$. In other words, even if T has a multiple eigenvalue it has now Jordan blocks, but the eigenpolynomial in the respective degree is no longer unique. A simple example of such situation occurs for $T = x^k \frac{d^k}{dx^k}$.

th:generalN

Theorem 16. *If T is exactly solvable and a non-negative integer n such that*
(i) *among the numbers $\Lambda_n^T = \{\lambda_0^T, \lambda_1^T, \dots, \lambda_n^T\}$ there exists a unique λ_ℓ^T with the maximal absolute value;*
(ii) *$\mathcal{I}_n^T$ contains no 1-point invariant sets,*
then there exists unique minimal nonempty closed set $\mathrm{M}_n^T \in \mathcal{I}_n^T$. The set M_n^T coincides with the intersection of all elements of $\mathcal{I}_n^T$.

Proof. Under the above assumptions take any invariant set $S \in \mathcal{I}_n^T$ in degree n. (Such invariant sets S obviously exist.) S contains at least two points which we denote by $u \neq v$. Consider the $(n+1)$-tuple of polynomials

$$q_j(x) \coloneqq (x-u)^j (x-v)^{n-j}, \ j = 0, 1, \dots, n. \tag{3.2}$$

{eq:basisUV}

Observe that each $T(q_j)$ has roots in S and that the $(n+1)$-tuple $\{q_0(x), \dots, q_n(x)\}$ spans $\mathbb{C}_n[x]$. Choose some basis $\{p_0^T, p_1^T, \dots, p_n^T\}$, $\deg p_i^T = i$ of eigenpolynomials of T in $\mathbb{C}_n[x]$. (Such a basis is not unique if there is a multiple eigenvalue, but it always exists since T considered in the monomial basis of $\mathbb{C}_n[x]$ is given by a triangular matrix.) Since $\{q_0(x), \dots, q_n(x)\}$ spans $\mathbb{C}_n[x]$ there exists $0 \le \kappa \le n$ such that the expansion of $q(x) \coloneqq q_\kappa(x) = \sum_{j=0}^n a_j p_j^T(x)$ has $a_\ell \neq 0$. Repeated application of T to q then gives

$$T^{\circ m}(q) = \sum_{j=0}^n a_j \lambda_j^m p_j^T(x) = \lambda_\ell^m \sum_{j=0}^n a_j \left(\frac{\lambda_j}{\lambda_\ell}\right)^m p_j^T(x). \tag{3.3}$$

{eqn:roots}

Obviously, the roots of $T^{\circ m}(q)$ belong to S. By our assumptions, $|\lambda_\ell| > |\lambda_j|$, for all $j \neq \ell$ which implies that the right-hand side of (3.3) equals $a_\ell \lambda_\ell^m p_\ell^T(x)$ plus some polynomial of degree at most n with coefficients tending to 0, as m tends to infinity. Since $S \in \mathcal{I}_n^T$ and $a_\ell \neq 0$, the roots of $p_\ell(x)$ must necessarily belong to S. Hence, the intersection of all subsets belonging to $\mathcal{I}_n^T$ is non-empty. This intersection gives the required unique minimal set M_n^T. Observe that if p_ℓ^T has at least two distinct roots, then so does M_n^T. (Below we will show that in this case M_n^T is infinite.) □

Reformulate this a bit?

Remark 17. If we keep the assumption (i) and drop the assumption (ii) in the formulation of Theorem 16, then the same proof provides the existence of the minimal set coinciding with the intersection of all closed invariant sets in $\mathcal{I}_n^T$ each of which contains at least 2 points. As in the case when (ii) is satisfied, we still denote this minimal set by M_n^T. The only difference with the case covered by Theorem 16 is that in the latter situation M_n^T will not be the unique minimal set contained in $\mathcal{I}_n^T$ since every 1-point invariant set is obviously minimal. We will study this situation in more details in Section 3.2 below.

prop:finite

Proposition 18. (i) For any linear operator $T : \mathbb{C}_n[x] \to \mathbb{C}_n[x]$, any set in $\mathcal{I}_n^T$ containing at least 3 points is infinite.
(ii) For any exactly solvable operator T, any set in $\mathcal{I}_n^T$ containing at least 2 points is infinite.

Proof. Assume that $T : \mathbb{C}_n[x] \to \mathbb{C}_n[x]$ is a linear operator and that $\mathfrak{I} = \{z_1, \dots, z_r\}$ is a finite invariant set of T in degree n with $r \geq 3$ points. Let $P = \{p_1, \dots, p_\ell\}$ be the *finite* set of all monic polynomials of degree n with roots in $\mathfrak{I}$.

For each $j \in \{1, \dots, \ell\}$, there exists $i_j \in \{1, \dots, \ell\}$ such that $T[p_j] = \mu_j p_{i_j}$, where μ_j is a complex number, possibly 0.

Note that μ_j can be zero, so T does not need to be an isomorphism!

Hence, there exists a positive integer k such that every $p \in P$ is an eigenvector of T^k, where $T^k : \mathbb{C}_n[x] \to \mathbb{C}_n[x]$ is the k-th power of T. Observe that T^k also has $\mathfrak{I}$ as an invariant set.

Without loss of generality we can assume that 0, 1 and α are three different points in $\mathfrak{I}$. Observe that the polynomials

$$q_j(x) \coloneqq x^j (x-1)^{n-j}, \; j = 0, 1, \dots, n \tag{3.4}$$

{eq:basis}

is a basis for $\mathbb{C}_n[x]$. Additionally, we have that $T^k[q_j] = \nu_j q_j$ for some $\nu_j \in \mathbb{C}$. Let us now express $(x-\alpha)^n$ in the basis defined by (3.4):

$$(x-\alpha)^n = a_0 q_0(x) + a_1 q_1(x) + \cdots + a_n q_n(x), \qquad a_j = (-1)^j \binom{n}{j} \alpha^{n-j} (\alpha - 1)^j.$$

By linearity of T^k,we have that

$$T^k[(x-\alpha)^n] = T^k \left[\sum_{j=0}^{n} a_j q_j(x) \right] = \sum_{j=0}^{n} \nu_j a_j q_j(x). \tag{3.5}$$

{eq:oneSide}

On the other hand,

$$T^k[(x-\alpha)^n] = \nu (x-\alpha)^n = \nu \sum_{j=0}^{n} a_j q_j(x) \tag{3.6}$$

{eq:otherSide}

for some ν. Combining (3.5) and (3.6) together with the fact that all a_j are non-zero implies that $\nu = \nu_j$ for all j, and we conclude that T^k act as a scaling with factor $\nu \neq 0$.

Therefore, T is diagonalizable with the q_j as eigenvectors such that $T[q_j] = \xi_j q_j$ for $j = 0, \dots, n$ where ξ_j is a kth root of ν. Repeating the above argument for T instead, imply that T itself must be a scaling.

For (ii), we can use the same technique and show that $T^k[q_j] = \nu_j q_j$, so that T^k acts as a diagonal matrix in the basis determined by Eq. (3.4), and the q_j's must be the eigenvectors of T. Notice now that

$$x^m = \sum_{j=m}^{n} (-1)^{n-j} \binom{n-m}{j-m} q_j(x) \tag{3.7}$$

{eq:basisChange}

and that $\operatorname{span}(x^m, \dots, x^n) = \operatorname{span}(q_m, \dots, q_n)$ for all $m = 0, \dots, n$. It follows that

$$T[x^m] \in \operatorname{span}(q_m, \dots, q_n) = \operatorname{span}(x^m, \dots, x^n).$$

Thus in order for T to be exactly solvable, we must have that all the x^m are eigenvectors of T. The triangular relation in Eq. (3.7) now implies that T must act as a scaling. □

CHECK THE ABOVE PROOF!

Let us now consider two situations not covered by Theorem 16. Namely, we want to understand what happens if λ_ℓ is still unique but $\mathcal{I}_n^T$ contains 1-point sets and what happens if there exist several λ_j's with the same maximal absolute value.

subsec:1-point

3.2. **1-point sets in $\mathcal{I}_n^T$.** Here we give the conditions which guarantee the existence of 1-point invariant sets in $\mathcal{I}_n^T$ and discuss how they affect the situation.

lm:deri

Lemma 19. *An exactly solvable operator T given by* (1.1) *has infinitely many* 1*-point invariant sets in $\mathcal{I}_n^T$ if and only if the truncation of T up to the derivatives of order n coincides with $\alpha\frac{d^m}{dx^m}$ for some constant $\alpha \neq 0$ and $m \leq n$ or it maps the whole space $\mathbb{C}_n[x]$ to* 0.

Proof. Since we are interested in the action of T on $\mathbb{C}_n[x]$ we can from the beginning truncate T up to the derivatives of order n since higher terms act trivially on $\mathbb{C}_n[x]$. If such truncation coincides with $\alpha\frac{d^m}{dx^m}$, then for any $z \in \mathbb{C}$, $T[(x-z)^n] = \alpha(n)_m(x-z)^{n-m}$ implying that each $\{z\}$ is a 1-point invariant set in $\mathcal{I}_n^T$. Let us show that the condition $T_{|\mathbb{C}_n[x]} = \alpha\frac{d^m}{dx^m}$ is necessary.

Given an exactly solvable $T = \sum_{j=0}^n Q_j(x)\frac{d^j}{dx^j}$, consider first $T[x^n] = \sum_{j=0}^n Q_j(x)(n)_j x^{n-j}$. Denoting $Q_i(x) = q_{i,i}x^i + q_{i,i-1}x^{i-1} + \cdots + q_{i,0}$, for $i = 0, 1, \ldots n$, we get that, for $j = 0, \ldots, n$, the coefficient of degree j in the expansion of $T[x^n]$ equals

$$A_j = n!q_{n,j} + (n)_{n-1}q_{n-1,j-1} + \cdots + (n)_{n-j}q_{n-j,0}. \tag{3.8}$$

{eq:coeffs}

Consider the system of $(n+1)$ linear homogeneous equations given by

$$A_n = 0,\ A_{n-1} = 0,\ \ldots, A_0 = 0, \tag{3.9}$$

{eq:basicsystem}

where each A_j is expressed as in (3.8). This system involves $\binom{n+2}{2}$ variables $q_{r,s}$ where $0 \leq s \leq r \leq n$. Observe additionally that each coefficient $q_{r,s}$ appears only once in the system, namely, in the expression for the coefficient A_{n-r+s}. This circumstance implies that the rank of the system (3.9) equals $n+1$ unless some of the equation vanishes identically, i.e., for some j, all the coefficients $q_{n-\ell,j-\ell},\ \ell = 0, \ldots, j$ in the right-hand side of (3.8) vanish.

Assume now that the origin is a 1-point invariant set in $\mathcal{I}_n^T$, i.e., either $T[x^n] = \beta x^m,\ \beta \neq 0,\ m \leq n$ or $T[x^n] = 0$. In the latter case, we need a solution of (3.9) which typically gives $n+1$ linearly independent conditions on $\binom{n+2}{2}$ variables $q_{r,s}$ where $0 \leq s \leq r \leq n$. In the former case, we need a solution of the system obtained from (3.9) by removing the equation $A_m = 0$ which typically gives n linearly independent conditions on the variables $q_{r,s}$.

Now keeping the same notation, consider $T[(x-z)^n]$. Changing the variable $t = x - z$, we obtain

$$T[(x-z)^n] = \sum_{j=0}^n Q_j(t+z)(n)_j t^{n-j}.$$

The expansion of $Q_i(t+z)$ in power of t is given by

$$Q_i(t+z) = q_{i,i}(t+z)^i + q_{i,i-1}(t+z)^{i-1} + \cdots + q_{i,0} = q_{i,i}t^i + (iq_{i,i}z + q_{i,i-1})t^{i-1} +$$
$$\left(\binom{i}{2}z^2q_{i,i} + (i-1)zq_{i,i-1} + q_{i,i-2}\right)t^{i-2} + \cdots + (q_{i,i}z^i + q_{i,i-1}z^{i-1} + \cdots + q_{i,0}).$$

Denote by $q_{i,\ell}(z)$ the coefficient of t^ℓ in the right-hand side of the latter formula. (For example, $q_{i,0}(z) = q_{i,i}z^i + q_{i,i-1}z^{i-1} + \cdots + q_{i,0}$.)

Using this notation, the coefficient of degree j in the expansion of $T[(x-z)^n]$ in powers of $(x-z)$ equals

$$A_j(z) = n!q_{n,j}(z) + (n)_{n-1}q_{n-1,j-1}(z) + \cdots + (n)_{n-j}q_{n-j,0}(z). \tag{3.10}$$

{eq:coeffsz}

Consider the system of $(n+1)$ equations given by

$$A_n(z) = 0,\ A_{n-1}(z) = 0,\ \ldots, A_0(z) = 0, \tag{3.11}$$

{eq:basicsystemz}

where each $A_j(z)$ is expressed as in (3.10). This system involves $\binom{n+2}{2}$ variables $q_{r,s}$ where $0 \le s \le r \le n$ and the additional variable z. For any fixed z, (3.11) is linear homogeneous in all $q_{r,s}$. We want to show that for fixed $q_{r,s}$, (3.11) has infinitely many solutions in z if and only if $T = \alpha \frac{d^m}{dx^m}$. This means that only the coefficient $q_{m,0} \neq 0$ and the rest of $q_{r,s}$ vanish. To see that observe all equations in (3.11) except one must vanish identically. This implies ....

□

3.3. Several eigenvalues with coinciding maximal absolute value.

This proposition is false!

prop:two

Proposition 20. *If T is exactly solvable and among the numbers $\Lambda_n^T = \{\lambda_0^T, \lambda_1^T, \dots, \lambda_n^T\}$ there exists at least two λ_i^T and λ_j^T with $i \neq j$ having the maximal absolute value among Λ_n^T, then the only invariant subset in $\mathcal{I}_n^T$ is $\mathbb{C}$.*

Proof. check!

Let us first assume that $\lambda_i^T = \lambda_j^T = \lambda$. Then the eigenspace $V_\lambda \subset \mathbb{C}_n[x]$ corresponding to λ is at least 2-dimensional. Fixing some basis $\{p_0(x), p_1(x), \dots, p_n(x)\}$ of eigenpolynomials we get similarly to (3.3)

$$T^{\circ m}(p) = \sum_{j=0}^{n} a_j \lambda_j^m p_j(x) = \lambda^m \sum_{\ell=0}^{n} a_\ell \left(\frac{\lambda_\ell}{\lambda}\right)^m p_\ell(x) \to \lambda^m (a_i p_i(x) + a_j p_j(x)). \tag{3.12}$$

{eqn:rootsM}

Thus if $\mathfrak{I} \in \mathcal{I}_n^T$, then it should contain the roots of $a_i p_i(x) + a_j p_j(x)$ for arbitrary complex numbers a_i and a_j.

Why is this true?

□

Example 21. Consider the exactly solvable operator

$$T = \frac{(1-i)x}{2}\frac{d}{dx} + i.$$

It has eigenvalues $\lambda_0 = i$, $\lambda_1 = \frac{1+i}{2}$ and $\lambda_2 = 1$. This operator for $n = 2$ seem to allow any closed disk D centered at the origin as invariant set. That is,

$$T[(x-\alpha)(x-\beta)] = x^2 - \frac{1+i}{2}(\alpha+\beta)x + i\alpha\beta$$

has all roots in the disk D if $\alpha, \beta \in D$.

Remark 22. The requirement that M_n^T contains at least 2 points is essential. Some exactly solvable operators T admit invariant sets in degree n consisting of a single point. As a concrete example take

$$T = (x^2 + ax + b)\frac{d^2}{dx^2} + (cx + d)\frac{d}{dx} + e$$

satisfying the conditions $b = 0,\ 2a + d = 0$. Then $T(x^3) = (6+3c+e)x^3$ which implies that $\mathcal{I}_3^T$ contains the invariant subset coinciding with the origin. Another example related to the same operator which satisfies the conditions $b = 0,\ 6 + 3c + e = 0$. Then $T(x^3) = (6a + 3d)x^2$ and the same argument holds. There should be an upper bound for the total number of one-point invariant subsets in terms of the order k of T.

Remark 23. Observe that the uniqueness of M_n^T also fails if T is not exactly solvable, see Example 41 below.

Remark 24. Observe that if $\ell < n$, then M_n^T can not be bounded. This follows from the fact that the additional polynomial in the right-hand side in (3.3) has degree n and its coefficients tend to zero which implies that some of the roots of the total right-hand side tend to infinity. On the other hand, by Theorem A, λ_n has the biggest absolute value among all $\lambda_j,\ j = 0, 1, \dots, n$, for n large enough.

Even in the case when if $|\lambda_n|$ is maximal in Λ_n^T we cannot make any conclusions regarding the boundedness of M_n^T. In Example 34 below, we construct an exactly solvable operator of degree 2, such that $|\lambda_0| < |\lambda_1| < |\lambda_2|$ but M_2^T is unbounded.

Some concrete examples of M_n^T can be found in subsequent sections.

ubsec:forwardIteration

3.4. **Forward iterations and alternative characterization of M_n^T.** Given an arbitrary set $\Omega \subseteq \mathbb{C}$, let $\mathcal{P}_n(\Omega)$ be the set of all polynomials of degree n with all zeros in Ω.

Definition 25. For an arbitrary set $\Omega \subseteq \mathbb{C}$ and an operator T given by (1.1), we define the new set

$$T_n(\Omega) \coloneqq \Omega \cup \{x \in \mathbb{C} | x \text{ is a root of } T(p(x)), \text{ for some } p \in \mathcal{P}_n(\Omega)\}. \tag{3.13}$$

{eq:forwardIterate}

Evidently, $S \in \mathcal{I}_n^T$ if and only if $T_n(S) \subseteq S$ and S is closed.

Given an arbitrary set $\Omega \subseteq \mathbb{C}$, we define $G_n(\Omega)$ as

$$G_n(\Omega) \coloneqq \bigcup_{j=0}^{\infty} T_n^{\circ j}(\Omega),$$

where $T_n^{\circ j}$ stands for the j-th iteration of the operation T_n. In other words, $G_n(\Omega)$ is the union of the iterations of T_n applied to Ω.

Lemma 26. *For any n and Ω, $\overline{G_n(\Omega)} \in \mathcal{I}_n^T$.*

Proof. Let $S \coloneqq \overline{G_n(\Omega)}$ We need to show the inclusion $T_n(S) \subseteq S$. But

$$T_n(S) = T_n\left(\overline{\bigcup_{j=0}^{\infty} T_n^{\circ j}(\Omega)}\right) \subseteq \overline{\bigcup_{j=1}^{\infty} T_n^{\circ j}(\Omega)} \subseteq \overline{\bigcup_{j=0}^{\infty} T_n^{\circ j}(\Omega)} = S$$

which proves the assertion. □

By construction, $G_n(\Omega)$ is the smallest (not necessarily closed) invariant set containing Ω. We can now make the equivalent definition of a minimal (closed) set in $\mathcal{I}_n^T$.

claim1 **Proposition 27.** *A set $S \in \mathcal{I}_n^T$ is minimal if and only if $\overline{G_n(S_1)} = \overline{G_n(S_2)}$ for any two non-empty subsets $S_1, S_2 \subseteq S$.*

Proof. Suppose that there are $S_1, S_2 \subseteq S$ such that $\overline{G_n(S_1)} \neq \overline{G_n(S_2)}$. Then the two latter sets are two different closed subsets of S, and S cannot be minimal.

In the other direction, suppose S is not minimal. Then there is some closed $S_1 \subset S$ which is strictly contained in S and invariant under T_n. It follows that $S_1 = \overline{G_n(S_1)}$ and $\overline{G_n(S)} = S$ so $\overline{G_n(S_1)} \neq \overline{G_n(S_2)}$. □

Proposition 28. *Every bounded and closed infinite minimal set $S \in \mathcal{I}_n^T$ is dense in itself.*

Proof. Let a be an accumulation point in S — such a point exists, since S is bounded and contains infinitely many pairwise distinct points. Since S is minimal, it follows from Proposition 27 that $S = \overline{G_n(\{a\})}$. Let $b \in S$ be any point and let $\Omega_{b,\epsilon}$ be an ϵ-neighborhood of b.

Using minimality of S, it follows that there is some j (depending on a, b and ϵ) such that the set $T_n^{\circ j}(\{a\})$ intersect $\Omega_{b,\epsilon/2}$. However, since a was chosen to be an accumulation point, there are infinitely many points of S which are in $\Omega_{b,\epsilon}$.

□

pr:M_n **Proposition 29.** *For any exactly solvable operator T and sufficiently large n, the minimal set M_n^T is given by $\overline{G_n(\{z\})}$, where z is any of the roots of the eigenpolynomial p_n^T.*

Proof. From the proof of Theorem 16 and Theorem A we know that whenever n is sufficiently large, every closed minimal set contains the roots of the eigenvalues of p_n^T. This implies that $\mathrm{M}_n^T = \overline{G_n(\{z\})}$ where z is any root of p_n^T. □

3.5. **Asymptotics of M_n^T when $n \to \infty$.** In the above notation, the next result is an analog of [ABS, Thm. 2, Part (ii)], see Figures 1–2.

th:M_nConv **Theorem 30.** *For any non-degenerate exactly solvable operator T, the sequence of sets $\{\mathrm{M}_n^T\}$ converges in the Hausdorff metric to $Conv(Q_k)$ as $n \to \infty$.*

Proof. It is clear that both M_n^T and $\mathrm{M}_{\geq n}^T$ contain the roots of the eigenpolynomial of degree n. It follows that $\mathrm{M}_n^T \subseteq \mathrm{M}_{\geq n}^T$, by taking forward iterates

(TODO–FORMULATION).

By using the same argument as for $\mathrm{M}_{\geq n}^T$, one can show that for every $\epsilon > 0$, for sufficiently large n, the set M_n^T contains points in an ϵ-neighborhood of $Conv(Q_k)$. Finally, convexity as $n \to \infty$ follows as before. □

The fact that $M_\infty^T := \lim_{n\to\infty} M_n^T$ equals $Conv(Q_k)$ follows from Theorem B which can be found in [Sha10, Theorem 9].

lm:collapse **Theorem B** (See [Sha10, Theorem 9])**.** Given a non-degenerate operator T as in (1.1) and $\epsilon > 0$, there exists a positive integer n_ϵ such that for any $n > n_\epsilon$ and any polynomial p of degree n with all roots in $Conv(Q_k)$, all roots of $T(p)$ lie in the ϵ-neighborhood of $Conv(Q_k)$.

4. Hutchinson-invariant sets

sec:hutchinson

Definition 31. Given an operator T, a set S is said to be *H-invariant* of degree d with respect to T if the zeros of $T[(x-z)^d]$ lie in S whenever $z \in S$.

thm:hutchinson **Theorem 32.** *Suppose $f_1, \ldots, f_d$ are maps $f_j : \mathbb{C} \to \mathbb{C}$ of the form $f_j(z) = a_j z + b_j$, with $|a_j| < 1$ and let M be a closed compact set such that $M = \cup_j f_j(M)$. Then there is a unique exactly solvable non-degenerate operator T for which M is H-invariant in degree d.*

Proof. A classical result by Hutchinson [Hut81] states that the set M exists and is uniquely determined by the f_j's. Thus we only need to construct an operator T with the desired properties.

Consider the system of equations defined via the identity

$$T[(x-z)^d] = \prod_{j=1}^{d} (x - (a_j z + b_j)), \tag{4.1}$$

{eq:completelyFactoriz

where T is symbolically computed with unknown $Q_0, \ldots, Q_d$ such that Q_j is of degree j. We will call such a map *completely factorizable*. We can then find T explicitly by comparing coefficients of the form $x^r z^s$ — this is a system of linear equations, with the unknowns being the coefficients of the Q_i's.

For example, the leading coefficient of Q_d is given by $\frac{1}{d!}\prod_j(x-a_j)$.

From the construction, it is clear that the roots of $T[(x-z)^d]$ are given by $f_j(z)$, $1 \le j \le d$, and it follows that the set M is H-invariant of degree d with respect to T. □

Prove?

Conjecture 33. *Let T be completely factorizable of degree d as in* (4.1)*. Then the first few eigenvalues $\lambda_0, \dots, \lambda_d$ are given as*

$$\lambda_j = \frac{e_{d-j}(a_1, \dots, a_d)}{\binom{d}{j}}$$

where $e_k(y_1, \dots, y_d)$ is the kth elementary symmetric function in d variables.

ex:hutchinson **Example 34.** In the case $d = 2$, one can easily show that

$$T[(x-z)^2] = (x - (a_1 z + b_1))(x - (a_2 z + b_2))$$

for an exactly solvable, non-degenerate T precisely when Q_0, Q_1 and Q_2 are given as

$$\begin{aligned} Q_0 &= a_1 a_2 \\ Q_1 &= -\frac{1}{2}\left((2a_1a_2 - a_1 - a_2)x + a_1 b_2 + a_2 b_1\right) \\ Q_2 &= \frac{1}{2}\left((a_1-1)(a_2-1)x^2 + (a_1b_2 + a_2b_1 - b_1 - b_2)x + b_1 b_2\right). \end{aligned}$$

If we choose $a_1 = -9/4$, $a_2 = 3/8$, $b_1 = b_2 = 1$, we have that the first eigenvalues of T are

$$\lambda_0 = -27/32, \qquad \lambda_1 = -15/16, \qquad \lambda_2 = 1$$

and

$$T = \left(\frac{65x^2}{64} - \frac{31x}{16} + \frac{1}{2}\right)\frac{d^2}{dx^2} + \left(-\frac{3x}{32} + \frac{15}{16}\right)\frac{d}{dx} - \frac{27}{32}.$$

Since $|a_1| > 1$, it is easy to see that there cannot be any bounded H-invariant set of degree 2, as such a set S must contain $a_1 S + 1$ as a subset, which is impossible for bounded sets. This example shows that strictly different eigenvalues as in Theorem 16 cannot say anything about the boundedness of M_n^T.

Note that H-invariant sets include classical fractals such as the Sierpinski triangle, the Cantor set, the Lévy curve (see Example 35) and the Koch curve.

ex:levy **Example 35.** For the differential operator $T = x(x+1)\frac{d^2}{dx^2} + i\frac{d}{dx} + 2$, we have that H_2^T is a Lévy curve. It is straightforward to show that the roots of $T[(x-\alpha)^2]$ are given by

$$\frac{1+i}{2}\alpha \qquad \text{and} \qquad \frac{1-i}{2}(\alpha - i).$$

The two maps

$$\alpha \mapsto \frac{1+i}{2}\alpha \qquad \text{and} \qquad \alpha \mapsto \frac{1-i}{2}(\alpha - i)$$

are both contracting maps which together produce a fractal Lévy curve as shown in Fig. 4. Both maps are indeed contracting affine maps, so in particular, every member of $\mathcal{I}_2^T$ must contain the fixpoints of these maps. In particular, we can be sure that M_2^T is non-empty, as it must be a superset of H_2^T.

The following result illustrates a relationship with Theorem 10. It shows that in the case when T defines a classical IFS via affine maps as in Theorem 32, it has an invariant disk in degree n precisely when the f_j are given by attracting (Lipschitz) maps.

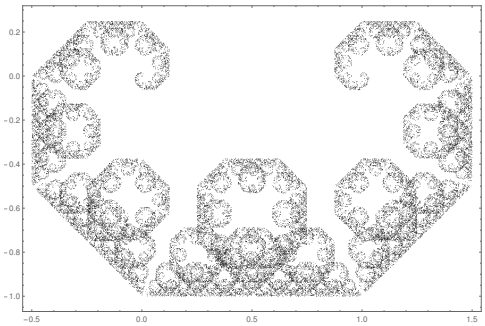

fig:levy

FIGURE 4. The Hutchinson-invariant set H_2^T, which is a Lévy curve.

thm:ifsVSinvariantDisk

Theorem 36. *Suppose $T[(x-z)^n] = \prod_{j=1}^n (x-(a_j z + b_j))$ for all $z \in \mathbb{C}$. This defines a non-degenerate operator $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$ with Fuchs index 0. For such T, we have that $P_0^n(x) = \prod_{j=1}^n (x + a_j)$.*

Proof. By linearity of T, we have that

$$G_T(x,y) = T[(1+xy)^n] = y^n T\left[\left(x + \frac{1}{y}\right)^n\right] \tag{4.2}$$

$$= y^n \prod_{j=1}^n \left(x + a_j y^{-1} - b_j\right) \tag{4.3}$$

where the latter is obtained by using $y = -1/z$ in the definition of T. Since $G_T(x,y) = \prod_j (xy + a_j - b_j y)$, it follows that $P_0^n(xy) = \prod_j (xy + a_j)$, which implies the assertion. □

Per: Add corollary, or concluding remark.

sec:julia

5. Differential operators and Julia sets

In this section, we completely characterize the geometry of M_1^T. Since we are only interested in the image of linear polynomials $x - \alpha$, the restriction of any differential operator T can be written as

$$T : (x-\alpha) \mapsto Q_1(x) + Q_0(x)(x-\alpha), \tag{5.1}$$

{eq:linearOperator}

where $Q_1(x)$ and $Q_0(x)$ are the coefficients at $\frac{d}{dx}$ and the constant term of T, respectively. Equivalently, T in (5.1) can also be described via the relation

$$T : (x-\alpha) \mapsto P_1(x) - \alpha P_0(x)$$

where $P_1(x) = Q_1(x) - xQ_0(x)$, and $P_0(x) = Q_0(x)$.

Definition 37. Given a rational function $f : \mathbb{C} \to \mathbb{C}$, its associated *Julia set* J_f may be defined as follows. First, we define the filled-in Julia set of the rational function f,

$$K_f := \{z \in \mathbb{C} : f^k(z) \text{ does not diverge to } \infty\}.$$

The Julia set J_f of f is the boundary of the filled-in Julia set, see [Fal04, Section 14.1].

lem:juliaProperties

Lemma 38 (See e.g. [Fal04, Bea00])**.** *The Julia set J_f is a completely invariant set under f, that is $J_f = f(J_f) = f^{-1}(J_f)$. Furthermore, the Julia set is the smallest, closed, completely invariant set containing at least three points.*

prop:juliaset

Proposition 39. *The unique infinite minimal invariant set* $\mathrm{M}_1^T \in \mathcal{I}_1^T$ *associated with the operator*

$$T : (x-\alpha) \mapsto P_1(x) - \alpha P_0(x) \tag{5.2}$$

{eq:juliaOperator}

coincide with the Julia set J_f*, where* $f(x) = P_1(x)/P_0(x)$*.*

Proof. First we show that $J_f \subseteq S$ for any infinite $S \in \mathcal{I}_1^T$.

Indeed take $\alpha \in S$. Then $T(x-\alpha) = P_1(x) - \alpha P_0(x)$, and therefore, all roots of $\alpha = P_1(x)/P_0(x)$ must belong to S. But this is equivalent to saying that $f^{-1}(\alpha) \subset S$, with f defined as above. By iterating this argument, we have that if $\alpha \in S$ then $\bigcup_j f^{-j}(\alpha) \subseteq S$. General theory of Julia sets then shows that J_f lie in the closure of $\bigcup_j f^{-j}(\alpha)$.

On the other hand, by Lemma 38, J_f is invariant under f^{-1}, so $J_f \in \mathcal{I}_1^T$. It then follows that J_f is unique minimal infinite closed set in $\mathcal{I}_1^T$. □

or:juliaSetEquivalence

Corollary 40. *For any differential operator* $T(p) = Q_1(x)p' + Q_0(x)p$*, the set* M_1^T *is a Julia set of the rational function* $f(x) := \frac{Q_1(x)+xQ_0(x)}{Q_0(x)}$*. Conversely, the Julia set of every univariate rational function can be realized as* M_1^T *for an appropriate linear differential operator* T*.*

woMinimalInvariantSets

Example 41. The differential operator $T = x(x-1)\frac{d}{dx} + 1$ admits *two* minimal sets for $n = 1$: either the one-point set $\{0\}$ or the unit circle. This is in line with Lemma 38 — some very special rational functions admit several completely invariant (Julia) sets with one or two points. The reason this case is so exceptional, is that the polynomial x is mapped to x^2, which has the same zeros as x. In general, such exceptional sets only show up in instances where there is some α, such that $x - \alpha$ is mapped to $c(x-\alpha)^k$ under T, see [Bea00].

The following proposition shows that invariant sets of higher degrees are also closely related to Julia sets.

prop:juliaInclusion

Proposition 42. *Let* $T = \sum_{j=0}^{k} Q_j(x)\frac{d}{dx}$ *be a differential operator and assume that* $S \in \mathcal{I}_n^T$*. Define polynomials* $P_0(\beta, x)$ *and* $P_1(\beta, x)$ *via the relation*

$$T((x-\beta)^{n-1}(x-\alpha)) = P_0(\beta, x) - \alpha P_1(\beta, x), \text{ and let } f_\beta(x) := \frac{P_0(\beta, x)}{P_1(\beta, x)}.$$

Then the Julia set J_β *determined by the rational function* f_β *is a subset of* S*, for all* $\beta \in S$*.*

Proof. We have that if $\alpha, \beta \in S$, then the roots of $P_0(\beta, x) - \alpha P_1(\beta, x)$ are in S, and hence the pre-image $f_\beta^{-1}(\alpha)$ must be a subset of S. It follows that the backward iterates

$$f_\beta^{-1} \circ f_\beta^{-1} \circ \cdots \circ f_\beta^{-1} \circ \alpha$$

are all subsets of S. However, it is also known [Bea00, Thm. 4.2.7] that (under very mild conditions on α)

$$J_\beta \subseteq \overline{\bigcup_{j=1}^{\infty} f_\beta^{\circ(-j)}(\alpha)}.$$

and it follows that $J_\beta \subseteq S$. □

In other words, every invariant set $S \in \mathcal{I}_n^T$ contains a family of Julia sets parametrized by $\beta \in S$.

The following lemma proves a different inclusion of Julia sets in minimal invariant sets.

lem:m1m2Inclusion

Lemma 43. *For the operators* $T_1 = Q_1(x)\frac{d}{dx} + Q_0(x)$ *and* $T_2 = \frac{1}{2}Q_1(x)\frac{d}{dx} + Q_0(x)$, *we have that* $\mathrm{H}_1^{T_1} = \mathrm{H}_2^{T_2}$ *and* $\mathrm{M}_1^{T_1} \subseteq \mathrm{M}_2^{T_2}$.

Proof. The operator T_2 acting on $(x-\alpha)^2$ is essentially the same as T_1 acting on $(x-\alpha)$.

Per : Elaborate a bit more.

□

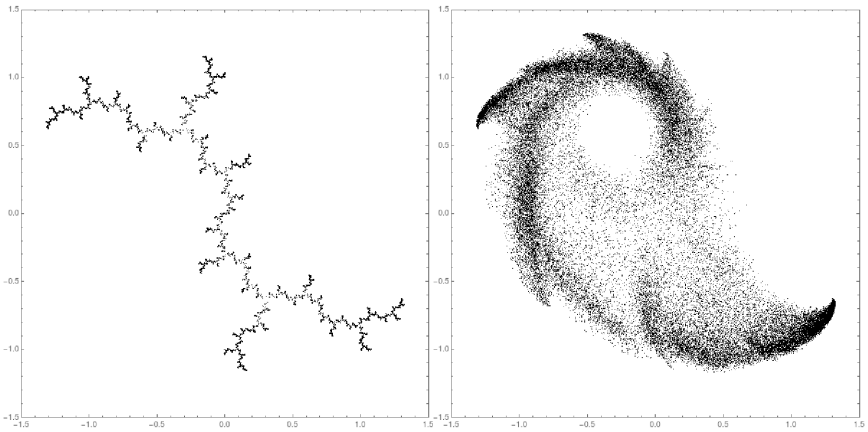

fig:julia (A) $\mathrm{M}_1^{T_1}$ fig:juliab (B) $\mathrm{M}_2^{T_2}$

fig:degreeTwoJulia FIGURE 5. The minimal sets $\mathrm{M}_1^{T_1}$ and $\mathrm{M}_2^{T_2}$.

sec:examples

6. Examples

We start this section with an example illustrating Lemma 43.

Example 44. The differential operator $T_1 = (x^2 - x + i)\frac{d}{dx} + 1$ gives the classical Julia set associated with $f(x) = x^2 + i$ as the unique minimal set $\mathrm{M}_1^{T_1}$, see Fig. 5a. For the differential operator $T_2 = \frac{1}{2}(x^2 - x + i)\frac{d}{dx} + 1$, we have that $\mathrm{M}_2^{T_2}$ is given by Fig. 5b. Note that $\mathrm{M}_1^{T_1}$ is a subset of $\mathrm{M}_2^{T_2}$ as implied by Lemma 43. The empty region in Fig. 5b is just an artefact of the computer algorithm used to generate the figure. The algorithm used is the one described in Section 3.4.

Example 45. Set

$$\delta = (x^2 - 1)\frac{d^2}{dx^2} + 2x\frac{d}{dx}.$$

The eigenpolynomials of δ are the Jacobi polynomials

$$\{P_n\}_{n=0}^{\infty} = \{1, \frac{3x^2 - 1}{2}, \dots\},$$

satisfying $\delta(P_n) = n(n+1)P_n$. The zeros of the Jacobi polynomials are located in $[-1, 1]$. Hence for any polynomial $Q(x) = \sum_{k=0}^{N} a_k x^k$, a finite order differential operator

$$T = \sum_{k=0}^{N} a_k \delta^k = Q(\delta),$$

has $\{P_n\}_{n=0}^{\infty}$ as eigenpolynomials with the eigenvalues $\{Q(n(n+1))\}_{n=0}^{\infty}$. Choose Q (and thus T) such that $Q(0) > Q(6) > 0$. For example, $Q(x) = 1 + (x+6)^2$. Then

$$T(3x^2/2) = T(P_2 + P_0/2) = Q(6)\frac{3x^2 - 1}{2} + Q(0)/2 = Q(6)\frac{3x^2}{2} + \frac{Q(0) - Q(6)}{2}$$

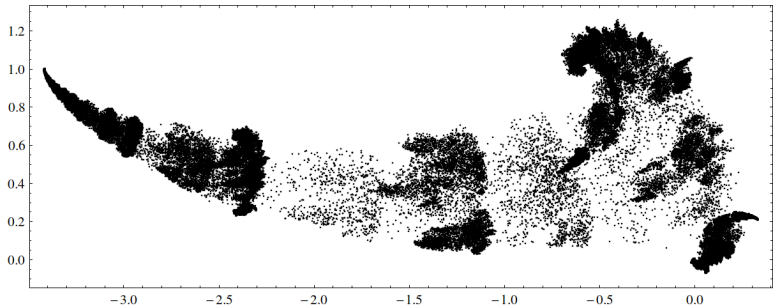

FIGURE 6. The minimal set M_2^T for the operator $(2x(x-i)(x+2)-x(x^2-1))\frac{d}{dx}+(x^2-1)$.

fig:degreeTwoJulia2

has non-real zeros. Thus T does not preserve the property of having all zeros in an interval, even though T has eigenpolynomials with all zeros in an interval.

Example 46. For the differential operator $T=(2x(x-i)(x+2)-x(x^2-1))\frac{d}{dx}+(x^2-1)$, we have that M_2^T is given by Fig. 6.

sec:final

7. Final Remarks

The present paper is just the very first step in the study of different types of invariant sets for exactly solvable operators. We just barely scratched the surface of this topic leaving dozens of natural questions untouched. Here is a very small sample of those.

Problem 47. *What can be said about the fractal properties of* M_n^T*?*

Problem 48. *What can be said about the compactness of* M_n^T*?*

Problem 49. *Is it true that* $Conv(\mathrm{M}_n^T)=\mathrm{M}_{\geq n}^T$*? Clearly,* $\mathrm{M}_n^T\subseteq\mathrm{M}_{\geq n}^T$.

Problem 50. *For* T *exactly solvable, when is* $\cap_n\mathrm{M}_n^T$ *non-empty?*

Problem 51. *Can one generalise the Hutchinson method to the case when the map* $T[(x-z)]^k$ *is not completely factorizable? What is a natural analog of contractibility of branches in the latter case?*

8. Appendix. Background on iterated function systems

An iterated function system (IFS), is a finite set of maps, in our case, $\mathbb{C}\to\mathbb{C}$. The most well-studied class is the set *of affine maps*, of the form $f(z)=az+b$ for constants a, b. It is clear that such a map has a *unique attracting fixed point* if $|a|<1$, in which case f is *contractive*. In other words, the iteration $f(z_0)$, $f(f(z_0))$ converges to the unique solution of $z=az+b$, independent of the starting point z_0.

In [Hut81], Hutchinson proved that if $f_1,\dots,f_n$ are such contractive affine maps, there is a unique closed non-empty *invariant set* S with the property that

$$S=\bigcup_{j=1}^{n}f_j(S).$$

Such sets S are usually fractal in nature, and classical examples of such sets are the Sierpinksi triangle, Barnsleys fern, the Koch curve, the Koch snowflake and so on.

The problem of an operator T preserving roots of $(x-z)^n$ for $z\in S$ can be phrased as follows:

We seek the smallest closed set S, such that if $z\in S$, then the roots of $P(x,z)=0$ are also in S, where P is a bivariate polynomial of degree n in x. Let $\rho_1(z),\dots,\rho_n(z)$

denote branches of the solutions of $P(x,z)=0$. Hence, we seek a set S with the property that

$$S=\bigcup_{j=1}^{n}\rho_j(S).$$

The problem of finding the minimal set S such that T preserves the roots of *all* polynomials with roots in S is translated as follows: Let $\mathbf{z}=(z_1,\dots,z_n)$ be a vector in $\mathbb{C}^n$, and define $P(x,\mathbf{z}) \coloneqq T[(x-z_1)\cdots(x-z_n)]$. Similar to previous setting, let $\rho_1(\mathbf{z}),\dots,\rho_n(\mathbf{z})$ denote branches of the solutions of $P(x,\mathbf{z})=0$. A minimal invariant set S then has the property that

$$S=\bigcup_{j=1}^{n}\rho_j(S,S,\dots,S).$$

Our setting therefore generalize the classical IFS literature in two ways — the maps we are considering take several inputs, and are in general not explicitly given.

Special cases of our setting include both the Julia setting, Möbius maps (studied in [MSW02]) and the affine IFS setting on $\mathbb{C}$ (when the maps are conformal).

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