

# INERTIA MINIMIZERS IN THE INVERSE MOMENT PROBLEM IN LOGARITHMIC POTENTIAL THEORY

YURII BURMAN AND BORIS SHAPIRO

ABSTRACT. Given an arbitrary probability measure  $\mu$  compactly supported in  $\mathbb{C}$ , we prove the existence and uniqueness of a probability measure whose moment of inertia with respect to the centre of mass of  $\mu$  is minimal within the class of probability measures having the same sequence of harmonic moments as  $\mu$ . We study the properties of the support of the latter inertia minimizer.

## 1. INTRODUCTION

**1.1. Crash course on the inverse moment problem in logarithmic potential theory.** (For the basic notions in the logarithmic potential theory in the complex plane consult e.g., [15].)

Let  $\mathcal{M}$  be the space of complex-valued Borel measures in  $\mathbb{R}^2 \simeq \mathbb{C}$  with compact support. For any  $\mu \in \mathcal{M}$ , denote by

$$\mathcal{U}_\mu(z) \stackrel{\text{def}}{=} \int_{\mathbb{C}} \ln |z - \xi| \mu(\xi)$$

its *logarithmic potential*, and by

$$\mathcal{C}_\mu(z) \stackrel{\text{def}}{=} \int_{\mathbb{C}} \frac{\mu(\xi)}{z - \xi} = \frac{\partial \mathcal{U}_\mu(z)}{\partial z},$$

its *Cauchy transform*. The Cauchy transform has the Taylor expansion at  $\infty$  of the form

$$\mathcal{C}_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \dots, \quad (1.1)$$

where

$$m_j(\mu) = \int_{\mathbb{C}} z^j \mu(z) \quad (1.2)$$

is the  $j$ -th *harmonic moment* of  $\mu$ .

We say that two positive measures  $\mu_1$  and  $\mu_2$  with compact supports in  $\mathbb{C}$  are *equipotential at  $\infty$*  if their logarithmic potentials coincide in some neighborhood of  $\infty \in \bar{\mathbb{C}}$ , where  $\bar{\mathbb{C}} := \mathbb{C} \cup \infty$  is the Riemann sphere. For equipotential measures  $\mu_1$  and  $\mu_2$ , if  $\text{supp } \mu_2 \subset \text{supp } \mu_1$ , then we say that  $\mu_1$  is obtained from  $\mu_2$  by a *balayage* (or, equivalently, that  $\mu_2$  is obtained from  $\mu_1$  by the *inverse balayage*.) If  $\mu_1$  and  $\mu_2$  are equipotential and  $\text{supp } \mu_2$  is contained in the convex hull of  $\text{supp } \mu_1$  then we say that  $\mu_2$  is *subordinated* to  $\mu_1$  (or, equivalently, that  $\mu_1$  *dominates*  $\mu_2$ ).

We call a holomorphic function  $f$  defined in a neighbourhood of  $\infty \in \bar{\mathbb{C}}$  *acceptable* if  $f(z) = a_0/z + o(1/z)$  as  $z \rightarrow \infty$ , for some  $a_0 > 0$ . Denote by  $\mathcal{M}_f$  the set of positive

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compactly supported measures on  $\mathbb{C}$  such that  $\mathcal{C}_\mu = f$  near  $\infty$ . Alternatively, a sequence of complex numbers  $\mathcal{A} = \{a_i\}_{i=0}^\infty$  is called *acceptable* if  $a_0 > 0$  and the series  $\sum_{i=0}^\infty a_i u^i$  has a positive convergence radius. For an acceptable sequence  $\mathcal{A}$ , denote by  $\mathcal{M}_\mathcal{A}$  the set of positive compactly supported measures  $\mu$  such that  $m_j(\mu) = a_j$ ,  $j = 0, 1, \dots$ . It follows from (1.1) and (1.2) that  $\mathcal{M}_\mathcal{A} = \mathcal{M}_f$  where  $f(z) \stackrel{\text{def}}{=} a_0/z + a_1/z^2 + \dots$ . By Theorem 1 of [2],  $\mathcal{M}_f$  is nonempty for any acceptable function  $f$  or, equivalently,  $\mathcal{M}_\mathcal{A}$  is nonempty for any acceptable sequence  $\mathcal{A}$ .

A typical inverse moment problem in the logarithmic potential theory asks to recover a measure belonging to a certain class from the information about its sequence of harmonic moments, or, equivalently, from the germ of its Cauchy transform at  $\infty$ . Inverse problems in the logarithmic potential theory have attracted substantial attention since the publication of the fundamental paper [13], in which P. S. Novikov proved that Lebesgue measure restricted to two different star-shaped (in particular, convex) domains in  $\Omega_1, \Omega_2 \subset \mathbb{C}$  cannot have the same logarithmic potential near  $\infty$  or, equivalently, the same sequence of harmonic moments. Mention also a fundamental formula by S. Natanzon [12] expressing a holomorphic diffeomorphism  $f : \Omega \rightarrow \{z \in \mathbb{C} \mid |z| < 1\}$ , for a star-shaped  $\Omega \subset \mathbb{C}$ , via harmonic moments of the Lebesgue measure of  $\Omega$ . ???

If  $\Omega$  is not star-shaped, then the uniqueness of its logarithmic potential near  $\infty$  is no longer true. Fig. 1 below shows an example of two polygons with the same logarithmic potential near  $\infty$ ; see [4, p. 333]. The problem of the uniqueness of the restriction of the Lebesgue measure to general polygons as well as to domains bounded by lemniscates has attracted a substantial attention in the literature. Several authors have also considered the class of measures whose density with respect to the Lebesgue measure is a bivariate polynomial.

*Example 1.* Consider two 6-tuples  $T = \{\pm\sqrt{3} \pm I, \pm 2I\}$  and  $T' = \{\pm \frac{1 \pm \sqrt{3}I}{2}, \pm 1\}$ . Let  $F \subset \mathbb{C}$  be the difference of the convex hull of  $T$  and the union of the set of 6 triangles obtained as the orbit of the triangle with nodes  $(\sqrt{3} + I, \sqrt{3} - I, 1)$  under the rotation by  $\frac{\pi}{3}$ , see Fig. 1. Let  $F' \subset \mathbb{C}$  be the difference of the convex hulls of  $T$  and of  $T'$ . Then  $F$  and  $F'$  have the same logarithmic potential.

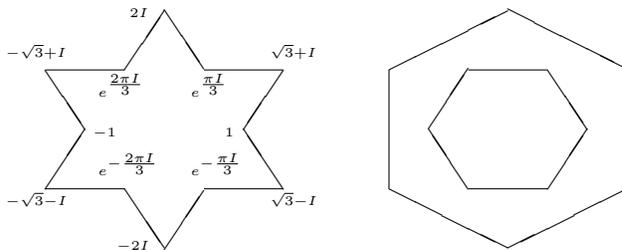


FIGURE 1. Two equipotential polygons:  $F$  on the left,  $F'$  on the right.

The main objective of the present paper is as follows. It is often desirable in the set of all pairwise equipotential and compactly supported probability measures to find the one whose support is minimal in an appropriate sense. This idea is closely related to the notion of a mother body of a given positive measure as pioneered by

D. Zidarov [16] and in several situations mathematically formalized by B. Gustafsson [9, 8]. To the best of our knowledge, there is currently no consensus of how a mother body should be defined, and its existence is unknown in many cases. Below we suggest an appropriate notion not for an individual measure, but for the whole class of compactly supported positive measures which are equipotential to each other. Our present setup is somewhat similar to that of [2].

Observe that the difference of two compactly supported measures which are equipotential at  $\infty$  is a compactly supported signed measure with all vanishing harmonic moments. Such measures typically form an infinite-dimensional linear space. Many examples of such measures can be found in [14].

## 1.2. Inertia minimizers.

**Definition 1.** Given a  $\mathbb{C}$ -valued measure  $\nu$  compactly supported in  $\mathbb{C}$ , define its *moment of inertia* as

$$J(\mu) \stackrel{\text{def}}{=} \int_{\mathbb{C}} |z|^2 d\mu(z).$$

Let  $\Omega \subset \mathbb{C}$  be an open set such that the closure  $\bar{\Omega} \subset \mathbb{C}$  is compact and has connected piecewise smooth boundary  $\partial\Omega$ . Let  $f$  be an acceptable holomorphic function on  $\bar{\mathbb{C}} \setminus \bar{\Omega}$ . We are going to consider an optimization problem

$$\begin{aligned} J(\mu) &\rightarrow \min \\ \mu &\geq 0; \\ \mathcal{C}_\mu &= f \quad \text{in } \bar{\mathbb{C}} \setminus \bar{\Omega}. \end{aligned} \tag{1.3}$$

Denote by  $\mathfrak{M}_f(\Omega)$  the set of  $\mathbb{R}$ -valued measures  $\mu$  with the support  $\text{supp } \mu \subset \bar{\Omega}$  such that conditions (1.3) hold. The optimization problem in question is well-posed by Theorem 1 below.

The solution  $\mu_f$  of the problem (1.3) will be called the *inertia minimizer* corresponding to the acceptable function  $f$  (or the corresponding acceptable sequence of moments), and its moment of inertia  $J(\mu_f)$  will be denoted  $J_f$  and called the *minimal moment of inertia* of  $f$ .

WE HOPE TO PROVE THAT FOR A GENERIC ACCEPTABLE SEQUENCE the inertia minimizer exists and is unique. Plus its support has many nice properties.

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## 2. MAIN RESULTS

Let  $\Omega \subset \mathbb{C}$  be an open set such that the closure  $\bar{\Omega} \subset \mathbb{C}$  is compact and has connected piecewise smooth boundary  $\partial\Omega$ . Let  $f$  be an acceptable holomorphic function on  $\bar{\mathbb{C}} \setminus \bar{\Omega}$ .

**Theorem 1.** *For any acceptable function  $f$ , the functional  $J$  achieves its minimum value on the set  $\mathfrak{M}_f(\Omega)$ .*

*Draft of the proof.* The set of  $\mathbb{R}$ -valued measures supported on  $\overline{\Omega}$  coincides with the dual space  $C^*(\overline{\Omega})$ , where  $C(\overline{\Omega})$  is the Banach space of continuous functions  $\overline{\Omega} \rightarrow \mathbb{R}$ .

If  $\mu \in \mathfrak{M}_f(\Omega)$  then  $\mu \geq 0$  and therefore  $\int_{\overline{\Omega}} |\mu| = \int_{\overline{\Omega}} \mu = a_0$ . So  $\mathfrak{M}_f(\Omega)$  is the intersection of the ball  $B_{a_0}(0) \subset C^*(\overline{\Omega})$  with the linear subspace  $\{\mu \mid \mathcal{C}_\mu = f\}$  and the cone  $\{\mu \mid \mu \geq 0\}$ . By the Banach–Alaoglu theorem, the ball is \*-weakly compact; the subspace and the cone are clearly \*-weakly closed. Hence  $\mathfrak{M}_f(\Omega)$  is \*-weakly compact. The functional  $J$  is \*-weakly continuous, so it achieves its minimum on  $\mathfrak{M}_f(\Omega)$  somewhere.  $\square$

**2.1. Dual operator and the explicit minimum for real rational Cauchy transform with positive residues.** Let  $\mathcal{H}_0^{\text{ext}}(\Omega)$  be the space of holomorphic functions  $f : \overline{\mathbb{C}} \setminus \overline{\Omega} \rightarrow \mathbb{C}$  such that

- (1)  $f$  can be continuously extended to the boundary  $\partial(\mathbb{C} \setminus \overline{\Omega}) = \partial\Omega$ ,
- (2)  $f(\infty) = 0$ .

The Cauchy transform is a continuous linear operator  $C^*(\overline{\Omega}) \rightarrow \mathcal{H}_0^{\text{ext}}(\Omega)$ . Consider now the space  $C(\partial\Omega)^\mathbb{C} \stackrel{\text{def}}{=} C(\partial\Omega) \otimes \mathbb{C}$  of  $\mathbb{C}$ -valued continuous functions on the boundary  $\partial\Omega$  and let  $\iota : \mathcal{H}_0^{\text{ext}}(\Omega) \rightarrow C(\partial\Omega)^\mathbb{C}$  be the restriction operator. By assumption,  $\partial\Omega = \gamma(S^1)$  for some continuous  $\gamma : S^1 \rightarrow \mathbb{C}$ , so the  $C(\partial\Omega)^\mathbb{C}$  is isomorphic to the codimension 1 subspace  $\{f \in C[0,1] \otimes \mathbb{C} \mid f(0) = f(1)\} \subset C[0,1] \otimes \mathbb{C}$ . The image of  $\iota$  is actually a (complex) codimension 1 subspace  $\mathcal{I} \subset C(\partial\Omega)^\mathbb{C}$ , due to condition 2 above.

The set  $\overline{\mathbb{C}} \setminus \overline{\Omega}$  is homeomorphic to a disk and therefore to  $\Omega$ . By the uniformization theorem there exists a holomorphic diffeomorphism  $D : \Omega \rightarrow \overline{\mathbb{C}} \setminus \overline{\Omega}$ . It can be extended to some diffeomorphism of the common boundary  $\partial\Omega$  of the two sets. In other words, there exists a diffeomorphism  $\varrho : S^1 \rightarrow S^1$  such that  $D(\gamma(\varrho(t))) = \gamma(t)$  for all  $t \in S^1$ . (This notion is related to fingerprints !!!)

Given  $f \in \mathcal{H}_0^{\text{ext}}(\Omega)$ , observe that  $g \stackrel{\text{def}}{=} f \circ D$  is a holomorphic function  $g : \Omega \rightarrow \mathbb{C}$  extendable to a continuous function  $g(\gamma(t)) = f(\gamma(\varrho(t)))$  at the boundary  $\partial\Omega$  and such that  $g(a) = 0$  where  $a \stackrel{\text{def}}{=} D(\infty)$ . Using Cauchy's formula, we get

$$0 = \int_0^1 \frac{g(\gamma(t))\gamma'(t)}{\gamma(t) - a} dt = \int_0^1 \frac{f(\gamma(\varrho^{-1}(t)))\gamma'(t)}{\gamma(t) - a} dt.$$

Changing the variable  $t = \varrho(s)$  gives  $0 = \int_0^1 \frac{f(\gamma(s))\gamma'(\varrho(s))\varrho'(s)}{\gamma(\varrho(s)) - a} dt$ . This equation on  $f$  gives an explicit description of the image  $\mathcal{I} \stackrel{\text{def}}{=} \iota(\mathcal{H}_0^{\text{ext}}(\Omega)) \subset C(\partial\Omega)^\mathbb{C}$ .

Let  $A \stackrel{\text{def}}{=} \iota \circ C : C^*(\overline{\Omega}) \rightarrow \mathcal{I}$ . The conjugate operator is  $A^* : \mathcal{I}^* \rightarrow C^{**}(\overline{\Omega})$ ; we will show later that its image actually lies in  $C(\overline{\Omega}) \subset C^{**}(\overline{\Omega})$ .

$\mathcal{I}^*$  is the conjugate to the codimension 1 space  $\mathcal{I} \subset C(\partial\Omega) \otimes \mathbb{C}$ , and therefore it is a quotient space of  $C^*(\partial\Omega)^\mathbb{C}$  by a 1-dimensional space  $V$ .  $C^*(\partial\Omega)^\mathbb{C}$  itself is the space of  $\mathbb{C}$ -valued measures on  $\partial\Omega$ ; the small print above shows that the subspace  $V$  is generated by the measure having the density  $\gamma'(\varrho(t))\varrho'(t)/(\gamma(\varrho(t)) - a)$  at the point  $\gamma(t) \in \partial\Omega$ . A measure  $\mu \in C^*(\partial\Omega)^\mathbb{C}$  acts on a continuous function  $f : \partial\Omega \rightarrow \mathbb{C}$  as  $\langle \mu, f \rangle = \text{Re} \int_{\partial\Omega} f(z)\overline{\mu}(z)$  (bar meaning complex conjugation). Hence one has for any ( $\mathbb{R}$ -valued) measure  $\nu \in C^*(\partial\Omega)$

$$\langle A^*\mu, \nu \rangle = \langle \mu, \iota(\mathcal{C}_\nu) \rangle = \text{Re} \int_{\partial\Omega} \overline{\mu(z)} \int_{\overline{\Omega}} \frac{\nu(\xi)}{z - \xi} = \int_{\overline{\Omega}} \nu(\xi) \text{Re} \int_{\partial\Omega} \frac{\overline{\mu(z)}}{z - \xi},$$

(because  $\nu$  is real-valued), so

$$A^*(\mu)(\xi) = \operatorname{Re} \int_{\partial\Omega} \frac{\overline{\mu(z)}}{z - \xi} = \operatorname{Re} \mathcal{N}_\mu(\xi).$$

where  $\mathcal{N}_\mu \stackrel{\text{def}}{=} -\mathcal{C}_{\overline{\mu}}$ . For a measure  $\mu \in C^*(\partial\Omega)^\mathbb{C}$  the function  $\mathcal{N}_\mu(\xi)$  is holomorphic outside the support of  $\overline{\mu}$ , in particular, it is holomorphic in  $\Omega$ . If  $\mu$  has continuous density in  $\partial\Omega$  then  $\overline{\mu(z)} = 2\pi i \lim_{\xi \rightarrow z} \mathcal{N}_\mu(\xi)$ ; if  $\mu$  has an atom at  $a \in \partial\Omega$  then  $\mathcal{N}_\mu(\xi) = O(1/(\xi - a))$  as  $\xi \rightarrow a$ . In other words, the operator  $\mathcal{N}$  is an isomorphism between  $C^*(\partial\Omega)^\mathbb{C}$  and the space  $\mathcal{H}_0^{\text{int}}(\Omega)$  of holomorphic functions on  $\Omega$  that can be extended to a continuous function  $\partial\Omega \rightarrow \mathbb{C} \cup \{\infty\}$ . **[this is not strict, and the definition of  $\mathcal{H}_0^{\text{int}}(\Omega)$  has to be made precise. Actually we are to prove that the optimal measure has no atoms on the boundary.]** The inverse isomorphism is done by the normalized conjugated restriction operator  $\iota^{\text{int}}(g)(z) = -\frac{1}{2\pi i} \lim_{\xi \rightarrow z} \overline{g(\xi)}$ ,  $z \in \partial\Omega$ .

Thus,  $A^*$  can be interpreted as an operator  $\mathcal{H}_0^{\text{int}}(\Omega) \rightarrow C^{**}(\overline{\Omega})$  sending a holomorphic function  $g$  to its real part. This real part is a harmonic, hence smooth, function, so the image of  $A^*$  lies in  $C(\overline{\Omega})$  **[again, if we prove that  $\mu$  has no atoms on the boundary]**.

Now it is possible to formulate a dual problem to (1.3): let  $\mu = \iota^{\text{int}}(g)$  where  $\mu \in C^*(\partial\Omega)^\mathbb{C}$  and  $g \in \mathcal{H}_0^{\text{int}}(\Omega)$ ; then  $(A^*\mu)(\xi) = \operatorname{Re}(2\pi i \overline{g(\xi)}) = -2\pi \operatorname{Im} g(\xi)$ . Thus, the dual problem looks like:

$$-\operatorname{Im} g(\xi) \leq |\xi|^2 / (2\pi) \quad \forall \xi \in \overline{\Omega}, \quad (2.1)$$

$$\langle f, g \rangle = \operatorname{Re} \int_{\partial\Omega} f(z) \overline{g(z)} dz \rightarrow \max.$$

Denote  $R(\Omega) \subset \mathcal{H}_0^{\text{int}}(\Omega)$  the set of functions such that (2.1) holds. Then by Kantorovich duality theorem [?] one has  $\langle f, g \rangle \leq J(\mu)$  for every measure  $\mu \in \mathfrak{M}_f(\Omega)$  (that is, if  $\operatorname{supp} \mu \subset \overline{\Omega}$ ,  $\mathcal{C}_\mu = f$  and  $\mu \geq 0$ ) and any  $g \in R(\Omega)$ . Also,

$$\inf_{\mu \in \mathfrak{M}_f(\Omega)} J(\mu) = \sup_{g \in R(\Omega)} \langle f, g \rangle,$$

and, by the complementary slackness condition, if  $\mu_f$  is the inertia minimizer (whose existence is guaranteed by Theorem 1) and  $g$ , the solution of the dual problem **[why does it exist?]** then  $\int_{\overline{\Omega}} (|\xi|^2 / (2\pi) + \operatorname{Im} g(\xi)) \mu_f(\xi) = 0$ , so that  $\operatorname{supp} \mu_f \subset \{z \in \overline{\Omega} \mid \operatorname{Im} g(z) = -|z|^2 / (2\pi)\}$ .

**Corollary.** *The support  $\operatorname{supp} \mu_f$  of the inertia minimizer has no internal points.*

*Draft of the proof.* Let  $g$  be a solution of the dual problem, and  $h(x, y) \stackrel{\text{def}}{=} -\operatorname{Im} g(x + iy)$ . Then  $h$  is harmonic. If  $(x, y)$  is an internal point of  $\operatorname{supp} \mu_f$  then it is an internal point of the set  $\{(x, y) \mid h(x, y) = (x^2 + y^2) / (2\pi)\}$  as well, and therefore  $0 = \Delta h(x, y) = \Delta((x^2 + y^2) / (2\pi)) = \frac{2}{\pi}$ .  $\square$

**Conjectural corollary.** *For any acceptable function  $f$  the measure  $\mu_f$  enjoys the following properties:*

- (1)  *$\operatorname{supp} \mu_f$  consists of finitely many compact real semi-analytic curves and points; in particular, it has zero Lebesgue measure.*
- (2) *the complement  $\mathbb{C} \setminus \operatorname{supp} \mu_f$  is connected.*

- (3)  $\text{supp } \mu_f$  must contain all singularities of the complete analytic continuation of the power series  $f$  (originally defined in the vicinity of  $\infty$ ).

*Draft of the proof.* Property 1 may follow from the fact that singularities of a harmonic function cannot be too wild. Is Property 2 indeed true? (recall that  $\mu_f$  may depend on  $\Omega$ !) How do we prove Property 3?  $\square$

*Example 2.* Let  $f = \sum_{j=1}^n \frac{a_j}{z-p_j}$  where  $a_j \geq 0$  and  $p_j \in \mathbb{R} \cap \Omega$  for all  $j$ . Then the measure  $\mu \stackrel{\text{def}}{=} \sum_{j=1}^n a_j \delta_{p_j}$  is real, positive, and  $\mathcal{C}_\mu(\xi) = f(\xi)$ , that is,  $\mu \in \mathcal{M}_f$ . One has  $J(\mu) = \sum_{j=1}^n a_j p_j^2$ .

For any  $g \in \mathcal{H}_0^{\text{int}}(\Omega)$  one has  $\langle f, g \rangle = \sum_{j=1}^n a_j \text{Re}(2\pi i g(\overline{p_j})) = -2\pi \sum_{j=1}^n a_j \text{Im} g(p_j)$ .

Take  $g(\xi) \stackrel{\text{def}}{=} -i\xi^2/(2\pi)$ . One has  $g \in \mathcal{H}_0^{\text{int}}(\Omega)$  for any domain  $\Omega$  containing all  $p_j$ , so  $\langle f, g \rangle = \sum_{j=1}^n a_j p_j^2 = J(\mu)$ . Also  $-\text{Im} g(\xi) \leq |g(\xi)| = |\xi|^2/(2\pi)$ . By the Kantorovich inequality,  $\mu$  is the solution of the optimization problem (1.3), and  $g$ , a solution of the dual problem (2.1).

In a similar way one can solve problem (1.3) for  $f = \sum_{j=1}^n \frac{a_j}{z-p_j}$  where  $a_j \geq 0$  and  $p_1, \dots, p_n \in \Omega$  all lie on one (real) line passing through the origin; here we should take  $g(\xi) = e^{i\varphi} \xi^2/(2\pi)$  with an appropriate  $\varphi$  depending on the line; the optimal measure here is again  $\mu = \sum_{j=1}^n a_j \delta_{p_j}$ . We can conjecture that the optimal measure is the same for any  $f = \sum_{j=1}^n \frac{a_j}{z-p_j}$  with  $a_j \geq 0$  and any  $p_1, \dots, p_n$ , but we do not have a proof for this yet.

### 3. EXAMPLE: CAUCHY TRANSFORM OF THE LEBESGUE MEASURE OF A POLYGON

Consider a sequence of moments  $a_n = \int_{\mathbb{C}} z^n \mu(z)$  of a measure  $\mu$  with a compact support on  $\mathbb{C} = \mathbb{R}^2$ . Define the power series

$$\Psi_\mu(z) = \sum_{n=0}^{\infty} \binom{n+2}{2} a_n z^n.$$

Recall that the Cauchy transform

$$\mathcal{C}_\mu(z) = \int_{\mathbb{C}} \frac{\mu(\xi)}{z-\xi} = \frac{1}{z} \int_{\mathbb{C}} \frac{\mu(\xi)}{1-\xi/z} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{\mathbb{C}} \xi^n \mu(\xi) = \sum_{n=0}^{\infty} a_n z^{-(n+1)};$$

therefore

$$\Psi_\mu(z) = \frac{d^2}{dz^2} (z \mathcal{C}_\mu(1/z)). \quad (3.1)$$

**Theorem 2** ([?]). *Let  $\mu$  be a Lebesgue measure of a triangle  $\Delta \subset \mathbb{C}$  with the vertices  $z_1, z_2, z_3$ . Then*

$$\Psi_\mu(u) = \frac{S_\Delta}{(1-z_1 u)(1-z_2 u)(1-z_3 u)},$$

where  $S_\Delta$  is the area of the triangle.

Now one has (let's omit  $\mu$  in indexes):

$$\begin{aligned} \Psi(z) &= \frac{\alpha_1}{1-z_1 u} + \frac{\alpha_2}{1-z_2 u} + \frac{\alpha_3}{1-z_3 u} \iff \\ S_\Delta &= \alpha_1(1-z_2 u)(1-z_3 u) + \alpha_2(1-z_1 u)(1-z_3 u) + \alpha_3(1-z_1 u)(1-z_2 u). \end{aligned}$$

Substitution  $u = 1/z_1$  gives  $\alpha_1 = \frac{z_1^2 S_\Delta}{(z_1 - z_2)(z_1 - z_3)}$ ; similarly for  $\alpha_2, \alpha_3$ , so that

$$\Psi(u) = \frac{z_1^2 S_\Delta}{(z_1 - z_2)(z_1 - z_3)} \frac{1}{1 - z_1 u} + \frac{z_2^2 S_\Delta}{(z_2 - z_1)(z_2 - z_3)} \frac{1}{1 - z_2 u} + \frac{z_3^2 S_\Delta}{(z_3 - z_1)(z_3 - z_2)} \frac{1}{1 - z_3 u}. \quad (3.2)$$

One has  $C(u) = a_0/u + o(1/u)$  as  $u \rightarrow \infty$ , so the Taylor series at  $u = 0$  for  $uC(1/u)$  starts with  $u^2$ . It follows from (3.1) and (3.2) that

$$\begin{aligned} \frac{d}{du}(uC(1/u)) &= -\frac{z_1 S_\Delta}{(z_1 - z_2)(z_1 - z_3)} \ln(1 - z_1 u) - \frac{z_2 S_\Delta}{(z_2 - z_1)(z_2 - z_3)} \ln(1 - z_2 u) \\ &\quad - \frac{z_3 S_\Delta}{(z_3 - z_1)(z_3 - z_2)} \ln(1 - z_3 u) \implies \end{aligned}$$

$$\begin{aligned} uC(1/u) &= ((z_1 u - 1) \ln(1 - z_1 u) - z_1 u) \frac{S_\Delta}{(z_1 - z_2)(z_1 - z_3)} \\ &\quad + ((z_2 u - 1) \ln(1 - z_2 u) - z_2 u) \frac{S_\Delta}{(z_2 - z_1)(z_2 - z_3)} \\ &\quad + ((z_3 u - 1) \ln(1 - z_3 u) - z_3 u) \frac{S_\Delta}{(z_3 - z_1)(z_3 - z_2)} \\ &= S_\Delta \left( \frac{(z_1 u - 1)}{(z_1 - z_2)(z_1 - z_3)} \ln(1 - z_1 u) + \frac{(z_2 u - 1)}{(z_2 - z_1)(z_2 - z_3)} \ln(1 - z_2 u) \right. \\ &\quad \left. + \frac{(z_3 u - 1)}{(z_3 - z_1)(z_3 - z_2)} \ln(1 - z_3 u) \right) \implies \\ C(u) &= S_\Delta \left( \frac{(z_1 - u) \ln(u - z_1)}{(z_1 - z_2)(z_1 - z_3)} + \frac{(z_2 - u) \ln(u - z_2)}{(z_2 - z_1)(z_2 - z_3)} + \frac{(z_3 - u) \ln(u - z_3)}{(z_3 - z_1)(z_3 - z_2)} \right) \end{aligned}$$

If the triangle with the vertices  $z_1, z_2, z_3$  is right-oriented then  $S_\Delta = \frac{1}{2} \operatorname{Im}(\overline{(z_1 - z_2)}(z_1 - z_3)) = -\frac{i}{4}(\overline{(z_1 - z_2)}(z_1 - z_3) - (z_1 - z_2)\overline{(z_1 - z_3)})$ . Similar formulas with the indices 1, 2, 3 cyclically shifted also hold; so one has

$$\begin{aligned} C(u) &= \frac{i}{4} \left( \left( \frac{z_1 - z_3}{z_1 - z_3} - \frac{z_1 - z_2}{z_1 - z_2} \right) (z_1 - u) \ln(u - z_1) \right. \\ &\quad + \left( \frac{z_2 - z_1}{z_2 - z_1} - \frac{z_2 - z_3}{z_2 - z_3} \right) (z_2 - u) \ln(u - z_2) \\ &\quad \left. + \left( \frac{z_3 - z_2}{z_3 - z_2} - \frac{z_3 - z_1}{z_3 - z_1} \right) (z_3 - u) \ln(u - z_3) \right). \quad (3.3) \end{aligned}$$

Let now  $\Pi \subset \mathbb{C}$  be a  $n$ -gon with the vertices  $z_1, \dots, z_n \in \mathbb{C}$  listed counterclockwise, and let  $\mu$  be its Lebesgue measure. Denote  $v_j \stackrel{\text{def}}{=} \frac{z_{j+1} - z_j}{|z_{j+1} - z_j|}$ ,  $j = 1, \dots, n \bmod n$  (a unit vector parallel to the  $j$ -th side of  $\Pi$ ). Then

**Theorem 3.**  $C_\mu(u) = \frac{i}{4} \sum_{j=1}^n (\bar{v}_j^2 - \bar{v}_{j-1}^2) (z_j - u) \ln(u - z_j)$ .

*Draft of the proof.* For a triangle ( $n = 3$ ) this formula coincides with (3.3). For any  $\Pi$  let us draw a set of diagonals cutting  $\Pi$  into triangles  $\Delta_1, \dots, \Delta_{n-2}$ . Since  $\mu = \mu_1 + \dots + \mu_{n-2}$  where  $\mu_i$  is the Lebesgue measure of  $\Delta_i$  we can apply (3.3) to each of the  $\mu_j$ . We get a sum over the vertices of  $\Pi$ , the coefficient at the  $j$ -th vertex containing  $\bar{v}_j^2$  and  $-\bar{v}_{j-1}^2$ . If  $jk$  is a diagonal of the triangulation and  $v \in \mathbb{C}$ ,  $|v| = 1$  is parallel to it, then this diagonal makes two contributions to the coefficient equal to  $\bar{v}^2$  and to  $-\bar{v}^2$ , so they cancel.  $\square$

**Conjectural corollary.** *Let  $\Omega \subset \mathbb{C}$  be a compact connected domain with connected piecewise smooth boundary  $\partial\Omega$ , and let  $\mu$  be its Lebesgue measure. Then*

$$\begin{aligned} \mathcal{C}_\mu(u) &= \frac{i}{4} \int_{\partial\Omega} \frac{d}{dz} \bar{v}^2(z)(z-u) \ln(u-z) dz = \frac{i}{2} \int_{\partial\Omega} v'(z)v(z)(z-u) \ln(u-z) dz \\ &= -\frac{1}{2} \int_{\partial\Omega} \kappa(z)v^2(z)(z-u) \ln(u-z) dz. \end{aligned}$$

where  $v(z) \in \mathbb{C}$  is parallel to the tangent line to  $\partial\Omega$  at the point  $z$ , and  $|v(z)| = 1$ ;  $\kappa(z)$  is the curvature of  $\partial\Omega$  at the point  $z$ .

*Draft of the proof.*  $v'(z)$  is normal to  $v(z)$  (since  $|v(z)| = \text{const.}$ ) and its absolute value is  $\kappa(z)$ , so  $v'(z) = i\kappa(z)v(z)$ .  $\square$

**[Both in Theorem 3 and in the corollary we need to say something about how we choose the branch of the logarithm.]**

#### 4. QUESTIONS AND CONJECTURES

- (1) Is it true that the motherbodies of convex polygons are inertia minimizing? Is inertia minimizer of a non-convex polytope supported on a union of straight segments?
- (2) Is it true that inertia minimizer of the moment sequence of some compactly supported measure is supported inside the convex hull of the support of the initial measure?
- (3) Prove that (the set of) inertia minimizers of a function depends continuously on the function.
- (4) In [8] there is an example of a measure (domain) which has no motherbody. It is the image of a unit disk under a map given by a quadratic polynomial. What could be the inertia minimizer for such a domain?
- (5) How does the inertia minimizer (and, in particular, its support)  $\mu_f \in \mathfrak{M}_f(\Omega)$  depend on the domain  $\Omega$ ? Is it true that there exists a maximal domain  $\Omega_0$  such that  $\mu_f(\Omega) = \mu_f(\Omega_0)$  for any  $\Omega \supset \Omega_0$ ?

#### 5. FINAL REMARKS

*Problem 1.* Study the space  $Null_h$ .

*Problem 2.* Can the complement  $\mathbb{C} \setminus \text{supp } \kappa_A$  be disconnected?

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NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS (NRU-HSE), RUSSIA, AND  
INDEPENDENT UNIVERSITY OF MOSCOW, B. VLASSIEVSKII PER., 11, MOSCOW, 119002, RUSSIA  
*E-mail address:* `burman@mccme.ru`

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, SE-106 91 STOCKHOLM, SWEDEN  
*E-mail address:* `shapiro@math.su.se`