PROBABILITY MEASURES IN $\mathbb{C}$ WHOSE CAUCHY TRANSFORMS HAVE IMAGINARY PERIODS

RIKARD BØGVAD AND BORIS SHAPIRO

To Alexander Vassiliev and Jan-Erik Björk, in memoriam

Abstract. We study compactly supported probability measures in the complex plane whose Cauchy transforms are branches of algebraic functions. To such a germ we associate the normalized spectral curve and the meromorphic differential $\Omega = w\,dz$. Under the assumptions that Ω has purely imaginary periods and satisfies a natural admissibility condition at its poles, we construct a positive mother body measure by means of the upper envelope of the single-valued harmonic function $\Re \int \Omega$. We also prove a uniqueness result for the canonical measure under an additional geometric hypothesis on the maximality graph. Finally, we recall the relation of our assumptions with real-normalized meromorphic differentials on compact Riemann surfaces.

1. Introduction

Let μ be a finite compactly supported complex measure in the complex plane $\mathbb{C}$. Define the *logarithmic potential* of μ as

$$\mathfrak{u}_\mu(z) := \int_{\mathbb{C}} \ln|z-\xi| d\mu(\xi)$$

and the *Cauchy transform* of μ as

$$\mathcal{C}_\mu(z) := \int_{\mathbb{C}} \frac{d\mu(\xi)}{z-\xi}.$$

Standard facts about the Cauchy transform include:

- $\mathcal{C}_\mu$ is locally integrable; in particular it defines a distribution on $\mathbb{C}$ and therefore can be acted upon by $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$.
- $\mathcal{C}_\mu$ is analytic in the complement in $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to the support of μ. For example, if μ is supported on the unit circle (which is the most classical case), then $\mathcal{C}_\mu$ is analytic both inside the open unit disc and outside the closed unit disc.
- with the standard convention $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$, the relations between μ, $\mathcal{C}_\mu$ and $\mathfrak{u}_\mu$ are as follows:

$$\mathcal{C}_\mu = 2\frac{\partial \mathfrak{u}_\mu}{\partial z} \quad \text{and} \quad \mu = \frac{1}{\pi}\frac{\partial \mathcal{C}_\mu}{\partial \bar{z}} = \frac{2}{\pi}\frac{\partial^2 \mathfrak{u}_\mu}{\partial z \partial \bar{z}} = \frac{1}{2\pi}\Delta \mathfrak{u}_\mu. \tag{1.1}$$

These identities are understood in the sense of distributions.

Date: May 8, 2026.
2020 *Mathematics Subject Classification.* Primary 31A15, 31A25; Secondary 30F30, 30G15, 35R30.
Key words and phrases. Algebraic functions, rational curves, Cauchy transform.

For more relevant information on the Cauchy transform we recommend a short and well-written treatise [6].

The following problem was formulated and discussed in some generality in [5].

Main problem. Given a germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i,\ a_0 \in \mathbb{R}$ of an algebraic (or, more generally, analytic) function near ∞, is it possible to find a compactly supported signed measure whose Cauchy transform coincides with (a branch of) the analytic continuation of $f(z)$ a.e. in $\mathbb{C}$? Additionally, for which $f(z)$ it is possible to find a positive measure with the above properties?

The present note is closely related to the theory of piecewise harmonic subharmonic functions and positive Cauchy transforms developed in [2, 3], to the mother-body problem for algebraic Cauchy transforms studied in [5], and to the appearance of Boutroux, or equivalently imaginary-period, curves in asymptotic zero distribution problems; see for example [4]. The terminology concerning real-normalized differentials comes from Krichever's work and its later developments [7, 8, 9].

Remark 1. Observe that the obvious necessary condition $a_0 > 0$ is sufficient for the existence of a compactly supported positive measure whose Cauchy transform coincides with any given germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i$ locally near ∞, see Theorem 1 of [5]. On the other hand, in general, the possibility to represent an analytic continuation of $f(z)$ as the Cauchy transform of a positive or signed measure almost everywhere in $\mathbb{C}$ imposes strong and at present unknown restrictions on $f(z)$, see relevant discussions in loc. cit.

If such a signed measure exists and additionally its support is a finite union of compact semi-analytic curves and isolated points we will call it *a real mother body measure* of the germ $f(z)$, see Definition 1 introduced in [5] below. If the corresponding measure is positive, then we call it *a positive mother body measure* of $f(z)$.

Definition 1. Given a germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i,\ a_0 > 0$ of an analytic function near ∞, we say that a signed measure μ_f is its *mother body* if:
(i) its support $supp(\mu_f)$ is the union of finitely many points and finitely many compact semi-analytic curves in $\mathbb{C}$;
(ii) The Cauchy transform of μ_f coincides in each connected component of the complement $\mathbb{C} \setminus supp(\mu_f)$ with some branch of the analytic continuation of $f(z)$.

(Here by a compact semi-analytic curve we mean a compact segment of a real-analytic curve in $\mathbb{C} \simeq \mathbb{R}^2$.)

Remark 2. The original notion of a mother body of a solid domain was apparently pioneered in the 1960's by a Bulgarian geophysicist D. Zidarov [13] and later mathematically developed by B. Gustafsson [11]. Although a number of interesting results about mother bodies was obtained in several special cases, [11], [13] there is still no consensus about its appropriate general definition. In particular, no general existence and/or uniqueness results are known at present.

Remark 3. Notice that by Theorem 1 of [3], if the Cauchy transform of a positive measure coincides a.e. in $\mathbb{C}$ with an algebraic function $f(z)$, then the support of this measure is a finite union of semi-analytic curves and isolated points. Therefore it is a mother body measure according to the above definition. Whether the latter result extends to signed measures is at present unknown.

Relations (1.1) suggest that a complex-valued measure μ in $\mathbb{C}$ should be considered as a 2-current (i.e. generalized complex-valued 2-form), its logarithmic potential u_μ as a distribution (i.e. a generalized function) harmonic outside the support of μ and its Cauchy transform $\mathcal{C}_\mu$ as a 1-current (i.e. generalized 1-form) holomorphic outside the support of μ. To emphasise this, we denote the measure μ by $\mu dz d\bar{z}$ and its Cauchy transform $\mathcal{C}_\mu$ by $\mathcal{C}_\mu dz$.

We consider the situation when $\mu dz d\bar{z}$ is a compactly supported probability measure which implies that $\mathcal{C}_\mu dz = (1/z + \sum_{i\geq 2} a_i/z^i)dz = -(1/Z + \sum_{i\geq 2} a_i Z^{i-2})dZ$, where $Z = 1/z$, i.e. it has a simple pole with residue -1.

Assume that $\mathcal{C}_\mu dz$ near ∞ equals $f(z)dz$ where $f(z)$ is a branch of an algebraic function, i.e. it is a branch of a plane algebraic curve $\Gamma_f \subset \mathbb{C}^2$ given by some bivariate polynomial

$$P_f(w, z) = 0,$$

where w stands for the variable corresponding to the Cauchy transform. Consider the standard compactification of $\mathbb{C}^2$ with coordinates (w, z) as a subset of $\mathbb{C}P^1 \times \mathbb{C}P^1$ by projectivising the coordinates w and z respectively. Denote by $\overline{\Gamma}_f \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ the compactification of $\Gamma_f \subset \mathbb{C}^2$; denote by $\widetilde{\Gamma}_f$ the normalisation of $\overline{\Gamma}_f$ and by $\nu : \widetilde{\Gamma}_f \to \overline{\Gamma}_f$ the standard normalisation map. The smooth compact Riemann surface $\widetilde{\Gamma}_f$ is birationally equivalent to Γ_f. Let us consider the meromorphic 1-form $\Omega_f = w dz$ on the curve Γ_f and take its pullback on $\widetilde{\Gamma}_f$. (By a slight abuse of notation, we denote the pullback of Ω_f to $\widetilde{\Gamma}_f$ by the same letter.)

Observe that we have a the standard branched covering $\pi_z : \widetilde{\Gamma}_f \to \mathbb{C}P^1$ along the w-axis onto the z-axis. The pullback $\pi_z^{-1}(\mathcal{C}_\mu dz)$ of the locally defined $\mathcal{C}_\mu(z)$ to the appropriate branch of $\widetilde{\Gamma}_f$, by definition, locally coincides with Ω_f.

Definition 2. We say that a germ of an algebraic function $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i$, $a_0 \neq 0$ near $\infty \in \mathbb{C}P^1$ *has purely imaginary periods* if the meromorphic 1-form Ω_f (which is globally defined on $\widetilde{\Gamma}$) has all imaginary periods.

Remark 4. The latter definition is equivalent to the fact that the multi-valued primitive function $\Psi(p) = \int_{p_0}^{p} \Omega_f$ which is well-defined on the universal covering of $\widetilde{\Gamma}_f \setminus Poles_f$, where $Poles_f \subset \widetilde{\Gamma}_f$ is the set of all poles of Ω_f and p_0 is some fixed base point has a uni-valued real part $\Re\Psi(p)$ which is well-defined on $\widetilde{\Gamma}_f \setminus Poles_f$. The periods of Ω_f are of two types: periods related to the poles of Ω_f and periods related to the 1-dimensional homology classes of $\widetilde{\Gamma}_f$. The first type of periods are purely imaginary if and only if all residues of Ω_f are real. The second type of periods is absent if and only if Γ_f has genus 0.

The main result of the present paper is as follows.

Theorem 1. *Let $f(z) = 1/z + \sum_{i\geq 2} a_i/z^i$ be a germ of an algebraic function at ∞, and let Γ, $X := \widetilde{\Gamma}$, $\pi : X \to \mathbb{C}P^1$ and $\Omega = w\,dz$ be as above. Assume that*

(1) *all periods of Ω are purely imaginary;*
(2) *the germ f determines a point $p_\infty \in X$ over $z = \infty$ at which Ω has residue -1;*
(3) *the real primitive $H = \Re \int \Omega$ has no positive blow-up away from p_∞: more precisely, for every pole $q \neq p_\infty$ of Ω one has $H(p) \to -\infty$ as $p \to q$, while near p_∞ one has $H(p) - \log|z(p)| = O(1)$.*

Then there exists a compactly supported positive probability measure μ_f on $\mathbb{C}$ such that its Cauchy transform coincides almost everywhere with a branch of the analytic continuation of f. Its support is contained in a finite union of real-analytic arcs and finitely many points. Thus μ_f is a positive mother body measure of the germ f.

2. Proofs

2.1. **Im-normalized differentials.** We shall use the following standard fact about meromorphic differentials with real, or after multiplication by i, imaginary periods. A meromorphic differential η on a compact Riemann surface X is usually called *real-normalized* if all its periods are real. Equivalently, $-i\Omega$ is real-normalized precisely when Ω has purely imaginary periods.

For fixed prescribed principal parts with real residues and total residue zero, there exists a unique real-normalized meromorphic differential. Indeed, start with any meromorphic differential with the prescribed principal parts. The real parts of its periods define a real linear functional on $H_1(X,\mathbb{R})$. By the non-degeneracy of the period pairing for holomorphic differentials, there is a unique holomorphic differential whose real periods cancel these real parts. Subtracting it gives the required real-normalized differential. Multiplication by i gives the corresponding statement for imaginary-normalized differentials. Thus the period condition in Theorem 1 is not an accidental constraint: it is the natural normalization which makes $\Re\int\Omega$ single-valued.

This class of differentials is classical in the algebro-geometric theory of integrable systems and in the geometry of moduli spaces. It appears, for example, in Krichever's work on real-normalized differentials and Arbarello's conjecture, and in the work of Grushevsky and Krichever on the universal Whitham hierarchy and the moduli space of pointed Riemann surfaces; see [7, 8]. In the present paper we use the equivalent imaginary-normalized convention because the real part of the primitive then becomes a globally defined harmonic function on $X \setminus \mathrm{Pol}(\Omega)$. This is exactly the feature needed for the logarithmic-potential construction below.

2.2. **The real primitive.** Let

$$X := \widetilde{\Gamma}, \qquad \pi : X \longrightarrow \mathbb{C}P^1$$

be the normalized compact curve associated with the algebraic function, and let

$$\Omega = w\,dz$$

be the corresponding meromorphic differential on X. Since all periods of Ω are purely imaginary, the function

$$H(p) := \Re\int_{p_0}^{p}\Omega$$

is single-valued on $X \setminus \mathrm{Pol}(\Omega)$, up to an additive constant. Changing the base point only adds a constant to H and will not change the measure constructed below after the normalization at infinity is fixed.

Near the distinguished point p_∞ corresponding to the germ $f(z) = 1/z + O(z^{-2})$ we have, with $Z = 1/z$,

$$\Omega = f(z)\,dz = \left(Z + O(Z^2)\right)\left(-\frac{dZ}{Z^2}\right) = -\frac{dZ}{Z} + O(1)\,dZ.$$

Thus $\operatorname{res}_{p_\infty} \Omega = -1$ and

$$H(p) = \log |z(p)| + O(1), \qquad p \to p_\infty.$$

By assumption, every other pole contributes only a negative logarithmic singularity for H; in particular it cannot appear in the upper envelope below.

2.3. **The upper envelope.** Define, for $z \in \mathbb{C}$,

$$U(z) := \max\{H(p) : \pi(p) = z\}.$$

The maximum is taken over finitely many points, counted without multiplicity. At ordinary points of the covering, U is the maximum of finitely many harmonic functions. At ramification points the local branches of H are still continuous harmonic functions after normalization of the curve. At poles different from p_∞ the relevant branches tend to $-\infty$ and hence do not affect the maximum. Consequently U is a well-defined continuous subharmonic function on $\mathbb{C}$ which is locally the maximum of finitely many harmonic functions.

The asymptotic normalization at p_∞ gives

$$U(z) = \log |z| + O(1), \qquad z \to \infty.$$

Indeed the branch corresponding to p_∞ gives the lower bound $U(z) \geq \log |z| + O(1)$, while the admissibility assumption rules out any other branch with larger growth at infinity or with a finite positive blow-up.

Now put

$$\mu_f := \frac{1}{2\pi} \Delta U = \frac{2}{\pi} \frac{\partial^2 U}{\partial z\, \partial \bar{z}}$$

in the sense of distributions. Since U is subharmonic, μ_f is a positive measure. Since $U(z) = \log |z| + O(1)$ at infinity, the total mass of μ_f is equal to 1. Hence μ_f is a positive probability measure.

The support of μ_f is contained in the locus where at least two branches of H have the same maximal value, together with possible logarithmic point masses coming from finite poles. Equalities of two distinct harmonic algebraic branches are locally real-analytic arcs, except at isolated singular points. Hence $\operatorname{supp} \mu_f$ is contained in a finite union of compact semi-analytic arcs and isolated points.

2.4. **The Cauchy transform.** On the open set where the maximum defining U is achieved by a unique branch $p = p(z)$, write locally

$$F(z) = \int^p \Omega, \qquad F'(z) = w(p(z)).$$

Since $U = \Re F$ on this open set, the standard complex derivative convention gives

$$2 \frac{\partial U}{\partial z} = F'(z) = w(p(z)).$$

By (1.1), the Cauchy transform of μ_f is

$$\mathcal{C}_{\mu_f}(z) = 2 \frac{\partial U}{\partial z}$$

outside $\operatorname{supp} \mu_f$. The exceptional set where the maximum is achieved by more than one branch is contained in the support of μ_f and has planar Lebesgue measure zero. Therefore $\mathcal{C}_{\mu_f}$ coincides almost everywhere in $\mathbb{C}$ with one of the branches of the algebraic function $w = f(z)$. Near infinity it is the original branch $1/z + O(z^{-2})$, because there $U(z) = \log |z| + O(1)$ is realized by the chosen point p_∞.

This proves the existence of a compactly supported positive mother body measure for the germ f.

3. Uniqueness of the canonical measure

Theorem 1 is an existence theorem. Nevertheless the construction used in its proof is canonical: after the additive constant in H is fixed by the normalization at infinity, the upper envelope

$$U(z) = \max_{p \in \pi^{-1}(z)} H(p)$$

and its Riesz measure are uniquely determined. The following result gives a precise uniqueness statement in a natural geometric class.

Let K denote the maximality graph of H, i.e. the compact set where either at least two branches of H attain the maximum in the definition of U, or where a logarithmic atom of the Riesz measure is located. Equivalently, $K = \operatorname{supp} \mu_f$ for the measure constructed above, after removing irrelevant arcs on which the jump of the normal derivative is zero.

Theorem 2 (Uniqueness on the canonical support)**.** *Assume, in addition to the hypotheses of Theorem 1, that the canonical support $K = \operatorname{supp} \mu_f$ is a compact finite union of real-analytic arcs and points and that $\mathbb{C} \setminus K$ is connected. Then μ_f is the unique finite positive measure ν satisfying the following two conditions:*

1. $\operatorname{supp} \nu \subseteq K$;
2. *in $\mathbb{C} \setminus K$, the Cauchy transform $\mathcal{C}_\nu$ coincides with the branch of the algebraic function selected by the upper envelope U.*

Equivalently, once the canonical support is prescribed, the upper-envelope construction determines a unique positive mother body measure.

Proof. Let ν be another measure satisfying (i)–(ii). The branch selected by the upper envelope is $1/z + O(z^{-2})$ at infinity. Hence ν has total mass one. Set $\sigma = \nu - \mu_f$. Then σ is a finite signed measure supported on K, and its Cauchy transform vanishes in $\mathbb{C} \setminus K$.

Expanding at infinity gives

$$\mathcal{C}_\sigma(z) = \sum_{j \geq 0} \frac{1}{z^{j+1}} \int_K \xi^j \, d\sigma(\xi) = 0$$

for sufficiently large $|z|$. Therefore all complex moments of σ vanish:

$$\int_K \xi^j \, d\sigma(\xi) = 0, \qquad j = 0, 1, 2, \ldots.$$

Since $\mathbb{C} \setminus K$ is connected, Mergelyan's theorem implies that polynomials are uniformly dense in $C(K)$. Thus $\int_K \varphi \, d\sigma = 0$ for every continuous function φ on K, and consequently $\sigma = 0$. Therefore $\nu = \mu_f$. □

Remark 5. *The support condition in Theorem 2 is essential. The inverse balayage problem and the general mother-body problem are not expected to have an automatic uniqueness theorem without specifying either a support class or an extremality condition. The theorem above should therefore be read as a uniqueness theorem on the canonical support produced by the upper-envelope construction, not as a claim that every positive measure with the same germ at infinity is necessarily equal to μ_f.*

Acknowledgements. We are sincerely obliged to the late Sasha Vassiliev and Jan-Erik Björk for numerous discussions of this topic.

4. Final remarks

Several questions remain open. First, it would be useful to find intrinsic criteria, stated only in terms of the spectral curve X and the divisor of Ω, guaranteeing that the upper-envelope measure has no atoms and is supported only on analytic arcs. Second, one would like to understand when the canonical support K coincides with a critical graph of a naturally associated quadratic differential. Finally, a more systematic comparison with classical mother bodies of algebraic domains and quadrature domains should clarify the relation between the present construction and the older theory initiated by Zidarov and developed by Gustafsson.

The admissibility condition in Theorem 1 deserves further study. The imaginary-period condition is classical, being equivalent after multiplication by i to real-normalization of meromorphic differentials. By contrast, the sign condition at finite poles is a positivity condition for the logarithmic potential. It rules out finite positive logarithmic blow-ups and is therefore essential for compactly supported positive probability measures.

References

[1] T. Bergkvist and H. Rullgård, *On polynomial eigenfunctions for a class of differential operators*, Math. Res. Rep. 2001:13, Stockholm University, 2001.

[2] J. Borcea, R.Bøgvad, *Piecewise harmonic subharmonic functions and positive Cauchy transforms.* Pacific J. Math., Vol. 240 (2009), No. 2, 231–265.

[3] J.-E. Björk, J. Borcea, R.Bøgvad, *Subharmonic Configurations and Algebraic Cauchy Transforms of Probability Measures.* Notions of Positivity and the Geometry of Polynomials Trends in Mathematics 2011, pp 39–62.

[4] R. Bøgvad, Ch. Hägg, B. Shapiro, Rodrigues' descendants of a polynomial and Boutroux curves, Constructive Approximation, Volume 59, pages 737–798, (2024), DOI 10.1007/s00365-023-09657-x.

[5] R. Bøgvad, B. Shapiro, On mother body measures with algebraic Cauchy transform, L'Enseignement Math., vol. 62, (2016) 117–142.

[6] J.B. Garnett, Analytic capacity and measure, LNM 297, Springer-Verlag, 1972, 138 pp.

[7] I. M. Krichever, *Real normalized differentials and Arbarello's conjecture.* Funct. Anal. Appl. 46 (2012), no. 2, 110–120.

[8] S. Grushevsky and I. M. Krichever, *The universal Whitham hierarchy and the geometry of the moduli space of pointed Riemann surfaces.* Surveys in Differential Geometry 14 (2009), 111–129.

[9] S. Grushevsky, I. M. Krichever and C. Norton, *Real-normalized differentials: limits on stable curves*, Russian Math. Surveys 74 (2019), no. 2, 265–324.

[10] I. M. Krichever, *The τ-function of the universal Whitham hierarchy, matrix models and topological field theories.* Comm. Pure Appl. Math. 47 (1994), 437–475.

[11] B. Gustafsson, *On Mother Bodies of Convex Polyhedra.* SIAM J Math. Anal., 29:5, (1998) 1106–1117.

[12] T. Sjödin, *Mother Bodies of Algebraic Domains in the Complex Plane.* Complex Variables and Elliptic Equations, 51:4 (2006) 357–369.

[13] D. Zidarov, Inverse Gravimetric Problem in Geoprospecting and Geodesy 1990, (Amsterdam: Elsevier.) vii+283 pp.

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
Email address: rikard@math.su.se, shapiro@math.su.se