ON POLYNOMIAL EIGENFUNCTIONS OF A HYPERGEOMETRIC-TYPE OPERATOR

G. Másson† and B. Shapiro‡

† deCODE Genetics Inc, Lynghals 1, IS-110, Reykjavik, Iceland, gisli.masson@decode.is
‡ Department of Mathematics, University of Stockholm,
S-10691, Sweden, shapiro@matematik.su.se

Abstract. Consider an operator $\mathfrak{d}_Q(f)=\frac{d^k}{dx^k}(Q(x)f(x))$ where $Q(x)$ is some fixed polynomial of degree k. One can easily see that $\mathfrak{d}_Q$ has exactly one polynomial eigenfunction $p_n(x)$ in each degree $n\ge 0$ and its eigenvalue $\lambda_{n,k}$ equals $\frac{(n+k)!}{n!}$. A more intriguing fact is that all zeros of $p_n(x)$ lie in the convex hull of the set of zeros to $Q(x)$. In particular, if $Q(x)$ has only real zeros then each $p_n(x)$ enjoys the same property. We formulate a number of conjectures on different properties of $p_n(x)$ based on computer experiments as, for example, the interlacing property, a formula for the asymptotic distribution of zeros etc. These polynomial egenfunctions might be thought as an interesting generalization of the classical Gegenbauer polynomials with the integer value of the parameter (which corresponds to the case $Q(x)=(x^2-1)^l$).

§1. Introduction

A real polynomial in one variable is called *hyperbolic* if all its zeros (counted with multiplicities) are real. Different properties of hyperbolic polynomials and criteria of hyperbolicity were extensively studied in the beginning of the century, see e.g. [PS], ch. 5-6. In 60's and 70's the interest to hyperbolic polynomials (mostly in the case of several variables) was revitalized due to the fundamental contributions of I. G. Petrovsky and L. Hörmander to the theory of linear partial differential equations with constant coefficients. But some new results were obtained even in the case of one variable, see e.g. [N]. In 80's V.Arnold and his students wrote a number of papers on hyperbolic polynomials motivated by their application to potential theory, see [Ar1-2], [Ko1-2].

A typical context in which hyperbolic polynomials appear is the spectral theory of Sturm-Liouville problems or, more specifically, the theory of orthogonal polynomials, see e.g. [C]. Namely, given some nonnegative supported on an interval $[a,b]$ weight function $w(x)$ with $\int_a^b w(x)dx>0$ one gets a family of monic polynomials $\{r_n(x)\}$, deg $r_n(x)=n$ satisfying a condition

$$\int_a^b w(x)r_i(x)r_j(x)dx=\delta_{i,j}, \qquad (1).$$

It is well known that each $r_i(x)$ is hyperbolic and all its zeros lie on $[a,b]$. The classical polynomial families such as Hermite, Laguerre, Jacobi polynomials (and their special cases

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX such as Tchebyschev, Legendre and Gegenbauer polynomials) arise in this way. Besides hyperbolicity among the main properties of orthogonal polynomials one should mention that

I) all the zeros of $r_i(x)$ are simple (and real);

II) the zeros of any pair of consecutive orthogonal polynomials $(r_i(x), r_{i+1}(x))$ are interlacing, i.e. $\bar{\alpha}_1 < \alpha_1 < \bar{\alpha}_2 < \alpha_2 < \ldots < \bar{\alpha}_{i-1} < \alpha_{i-1} < \bar{\alpha}_i$ where α_j (resp. $\bar{\alpha}_j$) is the jth smallest zero of r_i (resp. r_{j+1});

III) the density of the asymptotic distribution of zeros of $r_i(x)$ when $i \to \infty$ on the interval $[a,b]$ is independent on the weight function $w(x)$ (supported on $[a,b]$) and is given by $\frac{b-a}{2\pi\sqrt{(b-x)(x-a)}}$, see e.g. [Ne].

In this note we partially prove and mostly conjecture similar results for the polynomial eigenfunctions of the operator $\mathfrak{d}_Q(f) = \frac{d^k}{dx^k}(Q(x)f)$ where $Q(x)$ is a polynomial of degree k. This operator can be considered as a generalization of the famous hypergeometric operator

$$(x^2-1)f'' + (ax+b)f' + c,$$

see also Conjecture 14 in §2. Note that most of the abovementioned families of orthogonal polynomials are among its polynomial solutions.

The intriguing detail is that, in general, $\mathfrak{d}_Q$ is **neither** a positive-definite **nor** selfadjoint operator in which case the above properties are expected. (Unless mentioned explicitly in what follows Q is always a monic polynomial of degree k.)

The main results are as follows.

Theorem 1. a) For any $Q(x)$ the operator $\mathfrak{d}_Q$ has one polynomial eigenfunction $p_n(x)$ in each degree $n \geq 0$ and its eigenvalue $\lambda_{n,k}$ equals $\frac{(n+k)!}{n!}$.

b) for any $n \geq 0$ all the zeros of $p_n(x)$ belong to the convex hull Conv_Q of the set of zeros of $Q(x)$.

Corollary 2. If $Q(x)$ is hyperbolic then each $p_n(x)$ is also hyperbolic and its zeros lie between the minimal root α and the maximal root β of $Q(x)$.

Theorem 3. if $Q(x)$ is hyperbolic and has at least two pairwise different zeros (i.e. $\alpha < \beta$) then besides the property that they belong to (α, β) all the zeros to $p_n(x)$ are simple.

We now provide explicit determinantal formulas for the coefficients of $p_n(x)$. Set $Q(x) = x^k + q_{k-1}x^{k-1} \cdots + q_1 x + q_0$ and $p_n(x) = x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,1}x + a_{n,0}$, where

$$\frac{d^k}{dx^k}[Q(x)p_n(x)] = \lambda_{n,k}p_n(x), \qquad \lambda_{n,k} = \frac{(n+k)!}{n!}. \tag{2}$$

Consider the upper triangular $(n \times n)$-matrix

$$M_n = \begin{pmatrix} 1-\frac{\lambda_{n,k}}{\lambda_{0,k}} & q_{k-1} & q_{k-2} & \dots & q_0 & 0 & & \dots & 0 \\ 0 & 1-\frac{\lambda_{n,k}}{\lambda_{1,k}} & q_{k-1} & q_{k-2} & \dots & q_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ & & & & & & & & q_0 \\ & & & & & & & \ddots & \vdots \\ & & & & & & & \ddots & q_{k-2} \\ \vdots & & & & & & \ddots & \ddots & q_{k-1} \\ 0 & & \dots & & & & \dots & 0 & 1-\frac{\lambda_{n,k}}{\lambda_{n-1,k}} \end{pmatrix}.$$

Theorem 4. For every $n \geq 1$ the coefficients of $p_n(x)$ satisfy the linear system

$$M_n A = B \tag{3}$$

where

$$A = \begin{pmatrix} a_{n,0} \\ a_{n,1} \\ \vdots \\ \vdots \\ \vdots \\ a_{n,n-1} \end{pmatrix}, \qquad B = -\begin{pmatrix} 0 \\ \vdots \\ 0 \\ q_0 \\ q_1 \\ \vdots \\ q_{k-1} \end{pmatrix}.$$

Proposition 5. If $Q = (x^2-1)^l$ then the family $\{p_n(x)\}$ coincides (up to constant factors) with the family of Gegenbauer polynomials with the parameter value $\lambda = l + \frac{1}{2}$ (or, equivalently, with the Jacobi (l,l)-polynomials, see e.g. [C]).

The structure of the paper is as follows. The next (and apparently the most interesting) section contains a number of intriguing conjectures and §3 contains proofs of the formulated results.

Remark. At the moment of publication of this paper a number of conjectures presented in §2 were proven by the efforts of H. Rullgård, see [Ru] and S. Shadrin. The spectral properties of the operator $\mathfrak{d}_Q(f)$ in various functional spaces were recently studied in [Sh].

The authors are grateful to T. Ekedahl, H. Rullgård, H. Shapiro, S. Shadrin and T. Bergkvist for a number of stimulaing discussions of the project. The second author wants to acknowledge the hospitality and the financial support of Max-Planck Institute, Bonn during August-September 2000 and IHES, Paris during January 2001 when this article was prepared.

§2 Conjectures.

Let us now formulate a number of intriquing conjectures supported by extensive computer experiments with different choices of Q.

Let μ_n denote the discrete probability measure supported on the set of all zeros to $p_n(x)$ obtained by placing the mass equal to $\frac{\mathrm{mult}_i}{n}$ to each of the geometrically distinct roots x_i

of $p_n(x)$. Here mult_i is the multiplicity of x_i. Let $\mu_Q = \lim_{n\to\infty}\mu_n$ if it exists. Here convergence is understood as the weak convergence of measures. We call μ_Q *the asymptotic measure*. Many of the above conjectures describe the properties of the support of μ_Q and its density.

Case of hyperbolic $Q(x)$

By corollary 2 if $Q(x)$ is hyperbolic and α (resp. β) is its minimal (resp. maximal) zeros then all zeros of $p_n(x)$ belong to (α,β). Moreover computations strongly indicate

Conjecture 1. If $Q(x)$ is hyperbolic and has at least two distinct zeros then the zeros of any pair of consecutive polynomial eigenfunctions $p_i(x)$ and $p_{i+1}(x)$) are interlacing.

Remark. The proof of this conjecture obtained recently by S. Shadrin was added with his kind permission to the final version of the paper, see §4 below.

The numerical study of the distribution zeros to $p_n(x)$ for hyperbolic $Q(x)$ (see fig 1.) led us to

Conjecture 2. If $Q(x)$ is hyperbolic of degree k then the density of the asymptotic measure μ_Q on the interval $[\alpha,\beta]$ is proportional to $\frac{1}{\sqrt[k]{|Q(x)|}}$ where α is the minimal and β is the maximal root of $Q(x)$.

Important observation. If conjecture 2 holds then for any $Q(x)$ different from an integer power of a degree 2 polynomial then $\{p_n(x)\}$ is **not** a system of orthogonal polynomials for some weight function $w(x)$ supported on $[\alpha,\beta]$ or a family obtained by taking derivatives of such, since for such families the asymptotic density of the zeros coincides with $\frac{\beta-\alpha}{2\pi\sqrt{|(x-\alpha)(x-\beta)|}}$.

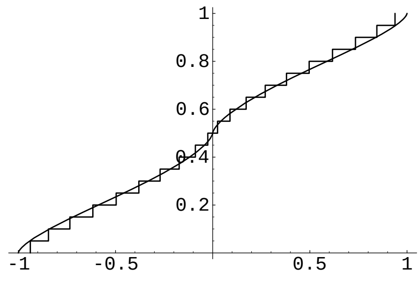

Fig. 1. Disribution of zeros of $p_{20}(x)$ for $Q_3(x) = x^3 - x$ and the asymptotic distribution function $\frac{\Gamma(\frac{4}{3})}{3\Gamma(\frac{2}{3})}\int_{-1}^{x}\frac{dx}{\sqrt[3]{|x^3-x|}}$.

Case of real $Q(x)$

For fixed k and every n let us define the discriminantal hypersurface $D_n \subset \mathrm{Pol}_k^{\mathbb{C}}$ consisting of polynomial Q for which the corresponding eigenpolynomial $p_n(x)$ has multiple zeros. Here $\mathrm{Pol}_k^{\mathbb{C}}$ (resp. $\mathrm{Pol}_k^{\mathbb{R}}$) is the space of all monic degree k polynomials with complex (resp. real) coefficients. Theorem 3 says that if $Q(x)$ is hyperbolic then every $p_n(x)$ is hyperbolic and has only simple zeros. Therefore we know that all the discriminants D_n do not intersect the domain of strictly hyperbolic polynomials in $\mathrm{Pol}_k^{\mathbb{R}}$. An example of these discriminants for the family $Q_3(x) = x^3 + ax + b$ is shown on Fig.2.

PROBLEM 3. Find equations for D_n and study their topological properties.

CONJECTURE 4. The domain of hyperbolic polynomials is the limit of the intersection of the connected components in the complement to all D_n containing it.

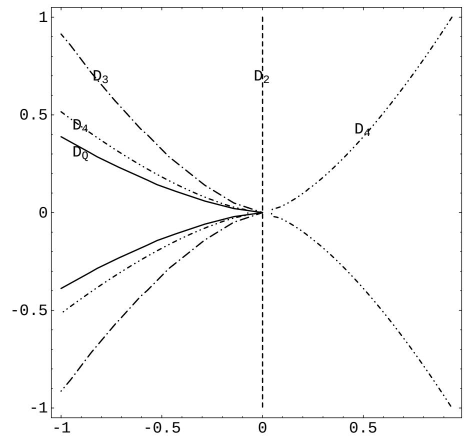

FIG. 2. DISCRIMINANTAL CURVES FOR $n \leq 4$ IN THE FAMILY $Q_3(x) = x^3 + ax + b$.

EXAMPLE. The explicit equations for the discriminants D_2, D_3, D_4 for the family $Q_3(x) = x^3 + ax + b$ are as follows:

$$D_2 = \frac{4\,a}{9}; \qquad D_3 = \frac{a^3}{16} + \frac{27\,b^2}{361};$$

$$D_4 = \frac{36864\,a^6}{15353125} + \frac{14336\,a^3\,b^2}{2042125} - \frac{6912\,b^4}{923521}$$

Note that the standard discriminant D_Q is given by

$$D_Q = 4\,a^3 + 27\,b^2.$$

Case of complex $Q(x)$

For a complex polynomial $Q(x)$ let us denote by $\mathcal{C}_Q$ the support of the asymptotic measure μ_Q and called it *the accumulation curve.*

CONJECTURE 5. The accumulation curve $\mathcal{C}_Q$ enjoys the following properties
a) $\mathcal{C}_Q$ is a planar tree imbedded in Conv_Q whose leaves are the roots of $Q(x)$;
b) for a generic $Q(x)$ the total number of vertices in $\mathcal{C}_Q$ equals $2k-2$, i.e. the number of internal vertices of $\mathcal{C}_Q$ equals $k-2$. (About the notion of genericity see Problem 12.)

Select a branch cut BC in $\mathbb{C}$ consisting of rays from all distinct roots of $Q(x)$ to infinity which do not intersect each other or the tree $\mathcal{C}_Q$ and select the unique branch of $1/\sqrt[k]{Q(x)}$ on the simply connected domain $\Omega_Q := \mathbb{C} \setminus BC$ which asymptotically coincides with $1/x$ near infinity. Finally choose a point x_0 in Ω_Q and consider the holomorphic mapping $\Psi_Q : \Omega_Q \to \mathbb{C}$ defined by the integral

$$\Psi_Q(x) = \int_{x_0}^{x} \frac{dx}{\sqrt[k]{Q(x)}}.$$

CONJECTURE 6. Ψ_Q maps $\mathcal{C}_Q$ onto a planar tree with straight edges, see Fig. 3. Furthermore, the angles and the densities of the asymptotic measure $\mu(x)$ along edges are uniquely determined by the combinatorics of the tree, see conjectures 7–8.

Note that for a generic Q the image $\Psi_Q(\mathcal{C}_Q)$ is independent of the choice of BC as above and different choices of a base point x_0 will simply translate the image. (With some straightforward modifications this fact holds for any Q.)

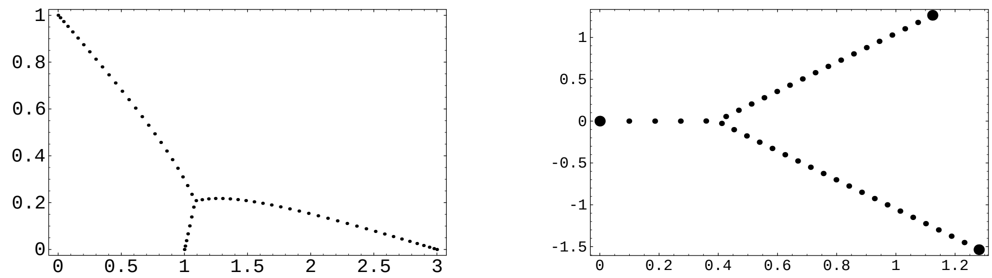

FIG. 3. ACCUMULATION CURVE FOR $Q_3(x) = (x-1)(x-3)(x-i)$ BEFORE AND AFTER THE MAPPING Ψ_Q.

(Both pictures in Fig. 3 are scaled which affects the angles between edges.)

Denote by $\Psi\mathcal{C}_Q$ the image $\Psi_Q(\mathcal{C}_Q)$ of the accumulation curve.

CONJECTURE 7. The angles at the vertices of $\Psi\mathcal{C}_Q$ are defined as follows. Fix a node $v \in \Psi\mathcal{C}_Q$. Consider the graph obtained from $\Psi\mathcal{C}_Q$ by removing all the edges adjacent to v. This graph is a forest whose subtrees are in 1-1 correspondence with the set of these edges. We set the weight $\sharp(e, v)$ of each such edge e wrt the vertex v equal to the number of leaves in its corresponding subtree. Then the angle $\epsilon(e_1, e_2)$ between the neighboring edges e_1 and e_2 adjacent to v equals

$$\epsilon(e_1, e_2) = \frac{\pi}{k}(\sharp(e_1, v) + \sharp(e_2, v)).$$

See an example on Fig.4.

Let $\Psi\mu_Q$ denote the image of the asymptotic measure μ_Q under the transformation Ψ_Q. Its support obviously coincides with $\Psi\mathcal{C}_Q$.

Properties of $\Psi\mu_Q$ are completely determined in the following conjecture.

CONJECTURE 8. The asymptotic distribution $\Psi\mu_Q$ of zeros on $\Psi\mathcal{C}_Q$ enjoys the following properties.

a) The density $\nu(e)$ of $\Psi\mu_Q$ is constant on every edge e of $\Psi\mathcal{C}_Q$ and therefore equals $\nu(e) = \frac{\mu(e)}{|e|}$ where $\mu(e)$ is the total mass of the edge e and $|e|$ is its length.

b) For a pair (v, e) where v is some vertex and e is an edge adjacent to v let $\bar{U}(v, e)$ denote the unit vektor parallell to e and pointing towards v. Then for any vertex v one has the following vector equation

$$\sum_{e \in E(v)} \nu(e)\bar{U}(v, e) = 0$$

where $E(v)$ is the index set of all edges adjacent to v.

Note that for a generic $Q(x)$ the equations above give $2(k-2)$ linear equations in $2k-3$ variables $\nu(e)$ which together with the condition $\sum_{e\in E}\nu(e)|e|=1$ allow us to determine the densities $\nu(e)$ uniquely.

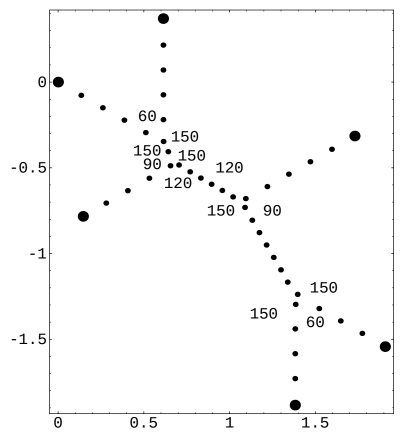

FIG. 4. HOW TO DETERMINE ANGLES.

With some straightforward modifications the same recipe works for all Q.

Assuming that conjectures 1–8 hold, it is clear that one gets in many respects the same asymptotic behaviour of $p_n(x)$ if $Q(x)$ is replaced by its arbitrary integer power $Q^r(x)$. Therefore it is reasonable to study the solutions of the equation one gets when taking $r=1/k$. In that case the "degree" of the polynomial is equal to 1 and the corresponding differential operator is thus of order 1. Furthermore $\lambda_{k,n}=(n+1)\dots(n+k)$ so for $k=1$ the "correct" eigenvalue for the n-th eigenfunction is $(n+1)$. In other words, we want to compare the eigenpolynomials p_n of the original operator to the solutions y_n of the differential equation

$$\partial(\sqrt[k]{Qy_n})=(n+1)y_n.$$

Multiplying through by $\sqrt[k]{Q}$ and setting $f_n=\sqrt[k]{Q}y_n$ transforms this equation into

$$f_n'=(n+1)Q^{-1/k}f_n$$

or

$$dlog\, f_n:=\frac{f_n'}{f_n}=(n+1)Q^{-1/k}$$

Now $f_n=Q^{1/k}y_n$ so $dlog\, f_n=dlog\, y_n+(1/k)\, dlog\, Q$ and therefore we get

$$\frac{1}{n}\, dlog\, y_n=\frac{n+1}{n}Q^{-1/k}-\frac{1}{kn}\, dlog\, Q$$

This implies that

$$\lim_{n\to\infty}\frac{1}{n}\, dlog\, y_n=Q^{-1/k}$$

Since y_n corresponds to p_n it is therefore natural to expect:

Conjecture 9.
In $\mathbb{C} \setminus \mathcal{C}_Q$ we have that

$$\frac{1}{n} \lim_{n\to\infty} \frac{p_n'(x)}{p_n(x)} = \frac{1}{\sqrt[k]{Q(x)}}.$$

uniformly in compact neighbourhoods.

Computer experiments suggest that this is indeed true.

By solving the first order equation above, one gets that

$$f_n(x) = e^{(n+1)\Psi_Q(x)}$$

or

$$y_n(x) = \frac{e^{(n+1)\Psi_Q(x)}}{\sqrt[k]{Q(x)}}$$

It is of course not reasonable to expect that $y_n(x)$ approaches $p_n(x)$ in any sense (note that $y_n(x)$ has no roots!). However, the ratio between two consequtive eigenfunctions may be asymptotically the same, i.e.,

Conjecture 10. In $\mathbb{C} \setminus \mathcal{C}_Q$ we have that

$$\lim_{n\to\infty} \frac{p_{n+1}(x)}{p_n(x)} = e^{\Psi_Q(x)}.$$

uniformly in compact neighbourhoods.

An even stronger form of this conjecture is as follows

Conjecture 11. In $\mathbb{C} \setminus \mathcal{C}_Q$ we have that

$$\lim_{n\to\infty} \frac{p_n(x)}{(e^{\Psi_Q(x)})^n} = \left(\frac{e^{\Psi_Q(x)}}{\sqrt[k]{Q}}\right)^{(k+1)/2}$$

uniformly in compact neighbourhoods.

Remark. Conjectures 2 and 5-11 are now proven, see [Ru].

Let us call a polynomial $Q \in \mathrm{Pol}_k^{\mathbb{C}}$ *degenerate* if its accumulation tree $\mathcal{C}_Q$ is not a three-valent graph with $2k-2$ distinct vertices. The set of all degenerate polynomial form a real hypersurface $\mathrm{Deg}_k \subset \mathrm{Pol}_k^{\mathbb{C}}$ containing the usual discriminant Disc_k consisting of all polynomials with multiple zeros. In the simplest case $k=3$ the discriminant Deg_3 consists of all 3rd degree polynomials whose zeros lie on some real affine line in $\mathbb{C}$.

Problem 12. Find the equation for Deg_k.

In the first nontrival case $k=4$ our computer experiments led us to the following:

Conjecture 13. Deg_4 is the analytic continuation of the set of all convex 4-tuples of points on $\mathbb{C}$ such that their convex hull admits an inscribed circle.

The last conjecture deals with a more general class of operators. Take a $(k+1)$-tuple of polynomials $Q_k, Q_{k-1}, ..., Q_0$ where Q_k is monic of degree k and every other Q_i is of degree at most i. Let us define the differential operator $\mathfrak{d}_{Q_k,Q_{k-1},...,Q_0} = \mathfrak{d}_{Q_k} + \mathfrak{d}_{Q_{k-1}} + ... + \mathfrak{d}_{Q_0}$.

Conjecture 14. Conjectures 5-13 hold for $\mathfrak{d}_{Q_k,Q_{k-1},...,Q_0}$ as well.

Some remarks on the above conjectures

1. The accumulation curve $\mathcal{C}_Q$ introduced above has a strong resemblance with Stokes lines occuring in the theory of differential equations with small parameter and asymptotic expansions, see [CL] and [Wa]. The role of a small parameter in our case is played by $\frac{1}{\lambda_{n,k}}$ when k is fixed and $n \to \infty$. Using the asymptotic expansion methods the first author was able to prove Conjecture 9 in some neighborhood of infinity.

2. As was mentioned to the second author by M. Kontsevich the integral $\Psi_Q(x) = \int_{x_0}^{x} \frac{dx}{\sqrt[k]{Q(x)}}$ considered on the plane curve $y^k = Q(x)$ apparently has properties similar to the properties of Strebel differentials and might serve as their generalization.

§3. Proofs

Proof of theorem 1.

Let $\mathrm{Pol}^{\mathbb{C}}_{\leq m}$ be the linear space of all polynomials with complex coefficients whose degree is less or equal m. In order to prove a) note that the action of $\mathfrak{d}_Q$ in the natural basis $x^m, x^{m-1}, ..., 1$ of $\mathrm{Pol}^{\mathbb{C}}_{\leq m}$ is given by an upper triangular matrix with nonvanishing diagonal entries $\lambda_{n,k} = (n+k)...(n+1) =$ for any $n \leq m$. Thus for each $n \geq 0$ there exists and unique monic polynomial $p_n(x)$ which is the eigenfunction of $\mathfrak{d}_Q$ corresponding to the eigenvalue $\lambda_{n,k} = \frac{(n+k)!}{n!}$.

In order to prove b) recall that by the well-known Gauss theorem the zeros of ϕ' lie in the convex hull Conv_ϕ of zeros to ϕ where ϕ is a polynomial in 1 variable with complex coefficients. Moreover, any zero of ϕ' lying on the boundary of Conv_ϕ is a multiple zero of ϕ. Let us now assume that $p_n(x)$ contains a zero not lying in Conv_Q. Then we can always choose such a zero κ of $p_n(x)$ which lie on the boundary of the convex hull Conv_{Qp_n} of the zeros of the product $Q(x)p_n(x)$. Since $p_n(x)$ satisfies (2) one gets that κ should have the same multiplicity both in $p_n(x)$ and $\frac{d^k}{dx^k}(Q(x)p_n(x))$ which is impossible by the assumption that $\kappa \notin \mathrm{Conv}_Q$. $\square$

Proof of corollary 2.

If $Q(x)$ is hyperbolic with the minimal root α and the maximal root β then $\mathrm{Conv}_Q = [\alpha, \beta]$. Therefore by Theorem 1 all zeros of $p_n(x)$ belong to $[\alpha, \beta]$. $\square$

In order to prove Theorem 3 we need a number of additional statements.

Recall that $p_n(x)$ denote a monic eigenpolynomial of degree n for the differential operator $\mathfrak{d}_Q = \frac{d^k}{dx^k}Q(x)$ where $Q(x)$ is a monic polynomial of degree k. This means that the polynomials Q and p_n satisfy the relation

$$\frac{d^k}{dx^k}(Qp_n) = \lambda_{k,n}p_n$$

where $\lambda_{n,k} := (n+k)!/n!$.

Lemma 3.1. Write $p_n = \overline{p_n}x^{n_0}$ and $Q = \overline{Q}x^{k_0}$ where $\overline{p_n}$ and $\overline{Q}$ have nonzero constant terms. If $k_0 < k$, then $k_0 + n_0 < k$ and the polynomial Qp_n is of the form

$$Qp_n = x^{k+n} + c_{k+n-1}x^{k+n-1} + \cdots + c_{k+n_0}x^{k+n_0} + c_{k-1}x^{k-1} + \cdots + c_{k_0+n_0}x^{k_0+n_0}$$

where $c_{k_0+n_0} \neq 0$.

Proof. If $n_0 = 0$ there is nothing to prove. We may therefore assume throughout the proof that $n_0 > 0$. Write

$$Qp_n = \overline{Q}\,\overline{p_n} x^{k_0+n_0} = c_{k_0+n_0} x^{k_0+n_0} + \text{higher order terms.}$$

By construction we have that $c_{k_0+n_0} \neq 0$.

Assume that $k \leq k_0 + n_0$. Then

$$\begin{aligned}
\lambda_{n,k}\overline{p_n}x^{n_0} &= \frac{d^k}{dx^k}(Qp_n) \\
&= \frac{d^k}{dx^k}(c_{k_0+n_0}x^{k_0+n_0} + \text{higher order terms}) \\
&= c_{k_0+n_0}\frac{(k_0+n_0)!}{(k_0+n_0-k)!}x^{k_0+n_0-k} + \text{higher order terms} \\
&= c_{k_0+n_0}\frac{(k_0+n_0)!}{(k_0+n_0-k)!}x^{n_0-(k-k_0)} + \text{higher order terms,}
\end{aligned}$$

which is impossible by the assumption $k > k_0$. Therefore we must have $k > k_0 + n_0$.

Choose an i with $0 \leq i < n_0$ and write $Qp_n = cx^{k+i} + \text{other terms}$. Then

$$\begin{aligned}
\lambda_{n,k}\overline{p_n}x^{n_0} &= \frac{d^k}{dx^k}Q(x)(Qp_n) \\
&= \frac{d^k}{dx^k}Q(x)(cx^{k+i} + \text{other terms}) \\
&= c\frac{(k+i)!}{i!}x^i + \text{other terms.}
\end{aligned}$$

Since $i < n_0$ while the left hand side is a multiple of x^{n_0} this implies that $c = 0$. This proves the second claim. □

Proposition 3.2. If Q has at least two different roots, then all the roots of p_n have multiplicity at most $k - 1$.

Proof. Select a root of p_n. Changing the independent variable $x \to x + \gamma$ we may assume that this root is equal to zero. Since Q has at least two different roots, $Q \neq x^k$. Thus lemma 3.1 applies and we have $n_0 \leq n_0 + k_0 < k$. □

Remark. Note that we get an even lower bound on the multiplicity in the case when the root of p_n is also a root of Q.

Proof of theorem 3.

By proposition 3.2. above we have that all the roots of p_n are real. Assume that p_n has a multiple root. Changing $x \to x + \gamma$ we may without restriction assume that this root equals zero. Lemma 3.1 thus implies that the polynomial Qp_n has at least two consequtive vanishing terms away from the ends. But since all the roots of Qp_n are real, this is impossible by lemma 3.3. below. □

The following lemma is almost certainly well known, but we prove it here for the convenience of the reader.

Lemma 3.3. Let $s = a_{n+m}x^{n+m} + \cdots + a_m x^m$ be a real polynomial with $a_{n+m} \neq 0 \neq a_m$. If $a_{m+i} = a_{m+i+1} = 0$ for some $i = 1, \ldots, n-2$ then s is not hyperbolic.

Proof. Without loss of generality we may assume that $m = 0$. First we treat the special case $s = a_n x^n + a_{n-r}x^{n-r} + \cdots + a_0$ where $r > 2$ and $a_{n-r} \neq 0$. Then $s^{(n-r)}$ is not hyperbolic and consequently s is not hyperbolic (since the derivative of a hyperbolic polynomial is hyperbolic).

The special case $s = a_n x^n + \cdots + a_r x^r + a_0$ where $r > 2$ and $a_r \neq 0$ can be reduced to the first case by looking at the polynomial $x^n s(1/x)$.

Finally the general case may be reduced to the second case by a suitable number of derivations. □

Proof of Theorem 4. For every polynomial $p(x)$ its Laplace transformation is given by

$$L[p](t) = \int_0^{+\infty} e^{-tx} p(x)dx, \qquad t > 0.$$

If a polynomial p_n satisfies the equation (2), then

$$L[\frac{d^k}{dx^k}(Qp_n)] = L[\lambda_n p_n].$$

By the standard properties of Laplace transformation (see e.g. [KoFo]), we have:

$$L[\lambda_{n,k} p_n] = \lambda_{n,k} L[p_n],$$

$$L[\frac{d^k}{dx^k}(Qp_n)] = t^k L[Qp_n] - \sum_{i=0}^{k-1} t^i \left.\frac{d^{(k-i-1)}}{dx^{(k-i-1)}}\right|_{x=0} (Qp_n),$$

$$L[Qp_n] = L[\sum_{i=0}^{k} q_i x^i p_n] = \sum_{i=0}^{k} q_i (-1)^i \frac{d^i}{dt^i} L[p_n] = Q\left(-\frac{d}{dt}\right)[L[p_n]],$$

$$L[p_n] = L[\sum_{i=0}^{n} a_{n,i} x^i] = \sum_{i=0}^{n} a_{n,i} \frac{i!}{t^{i+1}},$$

$$Q\left(-\frac{d}{dt}\right)[L[p_n]] = \sum_{i=0}^{n} a_{n,i} i! Q\left(-\frac{d}{dt}\right)\left[\frac{1}{t^{i+1}}\right] = \sum_{i=0}^{n} a_{n,i} \sum_{j=0}^{k} q_j \frac{(i+j)!}{t^{i+j+1}},$$

$$L[\frac{d^k}{dx^k}(Qp_n)] = t^k \sum_{\substack{0 \leqslant i \leqslant n, 0 \leqslant j \leqslant k \\ i+j \geqslant k}} a_{n,i} q_j \frac{(i+j)!}{t^{i+j+1}}.$$

Hence, the coefficients of the polynomial p_n satisfy the identity

$$\sum_{m=0}^{n} \frac{(m+k)!}{t^{m+1}} \sum_{\substack{m \leqslant i \leqslant n \\ i \leqslant m+k}} a_{n,i} q_{m+k-i} = \lambda_{n,k} \sum_{m=0}^{n} a_{n,m} \frac{m!}{t^{m+1}}.$$

It implies the equalities

$$\sum_{\substack{m\leqslant i\leqslant n\\ i\leqslant m+k}} a_{n,i}q_{m+k-i} = \frac{\lambda_{n,k}}{\lambda_{m,k}}a_{n,m}, \qquad m=0,\dots,n.$$

The equality for $m=n$ is trivial since $a_{n,n}=q_k=1$. The other equalities define the linear system (3). Since $\det M_n = \prod_{j=0}^{n-1}(1-\frac{\lambda_{n,k}}{\lambda_{j,k}})\neq 0$, this system has a unique solution. □

Applying Cramer's formulas to the above linear system we get

Corollary 3.4. The coefficients of the polynomial $p_n(x)$ equal to

$$a_{n,i} = -\frac{D_{n,i}}{\prod_{j=i}^{n-1}(1-\frac{\lambda_{n,k}}{\lambda_{j,k}})}, \qquad i=0,\dots,n-1$$

where $D_{n,i}$ is the determinant of the matrix obtained from the upper triangular $(n-i)\times(n-i)$-matrix $M_{n,i}$ given below substituting its first column by the numbers $(q_{k-1},q_{k-2},\dots,q_0,0,\dots,0)$ in the reverse order i.e. starting from the last entry

$$M_{n,i} = \begin{pmatrix} 1-\frac{\lambda_{n,k}}{\lambda_{i,k}} & q_{k-1} & q_{k-2} & \dots & q_0 & 0 & & \dots & 0\\ 0 & 1-\frac{\lambda_{n,k}}{\lambda_{i+1,k}} & q_{k-1} & q_{k-2} & \dots & q_0 & 0 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots\\ & & & & & & & \ddots & 0\\ & & & & & & & & q_0\\ & & & & & & & \ddots & \vdots\\ & & & & & & & \ddots & q_{k-2}\\ \vdots & & & & & & \ddots & \ddots & q_{k-1}\\ 0 & \dots & & & & & \dots & 0 & 1-\frac{\lambda_{n,k}}{\lambda_{n-1,k}} \end{pmatrix}.$$

Corollary 3.5. If the sum of roots of a polynomial $Q(x)$ vanishes then the same property holds for every $p_n(x)$ with $n>0$.

Proof. According to Corollary 3.3,

$$a_{n,n-1} = -\frac{q_{k-1}}{1-\frac{\lambda_{n,k}}{\lambda_{n-1,k}}}.$$

Hence, $a_{n,n-1}=0$ if $q_{k-1}=0$. □

Corollary 3.6. If a polynomial $Q(x)$ is even or odd, then the polynomials $p_n(x)$ with even (odd) n are even (odd, respectively).

Proof. According to Corollary 3.3, we have to prove that $D_{n,i}=0$ for all $i=n-1 \mod 2$ if $q_j=0$ for all $j=k-1 \mod 2$. But for such polynomial $Q(x)$ and odd $n-i$, the determinant $D_{n,i}$ is equal to

$$\prod_{\substack{i<j<n-1\\ j\equiv i+1 \mod 2}}(1-\frac{\lambda_{n,k}}{\lambda_{j,k}})\det\widetilde{M}_{n,i}$$

where the matrix $\widetilde{M}_{n,i}$ is obtained from $M_{n,i}$ by deletion of all even columns and rows. It remains to notice that the first column of the matrix $\widetilde{M}_{n,i}$ vanishes. $\square$

EXAMPLE. It is easy to calculate the coefficients of the polynomials $p_n(x)$ in the case when $Q(x)$ has only two terms. Indeed, let $Q(x) = x^k - \alpha x^{k-m}$ where $1 \le m \le k$. Then the system (3) can be easily solved by Gauss elimination method

$$a_{n,i} = 0 \text{ if } i \neq n \mod m,$$

$$a_{n,n-ml} = -\frac{\alpha^l}{\prod_{j=1}^{l}(1 - \frac{\lambda_{n,k}}{\lambda_{n-mj,k}})}$$

for all natural l such that $ml < n$.

PROOF OF PROPOSITION 5. We start with the following statement.

LEMMA 3.7. If $Q = (x^2 - 1)^l$ the eigenpolynomials $p_n(x)$ and $p_m(x)$, $n \neq m$ satisfy the orthogonality condition

$$\int_{-1}^{1} (x^2 - 1)^l p_n(x) p_m(x) dx = 0.$$

PROOF. Consider the polynomial family $\{q_n\}$ where $q_n = (x^2-1)^{2l} p_n$. One checks directly that $p_n = q_n^{(2l)}/\lambda_{n,2l}$ and that q_n satisfies the equation $\lambda_{n,2l} q_n(x) = (x^2-1)^l q_n^{(2l)}$. Take

$$I_{n,m} = \int_{-1}^{1} (x^2 - 1)^l p_n(x) p_m(x) dx.$$

In terms of q_i's the integral $I_{n,m}$ can be rewritten as

$$I_{n,m} = \lambda_{n,2l} \int_{-1}^{1} q_n^{(2l)}(x) q_m(x) dx.$$

Notice that both $q_n(x)$ and $q_m(x)$ have zeros of multiplicity at least l at ± 1 and therefore their derivatives of order at most $l-1$ vanish at these points. Applying integration by parts to the right-hand side of the above integral $2l$ times and using the previous remark we arrive at

$$I_{n,m} = \lambda_{n,2l} \int_{-1}^{1} q_n^{(2l)}(x) q_m(x) dx = \lambda_{n,2l} \int_{-1}^{1} q_n(x) q_m^{(2l)}(x) dx = \frac{\lambda_{n,2l}}{\lambda_{m,2l}} I_{n,m}.$$

Since $0 \neq \lambda_{m,2l} \neq \lambda_{n,2l} \neq 0$ for $m \neq n$ one has $I_{n,m} = 0$. Therefore the family $\{p_n(x)\}$ is orthogonal on the interval $[-1,1]$ w.r.t the nonnegative weight function $(x^2-1)^l$. $\square$

Proposition 5 follows since a system of polynomials orthogonal on $[-1,1]$ with the weight function $Q = (x^2 - 1)^l$ coincide (up to constant factors) with the Gegenbauer polynomials with the value of its parameter $\lambda = l + \frac{1}{2}$.

§4. Appendix. Proof of Conjecture 1 by S. Shadrin

The interlacing property of p_n's in the case when Q has only real zeros was recently proven by S. Shadrin. The proof consists of the following 3 lemmas. (Lemmas 4.1. and 4.3. are apparently well known.)

Lemma 4.1. If R_n and R_{n+1} are strictly hyperbolic polynomials of degrees n and $n+1$ resp. then $R_n + \epsilon R_{n+1}$ is hyperbolic for any sufficiently small ϵ.

Proof. For any sufficiently small ϵ the n real zeros of $R_n + \epsilon R_{n+1}$ are located in some small neighborhoods of the n simple real zeros of R_n and the $(n+1)$-th real zero has a very big absolute value. □

Lemma 4.2. If Q has only real zeros then any linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients of the polynomial egenfunctions of $\mathfrak{d}_Q$ is a hyperbolic polynomial.

Proof. Indeed, applying to $\alpha p_n + \beta p_{n+1}$ some high power $\mathfrak{d}_Q^{-N}$ of the inverse operator one gets

$$\mathfrak{d}_Q^{-N}(\alpha p_n + \beta p_{n+1}) = \frac{\alpha}{\lambda_{n,k}^N} p_n + \frac{\beta}{\lambda_{n+1,k}^N} p_{n+1} = \frac{\alpha}{\lambda_{n,k}^N}(p_n + \epsilon p_{n+1}),$$

where ϵ is arbitrarily small for the appropriate choice of N (since $0 < \lambda_{n,k} < \lambda_{n+1,k}$). Thus by lemma 4.1 the polynomial $\mathfrak{d}_Q^{-N}(\alpha p_n + \beta p_{n+1})$ is hyperbolic for sufficiently big N. Assume that $\alpha p_n + \beta p_{n+1}$ is nonhyperbolic and take the largest N_0 for which $R_{N_0} = \mathfrak{d}_Q^{-N_0}(\alpha p_n + \beta p_{n+1})$ is still nonhyperbolic. Then, obviously, $R_{N_0} = \mathfrak{d}_Q(R_{N_0+1})$ where $R_{N_0+1} = \mathfrak{d}_Q^{-N_0+1}(\alpha p_n + \beta p_{n+1})$. Note that R_{N_0+1} is hyperbolic and that if Q has only real zeros then the application of $\mathfrak{d}_Q$ to any hyperbolic polynomial results in a hyperbolic polynomial. Contradiction. □

Lemma 4.3. If R_n and R_{n+1} are any real polynomials of degrees n and $n+1$ resp. then their arbitrary linear combination $\alpha R_n + \beta R_{n+1}$ with real coefficients is hyperbolic if and only if

i) both R_n and R_{n+1} are hyperbolic;
ii) their roots are interlacing.

Proof. The condition i) is obviously necessary since in the opposite case one can choose $\alpha = 0$ or $\beta = 0$ to get a nonhyperbolic polynomial. Assume that both R_n and R_{n+1} are hyperbolic and $x_i < x_{i+1}$ are two consecutive zeros of, say, R_n not separated by a zero of R_{n+1}. Then considering $R_n + tR_{n+1}$ where $|t|$ is increasing and the sign of t is chosen so that tR_{n+1} and R_n have different signs on (x_i, x_{i+1}) we get that for sufficiently big $|t|$ the polynomial $R_n + tR_{n+1}$ looses its hyperbolicity.

On the other hand, i) and ii) are sufficient. (We consider only the case of simple zeros of R_{n+1} which implies the general case as well.) Indeed, any linear combination $\alpha R_n + \beta R_{n+1}$ with $\alpha \neq 0$ changes its sign at any two consecutive zeros of R_{n+1}. Therefore $\alpha R_n + \beta R_{n+1}$ has at least n real zeros and thus all its $n+1$ zeros are real. □

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