# ON HURWITZ–SEVERI NUMBERS

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Abstract. For a point $p \in \mathbb{CP}^2$ and a triple $(g, d, \ell)$ of non-negative integers we define a *Hurwitz–Severi number* $\mathfrak{H}_{g,d,\ell}$ as the number of generic irreducible plane curves of genus $g$ and degree $d+\ell$ having an $\ell$-fold node at $p$ and at most ordinary nodes as singularities at the other points, such that the projection of the curve from $p$ has a prescribed set of local and remote tangents and lines passing through nodes. In the cases $d+\ell \geq g+2$ and $d+2\ell \geq g+2 > d+\ell$ we express the above Hurwitz–Severi numbers via appropriate ordinary Hurwitz numbers. The remaining case $d + 2\ell < g + 2$ is still widely open.

## 1. Introduction and main results

In what follows we will always work over the field $\mathbb{C}$ of complex numbers, and by a genus $g$ of a (singular) curve $C$ we mean its geometric genus, i.e. the genus of its normalisation.

Fix a point $p \in \mathbb{CP}^2$ and denote by $\mathcal{W}_{g,d,\ell}$ the set consisting of all reduced irreducible plane curves of degree $d + \ell$, genus $g$, and having a $\ell$-fold node at the point $p$ (i.e. $\ell$ smooth local branches intersecting transversally at $p$; $\ell = 0$ means that $p$ does not belong to the curve). All the singularities outside $p$ (if any) are ordinary nodes. The set $\mathcal{W}_{g,d,\ell}$ is nonempty if and only if

$$g \leq \binom{d+\ell-1}{2} - \binom{\ell}{2}, \tag{1.1}$$

see [12].

$\mathcal{W}_{g,d,\ell}$ is usually referred to as the (open, generalized) *Severi variety*, the classical case corresponding to $\ell = 0$. The study of this variety was initiated by F. Severi [14] back in the 1920s. In a number of celebrated papers (see e.g., [6], [7], [12], [13]) it was proved that $\mathcal{W}_{g,d,\ell}$ is an irreducible algebraic variety of dimension $3d+2\ell+g-1$. Degrees of the Severi varieties are also well-studied; see [3].

A Hurwitz–Severi number $\mathfrak{H}_{g,d,\ell}$, which we define below, seems to be an equally natural characteristics of $\mathcal{W}_{g,d,\ell}$ as its degree, but at the moment we do not know how to calculate it for all triples $(g, d, \ell)$.

The set $\mathcal{W}_{g,d,\ell}$ is acted upon by the group $G \subset \mathrm{PGL}(3, \mathbb{C})$ of projective transformations of $\mathbb{CP}^2$ preserving $p$ and each line passing through $p$. Obviously, $G$ is a 3-dimensional Lie group that acts locally freely on $\mathcal{W}_{g,d,\ell}$. (In fact, unions of lines passing through $p$ are the only curves having positive-dimensional stabilizers under this action.) Denote the orbit space of this action by $\widetilde{\mathcal{W}}_{g,d,\ell} := \mathcal{W}_{g,d,\ell}/G$; it is smooth almost everywhere and its dimension equals $3d + 2\ell + g - 4$.

Let us denote by $\mathcal{C}$ a normalisation of a given plane curve $C$ and by $\kappa : \mathcal{C} \to C$, the normalisation map. For a curve $C \in \mathcal{W}_{g,d,\ell}$, one defines the associated meromorphic function of degree $d$

$$\alpha_C := \pi_p \circ \kappa : \mathcal{C} \to p^{\perp} \simeq \mathbb{CP}^1$$

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obtained by composing the normalisation map with the standard projection $\pi_p : \mathbb{CP}^2 \setminus p \to p^\perp$ from the point $p$ to the pencil $p^\perp \simeq \mathbb{CP}^1$ of lines passing through $p$.

For a generic $C \in \mathcal{W}_{g,d,\ell}$, there are $\ell$ distinct lines tangent to $C$ at $p$ (*local tangents*), and $2d+2g-2$ distinct lines passing through $p$ and tangent to $C$ elsewhere (*remote tangents*). Additionally, the curve $C$ has

$$\#_{\text{nodes}} = \binom{d+\ell-1}{2} - \binom{\ell}{2} - g = \binom{d-1}{2} + \ell(d-1) - g \geq 0$$

ordinary nodes (outside $p$), see e.g. [11]. A line passing through $p$ and a remote node will be called *node-detecting.*

For any set $X$, denote by $X^{(m)}$ its $m$-th symmetric power, i.e. the quotient of the Cartesian product $X \times X \times \cdots \times X$ of $m$ copies of $X$ by the natural action of the symmetric group $S_m$ permuting the copies. In the case when $X$ is a smooth complex curve, $X^{(m)}$ is naturally interpreted as the set of effective divisors of degree $m$ on $X$. Below we will denote elements of $X^{(m)}$ as divisors $z_1+\cdots+z_m$, where $z_1,\ldots,z_m \in X$ are not necessarily distinct. The $m$-th symmetric power $(\mathbb{CP}^1)^{(m)}$ is identified with $\mathbb{CP}^m$ by means of the standard map sending the divisor $[z_1{:}w_1] + \cdots + [z_m{:}w_m]$ to $[a_0{:}\ldots{:}a_m]$, where $\sum_{k=0}^m a_k t^k s^{m-k} := \prod_{k=1}^m (tz_k - sw_k)$.

We now define three natural maps

$$\begin{aligned}
\mathcal{RT} &: \mathcal{W}_{g,d,\ell} \to (p^\perp)^{(2d+2g-2)},\\
\mathcal{LT} &: \mathcal{W}_{g,d,\ell} \to (p^\perp)^{(\ell)},\\
\mathcal{ND} &: \mathcal{W}_{g,d,\ell} \to (p^\perp)^{(\#\text{nodes})},
\end{aligned}$$

where $\mathcal{RT}$ sends $C \in \mathcal{W}_{g,d,\ell}$ to the divisor of its remote tangents, $\mathcal{LT}$ sends $C \in \mathcal{W}_{g,d,\ell}$ to the divisor of its local tangents, and $\mathcal{ND}$ sends $C \in \mathcal{W}_{g,d,\ell}$ to the divisor of its node-detecting lines. Observe also that, for any $C \in W_{g,d,\ell}$, $\mathcal{RT}(C)$ coincides with the divisor of the critical values of the meromorphic function $\alpha_{\mathcal{C}}$.

$\mathcal{RT}$, $\mathcal{LT}$ and $\mathcal{ND}$ are obviously preserved by the action of the group $G$ and, therefore, can be considered as maps defined on $\widetilde{\mathcal{W}}_{g,d,\ell}$.

The triple of maps

$$\mathrm{Br}_{g,d,\ell} := (\mathcal{RT}, \mathcal{LT}, \mathcal{ND}) : \widetilde{\mathcal{W}}_{g,d,\ell} \to \mathcal{P}_{g,d,\ell}$$

where

$$\mathcal{P}_{g,d,\ell} := (p^\perp)^{(2d+2g-2)} \times (p^\perp)^{(\ell)} \times (p^\perp)^{(\#\text{nodes})},$$

is called the *branching morphism.*

**Definition 1.**

- A triple $(g,d,\ell)$ is called *bendable* if $\dim \widetilde{\mathcal{W}}_{g,d,\ell} \geq 2d+2g-2+\ell$. This is equivalent to $d+\ell \geq g+2$ and means that $\dim \widetilde{\mathcal{W}}_{g,d,\ell}$ is larger than or equal to the sum of the number of the branch points of $\alpha_C$ plus the number of the local tangents. A triple $(g,d,\ell)$ is called *strongly bendable* if $\dim \widetilde{\mathcal{W}}_{g,d,\ell} = 2d+2g-2+\ell+\#_{\text{nodes}}$, i.e. $\mathrm{Br}_{g,d,\ell}$ is a map between spaces of equal dimension.
- A triple $(g,d,\ell)$ is called *semi-bendable* if $\dim \widetilde{\mathcal{W}}_{g,d,\ell} < 2d+2g-2+\ell$, but $\dim \widetilde{\mathcal{W}}_{g,d,\ell} \geq 2d+2g-2$. This is equivalent to $d+\ell < g+2 \leq d+2\ell$ and means that $\dim \widetilde{\mathcal{W}}_{g,d,\ell}$ is larger than or equal to the number of the branch points of $\alpha_C$.
- Otherwise, a triple $(g,d,\ell)$ is called *unbendable.* It means that $\dim \widetilde{\mathcal{W}}_{g,d,\ell} < 2d+2g-2$ or, equivalently, that $d+2\ell < g+2$.

An easy calculation shows that for all triples $(g,d,\ell)$, one has

$$\begin{aligned}\dim \widetilde{\mathcal{W}}_{g,d,\ell} &= 3d+2\ell+g-4 = 2d+2g-2+\ell+\#_{\text{nodes}} - \frac{(d-2)(d+2\ell-3)}{2}\\ &\le 2d+2g-2+\ell+\#_{\text{nodes}} = \dim \mathcal{P}_{g,d,\ell}. \end{aligned} \tag{1.2}$$

The strongly bendable cases (where the equality takes place) are given by all triples $(g,2,\ell)$ with $g \le \ell$ (cf. (1.1)) and two exceptional triples: $(0,3,0)$ and $(1,3,0)$.

In view of (1.2), to get combinatorially meaningful quantities, we need to make the image space smaller so that its dimension would be equal to that of $\widetilde{\mathcal{W}}_{g,d,\ell}$.

**Definition 2.**

(1) Given a bendable triple $(g,d,\ell)$, let $\bar a = a_1+\dots+a_{2d+2g-2}$, $\bar b = b_1+\dots+b_\ell$, and $\bar x = x_1+\dots+x_m$ be generic divisors on $p^\perp$ of degrees $2d+2g-2$, $\ell$, and $m := \dim \widetilde{\mathcal{W}}_{g,d,\ell} - (2d+2g+\ell-2) = d+\ell-g-2$, respectively. Then the *Hurwitz–Severi number* $\mathfrak{H}_{g,d,\ell}$ is defined as the number of $G$-orbits $\mathfrak{O} \in \widetilde{\mathcal{W}}_{g,d,\ell}$ such that $\mathcal{RT}(\mathfrak{O}) = \bar a$, $\mathcal{LT}(\mathfrak{O}) = \bar b$, and $\mathcal{ND}(\mathfrak{O}) \ge \bar x$ (i.e. all the lines $x_1,\dots,x_m$ are node-detecting for any $C \in \mathfrak{O}$, but $C$ may have other node-detecting lines as well).

(2) Given a semi-bendable triple $(g,d,\ell)$, let $\bar a = a_1+\dots+a_{2d+2g-2}$ and $\bar b = b_1+\dots+b_m$ be generic divisors on $p^\perp$ of degrees $2d+2g-2$ and $m := \dim \widetilde{\mathcal{W}}_{g,d,\ell} - (2d+2g-2) = d+2\ell-g-2$, respectively. Then the *Hurwitz–Severi number* $\mathfrak{H}_{g,d,\ell}$ is defined as the number of $G$-orbits $\mathfrak{O} \in \widetilde{\mathcal{W}}_{g,d,\ell}$ such that $\mathcal{RT}(\mathfrak{O}) = \bar a$ and $\mathcal{LT}(\mathfrak{O}) \ge \bar b$ (i.e. all lines $b_1,\dots,b_m$ are local tangents for any $C \in \mathfrak{O}$, but $C$ may have other local tangents as well).

(3) Given an unbendable triple $(g,d,\ell)$, let $\bar a = a_1+\dots+a_m$ be a generic divisor on $p^\perp$ of degree $m := \dim \widetilde{\mathcal{W}}_{g,d,\ell}$. Then the *Hurwitz–Severi number* $\mathfrak{H}_{g,d,\ell}$ is defined as the number of orbits $\mathfrak{O} \in \widetilde{\mathcal{W}}_{g,d,\ell}$ such that $\mathcal{RT}(\mathfrak{O}) \ge \bar a$ (i.e. all the lines $a_1,\dots,a_m$ are remote tangents for any $C \in \mathfrak{O}$).

*Remark.* One can define a branching morphism and a Hurwitz-like number not only for Severi varieties, but also for many other natural families of plane algebraic curves. Given a generic curve $\gamma$ in such a family, take the divisor of all lines passing through a given point $p \in \mathbb{CP}^2$ which are not in general position with respect to $\gamma$. Then one can either define a branching morphism by just mapping $\gamma$ to this divisor or (as we did above) one can additionally split this divisor into several subdivisors keeping track of different singularities of the intersection of $\gamma$ with a given line. A Hurwitz-like number will be the number of preimages of a generic subspace of appropriate dimension in the image space under the branching morphism.

Our main results are formulas for the Hurwitz–Severi numbers in the bendable and semi-bendable cases. Consider the set of pairs $(\mathcal{C},\alpha)$, where $\mathcal{C}$ is a connected smooth curve of genus $g$ and $\alpha : \mathcal{C} \to \mathbb{CP}^1$ is the meromorphic function of degree $d$ with a prescribed set of simple critical values. Such pairs are considered up to an isomorphism: $(\mathcal{C},\alpha) \sim (\mathcal{C}',\alpha')$ if there exists a holomorphic homeomorphism $\phi : \mathcal{C} \to \mathcal{C}'$ such that $\alpha = \alpha' \circ \phi$. The number of the pairs is equal to $h_{g,1^d}/d!$, where $h_{g,1^d}$ is the ordinary Hurwitz number of genus $g$ and $1^d = (1,\dots,1)$ is a trivial partition of $d$; see [8] for the precise definition and an algorithm of computation of the ordinary Hurwitz numbers. (Note also that the quotient $h_{g,1^d}/d!$ is sometimes denoted by $h_{g,d}$ in the literature: it is the number of ramified covers of $\mathbb{C}P^1$ by a genus $g$ curve with a fixed generic branch divisor.)

**Theorem 1.** *Let $(g,d,\ell)$ be a bendable triple. Then the Hurwitz–Severi number $\mathfrak{H}_{g,d,\ell}$ is equal to* $\binom{d}{2}^{d+\ell-g-2} d^\ell h_{g,1^d}/d!$.

**Theorem 2.** *Let $(g, d, \ell)$ be a semi-bendable triple. Then the Hurwitz–Severi number $\mathfrak{H}_{g,d,\ell}$ is equal to $d^{d+2\ell-g-2}\binom{2g-d-\ell-1}{g-3}h_{g,1^d}/d!$.*

*Examples.*

(1) Projection of a smooth cubic from a point not lying on it. Here the triple $(g, d, \ell) = (1, 3, 0)$ is bendable. The ordinary Hurwitz number equals $h_{1,1^3} = 240$ by [8], so the Hurwitz–Severi number is $\mathfrak{H}_{1,3,0} = 40$; this result was obtained earlier in [11].

(2) Projection of a smooth cubic from a point lying on it. The triple $(g, d, \ell) = (1, 2, 1)$ is bendable. The ordinary Hurwitz number equals $h_{1,1^2} = 1$ by [8], so the Hurwitz–Severi number equals $\mathfrak{H}_{1,3,0} = 1$; this can be checked by a direct computation.

(3) Projection of a nodal cubic from an outside point corresponds to a bendable triple $(g, d, \ell) = (0, 3, 0)$. Here $h_{0,1^3} = 24$ by [8], implying $\mathfrak{H}_{0,3,0} = 12$. This answer can be checked directly using a computer algebra system.

(4) Projection of a nodal cubic from its smooth point corresponds to $(g, d, \ell) = (0, 2, 1)$. The ordinary Hurwitz number equals $h_{0,1^2} = 1$, so the Hurwitz–Severi number equals $\mathfrak{H}_{0,2,1} = 1$, which is easily checked by hand.

(5) Projection of a smooth quartic from its point corresponds to $(g, d, \ell) = (3, 3, 1)$, a semi-bendable triple. The ordinary Hurwitz number computed using the standard formulas of [8] equals $h_{3,1^3} = 19680$, so the Hurwitz–Severi number equals $\mathfrak{H}_{3,3,1} = 3280$.

(6) Projection of a smooth quartic from an outside point corresponds to $(g, d, \ell) = (3, 4, 0)$. This is an unbendable triple not covered by Theorems 1 and 2. This case was investigated by R. Vakil in [15] using different techniques. His answer is $\mathfrak{H}_{3,4,0} = 3762 \times 120$.

Theorems 1 and 2 give a complete description of Hurwitz–Severi numbers in the bendable and semi-bendable cases. Unlike them, the unbendable case seems to require completely new ideas. The only result in the unbendable case known to the authors at the time of writing is [15].

*Remark.* Let $u$ be a generic point in the image of the map $\mathrm{Br}_{g,d,\ell}$. By Theorems 1 and 2, the number of preimages $\mathrm{Br}^{-1}_{g,d,\ell}(u) \in \widetilde{\mathcal{W}}_{g,d,\ell}$ is the same for all $u$. This allows us to formulate

**Conjecture 1.** *For any triple $(g, d, \ell)$ the map $\mathrm{Br}_{g,d,\ell}$ is a branched covering of $\widetilde{\mathcal{W}}_{g,d,\ell}$ onto its image.*

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## 2. Proofs

The symmetric power $\mathcal{C}^{(k)}$ of the curve $\mathcal{C}$ is the set of effective divisors $D$ of degree $k$ on $\mathcal{C}$. Let $D = x_1c_1 + \cdots + x_mc_m \in \mathcal{C}^{(k)}$ where $c_1, \ldots, c_m \in \mathcal{C}$ are pairwise distinct, $x_1, \dots, x_m \in \mathbb{Z}_{>0}$, and $x_1 + \dots + x_m = k$. Let $U_1, \dots, U_m \subset \mathcal{C}$ be non-overlapping open sets such that $c_i \in U_i$ for all $i = 1, \dots, m$; equip each $U_i$ with a complex coordinate $z_i$ with $z_i(c_i) = 0$.

Let $U$ be the image of $U_1 \times \dots \times U_m$ under the standard projection $\mathcal{C}^k \to \mathcal{C}^{(k)}$ and $\tilde{D} = p_1 + \dots + p_k \in U$ (where $p_1, \dots, p_k$ are not necessarily distinct). Without loss of generality, it means that $p_1, \dots, p_{k_1} \in U_1$, $p_{k_1+1}, \dots, p_{k_2} \in U_2$, $\dots$, $p_{k_{m-1}+1}, \dots, p_k \in U_m$ for some $1 \le k_1 \le \dots \le k_{m-1} \le k$. Now for every $i = 1, \dots, m$ consider the principal part $F_i$ of a meromorphic function $f_i : U_i \to \mathbb{CP}^1$ having a pole at $p_i$ with the degree not exceeding the multiplicity of $p_i$ in the divisor $\tilde{D}$ and having no other poles in $U_i$. Set $F = (F_1, \dots, F_m)$. Using the local coordinates $z_i$, $i = 1, \dots, m$, one observes that every $F_i$ is the principal part of the function $f_i$ where

$$\begin{aligned} f_1(c) &= \frac{a_1 + a_2 z_1(c) + \dots + a_{k_1} z_1(c)^{k_1 - 1}}{(z_1(c) - z_1(p_1)) \dots (z_1(c) - z_1(p_{k_1}))} \\ &\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ f_m(c) &= \frac{a_{k_{m-1}+1} + a_{k_{m-1}+2} z_m(c) + \dots + a_k z_m(c)^{k - k_{m-1} - 1}}{(z_m(c) - z_m(p_{k_{m-1}+1})) \dots (z_m(c) - z_m(p_k))}. \end{aligned}$$

So, the vectors $F$ of the principal parts form a rank $k$ vector bundle over $\mathcal{C}^{(k)}$; the coefficients $a_1, \dots, a_k$ form a trivialisation of this bundle over the open set $U$. An immediate comparison of the transition maps shows that this bundle is isomorphic to the tangent bundle $T\mathcal{C}^{(k)}$.

Given a 1-form $\nu$ holomorphic in $U_i$ and a point $z \in U_i$, we define a linear functional $\nu_z$ on the space of the principal parts by the formula

$$\nu_z(F_i) = \mathrm{Res}_z\, F_i \nu.$$

For a divisor $D = x_1 c_1 + \dots + x_m c_m \in \mathcal{C}^{(k)}$, define $\nu_D := \sum_{i=1}^m \nu_{c_i}$. It follows from the above reasoning that $\nu_D$ is a section of the complex cotangent bundle $T^*\mathcal{C}^{(k)}$; cf. the fiber bundle of principal parts introduced in [4].

There exists a natural map $\Phi : \mathcal{O}(D) \to T_D \mathcal{C}^{(k)}$ sending a memomorphic function $f \in \mathcal{O}(D)$ to the $m$-tuple $F = (F_1, \dots, F_m)$ of its principal parts at the points $c_1, \dots, c_m$. By the Riemann–Roch theorem, $F = \Phi(f)$ for some $f$ if and only if $\nu_D(F) = 0$ for every holomorphic 1-form $\nu$ on $\mathcal{C}$. Fixing a basis $\nu_1, \dots, \nu_g$ of the space of holomorphic 1-forms on $\mathcal{C}$, we can calculate the dimension $h^0(D) = \dim \mathcal{O}(D)$ as $k + 1 - \dim\langle (\nu_1)_D, \dots, (\nu_g)_D \rangle$. For a meromorphic function $\alpha : \mathcal{C} \to \mathbb{C}P^1$ denote by $D_\alpha$ its divisor of poles.

To prove our main results, we need the following technical statement which is apparently well-known to the specialists, but we could not find it explicitly in the literature:

**Proposition 3.** *Take a pair $(\mathcal{C}, \alpha)$, where $\mathcal{C}$ is a smooth curve of genus $g$ and $\alpha : \mathcal{C} \to \mathbb{CP}^1$ is a meromorphic function of degree $d$, and suppose that $(\mathcal{C}, \alpha)$ is generic among such pairs. Set $D_\alpha := z_1 + \dots + z_d$, where all $z_i$ are pairwise distinct. If $m \ge g + 2 \ge d$ and $z_{d+1}, \dots, z_m$ are generic pairwise distinct points, then the divisor $z_1 + \dots + z_m$ is non-special.*

Generic divisors are never special, but $z_1 + \dots + z_m$ may be non-generic because $z_1 + \dots + z_m \ge D_\alpha$ for some $\alpha$ of degree $d$.

*Proof.* Let $\nu_1, \dots, \nu_g$ be a basis of holomorphic 1-forms on $\mathcal{C}$. Since $z_1 + \dots + z_d = D_\alpha$, one has $h^0(z_1 + \dots + z_d) \ge 2$. If there exists $\psi \in \mathcal{O}(D_\alpha)$ not proportional to $\alpha$, then there exists their non-constant linear combination with no pole at $z_d$. Since a generic $d$-gonal curve $\mathcal{C}$ is not $(d-1)$-gonal [5], this is impossible, and therefore $h^0(z_1+\cdots+z_d)=2$.

We now prove by induction that if $d+s\le g+1$ and the points $z_{d+1},\dots,z_{d+s}$ are in general position, then

$$\dim\langle(\nu_1)_{z_1+\dots+z_{d+s}},\dots,(\nu_g)_{z_1+\dots+z_{d+s}}\rangle = d+s-1.$$

Assume that starting with some $s$ the statement fails. It means that for $z_{d+1},\dots,z_{d+s-1}$ in general position,

$$\dim\langle(\nu_1)_{z_1+\dots+z_{d+s-1}},\dots,(\nu_g)_{z_1+\dots+z_{d+s-1}}\rangle = d+s-2,$$

but there exists a non-empty open set $\Omega\subset\mathcal{C}$ such that if $z_{d+s}\in\Omega$, then

$$\dim\langle(\nu_1)_{z_1+\dots+z_{d+s}},\dots,(\nu_g)_{z_1+\dots+z_{d+s}}\rangle = d+s-2$$

as well. In other words, vector $\vec{\nu}_{z_{d+s}} := ((\nu_1)_{z_{d+s}},\dots,(\nu_g)_{z_{d+s}})$ is a linear combination of $\vec{\nu}_{z_i}$, $i=1,\dots,d+s-1$.

The idea of the proof is to add *many* points from $\Omega$ keeping the divisor special. Let $q$ be any integer, and $z_{d+s},\dots,z_{d+s+q-1}\in\Omega$, then for $j=d+s,\dots,d+s+q-1$ every $\vec{\nu}_{z_j}$ is a linear combination of $\vec{\nu}_{z_i}$, $i=1,\dots,d+s-1$. Therefore

$$\dim\langle(\nu_1)_{z_1+\dots+z_{d+s+q-1}},\dots,(\nu_g)_{z_1+\dots+z_{d+s+q-1}}\rangle = d+s-2$$

for any $q$. Hence the divisor $z_{d+s}+\cdots+z_{d+s+q-1}$ is special for any collection $z_{d+s},\dots,z_{d+s+q-1}\in\Omega$, implying that the set of special divisors of any degree $q>g$ on $\mathcal{C}$ contains an open subset $\Omega^{(q)}\subset\mathcal{C}^{(q)}$. The latter claim is false since the set of special divisors of any sufficiently large degree is nowhere dense. $\square$

*Proof of Theorem 1.* Take $p=[0:1:0]$ and suppose without loss of generality that a curve $C\in\mathcal{W}_{g,d,\ell}$ does not contain the point $[1:0:0]$. Then the normalisation map $\kappa:\mathcal{C}\to C$ is given by

$$\kappa(z)=[\alpha(z):\beta(z):1],$$

where $\alpha,\beta:\mathcal{C}\to\mathbb{CP}^1$ are meromorphic functions of degrees $d$ and $d+\ell$, respectively, such that $D_\beta\ge D_\alpha$. In other words, $D_\beta-D_\alpha$ is an effective divisor on $\mathcal{C}$. For such choice of $p$ the action of the group $G$ (the group of projective transformations preserving $p$ and every element of $p^\perp$) on $\mathbb{C}P^2$ is given by the formulas

$$[x:y:1]\mapsto[x:px+qy+r:1] \tag{2.1}$$

where $q\ne0$ and $p,r$ are arbitrary.

Take a generic divisor $\bar{a}=a_1+\cdots+a_{2d+2g-2}$ on $\mathbb{CP}^1$. As it was noted above, there exist $h_{g,1^d}/d!$ pairs $(\mathcal{C},\alpha)$ such that $\bar{a}$ is the divisor of the critical values of $\alpha$. For any $\beta$, one can take $C:=\kappa(\mathcal{C})$, with the map $\kappa:\mathcal{C}\to\mathbb{CP}^2$ as above. Then $\mathcal{RT}(C)=\bar{a}$ regardless of the choice of $\beta$. Set $D_\alpha:=z_1+\cdots+z_d$ and notice that for generic $\bar{a}$ the points $z_1,\dots,z_d\in\mathcal{C}$ are pairwise distinct (i.e. $\alpha$ has only simple poles).

Now take a generic divisor $\bar{b}=b_1+\cdots+b_\ell$ on $\mathbb{CP}^1$ and choose points $z_{d+1},\dots,z_{d+\ell}\in\mathcal{C}$ such that $\bar{b}=\alpha(z_{d+1})+\cdots+\alpha(z_{d+\ell})$. Since the degree of $\alpha$ is $d$, there are $d^\ell$ ways to do this; the points $z_{d+1},\dots,z_{d+\ell}$ are pairwise distinct if $\bar{b}$ is in general position. For any such choice of $D_\beta=z_1+\cdots+z_{d+\ell}$ one has the equality $\mathcal{LT}(C)=\bar{b}$ for the curve $C=\kappa(\mathcal{C})$.

Assume now that $\bar{x}=x_1+\cdots+x_{d+\ell-g-2}$ is a generic divisor on $\mathbb{CP}^1$. For each $i$, take a pair of points $u_i\ne v_i\in\mathcal{C}$ such that $\alpha(u_i)=\alpha(v_i)=x_i$; there are $\binom{d}{2}^{d+\ell-g-2}$ ways to do this. For $i=1,\dots,d+\ell-g-2$, define the functionals $\rho_i:\mathcal{O}(z_1+\cdots+z_{d+\ell})\to\mathbb{C}$ by

$$\rho_i(\beta):=\beta(u_i)-\beta(v_i).$$

Apparently, $\rho_i(\beta) = 0$ if and only if the line $x_i$ is node-detecting for the curve $C = \kappa(\mathcal{C})$.

**Lemma 4.** *For a generic choice of $\alpha$ and $z_{d+1}, \dots, z_{d+\ell}$,, the functionals $\rho_i$, $i = 1, \dots, d + \ell - g - 2$ are linearly independent.*

*Proof.* By the Riemann–Roch theorem, $h^0(z_1 + \cdots + z_{d+\ell}) \geq d + \ell - g + 1$. Thus, for any $k \leq d + \ell - g - 2$, there exists a function $\beta \in \mathcal{O}(z_1 + \cdots + z_{d+\ell})$ such that $\rho_1(\beta) = \cdots = \rho_{k-1}(\beta) = 0$. If for generic $\alpha$ and $z_{d+1}, \dots, z_{d+\ell}$, the functional $\rho_k$ is a linear combination of $\rho_i$, $1 \leq i \leq k-1$, then there exists an open subset $\Omega \subset \mathbb{C}P^1$ with the following property: if $\alpha(u) = \alpha(v) = x \in \Omega$, then $\beta(u) = \beta(v)$.

In other words, $\beta(z)$ is a function of $\alpha(z)$ for $z \in \alpha^{-1}(\Omega)$. Therefore for any $z \in \alpha^{-1}(\Omega)$, the line $\alpha(z) \in p^{\perp}$ intersects the curve $C$ at exactly one point. But this is impossible for many reasons. For example, since $\alpha^{-1}(\Omega) \subset \mathcal{C}$ is open, there exists $z_* \in \alpha^{-1}(\Omega)$ such that the intersection in question is transversal. Thus the intersection index of the line $\alpha(z)$ with the curve $C$ is 1, but the degree of $C$ is $d > 1$. □

Now by Proposition 3, one has generically

$$h^0(z_1 + \cdots + z_{d+\ell}) = d + \ell - g + 1$$

implying that the solutions of the equations $\rho_1(\beta) = \cdots = \rho_{d+\ell-g-2}(\beta) = 0$ form a 3-dimensional space. It follows from (2.1) that the group $G$ acts transitively on this space of solutions, so Theorem 1 follows. □

Consider now a semi-bendable case. Here the dimension of $\widetilde{\mathcal{W}}_{g,d,\ell}$ is enough to fix all the remote tangents to the curve but only $g + 2 - d - \ell$ out of totally $\ell$ local tangents. The proof of Theorem 2 is based on the following statement:

**Proposition 5.** *Let $(g, d, \ell)$ be a semi-bendable triple. Then for a generic pair $(\mathcal{C}, \alpha)$, where $\mathcal{C}$ is a smooth curve and $\alpha : \mathcal{C} \to \mathbb{CP}^1$ is a degree $d$ meromorphic function, and for a generic divisor $D_0$ of degree $d + 2\ell - g - 2$ on $\mathcal{C}$, the set*

$$\mathcal{D} := \{D \in \mathcal{C}^{(g+2-d-\ell)} \mid h^0(D_\alpha + D_0 + D) \geq 3\}$$

*is finite and contains $\binom{2g-d-\ell-1}{g-3}$ elements. Additionally, for any $D \in \mathcal{D}$, one has $h^0(D_\alpha + D_0 + D) = 3$.*

*Proof.* Choose $D \in \mathcal{D}$ and $\beta \in \mathcal{O}(D_\alpha + D_0 + D)$ and define a plane curve $C_{D,\beta} := \kappa(\mathcal{C})$, where $\kappa(z) := [\alpha(z) : \beta(z) : 1]$. The group $G$ acts on $\mathcal{O}(D_\alpha + D_0 + D)$ by (2.1). Thus $G$ acts on the set of all $C_{D,\beta}$ with a fixed $D$.

Prove first that $\mathcal{D}$ is finite and that $h^0(D_\alpha + D_0 + D) = 3$ for any $D \in \mathcal{D}$. Consider the orbit space $\widetilde{\mathcal{D}} := \{(D, \beta) \mid D \in \mathcal{D}, \beta \in \mathcal{O}(D_\alpha + D_0 + D)\}/G$ and let $\delta := \dim \widetilde{\mathcal{D}}$ be its dimension. The pair $(\mathcal{C}, \alpha)$ is determined, up to a finite choice, by the divisor of the critical values of $\alpha$, which has degree $2d + 2g - 2$; so the set of all such pairs has dimension $2d + 2g - 2$. The dimension of the set of all divisors $D_0$ is equal to $\deg D_0 = d + 2\ell - g - 2$. The choice of $(D, \beta) \in \widetilde{\mathcal{D}}$ determines a curve $C_{D,\beta}$ up to the action of $G$, that is, it determines a point in $\widetilde{\mathcal{W}}_{g,d,\ell}$. On the other hand, for a given curve $C_{D,\beta} \in \mathcal{W}_{g,d,\ell}$, one can uniquely restore the divisor $D$ on the normalisation $\mathcal{C}$ of $C_{D,\beta}$ noticing that its points are the poles of $\beta$ or, equivalently, the points of $\mathcal{C}$ sent by the normalisation map to the base point $p \in C_{D,\beta}$. So, different choices of $D \in \mathcal{D}$ and different orbits of the $G$-action on $\mathcal{O}(D_\alpha + D_0 + D)$ for a fixed $D$ correspond to different points of $\widetilde{\mathcal{W}}_{g,d,\ell}$. This implies the inequality

$$\dim \widetilde{\mathcal{W}}_{g,d,\ell} \geq (2d + 2g - 2) + (d + 2\ell - g - 2) + \delta.$$

Since $\dim \widetilde{\mathcal{W}}_{g,d,\ell} = 3d + 2\ell + g - 4$ (see e.g., [6]), one gets that $\delta = 0$. Thus, $\mathcal{D}$ consists of a finite number of points, and for any such point $D$, the number of $G$-orbits in $\mathcal{O}(D_\alpha + D_0 + D)$ is finite, which means that $h^0(D_\alpha + D_0 + D) = 3$.

Count now the points $D \in \mathcal{D}$. Set $D_\alpha := z_1 + \dots + z_d$, $D_0 := z_{d+1} + \dots + z_{2d+2\ell-g-2}$, $D := z_{2d+2\ell-g-1} + \dots + z_{d+\ell}$ and denote

$$D' := D_\alpha + D_0 + D - z_d = z_1 + \dots + z_{d-1} + z_{d+1} + \dots + z_{d+\ell}.$$

As was shown above, $h^0(D_\alpha + D_0 + D) \geq 3$ if and only if

$$\dim\langle(\nu_1)_{D_\alpha+D_0+D}, \dots, (\nu_g)_{D_\alpha+D_0+D}\rangle \leq d + \ell - 2$$

or, equivalently, $\dim\langle\vec{\nu}_{z_1}, \dots, \vec{\nu}_{z_{d+\ell}}\rangle \leq d + \ell - 2$. Since $\mathcal{C}$ is a generic $d$-gonal curve, it is not $(d-1)$-gonal implying that $h^0(D_\alpha) = 2$. Thus the vector $\vec{\nu}_{z_d}$ is a linear combination of $\vec{\nu}_{z_1}, \dots, \vec{\nu}_{z_{d-1}}$ (see the proof of Proposition 3 for notation), which means that the last condition is equivalent to

$$\dim\langle\vec{\nu}_{z_1}, \dots, \vec{\nu}_{z_{d-1}}, \vec{\nu}_{z_{d+1}}, \dots, \vec{\nu}_{z_{d+\ell}}\rangle \leq d + \ell - 2,$$

i.e. $\dim\langle(\nu_1)_{D'}, \dots, (\nu_g)_{D'}\rangle \leq d + \ell - 2$.

For any $k$ denote by $S_k : \mathcal{C}^k \to \mathcal{C}^{(k)}$ the natural projection; for any vector $X \in \mathcal{C}^m$, denote by $\iota_X : \mathcal{C}^k \to \mathcal{C}^{m+k}$ the natural embedding (coordinates of $X$ are written before the coordinates of the argument). Take any point $Z = (z_1, \dots, z_{d-1}, z_{d+1}, \dots, z_{2d+2\ell-g-2})$ such that $S_{2d+2\ell-g-3}(Z) = D_0 + D_1 - z_d$ and consider the vector bundle $E = \iota_Z^* S_{d+\ell-1}^* T^*\mathcal{C}^{(d+\ell-1)}$ of rank $d+\ell-1$ on $\mathcal{C}^{g+2-d-\ell}$. (In other words, $Z$ is an arbitrary ordering of $z_1, \dots, z_{d-1}, z_{d+1}, \dots, z_{2d+2\ell-g-2}$.) The Riemann–Roch theorem implies that $D \in \mathcal{D}$ if and only if for any $W \in S_{g+2-d-\ell}^{-1}(D_1)$, one has

$$\dim\langle\iota_Z^* S_{d+\ell-1}^*(\nu_1)_{D'}(W), \dots, \iota_Z^* S_{d+\ell}^*(\nu_g)_{D'}\rangle \leq d + \ell - 2.$$

Now we are in a typical setting of the intersection theory. Namely, to describe the set of points at the base of a vector bundle such that values of some given sections at every such point span a subspace of the fiber having a given dimension (or smaller). The set of points in question is the variety $S_{g+2-d-\ell}^{-1}(\mathcal{D})$; we have proved that it has the expected dimension equal to 0. So by [9] its cardinality is determined by the Porteous formula:

$$\#S_{g+2-d-\ell}^{-1}(\mathcal{D})Y = \det\begin{pmatrix} c_1(E) & c_2(E) & \dots & c_{g+1-d-\ell}(E) & c_{g+3-d-\ell}(E) \\ 1 & c_1(E) & \dots & c_{g-d-\ell}(E) & c_{g+1-d-\ell}(E) \\ 0 & 1 & \dots & c_{g-1-d-\ell}(E) & c_{g-d-\ell}(E) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & c_1(E) \end{pmatrix}, \tag{2.2}$$

where $Y \in H^{2(g+2-d-\ell)}(\mathcal{C}^{g+2-d-\ell})$ is the generator of the top-dimensional cohomology, i.e. the Poincaré dual of a point.

Set $m := g+2-d-\ell$ for brevity, and denote by $\mathcal{N}_{k,m}$ the lower right $(k\times k)$-minor of (2.2). In particular, $\#S_{g+2-d-\ell}^{-1}(\mathcal{D})Y = \mathcal{N}_{m,m}$. Expanding the determinant by its first column, one obtains $\mathcal{N}_{m,m} = \sum_{k=1}^m (-1)^{k+1} c_k(E)\mathcal{N}_{m-k,m}$, implying

$$\sum_{k=0}^m \mathcal{N}_{k,m} = \left(\sum_{k=0}^m (-1)^k c_k(E)\right)^{-1}. \tag{2.3}$$

Denote by $x \in H^2(\mathcal{C}^{(k)})$ the class dual to the fundamental homology class of the diagonal $\{kz \mid z \in \mathcal{C}\} \subset \mathcal{C}^{(k)}$. It follows from the general formula of [10] that $c_k(T^*\mathcal{C}^{(m)}) = \binom{2g-2+k-m}{k} x^k$. One has $S_m^* x = x_1 + \dots + x_m := X$, where $x_i \in H^2(\mathcal{C}^m)$ is the class dual to the fundamental class of the $i$-th copy of $\mathcal{C}$ in the product $\mathcal{C} \times \cdots \times \mathcal{C} = \mathcal{C}^k$. Additionally, one has $\iota_Z^* x_i = x_{i-m} \in H^2(C^k)$ if $i > m$ and $\iota_Z^* x_i = 0$ if $i \le m$. Therefore $c_k(E) = \binom{g-3+m+k}{k} X^k$, implying that

$$\sum_{k=0}^{m} (-1)^k \binom{g-3+m+k}{k} X^k = (1+X)^{-(g-3+m)}.$$

Thus, (2.3) implies that $\sum_{k=0}^{m} \mathcal{N}_{k,m} = (1+X)^{(g-3+m)}$, giving $\mathcal{N}_{m,m} = \binom{g-3+m}{g-3} X^m$. Since $Y = X^m/m!$, then $\#S_m^{-1}(\mathcal{D}) = m!\binom{g-3+m}{g-3}$. By dimensional reasons, in generic situation all the elements $\mathcal{D}$ are sums of exactly $m$ distinct points, meaning that

$$\#\mathcal{D} = \frac{1}{m!}\#S_m^{-1}(\mathcal{D}) = \binom{g-3+m}{g-3} = \binom{2g-1-d-\ell}{g-3}.$$

□

*Proof of Theorem 2.* Similarly to the proof of Theorem 1, for a generic divisor $\bar{a} = a_1 + \cdots + a_{2d+2g-2}$, there are $h_{g,1^d}/d!$ ways to choose a curve $\mathcal{C}$ of genus $g$ and a degree $d$ meromorphic function $\alpha : \mathcal{C} \to \mathbb{CP}^1$ such that $\bar{a}$ is its divisor of critical values.

Let $D_\alpha$ be the pole divisor of $\alpha$; choose $d+2\ell-g-2$ points $z_{d+1}, \ldots, z_{2d+2\ell-g-2} \in \mathcal{C}$ such that $\bar{b} = \alpha(z_{d+1}) + \cdots + \alpha(z_{2d+2\ell-g-2})$. For generic $\bar{b}$, there are $d^{d+2\ell-g-2}$ ways to do that.

Similar to the proof of Proposition 5, denote $D_0 := z_{d+1} + \cdots + z_{2d+2\ell-g-2}$ for short, and denote by $\mathcal{D} \subset \mathcal{C}^{(g+2-d-\ell)}$ the set of effective divisors $D := z_{2d+2\ell-g-1} + \cdots + z_{d+\ell}$ of degree $g+2-d-\ell$ such that $h^0(D_\alpha + D_0 + D) \ge 3$. By Proposition 5, the set $\mathcal{D}$ is finite, and for any $D \in \mathcal{D}$, the space $\mathcal{O}(D_\alpha + D_0 + D)$ has dimension 3 and contains exactly one orbit of the group $G$. Therefore the number of points $C \in \widetilde{\mathcal{W}}_{g,d,\ell}$ with $\mathcal{RT}(C) = a$ and $\mathcal{LT}(C) \ge b$ is equal to

$$d^{d+2\ell-g-2} h_{g,1^d}/d!\#\mathcal{D} = d^{d+2\ell-g-2}\binom{2g-d-\ell-1}{g-3} h_{g,1^d}/d!.$$

□

## 3. Final Remarks

1. Our definition of the Hurwitz–Severi numbers given above can be easily extended from the class of Severi varieties $W_{g,d,\ell}$ to a somewhat broader class $W_{g,d,\ell,\mu}$ which appeared earlier in several papers of J. Harris and Z. Ran. Namely, one can additionally require that curves under consideration have a given set $\mu$ of tangency multiplicities to a given line passing through the point $p$. One might expect Theorems 1 and 2 to have straightforward analogs in this more general setup.

2. The problem of calculation of the Hurwitz–Severi numbers for the simplest unbendable case $g = (d-1)(d-2)/2, \ell = 0$, i.e. when a smooth plane curve of degree $d$ is projected from a point not lying on it, bears a strong resemblance with the problem of calculation of Zeuten's numbers, see [16]. Namely, in a special case, Zeuten's problem asks how many smooth plane curves of degree $d$ are tangent to a given set of $\frac{d(d+3)}{2}$ lines in general position. The Hurwitz–Severi number for the case $g = (d-1)(d-2)/2$ and $\ell = 0$ counts the number of $G$-orbits of smooth curves of degree $d$ which are tangent to a given set of $\frac{d(d+3)}{2} - 3$ generic lines passing through a given point $p$. To the best of our knowledge, both problems are unsolved at present and apparently are quite difficult.

3. One possible approach to the calculation of Hurwitz numbers in the unstable case (such as $((d-1)(d-2)/2, d, 0))$ might be the use of tropical algebraic geometry. For example, in [1] the authors studied tropical analogs of Zeuten's numbers and were able to recover some of the classical Zeuten numbers through their tropical analogs.

4. It would be interesting to study possible relation of the above Hurwitz–Severi numbers to the appropriate Gromov-Witten invariants of plane curves.

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