

ON SPECTRAL POLYNOMIALS OF THE HEUN EQUATION. II.

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ABSTRACT. The well-known *Heun equation* has the form

$$\left\{ Q(z) \frac{d^2}{dz^2} + P(z) \frac{d}{dz} + V(z) \right\} S(z) = 0,$$

where $Q(z)$ is a cubic complex polynomial, $P(z)$ and $V(z)$ are polynomials of degree at most 2 and 1 respectively. One of the classical problems about the Heun equation suggested by E. Heine and T. Stieltjes in the late 19-th century is for a given positive integer n to find all possible polynomials $V(z)$ such that the above equation has a polynomial solution $S(z)$ of degree n . Below we prove a conjecture of the second author, see [17] claiming that the union of the roots of such $V(z)$'s for a given n tends when $n \rightarrow \infty$ to a certain compact connecting the three roots of $Q(z)$ which is given by a condition that a certain natural abelian integral is real-valued, see Theorem 2. In particular, we prove several new results of independent interest about rational Strebel differentials.

1. INTRODUCTION AND MAIN RESULTS

The classical Heun equation

$$\left\{ Q(z) \frac{d^2}{dz^2} + P(z) \frac{d}{dz} + V(z) \right\} S(z) = 0, \quad (1.1)$$

where $Q(z)$ is a cubic polynomial, $P(z)$ is at most quadratic, and $V(z)$ is at most linear polynomials was and still is an object of active study, see [14]. Throughout this paper we always assume that $Q(z)$ is monic. The special case of (1.1) when $P(z) = Q'(z)/2$ is widely known as the *classical Lamé* equation. Below we study one aspect of the Heun equation suggested by E. Heine and T. Stieltjes, see [5], [18], and [21], ch. 23.

Problem 1 (Heine-Stieltjes). For a given pair of polynomials $Q(z)$ and $P(z)$ as above and a positive integer n find all polynomials $V(z)$ such that (1.1) has a polynomial solution $S(z)$ of degree n .

Polynomials $V(z)$ (resp. $S(z)$) are usually referred to as *Van Vleck* (resp. *Stieltjes*) polynomials. Already Heine and Stieltjes knew that for a generic pair $(Q(z), P(z))$ and any positive integer n there exist exactly $n + 1$ such distinct Van Vleck polynomials $V(z)$. Moreover in the case of the Lamé equation when one additionally assumes that the polynomial $Q(z)$ has three distinct real roots $\alpha < \beta < \gamma$ resp. Stieltjes was able to prove the following. The roots of any $V(z)$ and $S(z)$ belong to the interval (α, γ) and for a given n the $(n + 1)$ existing Stieltjes polynomials are distinguished by how many of their roots lie in the interval (α, β) . (The remaining roots lie in the interval (β, γ) , see [21], ch. 23, section 46.) Some further information of asymptotic character can be found in [3] and [12].

For a general Heun equation no essential results about the location of the roots of Van Vleck and Stieltjes polynomials seems to be previously known. One of the

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few exceptions is a classical proposition of Pólya, [13] claiming that if the rational function $P(z)/Q(z)$ has all positive residues then any root of any $V(z)$ as above and of any $S(z)$ as above lies within Δ_Q where Δ_Q is the convex hull of the set of all three roots of $Q(z)$.

The next statement is a specialization of the main result of [16] in the case of the Heun equation.

Theorem 1. *For any cubic polynomial $Q(z)$ and any polynomial $P(z)$ of degree at most 2 one has that*

- (1) *there exists N such that for any $n \geq N$ there exist exactly $n + 1$ linear polynomials $V(z)$ counted with appropriate multiplicity such that (1.1) has a polynomial solution $S(z)$ of degree exactly n ;*
- (2) *for any $\epsilon > 0$ there exists N_ϵ such that for any $n \geq N_\epsilon$ any root of any $V(z)$ having $S(z)$ of degree n as well as any root of this $S(z)$ lie in the ϵ -neighborhood of Δ_Q .*

Thus we can introduce the set \mathcal{V}_n consisting of polynomials $V(z)$ giving a polynomial solution $S(z)$ of (1.1) of degree n ; each such $V(z)$ appearing in \mathcal{V}_n the number of times equal to its multiplicity. (The exact definition of multiplicity of $V(z)$ is rather lengthy and is omitted here. An interested reader is recommended to consult [16] for details.) Then by the above results the set \mathcal{V}_n will contain exactly $n + 1$ linear polynomials for all sufficiently large n . It will be convenient to introduce a sequence $\{Sp_n(\lambda)\}$ of *spectral polynomials* where the n -th spectral polynomial is defined by

$$Sp_n(\lambda) = \prod_{j=1}^{n+1} (\lambda - t_{n,j}),$$

$t_{n,j}$ being the unique root of the j -th polynomial in \mathcal{V}_n in any fixed ordering. Notice that $Sp_n(\lambda)$ is well-defined for all sufficiently large n .

Associate to $Sp_n(\lambda)$ the finite measure

$$\mu_n = \frac{1}{n+1} \sum_{j=1}^{n+1} \delta(z - t_{n,j}),$$

where $\delta(z - a)$ is the Dirac measure supported at a . The measure μ_n obtained in this way is clearly a real probability measure which one usually refers to as the *root-counting measure* of the polynomial $Sp_n(\lambda)$.

Our main question is as follows.

Problem 2. Does the sequence $\{\mu_n\}$ converge (in the weak sense) to some limiting measure μ ? If the convergence takes place describe the limiting measure μ ?

Below we answer both parts of this question, see Theorem 2. With an essential contribution of the second author we were able to prove the existence of μ and to find the following elegant description of its support.

Denote the three roots of $Q(z)$ by a_1, a_2, a_3 . For $i \in \{1, 2, 3\}$ consider the curve γ_i given as the set of all $b \in \mathbb{C}$ satisfying the relation:

$$\gamma_i : \int_{a_j}^{a_k} \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}} dt \in \mathbb{R}, \quad (1.2)$$

here j and k are the remaining two indices in $\{1, 2, 3\}$ in any order and the integration is taken over the straight interval connecting a_j and a_k . One can see that a_i belongs to γ_i and show that these three curves connect all a_i 's with the unique

common point b_0 lying within Δ_Q , see Lemma 9, §4. Take the segment of γ_i connecting a_i with the common intersection point b_0 and denote this segment by Γ_i , see Fig. 1 and Fig. 7. Finally, denote the union of these three segments by Γ_Q .

Our first result is as follows.

Theorem 2. (i) For any equation (1.1) the sequence $\{\mu_n\}$ of the root-counting measures of its spectral polynomials converges to a probability measure μ depending only on the leading coefficient $Q(z)$;

(ii) The support of the limiting root-counting measure μ coincides with Γ_Q .

Remark 1. Knowing the support of μ it is also possible to define its density along the support using the linear differential equation satisfied by its Cauchy transform, see Theorem 4 of [17]. In case when $Q(z)$ has all real zeros the density is explicitly given in [3].

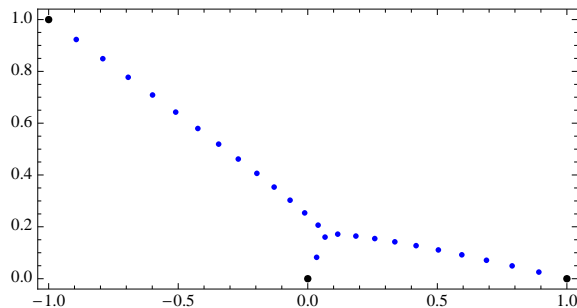


Figure 1. The roots of the spectral polynomial $Sp_{24}(\lambda)$ for the equation $Q(z)S''(z) + V(z)S(z) = 0$ with $Q(z) = z(z-1)(z+1-I)$.

An essential role in the proof of Theorem 2 is played by the description of the behavior of the Stokes lines of (1.1). Important contribution also comes from a generalization of the technique of [8]. In particular, in [17] using the latter technique we were able to find an additional probability measure which is easily described and from which the measure μ is obtained by the inverse balayage. In other words, the support of μ will be contained in the support of the measure which we construct and they will have the same logarithmic potential outside the support of the latter one. This measure is uniquely determined by the choice of a root of $Q(z)$ and thus we, in fact, construct three different measures having the same measure μ as their inverse balayage.

Our second result describes the asymptotic behavior of Stieltjes polynomials of increasing degrees when the sequence of their (normalized) Van Vleck polynomials has a limit. This result is a special case of a more general statement of [7] but we explain below in much more details the interaction of the limiting measure with the appropriate rational Strebel differential.

Namely, for a Heun equation (1.1) take any sequence $\{S_{n,i_n}(z)\}$, $\deg S_{n,i_n}(z) = n$ of its Stieltjes polynomials such that the sequence of normalized Van Vleck polynomials $\{\tilde{V}_{n,i_n}(z)\}$ converges to some monic linear polynomial $\tilde{V}(z)$. (Here by normalization we mean the division by the leading coefficient, i.e each $\tilde{V}_{n,i_n}(z)$ is the monic polynomial proportional to $V_{n,i_n}(z)$.)

Notice that since each $\tilde{V}_{n,i_n}(z)$ is linear for all sufficiently large n then the existence of the limiting polynomial $\tilde{V}(z)$ is the same as the existence of the limit of the sequence of (unique) roots $\{b_{n,i_n}\}$ of $\{V_{n,i_n}(z)\}$. Part 2 of Theorem 1 guarantees the existence of plenty of such converging sequences and Theorem 2 claims that the limit \tilde{b} of these roots must necessarily belong to Γ_Q .

Finally, denote by ν_{n,i_n} the root-counting measure of the corresponding Stieltjes polynomial $S_{n,j_n}(z)$.

Theorem 3. *In the above notation the sequence $\{\nu_{n,i_n}\}$ of the root-counting measures of the corresponding Stieltjes polynomials $\{S_{n,j_n}(z)\}$ weakly converges to the unique probability measure $\nu_{\tilde{V}}$ whose Cauchy transform $\mathcal{C}_{\tilde{V}}(z)$ satisfies almost everywhere in \mathbb{C} the equation*

$$\mathcal{C}_{\tilde{V}}^2(z) = \tilde{V}(z)/Q(z).$$

Typical behavior of the roots of $\{S_{n,j_n}(z)\}$ is illustrated on Fig. 2 below.

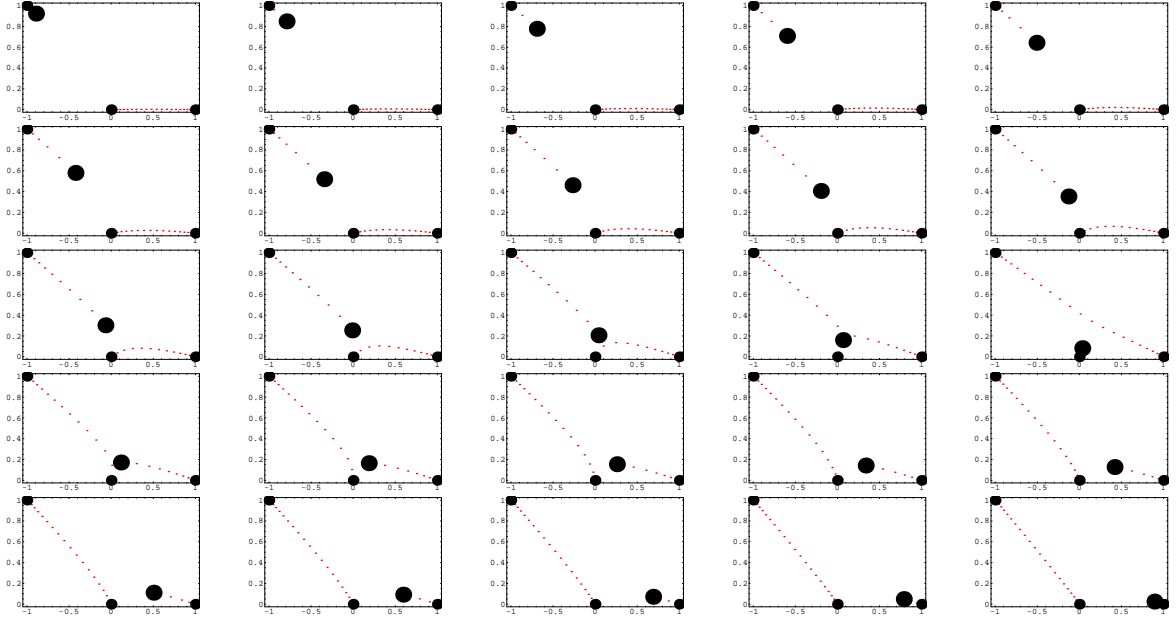


Figure 2. Zeros of 25 different Stieltjes polynomials of degree 24 for the equation $Q(z)S''(z) + V(z)S(z) = 0$ with $Q(z) = z(z-1)(z+1-I)$.

Explanation to Figure 2. The smaller dots on each of the 25 pictures above are the 24 zeros of the corresponding Stieltjes polynomial $S(z)$; the 3 average size dots are the zeros of $Q(z)$ and the single large dot is the (only) zero of the corresponding $V(z)$.

Recall that the Cauchy transform $\mathcal{C}_\nu(z)$ and the logarithmic potential $u_\nu(z)$ of a (complex-valued) measure ν supported in \mathbb{C} are by definition given by:

$$\mathcal{C}_\nu(z) = \int_{\mathbb{C}} \frac{d\nu(\xi)}{z - \xi} \quad \text{and} \quad u_\nu(z) = \int_{\mathbb{C}} \log |z - \xi| d\nu(\xi).$$

Obviously, $\mathcal{C}_\mu(z)$ is analytic outside the support of μ and has a number of important properties, e.g. that $\mu = \frac{1}{\pi} \partial \mathcal{C}_\mu(z) / \partial \bar{z}$ where the derivative is understood in the distributional sense. Detailed information about Cauchy transforms can be found in e.g. [4].

To formulate our further results we need to recall some information about quadratic differentials, see [19].

Definition 1. A (meromorphic) quadratic differential Ψ on a compact orientable Riemann surface Y without boundary is a (meromorphic) section of the tensor square $(T_{\mathbb{C}}^*Y)^{\otimes 2}$ of the holomorphic cotangent bundle $T_{\mathbb{C}}^*Y$. The zeros and the poles of Ψ constitute the set of *singular points* of Ψ denoted by $Sing_{\Psi}$. (Non-singular points of Ψ are usually called *regular*.)

Obviously, if Ψ is locally represented in two intersecting charts by $h(z)dz^2$ and by $\tilde{h}(\tilde{z})d\tilde{z}^2$ resp. with a transition function $\tilde{z}(z)$, then $h(z) = \tilde{h}(\tilde{z})(d\tilde{z}/dz)^2$. Any quadratic differential induces a canonical metric on its Riemann surface, whose length element in local coordinates is given by

$$|dw| = |h(z)|^{\frac{1}{2}}|dz|.$$

The above canonical metric $|dw| = |h(z)|^{\frac{1}{2}}|dz|$ on Y is closely related to two distinguished line fields given by the condition that $h(z)dz^2$ is either positive or negative. The first field is given by $h(z)dz^2 > 0$ and its integral curves are called *horizontal trajectories* of Ψ , while the second field is given by $h(z)dz^2 < 0$ and its integral curves are called *vertical trajectories* of Ψ . In what follows we will mostly use horizontal trajectories of rational quadratic differentials and reserve the term *trajectories* for the horizontal ones. In case we need vertical trajectories as in the § 4 we will mention this explicitly.

Since we only consider rational quadratic differentials then any such quadratic differential Ψ will be globally given in \mathbb{C} by $R(z)dz^2$, where $R(z)$ is a complex-valued rational function. (To study the behavior of Ψ at infinity one makes the variable change $\tilde{z} = 1/z$.)

Trajectories of Ψ can be naturally parameterized by their arclength. In fact, in a neighborhood of a regular point z_0 on \mathbb{C} one can introduce a local coordinate called a *canonical parameter* and given by

$$w(z) := \int_{z_0}^z \sqrt{R(\xi)}d\xi.$$

One can easily check that $dw^2 = R(z)dz^2$ implying that horizontal trajectories in the z -plane correspond to horizontal straight lines in the w -plane, i.e they are defined by the condition $\Im w = \text{const}$.

Definition 2. A trajectory of a meromorphic quadratic differential Ψ given on a compact Y without boundary is called *singular* if there exists a singular point of Ψ belonging to its closure.

Definition 3. A non-singular trajectory $\gamma_{z_0}(t)$ of a meromorphic Ψ is called *closed* if $\exists T > 0$ such that $\gamma_{z_0}(t+T) = \gamma_{z_0}(t)$ for all $t \in \mathbb{R}$. The least such T is called the *period* of γ_{z_0} .

Definition 4. A quadratic differential Ψ on a compact Riemann surface Y without boundary is called *Strebel* if the set of its closed trajectories covers Y up to a set of Lebesgue measure zero.

The following statement can be easily derived from results of Ch. 3, [19].

Lemma 1. *If a meromorphic quadratic differential Ψ is Strebel, then it has no poles of order greater than 2. If it has a pole of order 2, then the coefficient at leading term of Ψ at this pole is negative.*

In view of this lemma let us introduce the class $\mathcal{M}_{<2}^-$ of meromorphic quadratic differential on a Riemann surface Y satisfying the above restrictions, i.e. their poles are at most of order 2 and at each such double pole the leading coefficient is negative.

Definition 5. For a given quadratic differential $\Psi \in \mathcal{M}_{\leq 2}^-$ on a compact surface Y denote by $K_\Psi \subset Y$ the union of all its singular trajectories and singular points.

It is known that for a meromorphic Strebel differential Ψ given on a compact Riemann surface Y without boundary the set K_Ψ has several nice properties. In particular, it is a finite embedded multigraph on Y whose edges are singular trajectories connecting pairs of singular points of Ψ . (Here by a *multigraph* on a surface we mean a graph with possibly multiple edges and loops.) Unfortunately we were unable to find an appropriate result in the literature (although it should be obvious to the specialists in the field) and include its proof as Lemma 2. The most important circumstance is that each singular trajectory of a Strebel differential as above has **both** singular ends or, equivalently, a finite canonical length which is normally not the case for generic differentials.

Our next result relates a Strebel differential Ψ on \mathbb{CP}^1 with a double pole at ∞ to real-valued measures supported on K_Ψ and seems to be new in the theory of quadratic differentials. We suspect that it has a simple homological interpretation and can be extended to Riemann surfaces of higher genera.

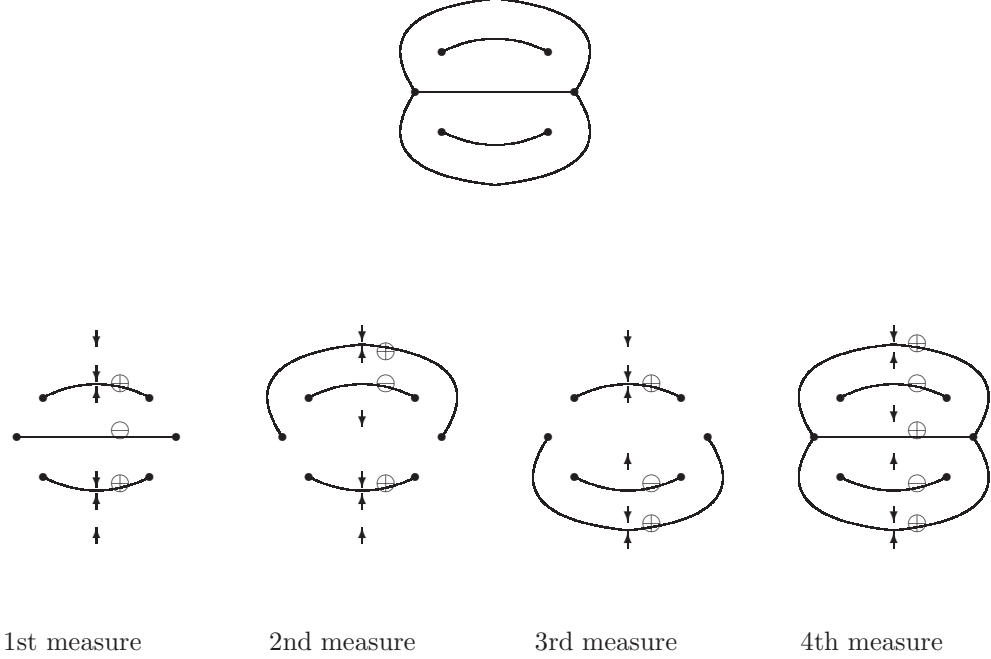


Figure 3. The compact K_Ψ of a Strebel differential Ψ and 4 related real measures

Theorem 4. Let $U_1(z)$ and $U_2(z)$ be arbitrary monic complex polynomials with $\deg U_2 - \deg U_1 = 2$. Then

- (1) the rational quadratic differential $\Psi = -U_1(z)dz^2/U_2(z)$ on \mathbb{CP}^1 is Strebel if and only if there exists a real and compactly supported in \mathbb{C} measure μ of total mass 1 (i.e. $\int_{\mathbb{C}} d\mu = 1$) whose Cauchy transform C_μ satisfies a.e. in \mathbb{C} the equation:

$$C_\mu^2(z) = U_1(z)/U_2(z). \quad (1.3)$$

- (2) for any Ψ as in (1) there exists exactly 2^{d-1} real measures whose Cauchy transforms satisfy (1.3) a.e. and whose support is contained in K_Ψ where d is the total number of connected components in $\mathbb{CP}^1 \setminus K_\Psi$ (including the unbounded component, i.e. the one containing ∞). These measures

are in 1 – 1-correspondence with 2^{d-1} possible choices of the branches of $\sqrt{U_1(z)/U_2(z)}$ in the union of $(d - 1)$ bounded components of $\mathbb{CP}^1 \setminus K_\Psi$. (In other words, we have to choose one of two possible branches of $\sqrt{U_1(z)/U_2(z)}$ in each of these connected components.)

Remark 2. The above theorem is illustrated on Fig. 3 above. Notice that we are **not** assuming here that μ as above is a positive measure **but** only real. (Such measures are called *signed*.)

Explanation to Fig. 3. The picture on top shows the union K_Ψ of singular trajectories and singular points of an appropriate rational Strebel differential $\Psi = -U_1(z)dz^2/U_2(z)$ with 2 simple zeros, 4 simple poles and a double pole at ∞ . (It is well-known that any multigraph with the properties formulated after Theorem 5 below is realizable as K_Ψ for a suitable rational Strebel differential Ψ .) The set $\mathbb{C} \setminus K_\Psi$ has two bounded connected component and one unbounded. Choosing a branch of square root $\sqrt{R(z)}$ in each of the two connected components we define the unique real measure supported on (a part of) K_Ψ . Four pictures in the second row show the actual support of these four measures where \oplus and \ominus indicate the sign of the (density of) the measure on the corresponding part of its support. Finally, arrows show the directions of the gradient of the logarithmic potential of the corresponding measures in respective components of $\mathbb{C} \setminus K_\Psi$. These arrows determine which of singular trajectories are present and which are not in the support of the constructed measure. (See more details in § 2.)

Remark 3. Notice that if we do not require the support of a real measure (whose Cauchy transform satisfy (1.3) a.e.) to be contained in K_Ψ then there exists plenty of such measures. In particular, their support can contain an arbitrary finite number of distinct closed trajectories of the quadratic differential $\Psi = -U_1(z)dz^2/U_2(z)$ near infinity.

Concerning possible positive measures we claim the following.

Theorem 5. *For any Strebel differential $\Psi = -U_1(z)dz^2/U_2(z)$ on \mathbb{CP}^1 (in the notation of Theorem 4) there exists at most one positive measure satisfying (1.3) a.e. in \mathbb{C} . Its support necessarily belongs to K_Ψ , and, therefore, among 2^{d-1} real measures described in Theorem 4 at most one is positive.*

Moreover, we can formulate an exact criterion of the existence of a positive measure in terms of rather simple topological properties of K_Ψ . To do this we need a few definitions. Notice that K_Ψ is a planar multigraph with the following properties. The vertices of K_Ψ are the finite singular points of Ψ (i.e. excluding ∞) and its edges are singular trajectories connecting these finite singular points. Each (open) connected component of $\mathbb{C} \setminus K_\Psi$ is homeomorphic to an (open) annulus. K_Ψ might have isolated vertices which are the finite double poles of Ψ . Vertices of K_Ψ having valency 1 (i.e. hanging vertices) are exactly the simple poles of Ψ . Vertices different from the isolated and hanging vertices are the zeros of Ψ . The number of edges adjacent to a given vertex minus 2 equals the order of the zero of Ψ at this point. Finally, the sum of the multiplicities of all poles (including the one at ∞) minus the sum of the multiplicities of all zeros equals 4.

Definition 6. By a *simple cycle* in a planar multigraph K_Ψ we mean any closed non-selfintersecting curve formed by the edges of K_Ψ . (Obviously, any simple cycle bounds an open domain homeomorphic to a disk which we call the *interior of the cycle*.)

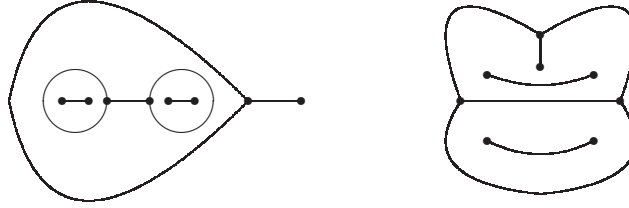


Figure 4. General examples of K_Ψ admitting and not admitting a positive measure

Proposition 1. *A Strebel differential $\Psi = -U_1(z)dz^2/U_2(z)$ admits a positive measure satisfying (1.3) if and only if no edge of K_Ψ is attached to a simple cycle from inside. In other words, for any simple cycle in K_Ψ and any edge not in the cycle but adjacent to some vertex in the cycle this edge does not belong to its interior. The support of the positive measure coincides with the forest obtained from K_Ψ after the removal of all its simple cycles.*

Remark 4. Notice that under the assumptions of Proposition 1 all simple cycles of K_Ψ are pairwise non-intersecting and, therefore, their removal is well-defined in an unambiguous way. It also seems very likely that results similar to Theorem 4 and 5 can be proved for Riemann surfaces of positive genera.

In particular, the compacta K_Ψ shown on top of Fig. 3 and on the right part of Fig. 4 admit no positive measure since both contain an edge cutting a simple cycle (the outer boundary) in two smaller cycles. The left picture on Fig. 4 has no such edges and, therefore, admits a positive measure whose support consists of the four horizontal edges of K_Ψ .

Now returning back to consideration of the asymptotics of Stieltjes polynomials in Theorem 3 and combining it with Theorem 4 and Proposition 1 we get the following consequence.

Corollary 1. *If the linear monic polynomial $\tilde{V}(z)$ is the limit of some sequence of normalized Van Vleck polynomials for (1.1) then the quadratic differential $\Phi = -\tilde{V}(z)dz^2/Q(z)$ is Strebel.*

We get the following result while applying Proposition 1 to generic Strebel differentials of the form $\Psi = -(z-b)dz^2/(z-a_1)(z-a_2)(z-a_3)$.

Corollary 2. *Of two possible planar multigraphs which appear as K_Ψ for generic Strebel differentials of the latter form the one on the left admits a positive measure and the one on the right does not, see Fig. 5.*

Remark 5. It is very important to mention that if we fix the monic cubic polynomial $Q(z) = (z-a_1)(z-a_2)(z-a_3)$ with, say, distinct roots and consider the family of all quadratic differentials of the form $\Psi = -(z-b)dz^2/Q(z)$, $b \in \mathbb{C}$ then the subset of Strebel differentials of this form is apparently dense in the complex plane of parameter b . We suspect that the subset of Strebel differentials is the union of countably many real semi-analytic curves. On the other hand, only polynomials $z-b$ with $b \in \Gamma_Q$ where Γ_Q was defined earlier can appear as the limits of sequences of normalized Van Vleck polynomials for (1.1). The same observation applies even if we restrict our consideration to the subset of Strebel differentials of the above form

admitting a positive measure (see the left picture on Fig. 5). In fact, Theorem 6 combined with Theorem 2 leads to the following statement.

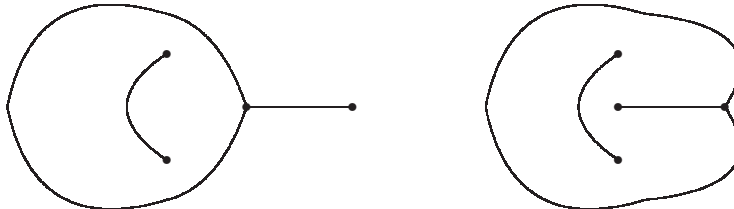


Figure 5. K_Ψ admitting and non-admitting positive measures for Ψ as above

Proposition 2. *Given $Q(z) = (z - a_1)(z - a_2)(z - a_3)$ one has that a polynomial $\tilde{V}(z) = z - b$ can be obtained as the limit of a sequence of Van Vleck polynomials for (1.1) if and only if the differential $\Psi = -\tilde{V}(z)dz^2/Q(z)$ admits a positive measure and, additionally, its two trajectories connecting pairs of vertices (i.e. K_Ψ with the loop removed) lie in the convex hull of the roots a_1, a_2, a_3 .*

Remark 6. To the best of our knowledge the latter condition related to the convex hull of the roots has never previously appeared in the theory of quadratic differentials. It might have a more transparent interpretation in terms of the related interval exchange transformation but is still quite unusual.

The structure of the paper is as follows. We will prove our results in the reverse order starting with Theorems 4, 5, then 3 and, finally, settling Theorem 2. This order is necessary since the convergence and unicity statements in Theorem 2 require some technique and facts from the former theorems.

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2. PROVING THEOREMS 4 AND 5

Lemma 2. *If Ψ is a meromorphic Strebel differential on a compact Riemann surface Y without boundary then its set K_Ψ is a finite embedded multigraph on Y .*

Proof. It is well-known that any closed trajectory γ of a Strebel differential Ψ is contained in the maximal connected domain D_γ , completely filled with closed

trajectories, such that any closed trajectory γ' is contained in D_γ if, and only if, it is homotopic to γ on the corresponding Riemann surface, see [19]. This implies, in particular, that any non-closed trajectory of a Strebel differential is a part of the boundary of one of these connected components consisting of closed trajectories. Such a boundary consists of singular points and singular trajectories since otherwise the boundary will be a closed trajectory itself, which clearly contradicts to the maximality of the corresponding domain. Hence the union K_Ψ of all singular trajectories and singular points of Ψ is contained in a compact set $K \subset Y$. Take an infinite sequence of points $\{z_n\}$, all lying on some (not necessarily the same) singular trajectory. By compactness of K , there is a converging subsequence $z_{n_i} \rightarrow z^*$. The point z^* can not lie on a closed trajectory, since it would then lie in an open domain free from singular trajectories. Hence it must lie on the boundary of a domain filled with closed trajectories. Thus z^* is either a singular point or lies on a singular trajectory and $K_\Psi = K$. \square

Remark 7. One can easily show that the fact that K_Ψ is an embedded graph of Y is equivalent to the fact that each singular trajectory of Ψ has finite canonical length.

To move further we need some information about compactly supported real measures and their Cauchy transform.

Lemma 3 (comp. Th. 1.2, Ch. II, [4]). *Suppose $f \in L^1_{loc}(\mathbb{C})$ and that $f(z) \rightarrow 0$ as $z \rightarrow \infty$ and let μ be a compactly supported measure in \mathbb{C} such that $\partial f / \partial \bar{z} = -\pi\mu$ in the sense of distributions. Then $f(z) = C_\mu(z)$ almost everywhere, where $C_\mu(z) = \int_{\mathbb{C}} d\mu(\xi)/(z - \xi)$ is the Cauchy transform of μ .*

Proof. It is clear that C_μ is locally integrable, analytic off the closure of the support of μ and vanishes at infinity. Considering $h = f - C_\mu$ and assuming that h is a locally integrable function vanishing at infinity and satisfying $\partial h / \partial \bar{z} = 0$ in the sense of distributions. We must show that $h \equiv 0$ almost everywhere. Let $\phi_r \in C_0^\infty(\mathbb{C})$ be an approximate to the identity, i.e. $\phi_r \geq 0$, $\int_{\mathbb{C}} \phi_r dx dy = 1$ and $\text{supp}(\phi_r) \subset \{|z| < r\}$, and consider the convolution

$$h_r(z) = h * \phi_r = \int_{\mathbb{C}} h(z - w)\phi_r(w) dx dy, \quad w = x + iy.$$

It is well known that $h_r \in C^\infty$ and that $\lim_{r \rightarrow 0} h_r \rightarrow h$ in $L^1(K)$ for any compact set K . Moreover

$$\frac{\partial h_r}{\partial \bar{z}} = \frac{\partial h}{\partial \bar{z}} * \phi_r = 0.$$

This shows that h_r is an entire function which vanishes at infinity, implying that $h_r \equiv 0$. Hence $h \equiv 0$ a.e. \square

Lemma 3 shows how given a Strebel differential $\Psi = -U_1(z)dz^2/U_2(z)$ to construct real measures whose support lies in K_Ψ and satisfying (1.3) by specifying branches of $\sqrt{U_1(z)/U_2(z)}$ in $\mathbb{C} \setminus K_\Psi$.

Proof of Theorem 4. To settle part (1) we first show that for a given Strebel differential $\Psi = -U_1(z)dz^2/U_2(z)$ one can construct real measures supported on K_Ψ with the required properties. To do this choose arbitrarily a branch of $\sqrt{U_1(z)/U_2(z)}$ in each bounded connected component of $\mathbb{C} \setminus K_\Psi$. (In the unbounded connected we have to choose the branch which behaves as $1/z$ at infinity.) Define now

$$\mu := \frac{\partial \sqrt{U_1(z)/U_2(z)}}{\partial \bar{z}}$$

in the sense of distributions. The distribution μ is evidently compactly supported on \mathfrak{F} . By lemma 3 we get that $C_\mu(z)$ satisfies (1.3) a.e in \mathbb{C} .

It remains to show that μ is a real measure. Take a point z_0 in the support of μ which is a regular point of $-U_1(z)dz^2/U_2(z)$ and take a small neighborhood N_{z_0} of z_0 which does not contain roots of $U_1(z)$ and $U_2(z)$. In this neighborhood we can choose a single branch $B(z)$ of $\sqrt{U_1(z)/U_2(z)}$. Notice that N_{z_0} is divided into two sets by the support of μ since it by construction consists of singular trajectories of μ . Denote these sets by M and M' resp. Choosing M and M' appropriately we can represent C_μ as $\chi_M B - \chi_{M'} B$ in N_{z_0} up to the support of μ , where χ_X denotes the characteristic function of the set X . By a theorem 2.15 in [6], ch.2 we have

$$\langle \mu, \phi \rangle = \langle \frac{\partial C_\mu}{\partial \bar{z}}, \phi \rangle = \langle \frac{\partial(\chi_M B - \chi_{M'} B)}{\partial \bar{z}} \rangle = i \int_{\partial M} B(z) \phi dz,$$

for any test function ϕ with compact support in N_{z_0} . Notice that the last equality holds because ϕ is identically zero in a neighborhood of ∂N_{z_0} , so it is only on the common boundary of M and M' that we get a contribution to the integral given in the last equality. This common boundary is the singular trajectory γ_{z_0} intersected with the neighborhood N_{z_0} . The integral

$$i \int_{\partial M} B(z) \phi dz$$

is real since the change of coordinate $w = \int_{z_0}^z iB(\xi)d\xi$ transforms the integral to the integral of ϕ over a part of the real line. This shows that μ is locally a real measure, which proves one implication in part (1) of the theorem.

To prove the converse implication, i.e. that a compactly supported real measure μ whose Cauchy transform satisfies (1.3) everywhere except for a set of measure zero produces the Strebel differential $\Psi = -U_1(z)dz^2/U_2(z)$ consider its logarithmic potential $u_\mu(z)$, i.e.

$$u_\mu(z) := \int_{\mathbb{C}} \log |z - \xi| d\mu(\xi).$$

The function $u_\mu(z)$ is harmonic outside the support of μ , and subharmonic in the whole \mathbb{C} . The following important relation:

$$\frac{\partial u_\mu(z)}{\partial z} = \frac{1}{2} C_\mu(z)$$

connects u_μ and C_μ . It implies that the set of level curves of $u_\mu(z)$ coincides with the set of horizontal trajectories of the quadratic differential $-C_\mu(z)^2 dz^2$. Indeed, the gradient of $u_\mu(z)$ is given by the vector field with coordinates $(\partial u_\mu / \partial x, \partial u_\mu / \partial y)$ in \mathbb{C} . Such a vector in \mathbb{C} coincides with the complex number $2\partial u_\mu / \partial \bar{z}$. Hence, the gradient of u_μ at z equals to $\bar{C}_\mu(z)$ (i.e. the complex conjugate of $C_\mu(z)$). But this is the same as saying that the vector $i\bar{C}_\mu(z)$ is orthogonal to (the tangent line to) the level curve of $u_\mu(z)$ at every point z outside the support of μ . Finally, notice that at each point z one has

$$-C_\mu^2(z)(i\bar{C}_\mu^2(z)) > 0,$$

which means that the horizontal trajectories of $-C_\mu(z)^2 dz^2 = -U_1(z)dz^2/U_2(z)$ are the level curves of $u_\mu(z)$ outside the support of μ . Notice that $u_\mu(z)$ behaves as $\log |z|$ near ∞ and is continuous except for possible second order poles where it has logarithmic singularities with a negative leading coefficient. This guarantees that almost all its level curves are closed and smooth implying that $-U_1(z)dz^2/U_2(z)$ is Strebel.

Let us settle part (2) of Theorem 4. Notice that if a real measure whose Cauchy transform satisfies (1.3) a.e. is supported on the compact set K_Ψ (which consists of finitely many singular trajectories and singular points) all one needs to determine it uniquely is just to prescribe which of the two branches of $\sqrt{U_1(z)/U_2(z)}$ one should choose as the Cauchy transform of this measure in each bounded connected

component of the complement $\mathbb{C} \setminus K_\Psi$. (The choice of a branch in the unbounded component is already prescribed by the requirement that it should behave as $\frac{1}{z}$ near ∞ .) Any such choice of branches in open domains leads to a real measure, see the above proof of part (1). \square

Remark 8. In order to explain the details shown on Fig. 3 notice that singular trajectories are level sets of the logarithmic potential of a real measure under consideration and, therefore, the gradient of this potential is perpendicular to any such trajectory. (More exactly, the logarithmic potential $u(z)$ is continuous at a generic point of any singular trajectory and its gradient has at least one-sided limits when z tends to such a generic point. These one-sided limits are necessarily perpendicular to the trajectory and they either coincide or are the opposite of each other.) So the choice of a branch of $\sqrt{U_1(z)/U_2(z)}$ in some connected component of $\mathbb{C} \setminus K_\Psi$ can be uniquely determined and restored from the choice of direction of the gradient of the logarithmic potential near some singular trajectory belonging to the boundary of this component. One can easily see that if the gradients on both sides of a certain singular trajectory belonging to K_Ψ have the same direction then one chooses the same branch of $\mathbb{C} \setminus K_\Psi$ on both sides and its $\partial\bar{z}$ -derivative vanishes on this singular trajectory leaving it outside the support of the corresponding measure. If these gradients have opposite directions then in case when they are both directed towards the trajectory the measure on this trajectory will be positive and in case when they are both directed away from the trajectory the measure on it will be negative. These observations explain how to obtain all supports and signs of measures appearing as the result of 2^{d-1} different choices of branches in $d-1$ bounded connected components.

We now settle Proposition 1.

Proof. Notice that given a finite measure supported on a finite union of curves with continuous density we get that its logarithmic potential will be a continuous function. As we have shown in the proof of part (1) of Theorem 4 in our situation the one-sided limits of the gradient of the logarithmic potential are orthogonal to the tangent lines of the curves in the support. In other words, the logarithmic potentials of our real measures attain constant values on each connected component of the support. (This phenomenon is characteristic for the so-called equilibrium measures of a given collection of curves, see [15].) Moreover, if the considered measure μ is positive (resp. negative) on a given curve in its support then its potential attains a local maximum (resp. minimum) on this curve. Thus for a positive measure as above its potential has no local minima except at ∞ . As a direct corollary of the latter observation we get that the support of a positive measure as above can not contain cycles. Indeed, let it contain a cycle. W.l.o.g. we can assume that this cycle is simple (i.e. has not self-intersections) since every cycle consists of simple cycles. Consider the interior of this simple cycle. On its boundary the potential is constant and is locally decreasing in the direction pointing inside this cycle. Therefore, the potential must have a local minimum in the interior which is impossible. Also notice that if a required positive measure exists then its potential should increase on each simple cycle in the direction of its interior.

Let us now show that the existence of a positive measure implies that no edge of K_Ψ is attached to a simple cycle from inside. Indeed, assume that such an edge exists. The potential should be constant on each connected component of K_Ψ and, in particular, on the one containing the considered cycle and the extra edge attached to it. Finally, it should increase in the direction of the interior of the cycle. But this immediately implies that the potential attains a local minimum on this

edge which means that the (density of the) measure on this extra edge is negative, see Fig. 6 where the arrows show the directions of the gradient.

Let us show the converse implication, i.e. that the absence of such edges implies the existence of a positive measure. Notice that the assumptions of Proposition 1 are equivalent to the fact that any connected component of the graph K_Ψ has the following property. No edge belonging to a component is located inside the interiors of the cycles belonging to this component (if they exist). Thus we can uniquely define the branch of $\sqrt{U_1(z)/U_2(z)}$ in each connected component of $\mathbb{C} \setminus K_\Psi$ so that on the boundary of each simple cycle in K_Ψ the gradient of the logarithmic potential points inside the cycle. (Recall that the gradient coincides with the complex conjugate of the chosen branch of $\sqrt{U_1(z)/U_2(z)}$ and is orthogonal to the edges of K_Ψ at each point except the vertices.) Since no simple cycles belonging to the same connected component of K_Ψ lie within each other this choice is unique and well-defined and it leads to the positive measure supported on the complement to the union of all simple cycles of K_Ψ . \square

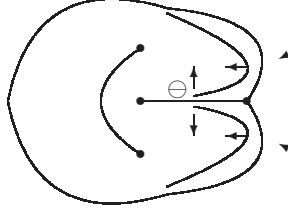


Figure 6. Illustration of the proof of Proposition 1

Remark 9. Theorem 5 is an immediate corollary of Proposition 1.

3. PROVING THEOREM 3

Our scheme follows roughly the scheme suggested in [2]. We need to prove that under the assumptions of Theorem 3 the sequence $\{\nu_{n,i_n}\}$ of root-counting measures of the sequence of Stieltjes polynomials $\{S_{n,i_n}(z)\}$ converges weakly to a probability measure $\nu_{\tilde{V}}$ whose Cauchy transform $\mathcal{C}_{\tilde{V}}(z)$ satisfies almost everywhere in \mathbb{C} the equation

$$\mathcal{C}_{\tilde{V}}^2(z) = \tilde{V}(z)/Q(z). \quad (3.1)$$

Since such a measure is positive it is unique by Theorem 5 which implies Corollary 1.

To simplify the notation we denote by $\{\tilde{S}_n(z)\}$ the chosen sequence $\{S_{n,i_n}(z)\}$ of Stieltjes polynomials whose sequence of normalized Van Vleck polynomials $\{\tilde{V}_{n,i_n}(z)\}$ converge to $\tilde{V}(z)$ and let $\{\tilde{\nu}_n\}$ denote the sequence of its root-counting measures. Also let $\tilde{\nu}_n^{(i)}$ be the root measure of the i th derivative $\tilde{S}_n^{(i)}(z)$. Assume now that NN is a subsequence of natural numbers such that

$$\tilde{\nu}^{(j)} = \lim_{n \rightarrow \infty, n \in NN} \tilde{\nu}_n^{(j)}$$

exists for $j = 0, 1, 2$. The next lemma shows that the Cauchy transform of $\tilde{\nu} = \tilde{\nu}^{(0)}$ satisfies the required algebraic equation.

Lemma 4. *The measures $\tilde{\nu}^0, \tilde{\nu}^1, \tilde{\nu}^2$ are all equal and the Cauchy transform $\mathcal{C}_{\tilde{V}}(z)$ of their common limit satisfies the equation (3.1) for almost every z .*

Proof. We have

$$\frac{\bar{S}_n^{(j+1)}(z)}{(n-j)\bar{S}_n^{(j)}(z)} \rightarrow \int \frac{d\bar{\nu}^{(j)}(\zeta)}{z-\zeta}$$

with convergence in L_{loc}^1 , and by passing to a subsequence again we can assume that we have pointwise convergence almost everywhere. From the relation

$$Q(z)\bar{S}_n''(z) + P(z)\bar{S}_n'(z) + V_n(z)\bar{S}_n(z) = 0$$

it follows that

$$\frac{Q(z)\bar{S}_n''(z)}{n(n-1)\bar{S}_n(z)} + \frac{V_n(z)}{n(n-1)} = -\frac{P(z)\bar{S}_n'(z)}{n(n-1)\bar{S}_n(z)}.$$

One can immediately check that $-V_n(z)/n(n-1) \rightarrow \tilde{V}(z)$, while the expression in the right-hand side converges pointwise to 0 almost everywhere in \mathbb{C} due to presence of the factor $n(n-1)$ in the denominator. Thus, for almost all $z \in \mathbb{C}$ one has

$$\frac{\bar{S}_n''(z)}{n(n-1)\bar{S}_n(z)} \rightarrow \frac{\tilde{V}(z)}{Q(z)}$$

when $n \rightarrow \infty$ and $n \in NN$. If $u^{(j)}$ denotes the logarithmic potential of $\bar{\nu}^{(j)}$, then one has

$$u^{(2)} - u^{(0)} = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \frac{\bar{S}_n''(z)}{n(n-1)\bar{S}_n(z)} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\log |\tilde{V}(z)| - \log |Q(z)| \right) = 0.$$

On the other hand, we have that $u^{(0)} \geq u^{(1)} \geq u^{(2)}$, see Lemma 5 below. Hence all the potentials $u^{(j)}$ are equal, and all $\nu_j = \Delta u^{(j)}/2\pi$ are equal as well. Finally we get

$$C_V^2(z) = \lim_{n \rightarrow \infty} \frac{\bar{S}_n'(z)}{n\bar{S}_n(z)} \cdot \frac{\bar{S}_n''(z)}{(n-1)\bar{S}_n(z)} = \lim_{n \rightarrow \infty} \frac{\bar{S}_n''(z)}{n(n-1)\bar{S}_n(z)} = \tilde{V}(z)/Q(z)$$

for almost all z .

Lemma 5 (see Lemma 8 of [2]). *Let $\{p_m\}$ be a sequence of polynomials, such that $n_m := \deg p_m \rightarrow \infty$ and there exists a compact set K containing the zeros of all p_m . Finally, let μ_m and μ'_m be the root-counting measures of p_m and p'_m resp. If $\mu_m \rightarrow \mu$ and $\mu'_m \rightarrow \mu'$ with compact support and u and u' are the logarithmic potentials of μ and μ' , then $u' \leq u$ in the whole \mathbb{C} . Moreover, $u = u'$ in the unbounded component of $\mathbb{C} \setminus \text{supp } \mu$.*

Proof. Assume wlog that p_m are monic. Let K be a compact set containing the zeros of all p_m . We have

$$u(z) = \lim_{m \rightarrow \infty} \frac{1}{n_m} \log |p_m(z)|$$

and

$$u'(z) = \lim_{m \rightarrow \infty} \frac{1}{n_m - 1} \log \left| \frac{p'_m(z)}{n_m} \right| = \lim_{m \rightarrow \infty} \frac{1}{n_m} \log \left| \frac{p'_m(z)}{n_m} \right|$$

with convergence in L_{loc}^1 . Hence

$$u'(z) - u(z) = \lim_{m \rightarrow \infty} \frac{1}{n_m} \log \left| \frac{p'_m(z)}{n_m p_m(z)} \right| = \lim_{m \rightarrow \infty} \frac{1}{n_m} \log \left| \int \frac{d\mu_m(\zeta)}{z-\zeta} \right|.$$

Now, if ϕ is a positive test function it follows that

$$\begin{aligned} \int \phi(z)(u'(z) - u(z))d\lambda(z) &= \lim_{m \rightarrow \infty} \int \phi(z) \log \left| \int \frac{d\mu_m(\zeta)}{z-\zeta} \right| d\lambda(z) \leq \\ &\leq \lim_{m \rightarrow \infty} \int \phi(z) \int \frac{d\mu_m(\zeta)}{|z-\zeta|} d\lambda(z) \leq \lim_{m \rightarrow \infty} \iint \frac{\phi(z)d\lambda(z)}{|z-\zeta|} d\mu_m(\zeta) \end{aligned}$$

where λ denotes Lebesgue measure in the complex plane. Since $\frac{1}{|z|}$ is locally integrable, the function $\int \phi(z)|z - \zeta|^{-1}d\lambda(z)$ is continuous, and hence bounded by a constant M for all z in K . Since $\text{supp } \mu_m \subseteq K$, the last expression in the above inequality is bounded by M/n_m , hence the limit when $m \rightarrow \infty$ equals to 0. This proves $u' \leq u$.

In the complement to $\text{supp } \mu$, u is harmonic and u' is subharmonic, hence $u' - u$ is a negative subharmonic function. Moreover, in the complement of K , $p'_n/(n_m p_m)$ converges uniformly on compact sets to the Cauchy transform $\mathcal{C}_\mu(z)$ of μ . Since $\mathcal{C}_\mu(z)$ is a non-constant holomorphic function in the unbounded component of $\mathbb{C} \setminus K$, then by the above $u' - u$ identically vanishes there. By the maximum principle for subharmonic functions it follows that $u' - u$ vanishes in the unbounded component of $\mathbb{C} \setminus \text{supp } \mu$ as well. \square

To accomplish the proof of Theorem 3 we need to show that we have the convergence for the whole sequence and not just for some subsequence. Assume now that the sequence $\bar{\nu}_n$ does not converge to $\bar{\nu}$. Then we can find a subsequence NN' such that $\bar{\nu}_n$ stay away from some fixed neighborhood of $\bar{\nu}$ in the weak topology, for all $n \in NN'$. Again by compactness, we can find a subsequence NN^* of NN' such that all the limits for root measures for derivatives exist for $j = 0, \dots, k$. But then $\bar{\nu}^{(0)}$ must coincide with $\bar{\nu}$ by the uniqueness and the latter lemma. We get a contradiction to the assumption that ν_n stay away from $\bar{\nu}$ for all $n \in NN'$ and hence all $n \in NN$. \square

4. ON STREBEL DIFFERENTIALS OF THE FORM

$$\Psi = (b - z)dz^2/(z - a_1)(z - a_2)(z - a_3)$$

The main result of this section is as follows.

Theorem 6. *For a given Strebel differential Ψ as in the title the union of its singular trajectories starting at a_1, a_2, a_3 is contained in the convex hull Δ_Q of these roots if and only if $b \in \Gamma_Q$ where $Q(z) = (z - a_1)(z - a_2)(z - a_3)$.*

Remark 10. For the definition of Γ_Q see Introduction. The proof below was suggested by the second author. A completely different proof was later found by Y. Baryshnikov based on his interpretation of interval exchange transformations for the above quadratic differentials.

Assume that the points a_1, a_2, a_3 are not collinear in the complex plane. Let i, j, k be a permutation of 1, 2, 3. Recall that we defined the curve γ_i , $i = 1, 2, 3$ by the condition:

$$\Im f_{j,k}(b) = 0 \quad \text{where} \quad f_{j,k}(b) = \int_{a_j}^{a_k} \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}} dt. \quad (4.1)$$

Lemma 6. *Each γ_i is smooth and they can only intersect transversally.*

Proof. Indeed, we have

$$f'_{j,k}(b) = \int_{a_j}^{a_k} \sqrt{\frac{-1}{(t-a_1)(t-a_2)(t-a_3)(t-b)}} dt. \quad (4.2)$$

Since the right-hand side in (4.2) is a complete elliptic integral, it represents a period of an elliptic curve which implies that the right-hand side is nonvanishing which in its turn implies the smoothness of γ_i . To show that γ_i and γ_j for $i \neq j$ can only intersect transversally notice the following. If they are tangent at some point

b^* then $f'_{jk}(b^*)/f'_{ik}(b^*) \in \mathbb{R}$ but this can never happen since the ratio of periods of a (non-singular) elliptic curve cannot be real. \square

Lemma 7. *The following 3 relations hold:*

$$\int_{a_j}^{a_k} \sqrt{\frac{a_i - t}{(t - a_1)(t - a_2)(t - a_3)}} dt = \int_{a_j}^{a_k} \sqrt{\frac{1}{(a_j - t)(t - a_k)}} dt = \pi, \quad (4.3)$$

$$\int_{a_j}^{a_k} \sqrt{\frac{b - t}{(t - a_1)(t - a_2)(t - a_3)}} dt + \int_b^{a_i} \sqrt{\frac{b - t}{(t - a_1)(t - a_2)(t - a_3)}} dt = \pi, \quad (4.4)$$

$$\left(\int_{a_j}^{a_k} + \int_{a_k}^{a_i} + \int_{a_i}^{a_j} \right) \sqrt{\frac{b - t}{(t - a_1)(t - a_2)(t - a_3)}} dt = 2\pi. \quad (4.5)$$

Proof. To prove the first relation make an affine change of variable $\tilde{z} = cz + d$ where $c \neq 0$. Set $\tilde{a}_i = ca_i + d$ and $\tilde{b} = cb + d$. If a_1, a_2, a_3, b correspond to $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{b}$ resp. then we have

$$\int_{z'}^z \sqrt{\frac{b - t}{(t - a_1)(t - a_2)(t - a_3)}} dt = \int_{\tilde{z}'}^{\tilde{z}} \sqrt{\frac{\tilde{b} - \tilde{t}}{(\tilde{t} - \tilde{a}_1)(\tilde{t} - \tilde{a}_2)(\tilde{t} - \tilde{a}_3)}} d\tilde{t}, \quad (4.6)$$

where $\tilde{t} = ct + d$ and $\tilde{z}' = cz' + d$. Hence we can always place two points a_j, a_k on the real axis and the third point a_i in the upper half plane. The second relation follows from the fact that the l.h.s. of the second relation equals to $\frac{1}{2} \int_C \sqrt{(b - t)/(t - a_1)(t - a_2)(t - a_3)} dt$ where C is any circle bounding a disk containing the triangle Δ_Q . To show that this integral equals π consider its limit when the radius of C tends to infinity. Similar considerations settle the third relation. \square

Lemma 8. *Let b be a point in the triangle Δ_Q and let l_b be the straight line passing through b and parallel to the side $\overline{a_j a_k}$. Then there exists a unique point b' on l_b such that*

$$\int_{a_j}^{a_k} \sqrt{\frac{b' - t}{(t - a_1)(t - a_2)(t - a_3)}} dt \in \mathbb{R}, \quad (4.7)$$

Moreover, $b' \in \Delta_Q \cap l_b$.

Proof. By (4.6) we can wlog assume that the points a_j, a_k lie on the real axis, $a_j < a_k$ and the point a_i lies in the upper half plane. Let us show that the imaginary part of $f_{j,k}(b)$ is a monotone decreasing function when b runs from left to right along the line l_b . Take $c_1, c_2 \in l_b$ such that $\Re c_1 < \Re c_2$, let t be a real number such that $a_j < t < a_k$. Decomposing them into the real and imaginary parts $c_1 = c_1^r + \sqrt{-1}c_1^i$, $c_2 = c_2^r + \sqrt{-1}c_2^i$, $a_i = d^r + \sqrt{-1}d^i$ we get $c_1^r - t < c_2^r - t$, $d^i > 0$ and $c_1^i = c_2^i$. Since $1/((a_k - t)(t - a_j)) > 0$ and

$$\frac{c_l - t}{a_i - t} = \frac{(c_l^r - t)(d^r - t) + c_l^i d^i + \sqrt{-1}(c_l^i(d^r - t) - (c_l^r - t)d^i)}{(d^r - t)^2 + (d^i)^2} \quad (l = 1, 2), \quad (4.8)$$

we get

$$\Im \left[\frac{c_1 - t}{(a_i - t)(a_k - t)(t - a_j)} \right] > \Im \left[\frac{c_2 - t}{(a_i - t)(a_k - t)(t - a_j)} \right].$$

Thus

$$\Im \sqrt{\frac{c_1 - t}{(a_i - t)(a_k - t)(t - a_j)}} > \Im \sqrt{\frac{c_2 - t}{(a_i - t)(a_k - t)(t - a_j)}}$$

and

$$\Im \int_{a_j}^{a_k} \sqrt{\frac{c_1 - t}{(t - a_1)(t - a_2)(t - a_3)}} dt > \Im \int_{a_j}^{a_k} \sqrt{\frac{c_2 - t}{(t - a_1)(t - a_2)(t - a_3)}} dt, \quad (4.9)$$

proving the required monotonicity. Notice that for any $a_j < t < a_k$ the imaginary part of $(c - t)/(a_i - t)$ is always positive if $c \in l_b$ is to the left of Δ_Q , and negative if $c \in l_b$ is to the right of Δ_Q . Hence condition (4.7) can not hold if $b' \notin \Delta_Q$. The results follows by the mean value theorem. \square

Remark 11. Thus the three curves $\gamma_1, \gamma_2, \gamma_3$ determined by (1.2) have to intersect the triangle Δ_Q . Relation (4.5) implies that if two of these curves meet at a certain point then the third curve also passes through the same point. By Lemma 8 any two of these curves meet at (at least) one point.

Lemma 9. *The curves $\gamma_1, \gamma_2, \gamma_3$ determined by (1.2) meet at exactly one point which lies inside Δ_Q .*

Proof. If Δ_Q is an equilateral triangle, then γ_i is the straight line which passes through a_i and is perpendicular to the side $\overline{a_j a_k}$. Assume that for some Δ_Q , two curves γ_i and γ_j meet at more than one point. Deform this Δ_Q into the equilateral triangle. During this deformation these two curves experience a deformation during which they should touch each other tangentially. But this contradicts to Lemma 6. \square

Notation 1. Let b_0 denote the point where $\gamma_i, \gamma_j, \gamma_k$ meet. Recall that we denote the segment of γ_i connecting a_i and b_0 by Γ_i . Let D_i be the domain bounded by Γ_j, Γ_k and by the side $\overline{a_j a_k}$, see Fig.6.

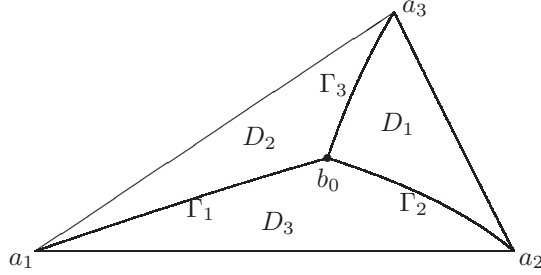


Figure 7. Used notation

Given $R(z) = (b - z)/(z - a_1)(z - a_2)(z - a_3)$ consider the quadratic differential $\Psi = R(z)dz^2$. Take its (horizontal) trajectory, i.e. a level curve:

$$\Im \int_{z_0}^z \sqrt{R(t)} dt = \text{const}, \quad (4.10)$$

where z_0 is some fixed point. Assume that b is located inside Δ_Q where as above $Q(z) = (z - a_1)(z - a_2)(z - a_3)$. If $R(z)$ is non-vanishing and regular at some $z = z^*$, then the curve $\mathfrak{H} : \Im \int_{z_0}^z \sqrt{R(t)} dt = \text{const}$ passing through z^* is analytic in a neighbourhood of z^* , and the tangential direction to \mathfrak{H} at z^* is given by $(\Re \sqrt{R(z^*)}, -\Im \sqrt{R(z^*)})$. To see this note that locally near z^* one has

$$\int_{z^*}^z \sqrt{R(t)} dt \sim \sqrt{R(z^*)}(z - z^*) + O((z - z^*)^2). \quad (4.11)$$

Analogously, the vertical trajectory \mathfrak{V} of Ψ (which is given by $\Re \int_{z_0}^z \sqrt{R(t)} dt = \text{const}$) passing through z^* is also analytic in a neighbourhood of z^* , and its tangential direction at z^* is given by $(\Im \sqrt{R(z^*)}, \Re \sqrt{R(z^*)})$. Note that the orientation of \mathfrak{H} and \mathfrak{V} depends on the choice of a branch of $\sqrt{R(z)}$.

Notation 2. For a fixed $z^* \in \mathbb{C} \setminus \{a_1, a_2, a_3, b\}$, denote by $\theta_1, \theta_2, \theta_3$, and ϕ the arguments of the complex numbers $a_1 - z^*, a_2 - z^*, a_3 - z^*$, and $b - z^*$ resp. Let $\theta_{j'j}, \phi_j$ be the arguments of $a_{j'} - a_j, b - a_j$. Finally, let ϕ_i be the argument of $a_i - b$.

The above formula for the tangent direction implies the following statement.

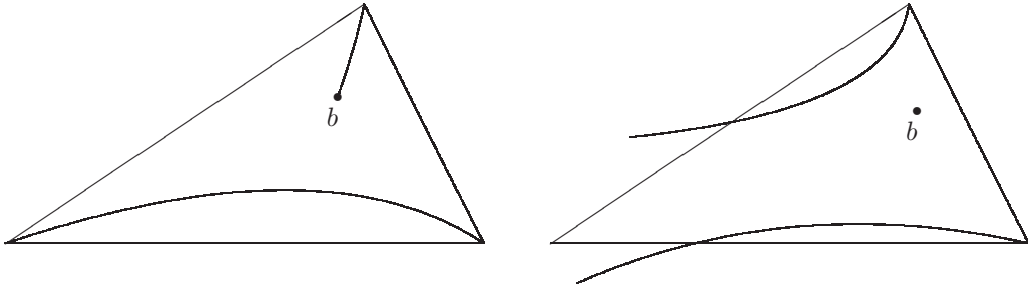
Lemma 10. *In the above notation*

- (1) *one singular horizontal trajectory emanates from each simple pole $z = a_j$ of $R(z)$; its tangent direction is given by $\theta_{ij} + \theta_{kj} - \phi_j$ and points inside Δ_Q ;*
- (2) *one singular vertical trajectory emanates from each simple pole $z = a_i$ of $R(z)$; its tangent direction is given by $\theta_{ij} + \theta_{kj} - \phi_j + \pi$ and points outside Δ_Q ;*
- (3) *three singular horizontal trajectories emanate from a simple zero $z = b$ and their tangent directions are given by $(\tilde{\phi}_1 + \tilde{\phi}_2 + \tilde{\phi}_3 + \pi(1 + 2m))/3$ ($m = 0, 1, 2$).*

Proof. The only thing we need to check is that if b is in Δ_Q , then we have $\min(\theta_{ij}, \theta_{kj}) \leq \phi_j \leq \max(\theta_{ij}, \theta_{kj})$ and $\min(\theta_{ij}, \theta_{kj}) \leq \theta_{ij} + \theta_{kj} - \phi_j \leq \max(\theta_{ij}, \theta_{kj})$. Hence the tangent direction of the horizontal trajectory emanating from any pole points inside Δ_Q . \square

Proposition 4.1.

- (i) If $b \notin \Delta_Q$ and a quadratic differential $\Psi = (b - z)dz^2/(z - a_1)(z - a_2)(z - a_3)$ is Strebel then at least one of its singular horizontal trajectories emanating from its poles a_1, a_2, a_3 leaves Δ_Q .
- (ii) If $b \in \Gamma_i \setminus \{b_0\}$ ($i = 1, 2, 3$), then a_j and a_k are connected by a singular horizontal trajectory γ and it does not contain the point $z = b$.
- (iii) If $b \in D_j$, then the singular horizontal trajectory γ' which starts at a_k (resp. a_i), goes inside the triangle Δ_Q , crosses the side $\overline{a_k a_i}$ (resp. $\overline{a_k a_j}$), and leaves the triangle Δ_Q .



Case $b \in \Gamma_3$

Case $b \in D_1$

Proof. Part (i) is completely obvious. Indeed, since K_Ψ is compact there should be a singular trajectory connecting one of the poles to the zero b . Since b is located outside Δ_Q the result follows.

To prove (ii) note that if b coincides with a_i , then the horizontal trajectory emanating from a_j is the straight segment $\overline{a_j a_k}$. Now take $b \in \Gamma_i$ sufficiently

close to a_i . Then the horizontal trajectory emanating from a_k passes close to a_j , because the direction of the horizontal trajectory changes continuously with b unless the horizontal trajectory hits a singular point. Assume that the horizontal trajectory emanating from a_k does not pass through the point a_j . Let $\theta_{j'j}$, ϕ_j , θ be the arguments of $a_{j'} - a_j$, $b - a_j$, $z - a_j$ resp. If z is sufficiently close to a_j , then the direction of the horizontal trajectory is approximately given by $(\theta + \theta_{ij} + \theta_{kj} - \phi_j)/2$. It follows from elementary affine geometry that the horizontal trajectories are approximately parabolas whose focus is a_j and the angle of the axis of the symmetry is $\theta_{ij} + \theta_{kj} - \phi_j$.

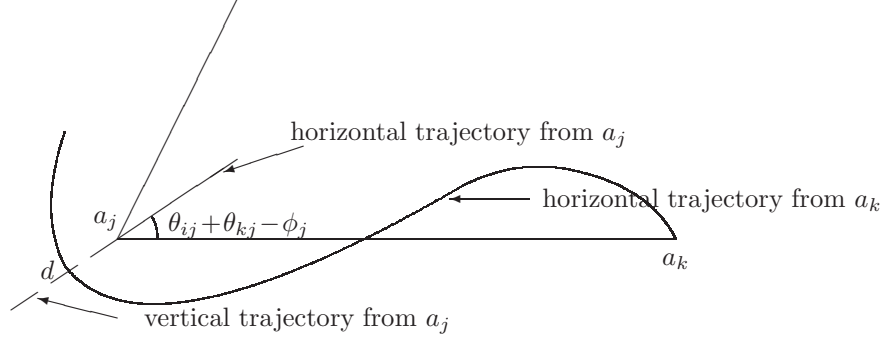


Figure 8. Behavior of horizontal and vertical trajectories near a_j .

Hence the horizontal trajectory emanating from a_k goes around the point a_j , and intersects the vertical trajectory emanating from a_j , see Fig. 8. Denote their intersection point by d and consider the integral

$$\left(\int_{a_j}^d + \int_d^{a_k} \right) \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}} dt, \quad (4.12)$$

where the path from a_j to d is taken along the vertical trajectory from a_j and the path from d to a_k is taken along the horizontal trajectory from a_k . By definition of horizontal and vertical trajectories, the value of the integral from d to a_k is real and the one from a_j to d is pure imaginary. Since the integration path does not hit a singular point, the imaginary part (resp. the real part) varies monotonely as the integration variable passes the vertical (resp. horizontal) trajectories. Hence the imaginary part of the value of (4.12) is not zero, but this contradicts to the definition of Γ_i .

Therefore we obtain that if $b \in \Gamma_i$ and b is sufficiently close to a_i , then a_j and a_k are connected by a smooth horizontal trajectory γ which does not hit the singular point $z = b$. Let us move the point b along Γ_i away from a_i . Then a_j and a_k are still connected by a smooth horizontal trajectory as long as the connecting horizontal trajectory does not hit the singular point $z = b$. On the other hand, if the horizontal trajectory passes through $z = b$, then a_j, a_k and b will be connected by a horizontal trajectory and the integrals:

$$\int_{a_j}^b \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}} dt, \quad \int_{a_i}^{a_k} \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}} dt \quad (4.13)$$

attain real values. Hence the point b is also contained in Γ_j and we conclude that $b = b_0$. Therefore, we have shown that a_j and a_k are connected by a smooth horizontal trajectory in case $b \in \Gamma_i$.

To prove (iii) choose $\tilde{b} \in \Gamma_i \setminus \{b_0\}$, and let $l_{\tilde{b}}$ be the straight line which passes through \tilde{b} and is parallel to the side $\overline{a_j a_k}$. Eq.(4.6) implies that we can wlog assume

that the points a_j, a_k lie on the real axis, $a_j < a_k$ and the point a_i lies in the upper half plane. Let $b \in l_{\tilde{b}} \cap \Delta_Q$ such that $\Re \tilde{b} < \Re b$. Then $b \in D_j$ by Lemma 8.

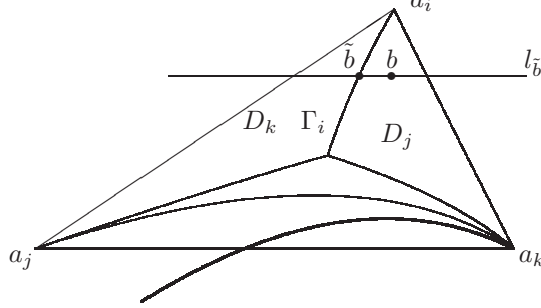


Figure 9. Why singular trajectories emanating from poles leave Δ_Q

Let z be a point such that $\Im z < \Im \tilde{b}$. By comparing the arguments of $b - z$ and $\tilde{b} - z$ one can easily conclude that the horizontal trajectory of the quadratic differential $\Psi = (b - z)dz^2/Q(z)$ emanating from a_k is located under the similar horizontal trajectory of $\tilde{\Psi} = (\tilde{b} - z)dz^2/Q(z)$, see Fig. 9. Since the horizontal trajectory of $\tilde{\Psi}$ emanating from a_k hits the point a_j , one has that the horizontal trajectory of Ψ emanating from a_k must intersect the side $\overline{a_j a_k}$ and leave the triangle Δ_Q . The intersection point of this trajectory with the side $\overline{a_j a_k}$ can not coincide with a_j since in that case the integral (1.2) will be real which contradicts Lemma 8. Let us vary b in D_j . The horizontal trajectory intersects the side $\overline{a_j a_k}$ as long as the intersection point does not coincide with a_j or a_k , or the horizontal trajectory hits the singular point b . Hence (iii) is settled for any $b \in D_i$. \square

Remark 12. One can show that part (i) of the latter Proposition holds independently of whether Ψ is Strebel or not but we do not need this fact.

5. PROVING THEOREM 2

For this proof we need Theorem 7 below whose formulation uses the definition of the following measures. Given $Q(z) = (z - a_1)(z - a_2)(z - a_3)$ choose one of three roots a_i and shift the variable $z = z - a_i$. (Abusing our notation we use the same letter for the shifted variable.) Then in this new coordinate one has $Q(z) = z^3 + v_i z^2 + w_i$, $i = 1, 2, 3$. Define the functions $\xi_i(\tau) = -v_i(1 - \tau)^2$, $\tau \in [0, 1]$ and $\psi_i(\tau) = -w_i(1 - (1 - \tau)^2)(1 - \tau)^2$, $\tau \in [0, 1]$. Let $\omega_i(\tau)$, $i = 1, 2, 3$, be the arcsine measure supported on the interval $[\xi_i(\tau) - 2\sqrt{\psi_i(\tau)}, \xi_i(\tau) + 2\sqrt{\psi_i(\tau)}]$ in the complex plane. Finally define the measure M_i , $i = 1, 2, 3$ by averaging

$$M_i = \int_0^1 \omega_i(\tau) d\tau.$$

Results of [17] claim that each M_i is supported on an ellipse which is uniquely determined by the triple of roots a_1, a_2, a_3 with the root a_i playing a special role. Moreover all these three measures have the property that their Cauchy transforms satisfy outside their respective supports one and the same linear inhomogeneous differential equation:

$$Q(z)C''(z) + Q'(z)C'(z) + \frac{Q''(z)}{8}C(z) + \frac{Q'''(z)}{24} = 0. \quad (5.1)$$

Recall that μ_n denotes the root-counting measure of the spectral polynomial $Sp_n(\lambda)$, see Introduction. The weak limit of the sequence $\{\mu_n\}$ (if it exists) is denoted by μ . In these terms the main result of [17] is as follows.

Theorem 7. *If in the above notation the measure μ exists then each of the measures M_i , $i \in \{1, 2, 3\}$ have μ as its inverse balayage, i.e. μ and M_i have the same logarithmic potential near infinity and the support of μ is contained inside the support of M_i .*

In fact the proof of Theorem 7 in [17] (as well as of the original Theorem 1.4 of [8]) extends without changing a single word in it to converging **subsequences** of the original sequence $\{\mu_n\}$.

Thus any two converging subsequences of measures from $\{\mu_n\}$ have the same limiting logarithmic potential near infinity. Notice additionally that the support of these measure must necessarily belong to Γ_Q which is the main result of § 4 above. Thus the limiting measures have the same logarithmic potential in the complement $\mathbb{C} \setminus \Gamma_Q$. But then they should coincide since both of them are \bar{z} -derivative of the same function.

Let us now prove Theorem 2. We show first that the whole sequence $\{\mu_n\}$ of root-counting measures for the whole sequence of $\{Sp_n(\lambda)\}$ converges. This argument resembles that of at the very end of § 3. Indeed, by part (2) of Theorem 1 for any $\epsilon > 0 \exists N_\epsilon$ such that for all $n \geq N_\epsilon$ all roots of all $Sp_n(\lambda)$ lie in the ϵ -neighborhood of Γ_Q . Therefore, by compactness the sequence $\{\mu_n\}$ contains a lot of (weakly) converging subsequences. Theorem 7 and the argument following it show that any two of such converging subsequences have the same (weak) limiting measure which we denote by μ . Let us show that then the whole sequence $\{\mu_n\}$ is converging to the same μ . Indeed, assume that $\{\mu_n\}$ is not converging to μ . Then we can find a subsequence N' of the natural numbers such that μ_n stays away from a fixed neighbourhood of μ in the weak topology for all $n \in N'$. Again by compactness we can find a subsequence N of N' such that the limit of $\{\mu_n\}$ over N exists and is equal to μ by the above argument. But this contradicts to the assumptions that μ_n stays away from μ for all $n \in N'$.

To show that the whole Γ_Q must be the support of the limiting measure μ notice that the Cauchy transform of μ satisfies (5.1) whose only singularities are a_1, a_2, a_3 and ∞ . One can check that the unique solution of (5.1) with the asymptotics $\frac{1}{z}$ near infinity has a nontrivial monodromy at each singularity a_1, a_2, a_3 . Notice that the Cauchy transform of μ coincides with this solution extended from infinity to the whole $\mathbb{C} \setminus \Gamma_Q$. But then the density of μ which is the \bar{z} -derivative of this solution restricted to $\mathbb{C} \setminus \Gamma_Q$ can not vanish at any generic point of Γ_Q , i.e. outside of b_0 . \square

6. FINAL REMARKS

1. A generalized Lamé equation has the form:

$$Q(z)S''(z) + P(z)S'(z) + V(z)S(z) = 0,$$

where $\deg Q(z) = l \geq 2$, $\deg P(z) \leq l - 1$, and $\deg V(z) \leq l - 2$. Fixing $Q(z)$ and $P(z)$ one looks for $V(z)$ of degree at most $l - 2$ such that the latter equation has a polynomial solution $S(z)$ of a given degree n , see many details in e.g. [16]. Typically for a given Lamé equation and a given positive integer n there exist $\binom{n+l-2}{n}$ such Van Vleck polynomials of degree $l - 2$. Moreover, they are exactly $\binom{n+l-2}{n}$ many for any given Lamé equation if they are counted with appropriate multiplicities and n is sufficiently large. Interesting computer experiments can be found in e.g. [1] and were also independently performed by the present authors. These experiments lead us to the following conclusion. Let \mathcal{V}_n be the finite set of all normalized Van Vleck polynomials (i.e monic polynomials proportional to Van Vleck polynomials) such that each of them has a Stieltjes polynomial $S(z)$ of degree exactly n counted with their multiplicities. In fact, \mathcal{V}_n can be interpreted as a finite probability measure in the space Pol_{l-2} of all monic polynomials of degree

$l - 2$ if we assign to each polynomial in \mathcal{V}_n a positive Dirac measure equal to its multiplicity divided by $\binom{n+l-2}{n}$.

Conjecture 1. *The sequence $\{\mathcal{V}_n\}$ of finite measures converges to a probability measure \mathcal{V}_Q in Pol_{l-2} which depends only on the leading coefficient $Q(z)$.*

2. Similar set-up was developed in [16] for linear differential operators of order exceeding 2 of the form $\mathfrak{q} = \sum_{i=1}^k Q_i(z) \frac{d^i}{dz^i}$ where $\deg Q_i(z) \leq i$ and $\deg Q_k(z) = k$. This topic was continued in [7] where it is shown that a very natural analog of the main object of the present paper, i.e. the quadratic differentials $\Psi = -\tilde{V}(z)dz^2/Q(z)$ appears for operators of higher order as well. It also has almost all closed trajectories which are continuous but, in general, only piecewise smooth. One needs to develop a notion of a Strebel differential for order > 2 which (to the best of our knowledge) is a completely open problem at the moment. So we pose our question in minimal possible generality.

Problem 3. How to define a notion of a rational cubic Strebel differential

$$\Psi = U_1(z)dz^3/U_2(z),$$

where $\deg U_1(z) = 1$ and $\deg U_2(z) = 4$.

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