“K-theoretic” analog of Postnikov–Shapiro algebra distinguishes graphs

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ABSTRACT

In this paper we study a filtered “K-theoretical” analog of a graded algebra associated to any loopless graph $G$ which was introduced in [5]. We show that two such filtered algebras are isomorphic if and only if their graphs are isomorphic. We also study a large family of filtered generalizations of the latter graded algebra which includes the above “K-theoretical” analog.

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1. Introduction

The following square-free algebra $C_G$ associated to an arbitrary vertex labeled graph $G$ was defined in [5], see also [1] and [2]. Let $G$ be a graph without loops on the vertex set $\{0, ..., n\}$. (Below we always assume that all graphs might have multiple edges, but no loops.) Throughout the whole paper, we fix a field $K$ of zero characteristic. Let $\Phi_G$ be the graded commutative algebra over $K$ generated by the variables $\phi_e, e \in G$, with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for any edge } e \in G.$$
Let $C_G$ be the subalgebra of $\Phi_G$ generated by the elements
\[ X_i = \sum_{e \in G} c_{i,e} \phi_e, \]
for $i = 1, \ldots, n$, where
\[ c_{i,e} = \begin{cases} 
1 & \text{if } e = (i,j), i < j; \\
-1 & \text{if } e = (i,j), i > j; \\
0 & \text{otherwise.} 
\end{cases} \tag{1} \]

In what follows we always assume that all algebras contain 1. For the reasons which will be clear soon, we call $C_G$ the \textit{spanning forests counting algebra} of $G$. Its Hilbert series and the set of defining relations were calculated in [6] following the initial paper [7]. Namely, let $J_G$ be the ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials
\[ p_I = \left( \sum_{i \in I} x_i \right)^{D_I+1}, \tag{2} \]
where $I$ ranges over all nonempty subsets in $\{1, \ldots, n\}$ and $D_I = \sum_{i \in I} d_I(i)$, where $d_I(i)$ is the total number of edges connecting a given vertex $i \in I$ with all vertices outside $I$.

Thus, $D_I$ is the total number of edges between $I$ and the complementary set of vertices $\overline{I}$. Set $B_G := \mathbb{K}[x_1, \ldots, x_n]/J_G$.

\textbf{Remark 1.} Observe that since $\sum_{i=0}^n X_i = 0$, we can define $C_G$ as the subalgebra of $\Phi_G$ generated by $X_0, X_1, \ldots, X_n$.

We can also define $B_G$ as the quotient algebra of $\mathbb{K}[x_0, \ldots, x_n]$ by the ideal generated by $p_I$, where $I$ runs over all subsets of $\{x_0, x_1, \cdots, x_n\}$. This follows from the relation
\[ p_I = \left( \sum_{i \in I} x_i \right)^{D_I+1} = \left( p_{\{0,1,\ldots,n\}} - \sum_{i \in \overline{I}} x_i \right)^{D_{\overline{I}}+1}. \]

To describe the Hilbert polynomial of $C_G$, we need the following classical notion going back to W.T. Tutte. Given a simple graph $G$, fix an arbitrary linear order of its edges. Now, given a spanning forest $F$ in $G$ (i.e., a subgraph without cycles which includes all vertices of $G$) and an edge $e \in G \setminus F$ in its complement, we say that $e$ is \textit{externally active} for $F$, if there exists a cycle $C$ in $G$ such that all edges in $C \setminus \{e\}$ belong to $F$ and $e$ is minimal in $C$ with respect to the chosen linear order. The total number of external edges is called the \textit{external activity} of $F$. Although the external activity of a given forest/tree in $G$ depends on the choice of a linear ordering of edges, the total number of forests/trees with a given external activity is independent of this ordering. Now we are ready to formulate the main result of [6].
Theorem 1 (Theorems 3 and 4 of [6]). For any simple graph $G$, the algebras $B_G$ and $C_G$ are isomorphic. The total dimension of these algebras (as vector spaces over $\mathbb{K}$) is equal to the number of spanning subforests in $G$. The dimension of the $k$-th graded component of these algebras equals the number of subforests $F$ in $G$ with external activity $|G| - |F| - k$. Here $|G|$ (resp. $|F|$) stands for the number of edges in $G$ (resp. $F$).

In the above notation, our main object will be the filtered subalgebra $K_G \subset \Phi_G$ defined by the generators:

$$Y_i = \exp(X_i) = \prod_{e \in G} (1 + c_{i,e} \phi_e), \quad i = 0, \ldots, n.$$ (Notice that we have one more generator here than in the previous case.)

Remark 2. Since $Y_i$ is obtained by exponentiation of $X_i$, we call $K_G$ the “$K$-theoretic” analog of $C_G$. The original generators $X_i$ are similar to the first Chern classes, see [7] while their exponentiations $Y_i$ are similar to the Chern characters which are the main object of $K$-theory.

Our first result is as follows. Define the ideal $\mathcal{I}_G$ in $\mathbb{K}[y_0, y_1, \ldots, y_n]$ as generated by the polynomials

$$q_I = \left( \prod_{i \in I} y_i - 1 \right)^{D_I + 1},$$

where $I$ ranges over all nonempty subsets in $\{0, 1, \ldots, n\}$ and the number $D_I$ is the same as in (2). Set $\mathcal{D}_G := \mathbb{K}[y_0, \ldots, y_n]/\mathcal{I}_G$.

Theorem 2. For any graph $G$, algebras $B_G$, $C_G$, $\mathcal{D}_G$ and $K_G$ are isomorphic as (non-filtered) algebras.

Moreover, the following stronger statement holds.

Theorem 3. For any graph $G$, algebras $\mathcal{D}_G$ and $K_G$ are isomorphic as filtered algebras.

Recall that in a recent paper [4] the first author has shown that $C_G$ contains all information about the matroid of $G$ and only it. Namely,

Theorem 4 (Theorem 2 of [4]). Given two graphs $G_1$ and $G_2$, algebras $C_{G_1}$ and $C_{G_2}$ are isomorphic if and only if the matroids of $G_1$ and $G_2$ coincide. (The latter isomorphism can be thought of either as graded or as non-graded, the statement holds in both cases.)

On the other hand, the filtered algebras $\mathcal{D}_G$ and $K_G$ contain complete information about $G$. 
Theorem 5. Given two graphs \( G_1 \) and \( G_2 \) without isolated vertices, \( \mathcal{K}_{G_1} \) and \( \mathcal{K}_{G_2} \) are isomorphic as filtered algebras if and only if \( G_1 \) and \( G_2 \) are isomorphic.

The structure of this paper is as follows. In § 2 we prove the new results formulated above. In § 3 we discuss Hilbert series of similar algebras defined by other sets of generators. In § 4 we discuss “K-theoretic” analogs of algebras counting spanning trees. Finally, in § 5 we present a number of open problems.

2. Proofs

To prove Theorem 2, we need some preliminary results.

Lemma 1. For any simple graph \( G \), the algebras \( C_G \) and \( \mathcal{K}_G \) coincide as subalgebras of \( \Phi_G \).

Proof. Since \((X_i)^{d_i+1} = 0\), where \( d_i \) is the degree of vertex \( i \), then

\[
Y_i = \exp(X_i) = 1 + \sum_{j=1}^{d_i} \frac{(X_i)^j}{j!}.
\]

Hence \( Y_i \in C_G \) which means that \( \mathcal{K}_G \subseteq C_G \subseteq \Phi_G \).

To prove the opposite inclusion, consider \( \tilde{Y}_i = Y_i - 1 = \exp(X_i) - 1 \). Since \( X_i | \tilde{Y}_i \), we get

\[(\tilde{Y}_i)^{d_i+1} = 0.\]

Using the relation \( X_i = \ln(1 + \tilde{Y}_i) = \sum_{j=1}^{d_i} \frac{(-1)^{j-1}(\tilde{Y}_i)^j}{j!} \), we conclude \( X_i \in \mathcal{K}_G \). Thus \( C_G \subseteq \mathcal{K}_G \), implying that \( C_G \) and \( \mathcal{K}_G \) coincide. \( \square \)

Lemma 2. For any simple graph \( G \), the algebras \( B_G \) and \( D_G \) are isomorphic as (non-filtered) algebras.

Proof. First we change the variables in \( D_G \) by using \( \tilde{y}_i = y_i - 1, \ i = 0, 1, \ldots, n \). The generators of ideal \( \mathcal{I}_G \) transform as

\[
\tilde{q}_I = \left( \prod_{i \in I} (\tilde{y}_i + 1) - 1 \right)^{D_I+1},
\]

for any subset \( I \subseteq \{0, 1, \ldots, n\} \).

Since for every vertex \( i = 0, 1, \ldots, n \),

\[
((\tilde{y}_i + 1) - 1)^{d_i+1} = \tilde{y}_i^{d_i+1},
\]

we conclude \( \mathcal{B}_G \subseteq \mathcal{D}_G \subseteq \Phi_G \). \( \square \)
we can consider \( \mathcal{D}_G \) as the quotient \( \mathbb{K}[[\tilde{y}_0, \ldots, \tilde{y}_n]]/\tilde{I}_G \) of the ring of formal power series (instead of the polynomial ring) factored by the ideal \( \tilde{I}_G \) generated by all \( \tilde{q}_I \).

Similarly we can consider \( \mathcal{B}_G \) as the quotient \( \mathbb{K}[[x_0, \ldots, x_n]]/\tilde{J}_G \) of the ring of formal power series by the ideal \( \tilde{J}_G \) generated by all \( p_I \).

Introduce the homomorphism \( \psi : \mathbb{K}[[\tilde{y}_0, \ldots, \tilde{y}_n]] \rightarrow \mathbb{K}[[x_0, \ldots, x_n]] \) defined by:

\[
\psi : \tilde{y}_i \rightarrow e^{x_i} - 1.
\]

In fact, \( \psi \) is an isomorphism, because \( \psi^{-1} \) is defined by \( x_i \rightarrow \ln(1 + \tilde{y}_i) \).

Let us look at what happens with the ideal \( \tilde{I}_G \) under the action of \( \psi \). For a given \( I \subset \{0, 1, \ldots, n\} \), consider the generator \( \tilde{q}_I \). Then,

\[
\psi(\tilde{q}_I) = \left( \prod_{i \in I} (\psi(\tilde{y}_i) + 1) - 1 \right)^{D_I+1} = \left( \prod_{i \in I} e^{x_i} - 1 \right)^{D_I+1} = \left( \exp \left( \sum_{i \in I} x_i \right) - 1 \right)^{D_I+1} = \left( \sum_{i \in I} x_i \right)^{D_I+1} \cdot \left( \frac{\exp \left( \sum_{i \in I} x_i \right) - 1}{\sum_{i \in I} x_i} \right)^{D_I+1}.
\]

The factor \( \frac{\exp \left( \sum_{i \in I} x_i \right) - 1}{\sum_{i \in I} x_i} \) is a formal power series starting with the constant term 1. Hence the last factor in the right-hand side of the latter expression is an invertible power series. Thus, the generator \( \tilde{q}_I \) is mapped by \( \psi \) to the product \( p_I \cdot \ast \), where \( \ast \) is an invertible series. This implies \( \psi(\tilde{I}_G) = \tilde{J}_G \). Hence the algebras \( \mathcal{D}_G \) and \( \mathcal{B}_G \) are isomorphic. \( \square \)

**Proof of Theorem 2.** By Lemmas 1, 2 and Theorem 1, we get that all four algebras are isomorphic to each other. Furthermore, by Theorem 1, we know that their total dimension over \( \mathbb{K} \) is the number of subforests in \( G \). \( \square \)

**Theorem 3** now follows from Theorem 2.

**Proof of Theorem 3.** Consider the surjective homomorphism \( h : \mathcal{D}_G \rightarrow \mathcal{K}_G \), defined by

\[
h(y_i) = Y_i, \ i = 0, 1, \ldots, n.
\]

(It is indeed a homomorphism because every relation \( q_I \) holds for \( Y_0, \ldots, Y_n \).) By Theorem 2 we know that these algebras have the same dimension, implying that \( h \) is an isomorphism. Since the filtrations in \( \mathcal{D}_G \) and \( \mathcal{K}_G \) are defined using \( y_i \)'s and \( Y_i \)'s respectively it is clear that \( h \) is a filtration preserving isomorphism. \( \square \)

2.1. **Proving Theorem 5**

We start with a few definitions.
Given a commutative algebra $A$, its element $t \in A$ is called reducible nilpotent if and only if there exists a presentation $t = \sum u_i v_i$, where all $u_i, v_i$ are nilpotents.

For a nilpotent element $t \in A$, define its degree $d(t)$ as the minimal non-negative integer for which there exists a reducible nilpotent element $h \in A$ such that

$$(t - h)^{d+1} = 0.$$ 

Given an element $R \in \Phi_G$, we say that an edge-element $\phi_e$ belongs to $R$, if the monomial $\phi_e$ has a non-zero coefficient in the expansion of $R$ as the sum of square-free monomials in $\Phi_G$.

Lemma 3. For any nilpotent element $R \in \mathcal{K}_G$, the degree $d(R)$ of $R$ equals the number of edges of $G$ belonging to $R$. (Observe that $d(R)$ is defined in $\mathcal{K}_G$.)

Proof. We can write $R$ in terms of $\{X_0, \ldots, X_n\}$. (Observe that $\mathcal{K}_G$ and $\mathcal{C}_G$ coincide as subsets of $\Phi_G$, but have different graded/filtered structures.) Now we can concentrate on the graded structure of $\mathcal{C}_G$. Select the part of $R$ which lies in the first graded component of $\mathcal{C}_G$. Thus

$$R = R_1 + R' = \sum_{i=0}^n a_i X_i + R',$$

where $R'$ is reducible nilpotent because it belongs to the linear span of other graded components. Thus $d(R) = d(R_1)$. The statement of Lemma 3 is obvious for $R_1$. Additionally by construction, an edge-element $\phi_e$ belongs to $R$ if and only if it belongs to $R_1$. \[\square\]

Lemma 4. Given a graph $G$, let $\{\tilde{Y}_0, \ldots, \tilde{Y}_n\}$ be the set of generators of $\mathcal{K}_G$ corresponding to the vertices (i.e., $\tilde{Y}_i = \exp(X_i) - 1$). Then

1. $\{\tilde{Y}_0, \ldots, \tilde{Y}_n\}$ are nilpotents;
2. $\sum_{i=0}^n \ln(1 + \tilde{Y}_i) = 0$;
3. for any subset $I \subseteq [0,n]$ and any set of pairwise distinct non-zero numbers $a_i \in \mathbb{K}$ ($i \in I$), the degree $d(\sum_{i \in I} a_i \tilde{Y}_i)$ is equal to the number of edges incident to at least one vertex belonging to $I$;
4. the number of edges between vertices $i$ and $j$ equals to $\frac{d(\tilde{Y}_i) + d(\tilde{Y}_j) - d(\tilde{Y}_i + \tilde{Y}_j)}{2}$.

Proof. Item (1) is obvious.

To settle (2), observe that $\ln(1 + \tilde{Y}_i) = X_i$ which implies

$$\sum_{i=0}^n \ln(1 + \tilde{Y}_i) = \sum_{i=0}^n X_i = 0.$$
To prove (3), notice that, by Lemma 3, the degree $d(\sum_{i \in I} a_i \tilde{Y}_i)$ is equal to the number of edges belonging to the sum $\sum_{i \in I} a_i \tilde{Y}_i$. Each edge belongs either to zero, to one or to two generators $\tilde{Y}_i$ from the latter sum. Moreover, if an edge belongs to two generators, then it has coefficients of opposite signs. Since all $a_i$ are different, an edge-element $\phi_e$ belongs to $\sum_{i \in I} a_i \tilde{Y}_i$ if and only if it belongs to at least one $\tilde{Y}_i$, for $i \in I$. Thus the degree $d(\sum_{i \in I} a_i \tilde{Y}_i)$ is the number of edges incident to at least one vertex from $I$.

To settle (4), notice that if $e$ is an edge between vertices $i$ and $j$, then $\phi_e$ belongs to $\tilde{Y}_i$ and to $\tilde{Y}_j$ with the opposite coefficients. Therefore $\phi_e$ does not belong to $(\tilde{Y}_i + \tilde{Y}_j)$. Using Lemma 3, we get that $d(\tilde{Y}_i) + d(\tilde{Y}_j) - d(\tilde{Y}_i + \tilde{Y}_j)$ equals twice the number of edges between $i$ and $j$. \(\Box\)

Our proof of Theorem 5 uses the following technical lemma which should be obvious to the specialists.

Lemma 5 (Folklore). Let $E$ be the set of edges of some graph $G$ without isolated vertices. If we know the following information:

(1) which pairs $e_i, e_j \in E$ of edges are multiple, i.e., connect the same pair of vertices;
(2) which pairs $e_i, e_j \in E$ of edges have exactly one common vertex;
(3) which triples $e_i, e_j, e_k \in E$ of edges form a triangle,

then we can reconstruct $G$ up to an isomorphism.

Proof. Assume the contrary, i.e., that there exist two non-isomorphic graphs $G$ and $G'$ such that there exists a bijection $\psi$ of their edge sets $E$ and $E'$ preserving (1)-(3). Assume that under this bijection an edge $e \in E$ corresponds to the edge $e' \in E'$. Additionally assume that $|V(G')| \geq |V(G)|$.

Now we construct an isomorphism between $G$ and $G'$. Let us split the vertices of $G$ into two subsets: $V(G) = \hat{V}(G) \cup \tilde{V}(G)$, where $\hat{V}(G)$ are all vertices with at least two distinct neighboring vertices.

Let us construct a bijection $\psi$ between the vertices of $G$ and $G'$, which extends the given bijection $\psi$ of edges, i.e., for any $e = uv = \in E$, $e' = \psi(e) = \psi(u)\psi(v)$.

At first we define it on $\hat{V}(G)$. Namely, given a vertex $v \in \hat{V}(G)$, choose two non-multiple edges $e_i$ and $e_j$ incident to it, and define $\psi(v)$ as the common vertex of $e'_i$ and $e'_j$. We need to show that $\psi(v)$ does not depend on the choice of $e_i$ and $e_j$. It is enough to check it for a pair $e_i$ and $e_k \neq e_j$, where $e_k$ is another edge incident to $v$. Indeed, if $e'_k$ has no common vertex with both $e'_i$ and $e'_j$, then $e'_i$, $e'_j$ and $e'_k$ form a triangle in $G'$ (because $e'_k$ has a common vertex with $e'_i$ and with $e'_j$). Hence, $e_i$, $e_j$ and $e_k$ form a triangle in $G$, but they have a common vertex $v$. Contradiction.

Now we need to extend $\psi$ to vertices belonging to $\tilde{V}(G)$. Note that each vertex $v \in \tilde{V}(G)$ has exactly one adjacent vertex. There are two possibilities.
1° The adjacent vertex $u$ of $v$ belongs to $\tilde{V}(G)$. Consider the edge $e_{uv} \in E$. (There might be several such edges, but this is not important, because in $G'$ they are also multiple.) Knowing the image $\psi(e_{uv})$ and the vertex $\psi(u)$, we define $\psi(v)$ as the vertex of $\psi(e_{uv})$ different from $\psi(u)$.

2° Adjacent vertex $u$ of $v$ belongs to $\tilde{V}(G)$. Consider the edge $e_{uv} \in E$ Knowing $\psi(e_{uv})$, we define $\psi(u)$ and $\psi(v)$ as the vertices of the edge $\psi(e_{uv})$ (not important which is mapped to which).

Since $G'$ has no isolated vertices and each edge $e'$ has exactly two incident vertices from $\psi(V)$, we get that $\psi : G \rightarrow G'$ is surjective. Hence, $\psi : G \rightarrow G'$ is an isomorphism (otherwise it must be non-injective on vertices and, hence, $|V(G)| > |V(G')|$). Therefore $G$ and $G'$ are isomorphic. □

Proof of Theorem 5. Let $G$ and $G'$ be a pair of graphs such that their filtered algebras $\mathcal{K}_G$ and $\mathcal{K}_{G'}$ are isomorphic. Without loss of generality, we can assume that $|E(G)| \leq |E(G')|$. Denote the numbers of vertices in $G$ and $G'$ by $n + 1$ and $n' + 1$ resp.

Consider $\mathcal{K}_G$ as a subalgebra in $\Phi_G$. The elements $\tilde{Y}_i = \exp(X_i) - 1$, $i \in [0, n]$ form a set of generators of $\mathcal{K}_G$. Since, by our assumptions, $\mathcal{K}_G$ and $\mathcal{K}_{G'}$ are isomorphic as filtered algebras, denote by $\tilde{Z}_i \in \mathcal{K}_G$, $i \in [0, n']$ the elements corresponding to the vertices of $G'$ under the latter isomorphism. The set $\{\tilde{Z}_i, i \in [0, n']\}$ is also a generating set for $\mathcal{K}_G$ which gives the same filtered structure and satisfies the assumptions of Lemma 4. (Indeed, the operations of taking the logarithm and calculating the “degree” of an element are well-defined inside $\mathcal{K}_G$ and $\mathcal{K}_{G'}$. Thus we do not need to use the ambient algebras $\Phi_G$ and $\Phi_{G'}$; while applying Lemma 4.) In order to avoid confusion, we call $\tilde{Y}_i$ the $i$-th vertex of graph $G$, and we call $\tilde{Z}_j$ the $j$-th vertex of graph $G'$.

Since $\tilde{Y}_i, i \in [0, n]$ and $\tilde{Z}_i, i \in [0, n']$ determine the same graded structure, then, in particular,

$$\text{span}\{1, \tilde{Y}_0, \ldots, \tilde{Y}_n\} = \text{span}\{1, \tilde{Z}_0, \ldots, \tilde{Z}_{n'}\}.$$  

Additionally, by Lemma 4, $\tilde{Y}_i, i \in [0, n]$ and $\tilde{Z}_i, i \in [0, n']$ are nilpotents, implying that

$$\text{span}\{\tilde{Y}_0, \ldots, \tilde{Y}_n\} = \text{span}\{\tilde{Z}_0, \ldots, \tilde{Z}_{n'}\}.$$  

Firstly, we need to show that each edge-element $\phi_e$ belongs to at most two different $\tilde{Z}_i$’s. Assume the contrary, i.e., that $\phi_e$ belongs to $\tilde{Z}_i, \tilde{Z}_j$ and $\tilde{Z}_k$. Then there exist three distinct non-zero coefficients $r_1, r_2, r_3 \in \mathbb{K}$ such that $\phi_e$ does not belong to $r_1 \tilde{Z}_i + r_2 \tilde{Z}_j + r_3 \tilde{Z}_k$. Moreover, for generic distinct non-zero coefficients $r'_1, r'_2, r'_3 \in \mathbb{K}$, element $\phi_{e'}$ ($e' \in E(G)$) belongs to $r'_1 \tilde{Z}_i + r'_2 \tilde{Z}_j + r'_3 \tilde{Z}_k$ if and only if $\phi_{e'}$ belongs to at least one of $\tilde{Z}_i, \tilde{Z}_j$ and $\tilde{Z}_k$. Hence by Lemma 3,

$$d(r_1 \tilde{Z}_i + r_2 \tilde{Z}_j + r_3 \tilde{Z}_k) < d(r'_1 \tilde{Z}_i + r'_2 \tilde{Z}_j + r'_3 \tilde{Z}_k).$$

But at the same time, by Lemma 4 (3), they should coincide, contradiction.
By Lemma 4, for any $i \in [0, n']$, the degree $d(\tilde{Z}_i)$ equals the valency of $\tilde{Z}_i$. Therefore,

$$2|E(G')| = \sum_{i=0}^{n'} d(\tilde{Z}_i) \leq 2|E(G)|,$$

because each edge-element is included in at most two $\tilde{Z}_i$. Since $|E(G)| \leq |E(G')|$, we conclude that $|E(G)| = |E(G')|$. Furthermore, by Lemma 4 (2), each element $\phi_e$, $e \in E(G)$ belongs exactly to two vertices from $\tilde{Z}_i$, $i \in [0, n']$ with the opposite coefficients. The number of edges between $\tilde{Z}_i$ and $\tilde{Z}_j$ equals $\frac{d(\tilde{Z}_i)+d(\tilde{Z}_j)-d(\tilde{Z}_i+\tilde{Z}_j)}{2}$ by Lemma 4; the number of common $\phi_e$'s equals the latter number by Lemma 3. Thus we obtain a natural bijection between the edges of $G$ and the edges of $G'$. Let us additionally assume that the number of pairs of non-multiple edges which have a common vertex in $G'$ is bigger than that in $G$.

So far we have constructed a bijection between the edges of $G$ and the edges of $G'$. We want to prove that this bijection provides a graph isomorphism. We will achieve this as a result of the 5 claims collected in the following proposition which is closely related to Lemma 5.

**Proposition 1.** The following facts hold.

1. If $\phi_{e_1}$ and $\phi_{e_2}$ have no common vertex in $G$, then they have no common vertex in $G'$ as well.
2. If $\phi_{e_1}$ and $\phi_{e_2}$ are multiple edges in $G$, then they are multiple edges in $G'$ as well.
3. If $\phi_{e_1}$ and $\phi_{e_2}$ have exactly one common vertex in $G$, then they have exactly one common vertex in $G'$ as well.
4. If $\phi_{e_1}$, $\phi_{e_2}$ and $\phi_{e_3}$ form a claw in $G$, then they form a claw in $G'$ as well. (Three edges form a claw if they have one common vertex and their three other ends are distinct.)
5. If $\phi_{e_1}$, $\phi_{e_2}$ and $\phi_{e_3}$ form a triangle in $G$, then they form a triangle in $G'$ as well.

**Proof.** To prove (1), assume the contrary, i.e., assume that $\phi_{e_1}$ and $\phi_{e_2}$ belong to $\tilde{Z}_j$ (and denote the corresponding coefficients by $a$ and $b$ resp.). Since elements $\tilde{Y}_0, \ldots, \tilde{Y}_n$ have no monomial $\phi_{e_1}\phi_{e_2}$, then $\tilde{Z}_0, \ldots, \tilde{Z}_{n'}$ have no monomial $\phi_{e_1}\phi_{e_2}$ as well (since their spans coincide). Then $\ln(1 + \tilde{Z}_j)$ contains the monomial $\phi_{e_1}\phi_{e_2}$ with the coefficient $-ab$.

By Lemma 4 (2), we have $\sum_{i=0}^{n'} \ln(1 + \tilde{Z}_i) = 0$, so there exists $k \in [0, n']$, $k \neq j$ such that $\ln(1 + \tilde{Z}_k)$ contains the monomial $\phi_{e_1}\phi_{e_2}$ with a non-zero coefficient. Then $\tilde{Z}_k$ must contain $\phi_{e_1}$ and $\phi_{e_2}$ (since $\tilde{Z}_k$ does not contain $\phi_{e_1}\phi_{e_2}$). Hence, $\tilde{Z}_k$ has $\phi_{e_1}$ and $\phi_{e_2}$ with coefficients $-a$ and $-b$ resp. Therefore $\ln(1 + \tilde{Z}_k)$ contains monomial $\phi_{e_1}\phi_{e_2}$ with the coefficient $-(-a)(-b) = -ab$. Thus the sum $\sum_{j=0}^{n'} \ln(1 + \tilde{Z}_j)$ contains $\phi_{e_1}\phi_{e_2}$ with coefficient $-2ab$, contradiction.

To prove (2), consider the map from span{\$\tilde{Y}_0, \ldots, \tilde{Y}_n\$} to $\mathbb{K}^2$, sending an element from the span to the pair of coefficients of $\phi_{e_1}$ and $\phi_{e_2}$ resp. Since edges $e_1$ and $e_2$ are multiple
in $G$, the image of this map has dimension 1. If $\phi_{e_1}$ and $\phi_{e_2}$ are not multiple in $G'$, then
the image of the map from $\text{span}\{\tilde{Z}_0, \ldots, \tilde{Z}_{n'}\} = \text{span}\{\tilde{Y}_0, \ldots, \tilde{Y}_n\}$ has dimension 2.

To prove (3), observe that we have already settled Claims 1 and 2, and also we
additional assumed that the number of pairs of edges which have a common vertex in
$G'$ is bigger than that in $G$. Then each such pair of edges from $G$ is mapped to the pair
of edges from $G'$ with the same property.

To prove (4), consider the map from $\text{span}\{\tilde{Y}_0, \ldots, \tilde{Y}_n\}$ to $\mathbb{K}^3$, sending an element in
the span to the triple of coefficients of $\phi_{e_1}, \phi_{e_2}$ and $\phi_{e_3}$ resp. The image of this map has
dimension 3. However if $\phi_{e_1}, \phi_{e_2}$ and $\phi_{e_3}$ form a triangle in $G'$, then the image of the
map from $\text{span}\{\tilde{Z}_0, \ldots, \tilde{Z}_{n'}\}$ has dimension 2.

Proof of (5) is similar to that of (4). □

Now applying Lemma 5 we finish our proof of Theorem 5. □

3. Further generalizations

In this section we will consider the Hilbert series of other filtered algebras similar to
$K_G$. (Recall that the Hilbert series of a filtered algebra is, by definition, the Hilbert series
of its associated graded algebra.)

Let $f$ be a univariate polynomial or a formal power series over $\mathbb{K}$. We define the
subalgebra $F[f]_G \subset \Phi_G$ as generated by 1 together with

$$f(X_i) = f \left( \sum c_{i,e} \phi_e \right), \ i = 0, \ldots, n.$$  

**Example 1.** For $f(x) = x$, $F[f]_G$ coincides with $C_G$. For $f(x) = \exp(x)$, $F[f]_G$ coincides
with $K_G$.

Obviously, the filtered algebra $F[f]_G$ does not depend on the constant term of $f$. From
now on, we assume that $f(x)$ has no constant term, since for any $g$ such that $f-g$
is constant, the filtered algebras $F[f]_G$ and $F[g]_G$ are the same.

**Proposition 2.** Let $f$ be any polynomial with a non-vanishing linear term. Then the
algebras $C_G$ and $F[f]_G$ coincide as subalgebras of $\Phi_G$.

**Proof.** The argument is the same as in the proof of Lemma 1. We only need to change
$\exp(x) - 1$ to $f(x)$ and $\ln(1 + y)$ to $f^{-1}(y)$. □

**Theorem 6.** Let $f$ be any polynomial with non-vanishing linear and quadratic terms.
Then given two simple graphs $G_1$ and $G_2$ without isolated vertices, $F[f]_{G_1}$ and $F[f]_{G_2}$
are isomorphic as filtered algebras if and only if $G_1$ and $G_2$ are isomorphic graphs.

**Proof.** Repeat the proof of Theorem 5. □
3.1. Generic functions $f$ and their Hilbert series

Since $X_i^{d_i+1} = 0$ for any $i$, we can always truncate any polynomial (or a formal power series) $f$ at degree $|G| + 1$ without changing $F[f]_G$. Therefore, for a given graph $G$, it suffices to consider $f$ as a polynomial of degrees less than or equal to $|G|$. To simplify our notation, let us write $HS_{f,G}$ instead of $HS_{F[f]_G}$.

Given a graph $G$, consider the space of polynomials of degree less than or equal to $|G|$ and the corresponding Hilbert series.

Proposition 3. In the above notation, for generic polynomials $f$ of degree at most $|G|$, the Hilbert series $HS_{f,G}$ is the same. This generic Hilbert series (denoted by $HS_G$ below) is maximal in the majorization partial order among all $HS_{g,G}$, where $g$ runs over the set of all formal power series with non-vanishing linear term.

Here (as usual) by generic polynomials of degree at most $|G|$ we mean polynomials belonging to some Zariski open subset in the linear space of all polynomials of degree at most $|G|$.

Recall that, by definition, a sequence $(a_0, a_1, \ldots)$ is bigger than $(b_0, b_1, \ldots)$ in the majorization partial order if and only if, for any $k \geq 0$,

$$\sum_{i=0}^{k} a_i \geq \sum_{i=0}^{k} b_i.$$

More information about the majorization partial order can be found in e.g. [3].

Proof. Note that, for a function $f$, the sum of the first $k + 1$ entries of its Hilbert series $HS_{f,G}$ equals the dimension of

$$\text{span}\ \{f^{a_0}(X_0)f^{a_1}(X_1)\cdots f^{a_n}(X_n) : \sum_{i=0}^{n} a_i \leq k\}.$$

It is obvious that, for a generic $f$, this dimension is maximal. Since all Hilbert series $HS_{f,G}$ are polynomials of degree at most $|G| + 1$, then the required property has to be checked only for $k \leq |G|$. Therefore it is obvious that, for generic $f$, their Hilbert series is maximal in the majorization order. \(\Box\)

Remark 3. We know that the Hilbert series of the graded algebra $C_G$ is a specialization of the Tutte polynomial of $G$. However we can not calculate the Hilbert series of $K_G$ from the Tutte polynomial of $G$, because there exists a pair of graphs $(G, G')$ with the same Tutte polynomial and different $HS_{K_G}$ and $HS_{K_{G'}}$, see Example 2.

Additionally, notice that, in general, $HS_{K_{G'}} := HS_{K_G} \neq HS_G$. Analogously we can not calculate generic Hilbert series $HS_G$ from the Tutte polynomial of $G$, see Example 2.
**Example 2.** Consider two graphs $G_1$ and $G_2$ presented in Fig. 1. It is easy to see that $G_1$ and $G_2$ have isomorphic matroids and hence, the same Tutte polynomial. Therefore, the Hilbert series of $C_{G_1}$ and $C_{G_2}$ coincide. Namely,

$$HS_{C_{G_1}}(t) = HS_{C_{G_2}}(t) = 1 + 3t + 6t^2 + 9t^3 + 8t^4 + 4t^5 + t^6.$$ 

However, the Hilbert series of their “K-theoretic” algebras are distinct. Namely

$$HS_{K_{G_1}}(t) = 1 + 4t + 10t^2 + 14t^3 + 3t^4,$$

$$HS_{K_{G_2}}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4.$$ 

Moreover their generic Hilbert series are also distinct and different from their “K-theoretic” Hilbert series. Namely,

$$HS_{G_1}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4,$$

$$HS_{G_2}(t) = 1 + 4t + 10t^2 + 16t^3 + t^4.$$ 

Putting our information together we get,

$$HS_{C_{G_1}} = HS_{C_{G_2}} < HS_{K_{G_1}} < HS_{K_{G_2}} = HS_{G_1} < HS_{G_2},$$

where $<$ denotes the majorization partial order.

**4. “K-theoretical” analog for spanning trees**

In this section we always assume that $G$ is connected. For an arbitrary loopless graph $G$ on the vertex set $\{0, ..., n\}$, let $\Phi^T_G$ be the graded commutative algebra over a given field $K$ generated by the variables $\phi_e, e \in G$, with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for any edge } e \in G;$$

$$\prod_{e \in H} \phi_e = 0, \quad \text{for any non-slim subgraph } H \subset G,$$

where a subgraph $H$ is called slim if its complement $G \setminus H$ is connected.
Let $C_T^G$ be the subalgebra of $\Phi_T^G$ generated by the elements

$$X_i^T = \sum_{e \in G} c_{i,e} \phi_e,$$

for $i = 1, \ldots, n$, where $c_{i,e}$ is given by (1). (Notice that $X_i^T$ and $X_i$ are defined by exactly the same formula but in different ambient algebras.)

Algebra $C_T^G$ will be called the *spanning trees counting algebra* of $G$ and is, obviously, the quotient of $C_G$ modulo the set of relations $\prod_{e \in H} \phi_e = 0$ over all non-slim subgraphs $H$. Its defining set of relations is very natural and resembles that of (2). Namely, define the ideal $J_T^G$ in $\mathbb{K}[x_1, \ldots, x_n]$ as generated by the polynomials:

$$p_I^T = \left(\sum_{i \in I} x_i\right)^{D_I},$$

where $I$ ranges over all nonempty subsets in $\{1, \ldots, n\}$ and the number $D_I$ is the same as in (2). Set $B_T^G := \mathbb{K}[x_1, \ldots, x_n]/J_T^G$. One of the results of [5] claims the following.

**Theorem 7** (Theorems 9.1 and Corollary 10.5 of [5]). For any simple graph $G$ on the set of vertices $\{0, 1, \ldots, n\}$, the algebras $B_T^G$ and $C_T^G$ are isomorphic. Their total dimension is equal to the number of spanning trees in $G$. The dimension $\dim B_T^G(k)$ of the $k$-th graded component of $B_T^G$ equals the number of spanning trees $T$ in $G$ with external activity $|G| - n - k$.

Similarly to the above, we can define the filtered algebra $\mathcal{K}_G^T \subset \Phi_G^T$ which is isomorphic to $C_G^T$ as a non-filtered algebra. Namely, $\mathcal{K}_G^T$ is defined by the generators:

$$Y_i^T = \exp(X_i^T) = \prod_{e \in G} (1 + c_{i,e} \phi_e), \quad i = 0, \ldots, n.$$

The first result of this section is as follows. Define the ideal $I_T^G \subset \mathbb{K}[y_0, y_1, \ldots, y_n]$ as generated by the polynomials:

$$q_I^T = \left(\prod_{i \in I} y_i - 1\right)^{D_I},$$

where $I$ ranges over all nonempty proper subsets in $\{0, 1, \ldots, n\}$ and the number $D_I$ is the same as in (2), together with the generator

$$q_{\{0,1,\ldots,n\}}^T = \prod_{i=0}^{n} y_i - 1.$$

Set $D_T^G := \mathbb{K}[y_0, \ldots, y_n]/I_T^G$.

We present two results similar to the case of spanning forests.
**Theorem 8.** For any simple graph $G$, algebras $B^T_G$, $C^T_G$, $D^T_G$ and $K^T_G$ are isomorphic as (non-filtered) algebras. Their total dimension is equal to the number of spanning trees in $G$.

**Proof.** The proof is similar to that of Theorem 2. Algebras $C^T_G$ and $K^T_G$ coincide as subalgebras of $\Phi^T_G$ (but they have different filtrations); algebras $C^T_G$ and $B^T_G$ are isomorphic by Theorem 7. The proof of the isomorphism between $D^T_G$ and $B^T_G$ is the same as above; we only need to add the variable $x_0 = -(\sum_{i=1}^n x_i)$ to $B^T_G$. □

**Theorem 9.** For any simple graph $G$, algebras $D^T_G$ and $K^T_G$ are isomorphic as filtered algebras.

**Proof.** Similar to the above proof of Theorem 3. □

To move further, we need to give a definition.

**Definition 1.** Let $G$ be a connected graph. We define its $\Delta$-subgraph $\hat{G} \subseteq G$ as the subgraph with the following edges and vertices:

- $e \in E(\hat{G})$, if $e$ is not a bridge (i.e., $G \setminus e$ is still connected),
- $v \in V(\hat{G})$, if there is an edge $e \in E(\hat{G})$ incident to $v$.

By the bridge-free matroid of $G$ we call the graphical matroid of $\hat{G}$.

In general, $\hat{G}$ contains more information about $G$ than its matroid, because there exist graphs with isomorphic matroids and non-isomorphic $\Delta$-subgraphs, see Fig. 2.

Recall that in a recent paper [4], the first author has shown that $C^T_G$ depends only on the bridge-free matroid of $G$. Namely,

**Proposition 4 (Proposition 16 of [4]).** For any two connected graphs $G_1$ and $G_2$ with isomorphic bridge-free matroids (matroids of their $\Delta$-subgraphs), algebras $C^T_{G_1}$ and $C^T_{G_2}$ are isomorphic.
Unfortunately, we can not prove the converse implication at present although we conjecture that is should hold as well, see Conjecture 6 in § 5. In case of filtered algebra $\mathcal{K}_{G_1}^T$ and $\mathcal{K}_{G_2}^T$, we can also prove an appropriate result only in one direction, see Proposition 5.

Similarly to § 3 we can to define $\mathcal{F}[f]_{\mathcal{G}}^T \subset \Phi_{\mathcal{G}}$. Let $f$ be a univariate polynomial or a formal power series over $\mathbb{K}$. We define the subalgebra $\mathcal{F}[f]_{\mathcal{G}}^T \subset \Phi_{\mathcal{G}}$ as generated by 1 and by

$$f(X_i^T) = f\left(\sum c_i,\phi_e\right), \ i = 0, \ldots, n.$$ \(\text{Proposition 5.}\)

For a univariate polynomial $f$ and any two connected graphs $G_1$ and $G_2$ with isomorphic $\Delta$-subgraphs $\tilde{G}_1$ and $\tilde{G}_2$, algebras $\mathcal{F}[f]_{\mathcal{G}_1}^T$ and $\mathcal{F}[f]_{\mathcal{G}_2}^T$ are isomorphic as filtered algebras. Additionally, $\mathcal{K}_{G_1}^T$ and $\mathcal{K}_{G_2}^T$ are isomorphic as filtered algebras.

\(\text{Proof.}\) Note that if $G$ has a bridge $e$, then filtered algebra $\mathcal{F}[f]_{\mathcal{G}}^T$ is the Cartesian product of filtered algebras $\mathcal{F}[f]_{\mathcal{G}'}^T$ and $\mathcal{F}[f]_{\mathcal{G}''}^T$, where $G'$ and $G''$ are connected components of $G \setminus e$.

Thus filtered algebra $\mathcal{F}[f]_{\mathcal{G}}^T$ is the Cartesian product of such filtered algebras corresponding to the connected components of the $\Delta$-subgraph of $G$.

Therefore if connected graphs $G_1$ and $G_2$ have isomorphic $\Delta$-subgraphs, then their filtered algebras $\mathcal{F}[f]_{\mathcal{G}_1}^T$ and $\mathcal{F}[f]_{\mathcal{G}_2}^T$ are isomorphic. \(\Box\)

\(\text{Remark 4.}\) In the general case we cannot prove that these algebras distinguish graphs with different $\Delta$-subgraphs. The proof of Theorem 5 does not work for two reasons. Firstly, $d(\tilde{Y}_i)$ is not the degree of the $i$-th vertex in $G$. Secondly, even if we can construct a similar bijection between edges, we do not have an analog of Proposition 1. Since in the proof we consider coefficients of monomial $\phi_{e_1}\phi_{e_2}$, in case when $e_1$ and $e_2$ are not bridges and when $\{e_1, e_2\}$ is a cut, this monomial can still lie in the ideal.

It is possible to construct such a bijection in a smaller set of graphs, namely for graphs such that, for any edge $e$ in the graph, there is another edge $e'$ which is multiple to $e$. For such graphs we do not have the second problem, because if $\{e_1, e_2\}$ is a cut, then $e_1$ and $e_2$ are multiple edges. So, instead of the actual converse of Proposition 5, we can prove the converse in the latter situation, but we do not present this result here.

\(\text{Proposition 6.}\) In the above notation, for generic polynomials $f$ of degree at most $|G|$, the Hilbert series $HS_{\mathcal{F}[f]_{\mathcal{G}}^T}$ is the same. This generic Hilbert series (denoted by $HS_{G^T}$ below) is maximal in the majorization partial order among $HS_{\mathcal{F}[g]_{\mathcal{G}}^T}$ for $g$ running over the set of power series with non-vanishing linear term.

\(\text{Proof.}\) See the proof of Proposition 3. \(\Box\)

\(\text{Example 3.}\) Consider two graphs $G_1$ and $G_2$, see Fig. 2. It is easy to check that subgraphs $\tilde{G}_1$ and $\tilde{G}_2$ have isomorphic matroids, implying that algebras $C_{G_1}^T$ and $C_{G_2}^T$ are isomorphic.
\[ HS_{C_1^T}(t) = HS_{C_2^T}(t) = 1 + 4t + 4t^2. \]

The Hilbert series of “K-theoretic” algebras are distinct, namely

\[ HS_{K_1^T}(t) = 1 + 5t + 3t^2, \]
\[ HS_{K_2^T}(t) = 1 + 6t + 2t^2. \]

These graphs are “small”, so their generic Hilbert series coincides with the “K-theoretic” one. Putting our information together, we get

\[ HS_{C_1^T} = HS_{C_2^T} \prec HS_{K_1^T} = HS_{G_1^T} \prec HS_{K_2^T} = HS_{G_2^T}. \]

5. Related problems

At first, we formulate several problems in case of spanning forests; their analogs for spanning trees are straightforward.

**Problem 1.** For which functions \( f \) besides \( a + bx \) and \( a + be^x \), can one present relations in \( F[f]_G \) for any graph \( G \) in a simple way? In other words, for which \( f \), can one define an algebra similar to \( B_G \) and \( D_G \)?

Since the Hilbert series \( HS_{K_G} \) and \( HS_G \) are not expressible in terms of the Tutte polynomial of \( G \), they contain some other information about \( G \).

**Problem 2.** Find combinatorial descriptions of \( HS_{K_G} \) and \( HS_G \).

**Problem 3.** For which graphs \( G \), do the Hilbert series \( HS_{K_G} \) and \( HS_G \) coincide? In other words, for which \( G \), is \( \exp \) a generic function?

**Problem 4.** Describe combinatorial properties of \( HS_{f,G} \) when \( f \) is a function starting with a monomial of degree bigger than 1, i.e. \( f(x) = x^k + \cdots \), \( k > 1 \)? In particular, calculate the total dimension of \( F[f]_G \).

The most delicate and intriguing question is as follows.

**Problem 5.** Do there exist non-isomorphic graphs \( G_1 \) and \( G_2 \) such that, for any polynomial \( f(x) \), the Hilbert series \( HS_{f,G_1} \) and \( HS_{f,G_2} \) coincide? In other words, does the collection of Hilbert series \( HS_{f,G} \) taken over all polynomials \( f \) determine \( G \) up to isomorphism?

The following problems deal with the case of spanning trees only.
Conjecture 6 (Comp. [4]). Algebras $\mathcal{C}^T_{G_1}$ and $\mathcal{C}^T_{G_2}$ for graphs $G_1$ and $G_2$ are isomorphic if and only if their bridge-free matroids are isomorphic, where the bridge-free matroid is the graphical matroid of the $\Delta$-subgraph.

Problem 7. Which class of graphs satisfies the property that if two graphs $G_1$ and $G_2$ from this class have isomorphic $\mathcal{K}^T_{G_1}$ and $\mathcal{K}^T_{G_2}$, then their $\Delta$-subgraphs are isomorphic. In other words, can one classify all pairs $(G_1, G_2)$ of connected graphs, which have isomorphic filtered algebras $\mathcal{K}^T_{G_1}$ and $\mathcal{K}^T_{G_2}$? (The same problem for $\mathcal{F}[f]_{G_1}$ and $\mathcal{F}[f]_{G_2}$, where $f(x) = x + ax^2 + \cdots$.)

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