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"K-theoretic" analog of Postnikov–Shapiro algebra distinguishes graphs

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ABSTRACT

In this paper we study a filtered "K-theoretical" analog of a graded algebra associated to any loopless graph G which was introduced in [5]. We show that two such filtered algebras are isomorphic if and only if their graphs are isomorphic. We also study a large family of filtered generalizations of the latter graded algebra which includes the above "K-theoretical" analog.

1. Introduction

The following square-free algebra $\mathcal{C}_G$ associated to an arbitrary vertex labeled graph G was defined in [5], see also [1] and [2]. Let G be a graph without loops on the vertex set $\{0, ..., n\}$. (Below we always assume that all graphs might have multiple edges, but no loops.) Throughout the whole paper, we fix a field $\mathbb{K}$ of zero characteristic. Let Φ_G be the graded commutative algebra over $\mathbb{K}$ generated by the variables $\phi_e, e \in G$, with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for any edge } e \in G.$$

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Let $\mathcal{C}_G$ be the subalgebra of Φ_G generated by the elements

$$X_i = \sum_{e \in G} c_{i,e}\phi_e,$$

for $i = 1, ..., n$, where

$$c_{i,e} = \begin{cases} 1 & \text{if } e = (i,j), i < j; \\ -1 & \text{if } e = (i,j), i > j; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In what follows we always assume that all algebras contain 1. For the reasons which will be clear soon, we call $\mathcal{C}_G$ the *spanning forests counting algebra* of G. Its Hilbert series and the set of defining relations were calculated in [6] following the initial paper [7]. Namely, let $\mathcal{J}_G$ be the ideal in $\mathbb{K}[x_1, \dots, x_n]$ generated by the polynomials

$$p_I = \left(\sum_{i \in I} x_i\right)^{D_I + 1}, \tag{2}$$

where I ranges over all nonempty subsets in $\{1, \dots, n\}$ and $D_I = \sum_{i \in I} d_I(i)$, where $d_I(i)$ is the total number of edges connecting a given vertex $i \in I$ with all vertices outside I. Thus, D_I is the total number of edges between I and the complementary set of vertices $\bar{I}$. Set $B_G := \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}_G$.

Remark 1. Observe that since $\sum_{i=0}^{n} X_i = 0$, we can define $\mathcal{C}_G$ as the subalgebra of Φ_G generated by $X_0, X_1, \dots, X_n$.

We can also define B_G as the quotient algebra of $\mathbb{K}[x_0, \dots, x_n]$ by the ideal generated by p_I, where I runs over all subsets of $\{x_0, x_1, \cdots, x_n\}$. This follows from the relation

$$p_I = \left(\sum_{i \in I} x_i\right)^{D_I + 1} = \left(p_{\{0,1,\dots,n\}} - \sum_{i \in \bar{I}} x_i\right)^{D_{\bar{I}} + 1}.$$

To describe the Hilbert polynomial of $\mathcal{C}_G$, we need the following classical notion going back to W.T. Tutte. Given a simple graph G, fix an arbitrary linear order of its edges. Now, given a spanning forest F in G (i.e., a subgraph without cycles which includes all vertices of G) and an edge $e \in G \setminus F$ in its complement, we say that e is *externally active* for F, if there exists a cycle C in G such that all edges in $C \setminus \{e\}$ belong to F and e is minimal in C with respect to the chosen linear order. The total number of external edges is called the *external activity* of F. Although the external activity of a given forest/tree in G depends on the choice of a linear ordering of edges, the total number of forests/trees with a given external activity is independent of this ordering. Now we are ready to formulate the main result of [6].

Theorem 1 *(Theorems 3 and 4 of [6]). For any simple graph G, the algebras B_G and $\mathcal{C}_G$ are isomorphic. The total dimension of these algebras (as vector spaces over $\mathbb{K}$) is equal to the number of spanning subforests in G. The dimension of the k-th graded component of these algebras equals the number of subforests F in G with external activity $|G|-|F|-k$. Here $|G|$ (resp. $|F|$) stands for the number of edges in G (resp. F).*

In the above notation, our main object will be the filtered subalgebra $\mathcal{K}_G \subset \Phi_G$ defined by the generators:

$$Y_i = \exp(X_i) = \prod_{e \in G} (1 + c_{i,e}\phi_e),\ i = 0, \dots, n.$$

(Notice that we have one more generator here than in the previous case.)

Remark 2. Since Y_i is obtained by exponentiation of X_i, we call $\mathcal{K}_G$ the "K-theoretic" analog of $\mathcal{C}_G$. The original generators X_i are similar to the first Chern classes, see [7] while their exponentiations Y_i are similar to the Chern characters which are the main object of K-theory.

Our first result is as follows. Define the ideal $\mathcal{I}_G$ in $\mathbb{K}[y_0, y_1, \dots, y_n]$ as generated by the polynomials

$$q_I = \left(\prod_{i \in I} y_i - 1\right)^{D_I+1}, \tag{3}$$

where I ranges over all nonempty subsets in $\{0, 1, \dots, n\}$ and the number D_I is the same as in (2). Set $\mathcal{D}_G := \mathbb{K}[y_0, \dots, y_n]/\mathcal{I}_G$.

Theorem 2. *For any graph G, algebras B_G, $\mathcal{C}_G$, $\mathcal{D}_G$ and $\mathcal{K}_G$ are isomorphic as (non-filtered) algebras.*

Moreover, the following stronger statement holds.

Theorem 3. *For any graph G, algebras $\mathcal{D}_G$ and $\mathcal{K}_G$ are isomorphic as filtered algebras.*

Recall that in a recent paper [4] the first author has shown that $\mathcal{C}_G$ contains all information about the matroid of G and only it. Namely,

Theorem 4 *(Theorem 2 of [4]). Given two graphs G_1 and G_2, algebras $\mathcal{C}_{G_1}$ and $\mathcal{C}_{G_2}$ are isomorphic if and only if the matroids of G_1 and G_2 coincide. (The latter isomorphism can be thought of either as graded or as non-graded, the statement holds in both cases.)*

On the other hand, the filtered algebras $\mathcal{D}_G$ and $\mathcal{K}_G$ contain complete information about G.

Theorem 5. *Given two graphs G_1 and G_2 without isolated vertices, $\mathcal{K}_{G_1}$ and $\mathcal{K}_{G_2}$ are isomorphic as filtered algebras if and only if G_1 and G_2 are isomorphic.*

The structure of this paper is as follows. In § 2 we prove the new results formulated above. In § 3 we discuss Hilbert series of similar algebras defined by other sets of generators. In § 4 we discuss "K-theoretic" analogs of algebras counting spanning trees. Finally, in § 5 we present a number of open problems.

2. Proofs

To prove Theorem 2, we need some preliminary results.

Lemma 1. *For any simple graph G, the algebras $\mathcal{C}_G$ and $\mathcal{K}_G$ coincide as subalgebras of Φ_G.*

Proof. Since $(X_i)^{d_i+1} = 0$, where d_i is the degree of vertex i, then

$$Y_i = \exp(X_i) = 1 + \sum_{j=1}^{d_i} \frac{(X_i)^j}{j!}.$$

Hence $Y_i \in \mathcal{C}_G$ which means that $\mathcal{K}_G \subset \mathcal{C}_G \subset \Phi_G$.

To prove the opposite inclusion, consider $\widetilde{Y}_i = Y_i - 1 = \exp(X_i) - 1$. Since $X_i | \widetilde{Y}_i$, we get

$$(\widetilde{Y}_i)^{d_i+1} = 0.$$

Using the relation $X_i = \ln(1 + \widetilde{Y}_i) = \sum_{j=1}^{d_i} \frac{(-1)^{j-1}(\widetilde{Y}_i)^j}{j!}$, we conclude $X_i \in \mathcal{K}_G$. Thus $\mathcal{C}_G \subset \mathcal{K}_G$, implying that $\mathcal{C}_G$ and $\mathcal{K}_G$ coincide. □

Lemma 2. *For any simple graph G, the algebras B_G and $\mathcal{D}_G$ are isomorphic as (non-filtered) algebras.*

Proof. First we change the variables in $\mathcal{D}_G$ by using $\widetilde{y_i} = y_i - 1$, $i = 0, 1, \ldots, n$. The generators of ideal $\mathcal{I}_G$ transform as

$$\widetilde{q}_I = \left(\prod_{i \in I} (\widetilde{y_i} + 1) - 1 \right)^{D_I + 1},$$

for any subset $I \subset \{0, 1, \ldots, n\}$.

Since for every vertex $i = 0, 1, \ldots, n$,

$$((\widetilde{y_i} + 1) - 1)^{d_i+1} = \widetilde{y_i}^{d_i+1},$$

we can consider $\mathcal{D}_G$ as the quotient $\mathbb{K}[[\widetilde{y}_0, \dots, \widetilde{y}_n]]/\widetilde{\mathcal{I}}_G$ of the ring of formal power series (instead of the polynomial ring) factored by the ideal $\widetilde{\mathcal{I}}_G$ generated by all $\widetilde{q}_I$.

Similarly we can consider B_G as the quotient $\mathbb{K}[[x_0, \dots, x_n]]/\widetilde{\mathcal{J}}_G$ of the ring of formal power series by the ideal $\widetilde{\mathcal{J}}_G$ generated by all p_I.

Introduce the homomorphism $\psi: \mathbb{K}[[\widetilde{y}_0, \dots, \widetilde{y}_n]] \mapsto \mathbb{K}[[x_0, \dots, x_n]]$ defined by:

$$\psi : \widetilde{y}_i \to e^{x_i} - 1.$$

In fact, ψ is an isomorphism, because ψ^{-1} is defined by $x_i \to \ln(1 + \widetilde{y}_i)$.

Let us look at what happens with the ideal $\widetilde{\mathcal{I}}_G$ under the action of ψ. For a given $I \subset \{0, 1, \dots, n\}$, consider the generator $\widetilde{q}_I$. Then,

$$\psi(\widetilde{q}_I) = \left(\prod_{i \in I} (\psi(\widetilde{y}_i) + 1) - 1 \right)^{D_I+1} = \left(\prod_{i \in I} e^{x_i} - 1 \right)^{D_I+1} =$$

$$= \left(\exp\left(\sum_{i \in I} x_i \right) - 1 \right)^{D_I+1} = \left(\sum_{i \in I} x_i \right)^{D_I+1} \cdot \left(\frac{\exp\left(\sum_{i \in I} x_i\right) - 1}{\sum_{i \in I} x_i} \right)^{D_I+1}.$$

The factor $\frac{\exp(\sum_{i \in I} x_i) - 1}{\sum_{i \in I} x_i}$ is a formal power series starting with the constant term 1. Hence the last factor in the right-hand side of the latter expression is an invertible power series. Thus, the generator $\widetilde{q}_I$ is mapped by ψ to the product $p_I \cdot *$, where $*$ is an invertible series. This implies $\psi(\widetilde{\mathcal{I}}_G) = \widetilde{\mathcal{J}}_G$. Hence the algebras $\mathcal{D}_G$ and B_G are isomorphic. □

Proof of Theorem 2. By Lemmas 1, 2 and Theorem 1, we get that all four algebras are isomorphic to each other. Furthermore, by Theorem 1, we know that their total dimension over $\mathbb{K}$ is the number of subforests in G. □

Theorem 3 now follows from Theorem 2.

Proof of Theorem 3. Consider the surjective homomorphism $h : \mathcal{D}_G \to \mathcal{K}_G$, defined by

$$h(y_i) = Y_i, \ i = 0, 1, \dots, n.$$

(It is indeed a homomorphism because every relation q_I holds for $Y_0, \dots, Y_n$.) By Theorem 2 we know that these algebras have the same dimension, implying that h is an isomorphism. Since the filtrations in $\mathcal{D}_G$ and $\mathcal{K}_G$ are defined using y_i's and Y_i's respectively it is clear that h is a filtration preserving isomorphism. □

2.1. Proving Theorem 5

We start with a few definitions.

Given a commutative algebra A, its element $t \in A$ is called *reducible nilpotent* if and only if there exists a presentation $t = \sum u_i v_i$, where all u_i, v_i are nilpotents.

For a nilpotent element $t \in A$, define its *degree* $d(t)$ as the minimal non-negative integer for which there exists a reducible nilpotent element $h \in A$ such that

$$(t - h)^{d+1} = 0.$$

Given an element $R \in \Phi_G$, we say that an edge-element ϕ_e *belongs to* R, if the monomial ϕ_e has a non-zero coefficient in the expansion of R as the sum of square-free monomials in Φ_G.

Lemma 3. *For any nilpotent element $R \in \mathcal{K}_G$, the degree $d(R)$ of R equals the number of edges of G belonging to R. (Observe that $d(R)$ is defined in $\mathcal{K}_G$.)*

Proof. We can write R in terms of $\{X_0, \dots, X_n\}$. (Observe that $\mathcal{K}_G$ and $\mathcal{C}_G$ coincide as subsets of Φ_G, but have different graded/filtered structures.) Now we can concentrate on the graded structure of $\mathcal{C}_G$. Select the part of R which lies in the first graded component of $\mathcal{C}_G$. Thus

$$R = R_1 + R' = \sum_{i=0}^{n} a_i X_i + R',$$

where R' is reducible nilpotent because it belongs to the linear span of other graded components. Thus $d(R) = d(R_1)$. The statement of Lemma 3 is obvious for R_1. Additionally by construction, an edge-element ϕ_e belongs to R if and only if it belongs to R_1. □

Lemma 4. *Given a graph G, let $\{\widetilde{Y}_0, \dots, \widetilde{Y}_n\}$ be the set of generators of $\mathcal{K}_G$ corresponding to the vertices (i.e., $\widetilde{Y}_i = \exp(X_i) - 1$). Then*

(1) $\{\widetilde{Y}_0, \dots, \widetilde{Y}_n\}$ *are nilpotents;*
(2) $\sum_{i=0}^{n} \ln(1 + \widetilde{Y}_i) = 0$*;*
(3) *for any subset $I \subset [0, n]$ and any set of pairwise distinct non-zero numbers $a_i \in \mathbb{K}$ ($i \in I$), the degree $d(\sum_{i \in I} a_i \widetilde{Y}_i)$ is equal to the number of edges incident to at least one vertex belonging to I;*
(4) *the number of edges between vertices i and j equals to $\frac{d(\widetilde{Y}_i) + d(\widetilde{Y}_j) - d(\widetilde{Y}_i + \widetilde{Y}_j)}{2}$.*

Proof. Item (1) is obvious.

To settle (2), observe that $\ln(1 + \widetilde{Y}_i) = X_i$ which implies

$$\sum_{i=0}^{n} \ln(1 + \widetilde{Y}_i) = \sum_{i=0}^{n} X_i = 0.$$

To prove (3), notice that, by Lemma 3, the degree $d(\sum_{i\in I} a_i\widetilde{Y}_i)$ is equal to the number of edges belonging to the sum $\sum_{i\in I} a_i\widetilde{Y}_i$. Each edge belongs either to zero, to one or to two generators $\widetilde{Y}_i$ from the latter sum. Moreover, if an edge belongs to two generators, then it has coefficients of opposite signs. Since all a_i are different, an edge-element ϕ_e belongs to $\sum_{i\in I} a_i\widetilde{Y}_i$ if and only if it belongs to at least one $\widetilde{Y}_i$, for $i \in I$. Thus the degree $d(\sum_{i\in I} a_i\widetilde{Y}_i)$ is the number of edges incident to at least one vertex from I.

To settle (4), notice that if e is an edge between vertices i and j, then ϕ_e belongs to $\widetilde{Y}_i$ and to $\widetilde{Y}_j$ with the opposite coefficients. Therefore ϕ_e does not belong to $(\widetilde{Y}_i + \widetilde{Y}_j)$. Using Lemma 3, we get that $d(\widetilde{Y}_i) + d(\widetilde{Y}_j) - d(\widetilde{Y}_i + \widetilde{Y}_j)$ equals twice the number of edges between i and j. □

Our proof of Theorem 5 uses the following technical lemma which should be obvious to the specialists.

Lemma 5 *(Folklore). Let E be the set of edges of some graph G without isolated vertices. If we know the following information:*

(1) *which pairs $e_i, e_j \in E$ of edges are multiple, i.e., connect the same pair of vertices;*
(2) *which pairs $e_i, e_j \in E$ of edges have exactly one common vertex;*
(3) *which triples $e_i, e_j, e_k \in E$ of edges form a triangle,*

then we can reconstruct G up to an isomorphism.

Proof. Assume the contrary, i.e., that there exist two non-isomorphic graphs G and G' such that there exists a bijection ψ of their edge sets E and E' preserving (1)–(3). Assume that under this bijection an edge $e \in E$ corresponds to the edge $e' \in E'$. Additionally assume that $|V(G')| \geq |V(G)|$.

Now we construct an isomorphism between G and G'. Let us split the vertices of G into two subsets: $V(G) = \widehat{V}(G) \cup \tilde{V}(G)$, where $\widehat{V}(G)$ are all vertices with at least two distinct neighboring vertices.

Let us construct a bijection ψ between the vertices of G and G', which extends the given bijection ψ of edges, i.e., for any $e = uv =\in E$, $e' = \psi(e) = \psi(u)\psi(v)$.

At first we define it on $\widehat{V}(G)$. Namely, given a vertex $v \in \widehat{V}(G)$, choose two non-multiple edges e_i and e_j incident to it, and define $\psi(v)$ as the common vertex of e'_i and e'_j. We need to show that $\psi(v)$ does not depend on the choice of e_i and e_j. It is enough to check it for a pair e_i and $e_k \neq e_j$, where e_k is another edge incident to v. Indeed, if e'_k has no common vertex with both e'_i and e'_j, then e'_i, e'_j and e'_k form a triangle in G' (because e'_k has a common vertex with e'_i and with e'_j). Hence, e_i, e_j and e_k form a triangle in G, but they have a common vertex v. Contradiction.

Now we need to extend ψ to vertices belonging to $\tilde{V}(G)$. Note that each vertex $v \in \tilde{V}(G)$ has exactly one adjacent vertex. There are two possibilities.

1° *The adjacent vertex u of v belongs to $\widehat{V}(G)$.* Consider the edge $e_{uv} \in E$. (There might be several such edges, but this is not important, because in G' they are also multiple.) Knowing the image $\psi(e_{uv})$ and the vertex $\psi(u)$, we define $\psi(v)$ as the vertex of $\psi(e_{uv})$ different from $\psi(u)$.

2° *Adjacent vertex u of v belongs to $\tilde{V}(G)$.* Consider the edge $e_{uv} \in E$ Knowing $\psi(e_{uv})$, we define $\psi(u)$ and $\psi(v)$ as the vertices of the edge $\psi(e_{uv})$ (not important which is mapped to which).

Since G' has no isolated vertices and each edge e' has exactly two incident vertices from $\psi(V)$, we get that $\psi : G \to G'$ is surjective. Hence, $\psi : G \to G'$ is an isomorphism (otherwise it must be non-injective on vertices and, hence, $|V(G)| > |V(G')|$). Therefore G and G' are isomorphic. □

Proof of Theorem 5. Let G and G' be a pair of graphs such that their filtered algebras $\mathcal{K}_G$ and $\mathcal{K}_{G'}$ are isomorphic. Without loss of generality, we can assume that $|E(G)| \leq |E(G')|$. Denote the numbers of vertices in G and G' by $n+1$ and $n'+1$ resp.

Consider $\mathcal{K}_G$ as a subalgebra in Φ_G. The elements $\widetilde{Y}_i = \exp(X_i) - 1,\ i \in [0, n]$ form a set of generators of $\mathcal{K}_G$. Since, by our assumptions, $\mathcal{K}_G$ and $\mathcal{K}_{G'}$ are isomorphic as filtered algebras, denote by $\widetilde{Z}_i \in \mathcal{K}_G,\ i \in [0, n']$ the elements corresponding to the vertices of G' under the latter isomorphism. The set $\{\widetilde{Z}_i, i \in [0, n']\}$ is also a generating set for $\mathcal{K}_G$ which gives the same filtered structure and satisfies the assumptions of Lemma 4. (Indeed, the operations of taking the logarithm and calculating the "degree" of an element are well-defined inside $\mathcal{K}_G$ and $\mathcal{K}_{G'}$. Thus we do not need to use the ambient algebras Φ_G and $\Phi_{G'}$ while applying Lemma 4.) In order to avoid confusion, we call $\widetilde{Y}_i$ the *i-th vertex of graph G*, and we call $\widetilde{Z}_j$ the *j-th vertex of graph G'*.

Since $\widetilde{Y}_i,\ i \in [0, n]$ and $\widetilde{Z}_i,\ i \in [0, n']$ determine the same graded structure, then, in particular,

$$\mathrm{span}\{1, \widetilde{Y}_0, \dots, \widetilde{Y}_n\} = \mathrm{span}\{1, \widetilde{Z}_0, \dots, \widetilde{Z}_{n'}\}.$$

Additionally, by Lemma 4, $\widetilde{Y}_i,\ i \in [0, n]$ and $\widetilde{Z}_i,\ i \in [0, n']$ are nilpotents, implying that

$$\mathrm{span}\{\widetilde{Y}_0, \dots, \widetilde{Y}_n\} = \mathrm{span}\{\widetilde{Z}_0, \dots, \widetilde{Z}_{n'}\}.$$

Firstly, we need to show that each edge-element ϕ_e belongs to at most two different $\widetilde{Z}_i$'s. Assume the contrary, i.e., that ϕ_e belongs to $\widetilde{Z}_i, \widetilde{Z}_j$ and $\widetilde{Z}_k$. Then there exist three distinct non-zero coefficients $r_1, r_2, r_3 \in \mathbb{K}$ such that ϕ_e does not belong to $r_1\widetilde{Z}_i + r_2\widetilde{Z}_j + r_3\widetilde{Z}_k$. Moreover, for generic distinct non-zero coefficients $r'_1, r'_2, r'_3 \in \mathbb{K}$, element $\phi_{e'}$ $(e' \in E(G))$ belongs to $r'_1\widetilde{Z}_i + r'_2\widetilde{Z}_j + r'_3\widetilde{Z}_k$ if and only if $\phi_{e'}$ belongs to at least one of $\widetilde{Z}_i, \widetilde{Z}_j$ and $\widetilde{Z}_k$. Hence by Lemma 3,

$$d(r_1\widetilde{Z}_i + r_2\widetilde{Z}_j + r_3\widetilde{Z}_k) < d(r'_1\widetilde{Z}_i + r'_2\widetilde{Z}_j + r'_3\widetilde{Z}_k).$$

But at the same time, by Lemma 4 (3), they should coincide, contradiction.

By Lemma 4, for any $i \in [0, n']$, the degree $d(\widetilde{Z}_i)$ equals the valency of $\widetilde{Z}_i$. Therefore,

$$2|E(G')| = \sum_{i=0}^{n'} d(\widetilde{Z}_i) \leq 2|E(G)|,$$

because each edge-element is included in at most two $\widetilde{Z}_i$. Since $|E(G)| \leq |E(G')|$, we conclude that $|E(G)| = |E(G')|$. Furthermore, by Lemma 4 (2), each element ϕ_e, $e \in E(G)$ belongs exactly to two vertices from $\widetilde{Z}_i$, $i \in [0, n']$ with the opposite coefficients. The number of edges between $\widetilde{Z}_i$ and $\widetilde{Z}_j$ equals $\frac{d(\widetilde{Z}_i)+d(\widetilde{Z}_j)-d(\widetilde{Z}_i+\widetilde{Z}_j)}{2}$ by Lemma 4; the number of common ϕ_e's equals the latter number by Lemma 3. Thus we obtain a natural bijection between the edges of G and the edges of G'. Let us additionally assume that the number of pairs of non-multiple edges which have a common vertex in G' is bigger than that in G.

So far we have constructed a bijection between the edges of G and the edges of G'. We want to prove that this bijection provides a graph isomorphism. We will achieve this as a result of the 5 claims collected in the following proposition which is closely related to Lemma 5.

Proposition 1. *The following facts hold.*

(1) *If ϕ_{e_1} and ϕ_{e_2} have no common vertex in G, then they have no common vertex in G' as well.*
(2) *If ϕ_{e_1} and ϕ_{e_2} are multiple edges in G, then they are multiple edges in G' as well.*
(3) *If ϕ_{e_1} and ϕ_{e_2} have exactly one common vertex in G, then they have exactly one common vertex in G' as well.*
(4) *If ϕ_{e_1}, ϕ_{e_2} and ϕ_{e_3} form a claw in G, then they form a claw in G' as well. (Three edges form a claw if they have one common vertex and their three other ends are distinct.)*
(5) *If ϕ_{e_1}, ϕ_{e_2} and ϕ_{e_3} form a triangle in G, then they form a triangle in G' as well.*

Proof. To prove (1), assume the contrary, i.e., assume that ϕ_{e_1} and ϕ_{e_2} belong to $\widetilde{Z}_j$ (and denote the corresponding coefficients by a and b resp.). Since elements $\widetilde{Y}_0, \dots, \widetilde{Y}_n$ have no monomial $\phi_{e_1}\phi_{e_2}$, then $\widetilde{Z}_0, \dots, \widetilde{Z}_{n'}$ have no monomial $\phi_{e_1}\phi_{e_2}$ as well (since their spans coincide). Then $\ln(1+\widetilde{Z}_j)$ contains the monomial $\phi_{e_1}\phi_{e_2}$ with the coefficient $-ab$.

By Lemma 4 (2), we have $\sum_{i=0}^{n'} \ln(1+\widetilde{Z}_i) = 0$, so there exists $k \in [0, n'], k \neq j$ such that $\ln(1+\widetilde{Z}_k)$ contains the monomial $\phi_{e_1}\phi_{e_2}$ with a non-zero coefficient. Then $\widetilde{Z}_k$ must contain ϕ_{e_1} and ϕ_{e_2} (since $\widetilde{Z}_k$ does not contain $\phi_{e_1}\phi_{e_2}$). Hence, $\widetilde{Z}_k$ has ϕ_{e_1} and ϕ_{e_2} with coefficients $-a$ and $-b$ resp. Therefore $\ln(1+\widetilde{Z}_k)$ contains monomial $\phi_{e_1}\phi_{e_2}$ with the coefficient $-(-a)(-b) = -ab$. Thus the sum $\sum_{j=0}^{n'} \ln(1+\widetilde{Z}_j)$ contains $\phi_{e_1}\phi_{e_2}$ with coefficient $-2ab$, contradiction.

To prove (2), consider the map from $\mathrm{span}\{\widetilde{Y}_0, \dots, \widetilde{Y}_n\}$ to $\mathbb{K}^2$, sending an element from the span to the pair of coefficients of ϕ_{e_1} and ϕ_{e_2} resp. Since edges e_1 and e_2 are multiple in G, the image of this map has dimension 1. If ϕ_{e_1} and ϕ_{e_2} are not multiple in G', then the image of the map from $\operatorname{span}\{\widetilde{Z}_0, \dots, \widetilde{Z}_{n'}\} = \operatorname{span}\{\widetilde{Y}_0, \dots, \widetilde{Y}_n\}$ has dimension 2.

To prove (3), observe that we have already settled Claims 1 and 2, and also we additionally assumed that the number of pairs of edges which have a common vertex in G' is bigger than that in G. Then each such pair of edges from G is mapped to the pair of edges from G' with the same property.

To prove (4), consider the map from $\operatorname{span}\{\widetilde{Y}_0, \dots, \widetilde{Y}_n\}$ to $\mathbb{K}^3$, sending an element in the span to the triple of coefficients of ϕ_{e_1}, ϕ_{e_2} and ϕ_{e_3} resp. The image of this map has dimension 3. However if ϕ_{e_1}, ϕ_{e_2} and ϕ_{e_3} form a triangle in G', then the image of the map from $\operatorname{span}\{\widetilde{Z}_0, \dots, \widetilde{Z}_{n'}\}$ has dimension 2.

Proof of (5) is similar to that of (4). □

Now applying Lemma 5 we finish our proof of Theorem 5. □

3. Further generalizations

In this section we will consider the Hilbert series of other filtered algebras similar to $\mathcal{K}_G$. (Recall that the Hilbert series of a filtered algebra is, by definition, the Hilbert series of its associated graded algebra.)

Let f be a univariate polynomial or a formal power series over $\mathbb{K}$. We define the subalgebra $\mathcal{F}[f]_G \subset \Phi_G$ as generated by 1 together with

$$f(X_i) = f\left(\sum c_{i,e}\phi_e\right),\ i = 0, \dots, n.$$

Example 1. For $f(x) = x$, $\mathcal{F}[f]_G$ coincides with $\mathcal{C}_G$. For $f(x) = \exp(x)$, $\mathcal{F}[f]_G$ coincides with $\mathcal{K}_G$.

Obviously, the filtered algebra $\mathcal{F}[f]_G$ does not depend on the constant term of f. From now on, we assume that $f(x)$ has no constant term, since for any g such that $f - g$ is constant, the filtered algebras $\mathcal{F}[f]_G$ and $\mathcal{F}[g]_G$ are the same.

Proposition 2. *Let f be any polynomial with a non-vanishing linear term. Then the algebras $\mathcal{C}_G$ and $\mathcal{F}[f]_G$ coincide as subalgebras of Φ_G.*

Proof. The argument is the same as in the proof of Lemma 1. We only need to change $\exp(x) - 1$ to $f(x)$ and $\ln(1 + y)$ to $f^{-1}(y)$. □

Theorem 6. *Let f be any polynomial with non-vanishing linear and quadratic terms. Then given two simple graphs G_1 and G_2 without isolated vertices, $\mathcal{F}[f]_{G_1}$ and $\mathcal{F}[f]_{G_2}$ are isomorphic as filtered algebras if and only if G_1 and G_2 are isomorphic graphs.*

Proof. Repeat the proof of Theorem 5. □

3.1. Generic functions f and their Hilbert series

Since $X_i^{d_i+1} = 0$ for any i, we can always truncate any polynomial (or a formal power series) f at degree $|G| + 1$ without changing $\mathcal{F}[f]_G$. Therefore, for a given graph G, it suffices to consider f as a polynomial of degrees less than or equal to $|G|$. To simplify our notation, let us write $HS_{f,G}$ instead of $HS_{\mathcal{F}[f]_G}$.

Given a graph G, consider the space of polynomials of degree less than or equal to $|G|$ and the corresponding Hilbert series.

Proposition 3. *In the above notation, for generic polynomials f of degree at most $|G|$, the Hilbert series $HS_{f,G}$ is the same. This generic Hilbert series (denoted by HS_G below) is maximal in the majorization partial order among all $HS_{g,G}$, where g runs over the set of all formal power series with non-vanishing linear term.*

Here (as usual) by generic polynomials of degree at most $|G|$ we mean polynomials belonging to some Zariski open subset in the linear space of all polynomials of degree at most $|G|$.

Recall that, by definition, a sequence $(a_0, a_1, \dots)$ is *bigger* than $(b_0, b_1, \dots)$ in the majorization partial order if and only if, for any $k \geq 0$,

$$\sum_{i=0}^{k} a_i \geq \sum_{i=0}^{k} b_i.$$

More information about the majorization partial order can be found in e.g. [3].

Proof. Note that, for a function f, the sum of the first $k+1$ entries of its Hilbert series $HS_{f,G}$ equals the dimension of

$$\operatorname{span}\,\{f^{\alpha_0}(X_0) f^{\alpha_1}(X_1) \cdots f^{\alpha_n}(X_n) : \sum_{i=0}^{n} \alpha_i \leq k\}.$$

It is obvious that, for a generic f, this dimension is maximal. Since all Hilbert series $HS_{f,G}$ are polynomials of degree at most $|G| + 1$, then the required property has to be checked only for $k \leq [G|$. Therefore it is obvious that, for generic f, their Hilbert series is maximal in the majorization order. $\square$

Remark 3. We know that the Hilbert series of the graded algebra $\mathcal{C}_G$ is a specialization of the Tutte polynomial of G. However we can not calculate the Hilbert series of $\mathcal{K}_G$ from the Tutte polynomial of G, because there exists a pair of graphs (G, G') with the same Tutte polynomial and different $HS_{\mathcal{K}_G}$ and $HS_{\mathcal{K}_{G'}}$, see Example 2.

Additionally, notice that, in general, $HS_{\exp,G} := HS_{\mathcal{K}_G} \neq HS_G$. Analogously we can not calculate generic Hilbert series HS_G from the Tutte polynomial of G, see Example 2.

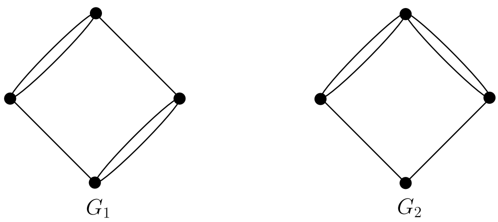

Fig. 1. Graphs with the same matroid and different "K-theoretic" and generic Hilbert series.

Example 2. Consider two graphs G_1 and G_2 presented in Fig. 1. It is easy to see that G_1 and G_2 have isomorphic matroids and hence, the same Tutte polynomial. Therefore, the Hilbert series of $\mathcal{C}_{G_1}$ and $\mathcal{C}_{G_2}$ coincide. Namely,

$$HS_{\mathcal{C}_{G_1}}(t) = HS_{\mathcal{C}_{G_2}}(t) = 1 + 3t + 6t^2 + 9t^3 + 8t^4 + 4t^5 + t^6.$$

However, the Hilbert series of their "K-theoretic" algebras are distinct. Namely

$$HS_{\mathcal{K}_{G_1}}(t) = 1 + 4t + 10t^2 + 14t^3 + 3t^4,$$
$$HS_{\mathcal{K}_{G_2}}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4.$$

Moreover their generic Hilbert series are also distinct and different from their "K-theoretic" Hilbert series. Namely,

$$HS_{G_1}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4,$$
$$HS_{G_2}(t) = 1 + 4t + 10t^2 + 16t^3 + t^4.$$

Putting our information together we get,

$$HS_{\mathcal{C}_{G_1}} = HS_{\mathcal{C}_{G_2}} \prec HS_{\mathcal{K}_{G_1}} \prec HS_{\mathcal{K}_{G_2}} = HS_{G_1} \prec HS_{G_2},$$

where $\prec$ denotes the majorization partial order.

4. "K-theoretical" analog for spanning trees

In this section we always assume that G is connected. For an arbitrary loopless graph G on the vertex set $\{0, ..., n\}$, let Φ_G^T be the graded commutative algebra over a given field $\mathbb{K}$ generated by the variables $\phi_e, e \in G$, with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for any edge } e \in G;$$
$$\prod_{e \in H} \phi_e = 0, \quad \text{for any non-slim subgraph } H \subset G,$$

where a subgraph H is called *slim* if its complement $G \setminus H$ is connected.

Let $\mathcal{C}_G^T$ be the subalgebra of Φ_G^T generated by the elements

$$X_i^T = \sum_{e \in G} c_{i,e}\phi_e,$$

for $i = 1, ..., n$, where $c_{i,e}$ is given by (1). (Notice that X_i^T and X_i are defined by exactly the same formula but in different ambient algebras.)

Algebra $\mathcal{C}_G^T$ will be called the *spanning trees counting algebra* of G and is, obviously, the quotient of $\mathcal{C}_G$ modulo the set of relations $\prod_{e \in H} \phi_e = 0$ over all non-slim subgraphs H. Its defining set of relations is very natural and resembles that of (2). Namely, define the ideal $\mathcal{J}_G^T$ in $\mathbb{K}[x_1, \ldots, x_n]$ as generated by the polynomials:

$$p_I^T = \left(\sum_{i \in I} x_i\right)^{D_I}, \tag{4}$$

where I ranges over all nonempty subsets in $\{1, \ldots, n\}$ and the number D_I is the same as in (2). Set $B_G^T := \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_G^T$. One of the results of [5] claims the following.

Theorem 7 *(Theorems 9.1 and Corollary 10.5 of [5]). For any simple graph G on the set of vertices $\{0, 1, \ldots, n\}$, the algebras B_G^T and $\mathcal{C}_G^T$ are isomorphic. Their total dimension is equal to the number of spanning trees in G. The dimension $\dim B_G^T(k)$ of the k-th graded component of B_G^T equals the number of spanning trees T in G with external activity $|G| - n - k$.*

Similarly to the above, we can define the filtered algebra $\mathcal{K}_G^T \subset \Phi_G^T$ which is isomorphic to $\mathcal{C}_G^T$ as a non-filtered algebra. Namely, $\mathcal{K}_G^T$ is defined by the generators:

$$Y_i^T = \exp(X_i^T) = \prod_{e \in G} (1 + c_{i,e}\phi_e),\ i = 0, \ldots, n.$$

The first result of this section is as follows. Define the ideal $\mathcal{I}_G^T \subseteq \mathbb{K}[y_0, y_1, \ldots, y_n]$ as generated by the polynomials:

$$q_I^T = \left(\prod_{i \in I} y_i - 1\right)^{D_I}, \tag{5}$$

where I ranges over all nonempty proper subsets in $\{0, 1, \ldots, n\}$ and the number D_I is the same as in (2), together with the generator

$$q_{\{0,1,\ldots,n\}}^T = \prod_{i=0}^{n} y_i - 1. \tag{6}$$

Set $\mathcal{D}_G^T := \mathbb{K}[y_0, \ldots, y_n]/\mathcal{I}_G^T$.

We present two results similar to the case of spanning forests.

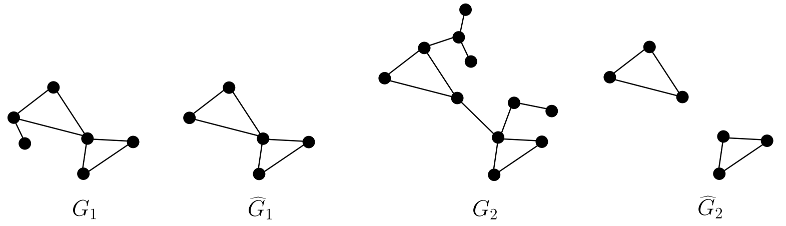

Fig. 2. Graphs and their Δ-subgraphs.

Theorem 8. *For any simple graph G, algebras B_G^T, $\mathcal{C}_G^T$, $\mathcal{D}_G^T$ and $\mathcal{K}_G^T$ are isomorphic as (non-filtered) algebras. Their total dimension is equal to the number of spanning trees in G.*

Proof. The proof is similar to that of Theorem 2. Algebras $\mathcal{C}_G^T$ and $\mathcal{K}_G^T$ coincide as subalgebras of Φ_G^T (but they have different filtrations); algebras $\mathcal{C}_G^T$ and B_G^T are isomorphic by Theorem 7. The proof of the isomorphism between $\mathcal{D}_G^T$ and B_G^T is the same as above; we only need to add the variable $x_0 = -(\sum_{i=1}^n x_i)$ to B_G^T. $\square$

Theorem 9. *For any simple graph G, algebras $\mathcal{D}_G^T$ and $\mathcal{K}_G^T$ are isomorphic as filtered algebras.*

Proof. Similar to the above proof of Theorem 3. $\square$

To move further, we need to give a definition.

Definition 1. Let G be a connected graph. We define its Δ-subgraph $\widehat{G} \subseteq G$ as the subgraph with the following edges and vertices:

- $e \in E(\widehat{G})$, if e is not a bridge (i.e., $G \setminus e$ is still connected),
- $v \in V(\widehat{G})$, if there is an edge $e \in E(\widehat{G})$ incident to v.

By the bridge-free matroid of G we call the graphical matroid of $\widehat{G}$.

In general, $\widehat{G}$ contains more information about G than its matroid, because there exist graphs with isomorphic matroids and non-isomorphic Δ-subgraphs, see Fig. 2.

Recall that in a recent paper [4], the first author has shown that $\mathcal{C}_G^T$ depends only on the bridge-free matroid of G. Namely,

Proposition 4 *(Proposition 16 of [4]). For any two connected graphs G_1 and G_2 with isomorphic bridge-free matroids (matroids of their Δ-subgraphs), algebras $\mathcal{C}_{G_1}^T$ and $\mathcal{C}_{G_2}^T$ are isomorphic.*

Unfortunately, we can not prove the converse implication at present although we conjecture that is should hold as well, see Conjecture 6 in § 5. In case of filtered algebra $\mathcal{K}_{G_1}^T$ and $\mathcal{K}_{G_2}^T$ we can also prove an appropriate result only in one direction, see Proposition 5.

Similarly to § 3 we can to define $\mathcal{F}[f]_G^T \subset \Phi_G$. Let f be a univariate polynomial or a formal power series over $\mathbb{K}$. We define the subalgebra $\mathcal{F}[f]_G^T \subset \Phi_G$ as generated by 1 and by

$$f(X_i^T) = f\left(\sum c_{i,e}\phi_e\right),\ i = 0, \dots, n.$$

Proposition 5. *For a univariate polynomial f and any two connected graphs G_1 and G_2 with isomorphic Δ-subgraphs $\widehat{G}_1$ and $\widehat{G}_2$, algebras $\mathcal{F}[f]_{G_1}^T$ and $\mathcal{F}[f]_{G_2}^T$ are isomorphic as filtered algebras. Additionally, $\mathcal{K}_{G_1}^T$ and $\mathcal{K}_{G_2}^T$ are isomorphic as filtered algebras.*

Proof. Note that if G has a bridge e, then filtered algebra $\mathcal{F}[f]_G^T$ is the Cartesian product of filtered algebras $\mathcal{F}[f]_{G'}^T$ and $\mathcal{F}[f]_{G''}^T$, where G' and G'' are connected components of $G \setminus e$.

Thus filtered algebra $\mathcal{F}[f]_G^T$ is the Cartesian product of such filtered algebras corresponding to the connected components of the Δ-subgraph of G.

Therefore if connected graphs G_1 and G_2 have isomorphic Δ-subgraphs, then their filtered algebras $\mathcal{F}[f]_{G_1}^T$ and $\mathcal{F}[f]_{G_2}^T$ are isomorphic. $\square$

Remark 4. In the general case we cannot prove that these algebras distinguish graphs with different Δ-subgraphs. The proof of Theorem 5 does not work for two reasons. Firstly, $d(\widetilde{Y}_i)$ is not the degree of the i-th vertex in G. Secondly, even if we can construct a similar bijection between edges, we do not have an analog of Proposition 1. Since in the proof we consider coefficients of monomial $\phi_{e_1}\phi_{e_2}$, in case when e_1 and e_2 are not bridges and when $\{e_1, e_2\}$ is a cut, this monomial can still lie in the ideal.

It is possible to construct such a bijection in a smaller set of graphs, namely for graphs such that, for any edge e in the graph, there is another edge e' which is multiple to e. For such graphs we do not have the second problem, because if $\{e_1, e_2\}$ is a cut, then e_1 and e_2 are multiple edges. So, instead of the actual converse of Proposition 5, we can prove the converse in the latter situation, but we do not present this result here.

Proposition 6. *In the above notation, for generic polynomials f of degree at most $|G|$, the Hilbert series $HS_{\mathcal{F}[f]_G^T}$ is the same. This generic Hilbert series (denoted by HS_{G^T} below) is maximal in the majorization partial order among $HS_{\mathcal{F}[g]_G^T}$ for g running over the set of power series with non-vanishing linear term.*

Proof. See the proof of Proposition 3. $\square$

Example 3. Consider two graphs G_1 and G_2, see Fig. 2. It is easy to check that subgraphs $\widehat{G}_1$ and $\widehat{G}_2$ have isomorphic matroids, implying that algebras $\mathcal{C}_{G_1}^T$ and $\mathcal{C}_{G_2}^T$ are isomorphic.

$$HS_{\mathcal{C}^T_{G_1}}(t) = HS_{\mathcal{C}^T_{G_2}}(t) = 1 + 4t + 4t^2.$$

The Hilbert series of "K-theoretic" algebras are distinct, namely

$$HS_{\mathcal{K}^T_{G_1}}(t) = 1 + 5t + 3t^2,$$
$$HS_{\mathcal{K}^T_{G_2}}(t) = 1 + 6t + 2t^2.$$

These graphs are "small", so their generic Hilbert series coincides with the "K-theoretic" one. Putting our information together, we get

$$HS_{\mathcal{C}^T_{G_1}} = HS_{\mathcal{C}^T_{G_2}} \prec HS_{\mathcal{K}^T_{G_1}} = HS_{G_1{}^T} \prec HS_{\mathcal{K}^T_{G_2}} = HS_{G_2{}^T}.$$

5. Related problems

At first, we formulate several problems in case of spanning forests; their analogs for spanning trees are straightforward.

Problem 1. *For which functions f besides $a + bx$ and $a + be^x$, can one present relations in $\mathcal{F}[f]_G$ for any graph G in a simple way? In other words, for which f, can one define an algebra similar to B_G and $\mathcal{D}_G$?*

Since the Hilbert series $HS_{\mathcal{K}_G}$ and HS_G are not expressible in terms of the Tutte polynomial of G, they contain some other information about G.

Problem 2. *Find combinatorial descriptions of $HS_{\mathcal{K}_G}$ and HS_G.*

Problem 3. *For which graphs G, do the Hilbert series $HS_{\mathcal{K}_G}$ and HS_G coincide? In other words, for which G, is* exp *a generic function?*

Problem 4. *Describe combinatorial properties of $HS_{f,G}$ when f is a function starting with a monomial of degree bigger than 1, i.e. $f(x) = x^k + \cdots,\ k > 1$? In particular, calculate the total dimension of $\mathcal{F}[f]_G$.*

The most delicate and intriguing question is as follows.

Problem 5. *Do there exist non-isomorphic graphs G_1 and G_2 such that, for any polynomial $f(x)$, the Hilbert series HS_{f,G_1} and HS_{f,G_2} coincide? In other words, does the collection of Hilbert series $HS_{f,G}$ taken over all polynomials f determine G up to isomorphism?*

The following problems deal with the case of spanning trees only.

Conjecture 6 *(Comp. [4]). Algebras $\mathcal{C}_{G_1}^T$ and $\mathcal{C}_{G_2}^T$ for graphs G_1 and G_2 are isomorphic if and only if their bridge-free matroids are isomorphic, where the bridge-free matroid is the graphical matroid of the Δ-subgraph.*

Problem 7. *Which class of graphs satisfies the property that if two graphs G_1 and G_2 from this class have isomorphic $\mathcal{K}_{G_1}^T$ and $\mathcal{K}_{G_2}^T$, then their Δ-subgraphs are isomorphic. In other words, can one classify all pairs (G_1, G_2) of connected graphs, which have isomorphic filtered algebras $\mathcal{K}_{G_1}^T$ and $\mathcal{K}_{G_2}^T$? (The same problem for $\mathcal{F}[f]_{G_1}^T$ and $\mathcal{F}[f]_{G_2}^T$, where $f(x) = x + ax^2 + \cdots$.)*

Acknowledgments

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References

[1] F. Ardila, A. Postnikov, Combinatorics and geometry of power ideals, Trans. Amer. Math. Soc. 362 (8) (2010) 4357–4384.

[2] O. Holtz, A. Ron, Zonotopal algebra, Adv. Math. 227 (2011) 847–894.

[3] A.W. Marshall, I. Olkin, B. Arnold, Inequalities: Theory of Majorization and Its Applications, second edition, Springer Series in Statistics, Springer, New York, 2011.

[4] G. Nenashev, On Postnikov–Shapiro algebras and their generalizations, arXiv:1509.08736.

[5] A. Postnikov, B. Shapiro, Trees, parking functions, syzygies, and deformations of monomial ideals, Trans. Amer. Math. Soc. 356 (8) (2004) 3109–3142.

[6] A. Postnikov, B. Shapiro, M. Shapiro, Algebras of curvature forms on homogeneous manifolds, in: Differential Topology, Infinite-Dimensional Lie Algebras, and Applications, in: Amer. Math. Soc. Transl. Ser. 2, vol. 194, Amer. Math. Soc., Providence, RI, 1999, pp. 227–235.

[7] B. Shapiro, M. Shapiro, On ring generated by Chern 2-forms on SL_n/B, C. R. Acad. Sci. Paris Sér. I Math. 326 (1) (1998) 75–80.