REAL POLYNOMIALS WITH CONSTRAINED REAL DIVISORS. I. FUNDAMENTAL GROUPS

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Abstract. In the late 80s, V. Arnold and V. Vassiliev initiated the topological study of the space of real univariate polynomials of a given degree d and with no real roots of multiplicity exceeding a given positive integer. Expanding their studies, we consider the spaces $\mathcal{P}_d^{\mathbf{c}\Theta}$ of real monic univariate polynomials of degree d whose real divisors avoid sequences of root multiplicities, taken from a given poset Θ of compositions which is closed under certain natural combinatorial operations.

In this paper, we concentrate on the fundamental group of $\mathcal{P}_d^{\mathbf{c}\Theta}$ and of some related topological spaces. We find explicit presentations for the groups $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ in terms of generators and relations and show that in a number of cases they are free with rank bounded from above by a quadratic function in d. We also show that $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ stabilizes for d large.

The mechanism that generates $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ has similarities with the presentation of the braid group as the fundamental group of the space of complex monic degree d polynomials with no multiple roots and with the presentation of the fundamental group of certain ordered configuration spaces over the reals which appear in the work of Khovanov.

We further show that the groups $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ admit an interpretation as special bordisms of immersions of 1-manifolds into the cylinder $S^1 \times \mathbb{R}$, whose images avoid the tangency patterns from Θ with respect to the generators of the cylinder.

1. Introduction

1.1. **Motivation and Outline of Results.** In [Ar], V. Arnold proved the following Theorems A–D, which were later generalized by V. Vassiliev, see [Va]. These results are the main source of motivation and inspiration for our study. In the formulations of these theorems, we keep the original notation of [Ar], which we will abandon later on. In what follows, theorems, conjectures, etc., labeled by letters, are borrowed from the existing literature, while those labeled by numbers are hopefully new.

Theorem A. *The fundamental group of the space of smooth functions $f : S^1 \to \mathbb{R}$ without critical points of multiplicity higher than 2 on a circle S^1 is isomorphic to the group of integers $\mathbb{Z}$.*

The space of smooth functions $f : \mathbb{R} \to \mathbb{R}$ without critical points of multiplicity higher than 2 and which, for arguments $|x| > 1$, coincide either with x or with x^2 also have the fundamental group $\mathbb{Z}$.

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Theorem B. *The latter fundamental group is naturally isomorphic to the group $\mathcal{B}$ of A_3-cobordism classes of embedded closed plane curves without vertical* [1] *tangential inflections.*

The generator of $\mathcal{B}$ is shown as the kidney-shaped loop in Figure 7(a).

Remark 1.1. The multiplication of the cobordism classes in $\mathcal{B}$ is defined as the disjoint union of curves, embedded in the half-planes $\{(t,x)|\, t<0\}$ and $\{(t,x)|\, t>0\}$, and the inversion is the change of sign of t.

For $1 \le k \le d$, let G_k^d be the space of real monic polynomials $x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{R}[x]$ with no real roots of multiplicity greater than k.

Theorem C. *If $k < d < 2k+1$, then G_k^d is diffeomorphic to the product of a sphere S^{k-1} by an Euclidean space. In particular, for all i and $k < d < 2k+1$,*

$$\pi_i(G_k^d) \simeq \pi_i(S^{k-1})$$

An analogous result holds for the space of polynomials whose sum of roots vanishes, i.e., polynomials with the vanishing coefficient a_{d-1}.

Theorem D. *The homology groups with integer coefficients of the space G_k^d are nonzero only for dimensions which are the multiples of $k-1$ and less or equal to d. For $(k-1)r \le d$, we have*

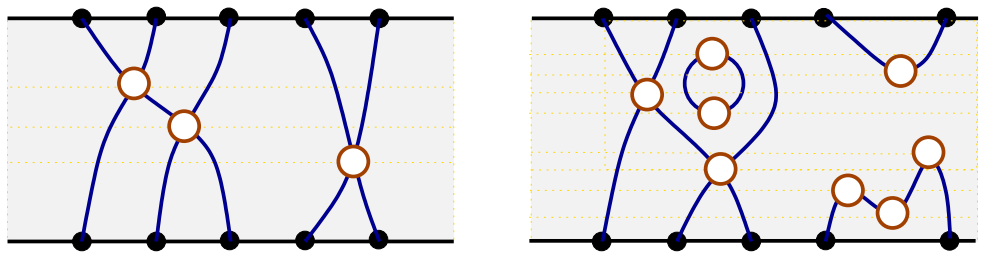
$$H_{r(k-1)}(G_k^d) \simeq \mathbb{Z}.$$

The main goal of this paper and its sequel [KSW] is to generalize Theorems A–D to the situation where the multiplicities of the real roots *avoid a given set of patterns* Θ. In our more general situation, the fundamental group of such polynomial spaces can be non-trivial and deserves a separate study, which is carried out below. We will see that the mechanism by which these fundamental groups are generated is in some sense similar to the one that produces the braid groups as the fundamental groups of spaces of complex degree d monic polynomials with no multiple roots. The mechanism is also related to Khovanov's paper [Kh], which studies the topology of spaces K_d, obtained from the space $\mathbb{R}^d$ by removing vectors $(x_1, \ldots, x_d)$ such that $x_j = x_k = x_l$ for some distinct j, k, l, or/and vectors of the form $x_j = x_k$ and $x_l = x_m$ for some distinct j, k, l, m.

This space is an ordered analog of the space of real polynomials containing the polynomials with only real roots, no roots of multiplicity 3 or higher, and no two different roots both of multiplicity 2 or higher. Figure 1 shows two contrasting images of objects from [Kh] and from the present paper, each representing a *loop* in the relevant configuration space.

Besides the studies of V. Arnold [Ar] and V. Vassiliev [Va], the second major motivation for this paper comes from results of the first author connecting the cohomology of spaces of real polynomials that avoid certain patterns of root multiplicities with characteristic classes, arising in the theory of traversing flows, see [Ka], [Ka1], and [Ka3]. For traversing vector flows on compact manifolds X with boundary ∂X and with a priori forbidden

[1] In our convention, the curves in the tx-plane do not have inflections with respect to the coordinate line $\{t = const\}$.



(a) (b)

Figure 1. (a) A graphic representation of a loop in the Khovanov's space K_5 and (b) in the space $\mathcal{P}_7^{\mathbf{c}\Theta}$ of real degree 7 polynomials with no real roots of multiplicity ≥ 3 and no pairs of real roots of multiplicity ≥ 2.

tangency patterns of their trajectories to ∂X, the spaces of polynomials avoiding the same patterns play a fundamental role which is similar to the one played by Graßmannians in the category of vector bundles.

Let $\mathcal{P}_d$ denote the space of real monic univariate polynomials of degree d. Given a polynomial $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathcal{P}_d$, we define its *real divisor* $D_{\mathbb{R}}(P)$ as the multiset

$$\underbrace{x_1 = \cdots = x_{i_1}}_{\omega_1 = i_1} < \underbrace{x_{i_1+1} = \cdots = x_{i_1+i_2}}_{\omega_2 = i_2 - i_1} < \cdots < \underbrace{x_{i_{\ell-1}+1} = \cdots = x_{i_\ell}}_{\omega_\ell = i_\ell - i_{\ell-1}}$$

of the real roots of $P(x)$. The tuple $\omega = (\omega_1, \ldots, \omega_\ell)$ is called the (ordered) *real root multiplicity pattern* of $P(x)$. Let $\mathring{\mathsf{R}}_d^\omega$ be the set of all polynomials with root multiplicity pattern ω, and let R_d^ω be the closure of $\mathring{\mathsf{R}}_d^\omega$ in $\mathcal{P}_d$.

For a given collection Θ of root multiplicity patterns, we consider the union $\mathcal{P}_d^\Theta$ of the subspaces $\mathring{\mathsf{R}}_d^\omega$ over all $\omega \in \Theta$. We denote by $\mathcal{P}_d^{\mathbf{c}\Theta} := \mathcal{P}_d \setminus \mathcal{P}_d^\Theta$ its complement. We restrict our studies to the case when $\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$ and call such Θ *closed*.

We observe in Lemma 2.1 that, for any Θ containing the minimal element (d) (and thus for any closed Θ), the space $\mathcal{P}_d^\Theta$ is contractible. Thus it makes more sense to consider its one-point compactification $\bar{\mathcal{P}}_d^\Theta$. For a closed Θ, the latter is the union of the one-point compactifications $\bar{\mathsf{R}}_d^\omega$ of the R_d^ω for $\omega \in \Theta$ with the identified points at infinity. By Alexander duality in $\bar{\mathcal{P}}_d \cong S^d$, we get the relation

$$\bar{H}^j(\mathcal{P}_d^{\mathbf{c}\Theta}; \mathbb{Z}) \approx \bar{H}_{d-j-1}(\bar{\mathcal{P}}_d^\Theta; \mathbb{Z})$$

in reduced (co)homology. This implies that the spaces $\mathcal{P}_d^{\mathbf{c}\Theta}$ and $\bar{\mathcal{P}}_d^\Theta$ carry equivalent (co)homological information.

Example 1.2. For Θ comprising all ω's with at least one component greater than or equal to k, we have that Θ is closed and $\mathcal{P}_d^{\mathbf{c}\Theta} \cong G_k^d$ (see Theorem C and Theorem D).

In this paper and its sequel [KSW], for a closed Θ, we aim at describing the topology of $\bar{\mathcal{P}}_d^\Theta$ and $\mathcal{P}_d^{\mathbf{c}\Theta}$ in terms of combinatorial properties of Θ.

Outline of the results: In § 2 we prove our main results, namely, Proposition 2.10, Theorem 2.11 and Theorem 2.15, generalizing Theorem C in the case $k = 2$. Proposition 2.10 gives a presentation of $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ by generators and relations while Theorem 2.11 states that under certain conditions on Θ, $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is a *free group*. In Theorem 2.15 we show that $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ stabilizes for d large. In Corollary 2.18, we provide an analog of Theorem A and show that the fundamental group of the space of real monic polynomials of a fixed odd degree $d > 1$ with no real critical points of multiplicity higher than 2 is isomorphic to $\mathbb{Z}$.

Additionally, when Θ consists of all ω's with the property $|\omega|' \geq 2$, we show in Theorem 2.4, without using the results from [KSW], that the number of generators of the free group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is equal to $\frac{d(d-2)}{4}$ for an even d, and to $\frac{(d-1)^2}{4}$ for an odd d. Moreover, we show that in this case $\mathcal{P}_d^{\mathbf{c}\Theta}$ is homotopy equivalent to a wedge of circles and hence is a $K(\pi, 1)$-space.

In § 3 a special cobordism theory which allows us to formulate and prove an analog of Theorem B is developed. Theorem 3.3 claims that the free group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$, where Θ consists of all ω's with the property $|\omega|' \geq 2$, admits an interpretation as special *bordisms of immersions* of 1-manifolds into the cylinder $S^1 \times \mathbb{R}$, i.e., immersions whose images avoid the tangency patterns from Θ with respect to the generators of the cylinder.

We close the paper in § 4 with open questions and a description of further research direction.

1.2. **Cell structure on the space of real univariate polynomials.** First we recall a well-known stratification of the space of real univariate polynomials of a given degree.

For any real polynomial $P(x)$, we have already defined its real divisor $D_{\mathbb{R}}(P)$, i.e. the ordered set of its real zeros, counted with their multiplicities. We have also associated to a polynomial $P(x) \in \mathbb{R}[x]$ and $D_{\mathbb{R}}(P)$ its real root multiplicity pattern $\omega_P := (\omega_1, \ldots, \omega_\ell)$. The combinatorics of multiplicity patterns will play the key role in our study. Let us fix some terminology and notation for multiplicity patterns.

Definition 1.3. A sequence $\omega = (\omega_1, \ldots, \omega_\ell)$ of positive integers is called a *composition* of the number $|\omega| := \omega_1 + \cdots + \omega_\ell$. (We allow the *empty composition* $\omega = ()$ of the number $|()| = 0$).

We denote by $\mathcal{P}_d^\omega$ the set of real monic polynomials of degree d, whose real roots generate the combinatorial pattern ω. Evidently, for a given composition ω, the stratum $\mathcal{P}_d^\omega$ is empty if and only if either $|\omega| > d$, or $|\omega| \leq d$ and $|\omega| \not\equiv d \mod 2$.

Notation 1.4. We denote by $\mathbf{\Omega}$ the set of all compositions of natural numbers. For a given positive integer d, we write $\mathbf{\Omega}_{\langle d]}$ for the set of all $\omega \in \mathbf{\Omega}$, such that $|\omega| \leq d$ and $|\omega| \equiv d \mod 2$. We denote by $\mathbf{\Omega}_{\langle d], |\sim|' \geq \ell}$ the subset of $\mathbf{\Omega}_{\langle d]}$, consisting of all compositions $\omega \in \mathbf{\Omega}_{\langle d]}$ with $|\omega|' \geq \ell$. Analogously, we define $\mathbf{\Omega}_{\langle d], |\sim|' = \ell}$ as the subset of $\mathbf{\Omega}_{\langle d]}$, consisting of all compositions $\omega \in \mathbf{\Omega}_{\langle d]}$ with $|\omega|' = \ell$.

Now we define two (sequences of) operations on $\mathbf{\Omega}$ that will govern our subsequent considerations, see also [Ka].

The *merge operation* $\mathsf{M}_j : \mathbf{\Omega} \to \mathbf{\Omega}$ sends $\omega = (\omega_1, \dots, \omega_\ell)$ to the composition

$$\mathsf{M}_j(\omega) = (M_j(\omega)_1, \dots, M_j(\omega)_{\ell-1}),$$

where, for any $j \geq \ell$, one has $\mathsf{M}_j(\omega) = \omega$, and for $1 \leq j < \ell$, one has

$$\begin{aligned} \mathsf{M}_j(\omega)_i &= \omega_i \text{ if } i < j, \\ \mathsf{M}_j(\omega)_j &= \omega_j + \omega_{j+1}, \\ \mathsf{M}_j(\omega)_i &= \omega_{i+1} \text{ if } i+1 < j \leq \ell - 1. \end{aligned} \tag{1.1}$$

Similarly, we define the *insertion operation* $\mathsf{I}_j : \mathbf{\Omega} \to \mathbf{\Omega}$ that sends $\omega = (\omega_1, \dots, \omega_\ell)$ to the composition $\mathsf{I}_j(\omega) = (I_j(\omega)_1, \dots, I_j(\omega)_{\ell+1})$, where for any $j > \ell + 1$, one has $\mathsf{I}_j(\omega) = \omega$, and for $1 \leq j \leq \ell + 1$, one has

$$\begin{aligned} \mathsf{I}_j(\omega)_i &= \omega_i \text{ if } i < j, \\ \mathsf{I}_j(\omega)_j &= 2, \\ \mathsf{I}_j(\omega)_i &= \omega_{i-1} \text{ if } j \leq i \leq \ell + 1. \end{aligned} \tag{1.2}$$

The next proposition collects some basic properties of R_d^ω, see [Ka, Theorem 4.1] for details.

Proposition E. *Take $d \geq 1$ and $\omega = (\omega_1, \dots, \omega_\ell) \in \mathbf{\Omega}_{\langle d]}$. Then $\mathring{\mathsf{R}}_d^\omega \subset \mathcal{P}_d$ is an (open) cell of codimension $|\omega|'$. Moreover, R_d^ω is the union of the cells $\{\mathring{\mathsf{R}}_d^{\omega'}\}_{\omega'}$, taken over all ω' that are obtained from ω by a sequence of merging and insertion operations. In particular,*

(a) *the cell $\mathring{\mathsf{R}}_d^\omega$ has (maximal) dimension d if and only if $\omega = (\underbrace{1, 1, \dots, 1}_{\ell})$ for $0 \leq \ell \leq d$ and $\ell \equiv d \mod 2$,*

(b) *the cell $\mathring{\mathsf{R}}_d^\omega$ has dimension 1 if and only if $\omega = (d)$. In this case, $\mathring{\mathsf{R}}^{(d)} = \mathsf{R}^{(d)} = \{(x-a)^d \mid a \in \mathbb{R}\}$.*

Geometrically speaking, if a point in $\mathring{\mathsf{R}}_d^\omega$ approaches the boundary $\mathsf{R}_d^\omega \setminus \mathring{\mathsf{R}}_d^\omega$, then either there is at least one value of the index j such that the distance between the jth and $(j+1)^{\text{st}}$ largest root in $D_{\mathbb{R}}(P)$ goes to 0, or there are two complex-conjugate real roots that converge to a double real root, which then either is j^{th} largest or adds 2 to the multiplicity of the jth largest real root.

The first situation corresponds to the application of the merge operation M_j to ω, and the second either to the application of the insertion I_j or to the application of I_j, followed by M_j.

Note that $|\sim|$ is preserved under the merge operations, while the insert operations increase $|\sim|$ by 2 and thus preserve its parity.

By Proposition E, the merge and the insert operations can be used to define a *partial order* "$\succ$" on the set $\mathbf{\Omega}$ of all compositions, reflecting the adjacency of the non-empty open cells $\mathring{\mathsf{R}}_d^\omega$.

Definition 1.5. For $\omega, \omega' \in \mathbf{\Omega}$, we say that ω' *is smaller than* ω (notation "$\omega \succ \omega'$" or "$\omega' \prec \omega$"), if ω' can be obtained from ω by a sequence of merge and insert operations $\{\mathsf{M}_j\}$, $j \geq 1$, and $\{\mathsf{I}_j\}$, $j \geq 0$.

From now on, we will consider a subset $\Theta \subseteq \mathbf{\Omega}$ as a *poset*, ordered by $\succ$. As an immediate consequence of Proposition E, we get the following statement.

Corollary F. *For* $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$,

(i) $\mathcal{P}_d^\Theta$ *is closed in* $\mathcal{P}_d$ *if and only if, for any* $\omega \in \Theta$ *and* $\omega' \in \mathbf{\Omega}_{\langle d]}$, *the relation* $\omega' \prec \omega$ *implies* $\omega' \in \Theta$*;*

(ii) *if* $\mathcal{P}_d^\Theta$ *is closed in* $\mathcal{P}_d$*, then the one-point compactification* $\bar{\mathcal{P}}_d^\Theta$ *carries the structure of a compact CW-complex with open cells* $\{\mathring{\mathsf{R}}_d^\omega\}_{\omega\in\Theta}$*, labeled by* $\omega \in \Theta$*, and the unique* 0*-cell, represented by the point* $\bullet$ *at infinity.*

Recall that we call $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$ closed if $\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$. Hence F has the following immediate reformulation.

$$\Theta \subseteq \mathbf{\Omega}_{\langle d]} \text{ is closed}$$
$$\Updownarrow$$
$$\text{for any } \omega \in \Theta \text{ and } \omega' \in \mathbf{\Omega}_{\langle d]}, \text{ the relation } \omega' \prec \omega \text{ implies } \omega' \in \Theta.$$

Now, we are in position to give a precise formulation of the main questions discussed in this paper and its sequel [KSW]:

Problem 1.6. For a given closed poset $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$,

- ▷ calculate the homotopy groups $\pi_i(\bar{\mathcal{P}}_d^\Theta)$ and $\pi_i(\mathcal{P}_d^{\mathbf{c}\Theta})$ in terms of the combinatorics of Θ;[2]
- ▷ calculate the integer homology of $\bar{\mathcal{P}}_d^\Theta$ or, equivalently, the integer cohomology of $\mathcal{P}_d^{\mathbf{c}\Theta}$ in terms of the combinatorics of Θ.

In this paper we concentrate on the fundamental groups of $\bar{\mathcal{P}}_d^\Theta$ and $\mathcal{P}_d^{\mathbf{c}\Theta}$. Questions about the (co)homology of $\bar{\mathcal{P}}_d^\Theta$ and $\mathcal{P}_d^{\mathbf{c}\Theta}$ will be addressed in [KSW].

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[2]In full generality, this goal is as illusive as computing the homotopy groups of spheres; in fact, as testified by Theorem C, for special Θ's, the two problems are intimately linked.

2. Computing $\pi_1(\bar{\mathcal{P}}_d^\Theta)$ and $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$

2.1. **Homotopy type of $\mathcal{P}_d^\Theta$ and the fundamental group $\pi_1(\bar{\mathcal{P}}_d^\Theta)$.** The following simple statement gives us a start on the homotopy type of the polynomial spaces under consideration. In the proof, we use the map $q : \mathcal{P}_d \times [0,\infty) \to \mathcal{P}_d$ which sends each pair $(P(x),\lambda)$, where $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$ and $\lambda \in [0,1]$, to the polynomial $x^d + a_{d-1}\,\lambda\, x^{d-1} + \cdots + a_0\,\lambda^d$. Hence, this transformation amounts to the multiplication of all roots of $P(x)$ by λ.

Lemma 2.1. *For any poset $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$ that contains (d), the space $\mathcal{P}_d^\Theta \subseteq \mathcal{P}_d$ is contractible. In particular, for any closed poset $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$, the space $\mathcal{P}_d^\Theta$ is contractible.*

Proof. For $P(x) \in \mathcal{P}_d^\Theta$ and $\lambda \geq 0$, the roots of $q(P(x),\lambda)$ are the roots of $P(x)$, being multiplied by λ. In particular, we have $q(P(x),0) = x^d \in \mathcal{P}_d^{(d)}$. Thus by $(d) \in \Theta$ we have $q(P(x),\lambda) \in \mathcal{P}_d^\Theta$ for $P(x) \in \mathcal{P}_d^\Theta$. Then the restriction of q to $[0,1] \times \mathcal{P}_d^\Theta$ is a well-defined homotopy between the identity map and the constant map that sends $\mathcal{P}_d^\Theta$ to x^d. The assertion now follows. □

In contrast to $\mathcal{P}_d^\Theta$, its one-point compactification $\bar{\mathcal{P}}_d^\Theta$ often has non-trivial topology for closed posets Θ. A simple example of such a situation is $\bar{\mathcal{P}}_d^{\mathbf{\Omega}_{\langle d]}} = \bar{\mathcal{P}}_d \cong S^d$. Other examples, including the case treated in Theorem C and Theorem D, show that $\mathcal{P}_d^{\mathbf{c}\Theta}$ can have non-trivial topology as well.

Besides the map q, the following map p has been frequently used in the literature on the topology of spaces of univariate polynomials. The map $p : \mathcal{P}_d \to \mathcal{P}_d$ sends $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{R}[x]$ to $P(x - \frac{a_{d-1}}{d})$. The map p preserves the stratification $\{\mathring{\mathsf{R}}_d^\omega\}_\omega$ and is a fibration with the fiber $\mathbb{R}$. Thus, for a closed poset $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$, the restriction $p|_{\mathcal{P}_d^\Theta} : \mathcal{P}_d^\Theta \to \mathcal{P}_d^\Theta$ is also a fibration with fiber $\mathbb{R}$. Its image $\mathcal{P}_{d,0}^\Theta$ consists of all polynomials in $\mathcal{P}_d^\Theta$ with the vanishing coefficient at x^{d-1}, i.e. with vanishing root sum. Therefore, we get a homeomorphism $\bar{\mathcal{P}}_d^\Theta \cong \tilde{\Sigma}\bar{\mathcal{P}}_{d,0}^\Theta$. Here $\tilde{\Sigma}\, X$ denotes the reduced suspension of a space X.

Since $q(P(x),\lambda)$ amounts to multiplying the roots of $P(x)$ by λ, their sum is multiplied by λ as well. In particular, q preserves $\mathcal{P}_{d,0}^\Theta$. Using the map

$$a_0 + \cdots + a_{d-1}x^{d-1} + x^d \;\mapsto\; (a_0,\ldots,a_{d-1})$$

we identify $\mathcal{P}_d$ with Euclidean d-space with respect to its metric structure. Clearly, for $\lambda \geq 0$ and fixed $0 \neq P(x) \in \mathcal{P}_d$, the norm $\|q(P(x),\lambda)\|$ is strictly monotone in λ. Since $\|q(P(x),0)\| = 0$, there is a unique $\lambda_P > 0$ for which $\|q(P(x),\lambda_P)\| = 1$. Let $S^{d-1} \subset \mathcal{P}_d$ be the unit sphere. Therefore, on $\mathcal{P}_{d,0}^\Theta \setminus \{0\}$ the map $P(x) \mapsto q(P(x),\lambda_P)$ is a deformation retraction to the closed subspace $S^{d-1} \cap \mathcal{P}_{d,0}^\Theta$ of S^{d-1}. Thus, we get a homeomorphism $\bar{\mathcal{P}}_{d,0}^\Theta \cong \Sigma(S^{d-1} \cap \mathcal{P}_{d,0}^\Theta)$. Note, that we consider $\Sigma\,\emptyset$ as the discrete two-point space. This analysis implies the following claim.

Theorem 2.2. *For any closed poset $\Theta \subseteq \mathbf{\Omega}_{\langle d]}$, we get $\pi_1(\bar{\mathcal{P}}_d^\Theta) = 0$, unless $\Theta = \{(d)\}$. If $\Theta = \{(d)\}$ then $\bar{\mathcal{P}}_d^\Theta \cong S^1$.*

Proof. By the arguments preceding the theorem, we have $\bar{\mathcal{P}}_d^{\Theta} \cong \tilde{\Sigma}\Sigma(S^{d-1} \cap \mathcal{P}_{d,0}^{\Theta})$. Therefore, $\bar{\mathcal{P}}_d^{\Theta}$ is simply connected, unless $S^{d-1} \cap P_{d,0}^{\Theta}$ is empty. But this can only happen if $\Theta = \{(d)\}$. It is easily seen that $\mathring{\mathsf{R}}_d^{(d)} = \{(x-\alpha)^d \mid \alpha \in \mathbb{R}\} \cong \mathbb{R}$. Hence its one-point compactification is S^1. □

Note, that the argument employed in the proof of Theorem 2.2 also implies the following isomorphism in homology

$$\bar{H}_i(S^{d-1} \cap P_{d,0}^{\Theta}; \mathbb{Z}) \approx \bar{H}_{i+2}(\bar{\mathcal{P}}_d^{\Theta}; \mathbb{Z})$$

for all $i \geq 0$.

2.2. **The fundamental group of the complement of the codimension two skeleton of the space $\mathcal{P}_d$.** In this section, we describe the fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}})$. Recall, that we collect in $\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}$ all $\omega \in \mathbf{\Omega}_{\langle d]}$ such that $|\omega|' \geq 2$. Thus $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$ is the complement of the codimension 2 skeleton of $\mathcal{P}_d$ in our celluation.

The outline of the arguments below is as follows. The space $\mathcal{P}_d^{\mathbf{c}\Theta}$ is an open d-dimensional manifold without boundary, stratified into cells of dimensions d and $d-1$. Moreover, for any pair of top-dimensional cells, there exists at most one $(d-1)$-dimensional cell which separates them. Any such manifold is homotopy equivalent to a graph whose vertices are top-dimensional cells and whose edges connect pairs of adjacent top-dimensional cells.

We associate a graph $\mathfrak{G}_d$ with the space $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$, subdivided into open d-cells by the $(d-1)$-cells. The set of vertices of $\mathfrak{G}_d$ is the union of the sets $\mathbf{\Omega}_{\langle d],\,|\sim|'=1}$ and $\mathbf{\Omega}_{\langle d],\,|\sim|'=0}$. We connect vertices $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$ and $\omega' \in \mathbf{\Omega}_{\langle d],\,|\sim|'=0}$ by an edge $\{\omega, \omega'\}$, if the $(d-1)$-cell $\mathring{\mathsf{R}}_d^{\omega}$ lies in the boundary of the closure $\mathsf{R}_d^{\omega'}$ of $\mathring{\mathsf{R}}_d^{\omega'}$. In particular, the edges of $\mathfrak{G}_d$ correspond to single insertion and merging operations, applied to compositions from $\mathbf{\Omega}_{\langle d],\,|\sim|'=0}$. As usual, we identify the graph $\mathfrak{G}_d$ with the 1-dimensional simplicial complex, defined by its vertices and edges (see Figure 5 for the example of $\mathfrak{G}_6$).

We embed the graph $\mathfrak{G}_d$ in $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$ by mapping the vertex of $\mathfrak{G}_d$, labeled by $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$, to a preferred point w_ω in the $(d-1)$-cell $\mathring{\mathsf{R}}_d^{\omega}$ and the vertex of $\mathfrak{G}_d$, labeled by $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=0}$, to a preferred point e_ω in the d-cell $\mathring{\mathsf{R}}_d^{\omega}$. Then we identify each edge $\{\omega, \omega'\}$ of $\mathfrak{G}_d$, where $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$, $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=0}$, with a smooth path $[w_\omega, e_{\omega'}]$ such that the semi-open segment $(w_\omega, e_{\omega'}] \subset \mathring{\mathsf{R}}_d^{\omega'}$. In addition, we can choose the paths so that $[w_{\omega_1}, e_{\omega'}] \cap [w_{\omega_2}, e_{\omega'}] = e_{\omega'}$ for any pair $\omega_1, \omega_2 \prec \omega'$ and $\omega_1 \neq \omega_2$. Moreover, for each $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$, we may arrange for the two paths, $[w_\omega, e_{\omega_1'}]$ and $[w_\omega, e_{\omega_2'}]$, to share the tangent vector at their common end w_ω, so that the path $[e_{\omega_1'}, e_{\omega_2'}]$ is transversal[3] to the hypersurface $\mathring{\mathsf{R}}_d^{\omega}$ at w_ω. This construction produces an embedding $\mathcal{E} : \mathfrak{G}_d \to \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$. In what follows, we do not distinguish between $\mathfrak{G}_d$ and its image $\mathcal{E}(\mathfrak{G}_d) \subset \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$.

[3]The transversality is needed to insure the stability under perturbations of the map $\mathcal{E}$, defined below, with respect to the hypersurfaces $\{\mathring{\mathsf{R}}_d^{\omega}\}$.

Lemma 2.3. *The graph $\mathfrak{G}_d$ is homotopy equivalent to a wedge of $\frac{d(d-2)}{4}$ circles if d is even, and of $\frac{(d-1)^2}{4}$ circles if d is odd.*

Proof. A simple calculation shows that $|\mathbf{\Omega}_{\langle d],\,|\sim|'=0}| = \lfloor \frac{d}{2} \rfloor + 1$, and

$$|\mathbf{\Omega}_{\langle d],\,|\sim|'=1}| = \sum_{k=1}^{\lfloor \frac{d}{2} \rfloor} (2k-1) = \left\lfloor \frac{d}{2} \right\rfloor^2.$$

For each $k \in \{2, \dots, d\}$ with the same parity as d, the vertex $(\underbrace{1, \dots, 1}_{k}) \in \mathbf{\Omega}_{\langle d],\,|\sim|'=0}$ is contained in the $k-1$ edges that lead to the vertices $(\underbrace{1, \dots, 1}_{s}, 2, \underbrace{1, \dots, 1}_{k-2-s}) \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$, where $0 \le s \le k-2 \le d-2$, and in the $k+1$ edges that lead to the vertices $(\underbrace{1, \dots, 1}_{s}, 2, \underbrace{1, \dots, 1}_{k-s}) \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$, where $0 \le s \le k \le d-1$. For d even, each case yields $\frac{d^2}{4}$ edges. It is easily seen that the graph $\mathfrak{G}_d$ is connected and hence it is homotopy equivalent to a wedge of circles. Now a simple calculation of the Euler characteristic $\chi(\mathfrak{G}_d)$ yields $1 - (\frac{d}{2} + 1 + \frac{d^2}{4}) + 2\frac{d^2}{4} = \frac{d(d-2)}{4}$ circles. The calculation for odd d is analogous. □

Our next result is inspired by Arnold's Theorem A. In Figure 2, for $d = 6$, we illustrate it by exhibiting the cell structure in $\mathcal{P}_6^{\mathbf{c\Omega}_{\langle 6],\,|\sim|'\ge 2}}$ and its graph $\mathfrak{G}_6$.

Theorem 2.4. *The space $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\ge 2}}$ is homotopy equivalent to a wedge of $\frac{d(d-2)}{4}$ circles for d even, and to a wedge of $\frac{(d-1)^2}{4}$ circles for d odd.*

In particular, the fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\ge 2}})$ is the free group on $\frac{d(d-2)}{4}$ generators for even d and on $\frac{(d-1)^2}{4}$ generators for odd d, and $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\ge 2}}$ is the corresponding $K(\pi, 1)$-space.

Proof. As an open subset of $\mathbb{R}^d$, the space $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\ge 2}}$ is paracompact. Now consider a finite open cover $\mathcal{X} := \{X_\omega\}_{\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}}$ of the space $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\ge 2}}$, where X_ω is the union of the $(d-1)$-cell $\mathring{\mathsf{R}}_d^\omega$ with the two adjacent d-cells that contain $\mathring{\mathsf{R}}_d^\omega$ in their boundary.

Each X_ω is open in $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\ge 2}}$. Indeed, any point $x \in X_\omega$ either (1) lies in one of the two d-cells and thus has an open neighborhood in $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\ge 2}}$ (contained in that d-cell), or (2) $x \in \mathring{\mathsf{R}}_d^\omega$, in which case it has on open neighborhood X_ω in $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\ge 2}}$.

By [Ka, Lemma 2.4] the attaching maps $\phi : D^d \to \mathcal{P}_d$ of the d-cells D^d are injective on the ϕ-preimage of each open $(d-1)$ cell in $\mathcal{P}_d$. This implies that X_ω retracts to the $\mathring{\mathsf{R}}_d^\omega$, which, in turn, is contractible. For $i \ge 2$ and for pairwise distinct compositions $\omega_1, \dots, \omega_i \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$, the intersection $\bigcap_{j=1}^i X_{\omega_i}$ is either empty, or is one of the open d-cells $\mathring{\mathsf{R}}_d^\omega$ for some $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=0}$. It follows that, for $i \ge 1$ and for compositions $\omega_1, \dots, \omega_i \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$, the intersection $\bigcap_{j=1}^i X_{\omega_i}$ is either empty or contractible.

The preceding arguments show that the assumptions of [Ha, Corollary 4G.3] are satisfied for the open covering $\mathcal{X}$ of $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$. Hence $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$ is homotopy equivalent to the nerve $N_{\mathcal{X}}$ of the covering $\mathcal{X}$. We can identify $N_{\mathcal{X}}$ with the simplicial complex, whose simplices are the non-empty subsets A of $\mathbf{\Omega}_{\langle d],\,|\sim|'=1}$ such that $\bigcap_{\omega\in A} X_\omega$ is non-empty. So the maximal simplices of the nerve $N_{\mathcal{X}}$ are in bijection with the elements of $\mathbf{\Omega}_{\langle d],\,|\sim|'=0}$.

The maximal simplex, corresponding to $\omega' \in \mathbf{\Omega}_{\langle d],\,|\sim|'=0}$, contains all $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$ for which $\mathring{\mathsf{R}}_d^{\omega'} \subseteq X_\omega$. The intersection of the two maximal simplices, corresponding to $\omega', \omega'' \in \mathbf{\Omega}_{\langle d],\,|\sim|'=0}$, is labeled by all $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$ for which there are edges from ω' to ω and from ω'' to ω in $\mathfrak{G}_d$.

The graph $\mathfrak{G}_d$ can be covered by 1-dimensional subcomplexes Y_ω , $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$, where Y_ω is the union of the two edges in $\mathfrak{G}_d$, containing ω. It is easy to check that the nerve of this covering $\mathcal{Y} := \{Y_\omega\}_{\omega\in\mathbf{\Omega}_{\langle d],\,|\sim|'=1}}$ is again $N_{\mathcal{X}}$. In fact, under the embedding $\mathcal{E} : \mathfrak{G}_d \to \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$, one has $Y_\omega = X_\omega \cap \mathcal{E}(\mathfrak{G}_d)$.

By [Bj, Theorem 10.6], the nerve $N_{\mathcal{Y}}$ and the graph $\mathfrak{G}_d$ are homotopy equivalent.

Moreover, by the proof of [Ha, Corollary 4G.3], the following claim is valid. Consider an embedding $Y \hookrightarrow X$ of a paracompact space Y into a paracompact space X and a locally finite open covering $\mathcal{X} = \{X_\alpha\}_\alpha$ of X. Put $\mathcal{Y} = \{Y_\alpha := X_\alpha \cap Y\}_\alpha$. If, for any nonempty intersection $\cap_i X_{\alpha_i}$, the intersection $\cap_i Y_{\alpha_i} \neq \emptyset$, and both intersections are contractible, then the nerves $N_{\mathcal{X}}$ and $N_{\mathcal{Y}}$ are naturally isomorphic as simplicial complexes, and $Y \hookrightarrow X$ is a homotopy equivalence.

We conclude that the embedding $\mathcal{E} : \mathfrak{G}_d \to \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$ is a homotopy equivalence. The result now follows from Lemma 2.3. □

Instead of $\mathfrak{G}_d$ we can also consider the graph $\mathfrak{G}'_d$ in which any pair of edges e, e', forming a path that joins two different nodes labeled by elements from $\mathbf{\Omega}_{\langle d],\,|\sim|'=0}$, is replaced by a single edge labeled by the unique $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$ in the intersection of e and e'. Thus $\mathfrak{G}_d$ is the graph subdivision of $\mathfrak{G}'_d$ and in particular, we have $\pi_1(\mathfrak{G}_d) \cong \pi_1(\mathfrak{G}'_d)$ when both are considered as 1-dimensional simplicial complexes. We will use the graph $\mathfrak{G}'_d$ in a second slightly different and perhaps more natural approach to the calculation of $\pi_1(\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}})$.

For that we introduce the alphabet A with letters $\omega_{ij} = (\underbrace{1,\ldots,1}_{i}, 2, \underbrace{1,\ldots,1}_{j})$, where $0 \leq i, j,\ i+j \leq d-2$ and $i+j \equiv d \mod 2$.

(See illustrations in Figure 2, Figure 3 and Figure 4).

We consider now the free group, generated by the letters from A. The letters are in obvious bijection with the edges of the graph $\mathfrak{G}'_d$. For a letter $\omega \in \mathsf{A}$, we use $\omega^+ = \omega$ or its inverse $\omega^- = \omega^{-1}$ to capture the orientations of these edges and to represent elements of the free group generated by A by words. We write $\mathsf{A}^\pm$ for the set of letters $\omega^\pm$ where $\omega \in \mathsf{A}$. We use the convention that the sign "+" corresponds to the orientation from the

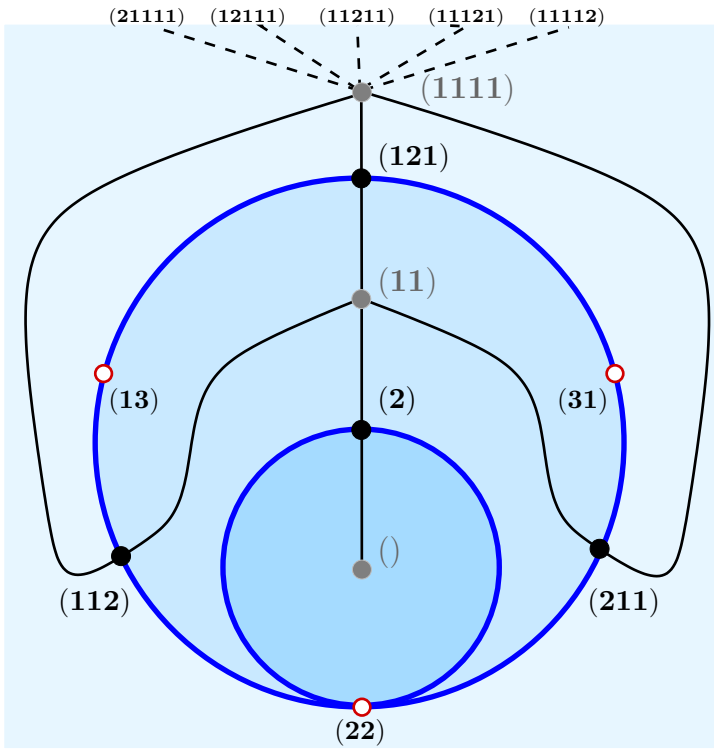

Figure 2. A slice through the celluation of $\mathcal{P}_6^{\mathbf{c\Omega}_{\langle 6],\,|\sim|'\geq 2}}$ together with the graph $\mathfrak{G}_6$, dual to the celluation (shown by black curves). Patterns $(22), (13), (31)$ label the strata of codimension 2, while the arcs, labeled by $(2), (211), (121), (112)$, represent the strata of codimension 1.

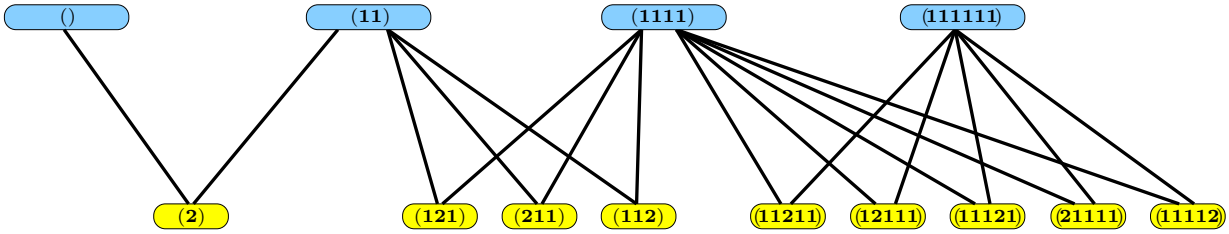

Figure 3. The graph $\mathfrak{G}_6$ as part of the poset $\mathbf{\Omega}_{\langle 6]}$

vertex with less ones towards the vertex with more ones, and the sign "-" corresponds to the opposite orientation.

We choose () as the basepoint of $\mathfrak{G}'_d$ if d is even, and (1) as the basepoint if d is odd. In particular, we can consider $\pi_1(\mathfrak{G}'_d)$ as a subgroup of the free group over A.

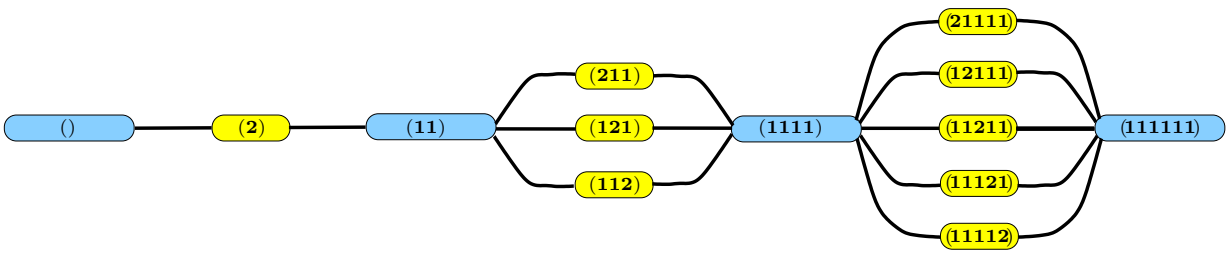

Figure 4. The graph $\mathfrak{G}_6$, drawn as it is embedded in $\mathcal{P}_6^{\mathbf{c}\boldsymbol{\Omega}_{\langle 6],\,|\sim|'\leq 2}}$

Next, we introduce a class of words which will be used to define canonical representatives of elements from $\pi_1(\mathfrak{G}'_d)$.

Definition 2.5. For $d \geq 1$, we say that a word w in the alphabet $\mathsf{A}^{\pm}$ is *admissible* if it is either empty or satisfies the following two conditions:

(a_e) if d is even, then w starts with the letter $(2)^+$ and ends with the letter $(2)^-$;

(a_o) if d is odd, then w starts with either the letter $(12)^+$ or with the letter $(21)^+$ and ends either with $(12)^-$ or with $(21)^-$.

(b) any pair $(\omega_1^{\pm}, \omega_2^{\pm})$ of two consecutive letters has the property that the number of 1's in ω_1 and ω_2 either coincide or differs by two. In the former case, the signs of the letters are different, and in the latter case, the signs should be as follows. If ω_1 has less ones than ω_2, then we only allow the pair (ω_1^+, ω_2^+); if ω_1 has more ones than ω_2, then we only allow the pair (ω_1^-, ω_2^-).

Clearly, admissible words are in bijection with based loops in the graph $\mathfrak{G}'_d$. Here are two examples of admissible words representing based loops, the first one for even $d \geq 6$ and second one for odd $d \geq 7$:

$$w_1 = (2)^+\,(112)^+\,(121)^-\,(211)^+\,(11112)^+\,(12111)^-\,(121)^-\,(2)^-,$$

$$w_2 = (21)^+\,(1112)^+\,(112111)^+\,(111211)^-\,(1211)^-\,(12)^-.$$

We observe that a word is admissible if and only if its reduction (in the sense of combinatorial group theory) is admissible.

Let $\mathcal{G}_d$ be the set of reduced admissible words over $\mathsf{A}^{\pm}$. To each $\omega_{ij}^+ \in \mathsf{A}^{\pm}$, we associate the word γ_{ij} given by

- $$\gamma_{ij} = \omega_{0,0}^+\,\omega_{0,2}^+\,\cdots\,\omega_{0,i+j-2}^+\,\omega_{ij}^+\,\omega_{0,i+j}^-\,\cdots\,\omega_{0,2}^-\,\omega_{0,0}^-$$
  if d is even and
- $$\gamma_{ij} = \omega_{0,1}^+\,\omega_{0,3}^+\,\cdots\,\omega_{0,i+j-2}^+\,\omega_{ij}^+\,\omega_{0,i+j}^-\,\cdots\,\omega_{0,3}^-\,\omega_{0,1}^-$$
  if d is odd.

All γ_{ij} are admissible, and if $i \neq 0$ then γ_{ij} is reduced. We notice that each γ_{ij} represents a single *loop* in $\mathfrak{G}'_d$.

The following is then a standard fact about graphs considered as 1-dimensional CW-complexes.

Lemma 2.6. *The set $\mathcal{G}_d$ is a subgroup of the free group over the alphabet A, isomorphic to $\pi_1(\mathfrak{G}'_d)$. In particular, $\mathcal{G}_d$ is a free group itself. The γ_{ij} for $0 \leq i, j$, $i \neq 0$ and $i + j \leq d - 2$ form a minimal generating set of $\mathcal{G}_d$.*

Note each $\gamma_{0,j}$ is 1 in $\mathcal{G}_d$. But for our geometric picture and for the sake of a less technical presentation, we will continue working with the $\gamma_{0,j}$.

Now we would like to relate this representation of $\pi_1(\mathfrak{G}'_d)$ to $\pi_1(\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \geq 2}})$. We use the notations from the proof of Theorem 2.4.

For $\omega_{ij} \in \mathsf{A}$, consider the codimension one "wall" $\mathring{\mathsf{R}}_d^{\omega_{ij}}$. The walls divide $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \geq 2}}$ into a set of open d-cells $\{\mathring{\mathsf{R}}_d^{\omega}\}$ for $\omega = (\underbrace{1, \ldots, 1}_{i}) \in \mathbf{\Omega}_{\langle d], |\sim|' = 0}$, where $0 \leq i \leq d$ is even when d is even and odd when d is odd.

We orient each wall $\mathring{\mathsf{R}}_d^{\omega_{ij}}$ in such a way that crossing it in the preferred direction increases the number of simple real roots by 2. We consider this as a "+"-crossing and the crossing in the opposite direction as a "−"-crossing. This notation goes along with our convention for orienting the edges in $\mathfrak{G}'_d$.

Consider an oriented loop $\gamma : S^1 \to \mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \geq 2}}$, based in $\mathring{\mathsf{R}}_d^{()}$ if d is even and in $\mathring{\mathsf{R}}_d^{(1)}$ if d is odd. By the general position arguments, we may assume that γ is smooth and transversal to each wall. In particular, since S^1 is compact, the (transversal) intersection of $\gamma(S^1)$ with each wall is a finite set. As we move along γ, we record each transversal crossing $\gamma \cap \mathring{\mathsf{R}}_d^{\omega_{ij}}$ and the direction of the crossing by the corresponding letter in $\mathsf{A}^{\pm}$.

Lemma 2.7. *The homotopy classes of based loops $\gamma \subset \mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \geq 2}}$ and the homotopy classes of based loops in $\mathfrak{G}'_d$ are in one-to-one correspondence with the reduced admissible words over the alphabet $\mathsf{A}^{\pm}$. In addition, concatenation of loops corresponds to the concatenation and reduction of words.*

In particular, $\pi_1(\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \geq 2}}) \simeq \mathcal{G}_d \simeq \pi_1(\mathfrak{G}'_d)$ is a free group.

Sketch of Proof: For $\pi_1(\mathfrak{G}'_d)$ the assertion already follows from Lemma 2.6 and the arguments preceding the lemma.

Our choice of basepoint in $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \geq 2}}$ corresponds exactly to the choice of the base vertices () for d even and (1) for d odd for admissible words in $\mathfrak{G}'_d$. The above arguments and the conditions for admissibility of words over the alphabet $\mathsf{A}^{\pm}$ show that based loops in $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \geq 2}}$ are represented by admissible words. Reduction is easily seen to be a homotopy equivalence. Moreover, loops corresponding to different reduced admissible words are not homotopy equivalent. Since the concatenation of admissible words is admissible, the remaining assertion now follows. □

Remark 2.8. Let X be a compact path-connected Hausdorff space and $C(X)$ the space of continuous functions on X. Let $P(u,x) = u^d + a_{d-1}u^{d-1} + \ldots + a_1 u + a_0$ be a polynomial in u with coefficients $a_i \in C(X)$. Assume, that for each $x \in X$, $P(u,x) \in \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$; that is $P(u,x)$ has only simple real roots and no more than a single root of multiplicity 2.

Thus every such $P(u,x)$ generates a continuous map $\alpha_P : X \to \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$. Since a subgroup of a free group is free or trivial, it follows that when $\pi_1(X)$ does not admit an epimorphism with the free target group, then the induced map

$$(\alpha_P)_* : \pi_1(X) \to \pi_1(\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}})$$

must be trivial. For example, this is the case for any X with a finite fundamental group.

Here is one implication of the triviality of $(\alpha_P)_*$: for any loop $\delta : S^1 \to X$, the loop $(\alpha_P)_*(\delta)$ hits the nonsingular part of the discriminant variety $\mathcal{D}_d \subset \mathcal{P}_d$ so that the admissible word in the alphabet $\mathsf{A}^{\pm}$, generated by $(\alpha_P)_*(\delta)$, can be reduced to the identity 1.

2.3. **The fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ for any closed poset Θ with the locus $\mathcal{P}_d^{\Theta} \subset \mathcal{P}_d$ of codimension ≥ 2.** This section describes how the fundamental group changes if we add to the space $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$ some strata of codimension 2. Each of these strata will create a relation in the fundamental group of the new space under construction. We provide an explicit description of these relations and show that, in many cases, the resulting quotient is still a *free* group.

We start with the following simple statement.

Lemma 2.9. *Let $\Theta \subset \mathbf{\Omega}_{\langle d]}$ be a closed poset such that $\Theta \subset \mathbf{\Omega}_{\langle d],\,|\sim|'\geq k}$. Then the homotopy groups $\pi_i(\mathcal{P}_d^{\mathbf{c}\Theta})$ vanish for all $i < k-1$. In particular, $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ vanishes, provided $\Theta \subset \mathbf{\Omega}_{\langle d],\,|\sim|'\geq 3}$. As a special case, we have $\pi_1(\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 3}}) = 0$.*

Proof. We observe that if $\Theta \subset \mathbf{\Omega}_{\langle d],\,|\sim|'\geq k}$, then $\operatorname{codim}(\bar{\mathcal{P}}_d^{\Theta}, \bar{\mathcal{P}}_d) \geq k$. Therefore, by the general position argument, $\pi_i(\mathcal{P}_d^{\mathbf{c}\Theta}) = 0$ for all $i < k-1$. In particular, $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta}) = 0$, provided that $\Theta \subset \mathbf{\Omega}_{\langle d],\,|\sim|'\geq 3}$. □

Further, we observe that, by the Alexander duality and the Hurewicz Theorem, for any closed $\Theta \subset \mathbf{\Omega}_{\langle d]}$, a minimal generating set of $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ contains at least $\mathsf{rank}(\bar{H}_{d-2}(\bar{\mathcal{P}}_d^{\Theta};\mathbb{Z}))$ elements.

Given a closed poset $\Theta \subseteq \mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}$, we consider two disjoint sets:

$$\Theta_{=2} = \mathbf{\Omega}_{\langle d],\,|\sim|'=2} \cap \Theta, \quad \mathbf{c}\Theta_{=2} = \mathbf{\Omega}_{\langle d],\,|\sim|'=2} \setminus \Theta. \tag{2.1}$$

By definition, $\Theta_{=2}$ and $\mathbf{c}\Theta_{=2}$ both consist of ω's which have some parts 1 and either a *single* entry 3, or *two* 2s.

Now consider a loop γ in $\mathcal{P}_d^{\mathbf{c}\Theta}$. It bounds a 2-disk D in $\mathcal{P}_d$. By a general position argument, we may assume that D avoids all strata R_d^{ω} with $|\omega|' \geq 3$ and that if γ intersects a stratum $\mathring{\mathsf{R}}_d^{\omega}$ for some $|\omega|' = 2$, then this intersection is transversal.

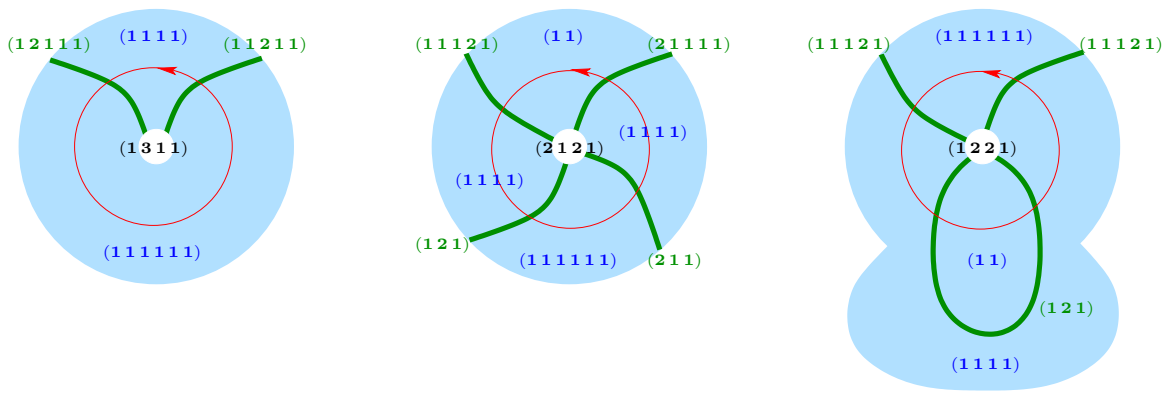

Figure 5. The normal disks to the strata $\mathcal{P}_6^{(1311)}$ (left), $\mathcal{P}_6^{(2121)}$ (middle), and $\mathcal{P}_6^{(1221)}$ (right) and the loops bounding them (in red).

Consider a fixed ω with $|\omega|' = 2$ such that γ intersects $\mathring{\mathsf{R}}_d^\omega$. For $x \in \mathring{\mathsf{R}}_d^\omega$, consider a small 2-disk $x \in D_x \subseteq \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 3}}$ transversal to $\mathring{\mathsf{R}}_d^\omega$ which intersects only cells $\mathring{\mathsf{R}}_d^{\omega'}$ for $\omega' \succeq \omega$. Let $\kappa_x := \partial D_x \subset \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$ be the loop bounding D_x (see Figure 5). Since $\mathring{\mathsf{R}}_d^\omega$ is contractible, for each $x \in D \cap \mathring{\mathsf{R}}_d^\omega$, the homotopy class of κ_x depends only on ω (up to inversion). Hence we can speak of κ_ω.

We choose a path $\beta \subset \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$ that connects our base point in $\mathring{\mathsf{R}}_d^{()}$ or $\mathring{\mathsf{R}}_d^{(1)}$ with κ_ω. By a general position argument we can again assume that β intersects all codimension 1 strata transversally. Let $\kappa_\omega^\bullet := \beta^{-1} \circ \kappa_\omega \circ \beta$ be the loop that starts at the base point, follows a path β, then traverses κ_ω once, and returns to the base point following β^{-1}. By recording the intersections of $\kappa_\omega^\bullet$ with the codimension 1 strata, indexed by the ω_{ij}, together with the direction of the intersections we obtain an admissible word in $\mathsf{A}^\pm$. By Lemma 2.9, the homotopy class of $\kappa_\omega^\bullet$ in $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$ depends only on this word. In particular, we can express this word as a product of words γ_{ij} or their inverses for the ω_{ij} intersected by κ_ω.

Evidently, if $\omega \in \mathbf{c}\Theta_{=2}$, the loop κ_ω and thus the loop $\kappa_\omega^\bullet$ are contractible in $\mathcal{P}_d^{\mathbf{c}\Theta}$. The following lemma gives a precise analysis of the contractible $\kappa_\omega^\bullet$ and shows that the triviality of the corresponding words provides a presentation of $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ generalizing Theorem 2.4.

Proposition 2.10. *For any closed poset $\Theta \subset \mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}$, the fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is a quotient of the free group $\mathcal{G}_d$ by the normal subgroup, generated by the following relations, one relation for every $\omega \in \mathbf{c}\Theta_{=2}$.*

(3) *For each $\omega = (\underbrace{1\cdots1}_{i}3\underbrace{1\cdots1}_{j}) \in \mathbf{c}\Theta_{=2}$ the corresponding relation is*

$$\gamma_{i,j+1}\,\gamma_{i+1,j}^{-1} = 1$$

(22) *For each* $\omega = (\underbrace{1\cdots 1}_{i}2\underbrace{1\cdots 1}_{j}2\underbrace{1\cdots 1}_{\ell}) \in \mathbf{c}\Theta_{=2}$ *the corresponding relation is*

$$\gamma_{i+j,\ell}\,\gamma_{i+j+2,\ell}\,\gamma_{i,j+\ell+2}^{-1}\,\gamma_{i,j+\ell}^{-1} = 1.$$

Recall, that whenever a $\gamma_{0,j}$ appears in one of the relations that it can be omitted by $\gamma_{0j} = 1$. Note also that, in case (22), when $j = 0$, the relation $\gamma_{i+j,\ell}\,\gamma_{i+j+2,\ell}\,\gamma_{i,j+\ell+2}^{-1}\,\gamma_{i,j+\ell}^{-1}$ is conjugate to the relation $\gamma_{i+j+2,\ell}\,\gamma_{i,j+\ell+2}^{-1}$.

Proof. First we show that the relations (3) and (22) lie in the kernel of the homomorphism

$$\pi_1(\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\le 2}}) \to \pi_1(\mathcal{P}_d^{\mathbf{c}\Theta}) \tag{2.2}$$

induced by the inclusion of spaces. By the arguments preceding the lemma, it suffices to show that each relation corresponds to an admissible word representing $\kappa_\omega^\bullet$ for some $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=2}$.

(3) For each $\omega = (\underbrace{1,\cdots,1}_{i},3,\underbrace{1,\cdots,1}_{j}) \in \mathbf{c}\Theta_{=2}$, we notice that $\mathring{\mathsf{R}}_d^\omega$ lies only in the boundaries of the codimension one cells $\mathring{\mathsf{R}}_d^{\omega_{i+1,j}}$ and $\mathring{\mathsf{R}}_d^{\omega_{i,j+1}}$, and of the codimension 0 cells corresponding to $(\underbrace{1,\ldots,1}_{i+j+3})$ and to $(\underbrace{1,\ldots,1}_{i+j+1})$.

Let us traverse the loop κ_ω, see the left diagram in Figure 5. When κ_ω enters the cell, labelled by $(\underbrace{1,\ldots,1}_{i+j+3})$, from a point on $\mathring{\mathsf{R}}_d^{\omega_{i,j+1}}$ the $(i+1)^{\text{st}}$ largest real root splits into 2 distinct real roots. On its way towards $\mathring{\mathsf{R}}_d^{\omega_{i+1,j}}$, the smaller of these two roots approaches the $(i+1)^{\text{st}}$ largest real root and at the end merges with it. Then, when entering the cell corresponding to $(\underbrace{1,\ldots,1}_{i+j+1})$, the real double root splits into two conjugate complex roots with small imaginary part and with real part between the $(i+1)^{\text{st}}$ and $(i+2)^{\text{nd}}$ largest real roots. While traversing κ_ω, the value of the real part passes the $(i+1)^{\text{st}}$ real root and then, when the imaginary parts vanish, κ_ω is back in $\mathring{\mathsf{R}}_{i,j+1}^\omega$. Thus $\kappa_\omega^\bullet$ corresponds to the relation $\gamma_{i,j+1}\,\gamma_{i+1,j}^{-1}$.

(22) For $\omega = (\underbrace{1,\cdots,1}_{i},2,\underbrace{1,\cdots,1}_{j},2,\underbrace{1\cdots 1}_{\ell}) \in \mathbf{c}\Theta_{=2}$, the codimension 2 cell $\mathring{\mathsf{R}}_d^\omega$ lies only in the boundary of the codimension 1 cells $\mathring{\mathsf{R}}_d^{\omega_{i+2+j,\ell}}$, $\mathring{\mathsf{R}}_d^{\omega_{i,j+\ell+2}}$, $\mathring{\mathsf{R}}_d^{\omega_{i+j,\ell}}$ and $\mathring{\mathsf{R}}_d^{\omega_{i,j+\ell}}$. (Note that the last two coincide when $j = 0$). It also lies in the boundary of the codimension 0-cells $(\underbrace{1,\ldots,1}_{k})$ for $k = i+j+\ell$, $i+j+\ell+2$, $i+j+\ell+4$.

Again, let us traverse the loop κ_ω, see the middle and the right diagrams in Figure 5. We start in $\mathring{\mathsf{R}}_d^{\omega_{i+j+2,\ell}}$ and, when entering the cell labelled by $(\underbrace{1,\ldots,1}_{i+j+\ell+4})$, we split the $(i+j+2)^{\text{nd}}$ root into two distinct real roots. Then, on the way through this cell, the $(i+1)^{\text{st}}$ and $(i+2)^{\text{nd}}$ largest real roots approach each other and finally

merge at a point belonging to the wall $\mathring{\mathsf{R}}_d^{\omega_{i,j+\ell+2}}$. After passing $\mathring{\mathsf{R}}_d^{\omega_{i,j+\ell+2}}$, the $(i+1)^{\text{st}}$ largest real root splits into two complex conjugate roots. Next, we move towards $\mathring{\mathsf{R}}_d^{\omega_{i,j+\ell}}$ by bringing the ith and $(i+1)^{\text{st}}$ largest real roots together. Then we split this double root into two complex conjugate roots inside the d-cell labelled by $(\underbrace{1,\dots,1}_{i+j+\ell})$; further we move the real part of these roots to a value between the $(i+j)^{\text{th}}$ and the $(i+j+1)^{\text{st}}$ real roots, while letting the imaginary part to approach 0. This leads us to $\mathring{\mathsf{R}}^{\omega_{i+j,\ell}}$, from where we follow the analogous route back to $\mathring{\mathsf{R}}_d^{\omega_{i+j+2,\ell}}$. Note that along the way, we made certain choices that determine which codimension 1 cell we want to approach next. The arguments, preceding the lemma, show that all such choices lead to homotopic paths/loops.

As a consequence, the loop $\kappa_\omega^\bullet$ translates into the relation

$$\gamma_{i+j,\ell}\,\gamma_{i+j+2,\ell}\,\gamma_{i,j+\ell+2}^{-1}\,\gamma_{i,j+\ell+2}^{-1}=1.$$

Next we show that the relations (3) and (22) generate the kernel of the homomorphism (2.2). For each ω with $|\omega|'=2$, consider a closed regular neighborhood U_ω of the cell $\mathring{\mathsf{R}}_d^\omega$ in the space $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}\cup\mathring{\mathsf{R}}_d^\omega$. We may assume that, for distinct ω's, these neighborhoods are disjoint. The space U_ω fibers over $\mathring{\mathsf{R}}_d^\omega$ with fibers being 2-disks. Since the base $\mathring{\mathsf{R}}_d^\omega$ is contractible, the fibration $p_\omega: U_\omega\to\mathring{\mathsf{R}}_d^\omega$ is trivial.

Set $X:=\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}}$. Adding to X a cell $\mathring{\mathsf{R}}_d^\omega$, where $\omega\in\mathbf{c}\Theta_{=2}$ produces a new space Y. Its homotopy type is the result of attaching a 2-handle to X along its boundary. The spaces X, U_ω, and $X\cap U_\omega$ are path-connected. Thus, by the Seifert–van Kampen Theorem,

$$\pi_1(Y)\approx\pi_1(X)*_{\pi_1(X\cap U_\omega)}\pi_1(U_\omega),$$

which is the free product $*$ of the groups $\pi_1(X)$ and $\pi_1(U_\omega)$, amalgamated over $\pi_1(X\cap U_\omega)$. Since U_ω is homotopy equivalent to the cell $\mathring{\mathsf{R}}_d^\omega$, we get $\pi_1(U_\omega)=0$. At the same time, using the triviality of p_ω, $X\cap U_\omega=U_\omega\setminus\mathring{\mathsf{R}}_d^\omega$ is homotopy equivalent to a loop κ_ω, which bounds a small 2-disk normal to the stratum $\mathring{\mathsf{R}}_d^\omega$. Therefore, $\pi_1(Y)\approx\pi_1(X)/[\kappa_\omega^\bullet]$. Thus $\pi_1(Y)$ arises from $\pi_1(X)$ by factoring modulo one of the relation of type (3) or (22).

Since the U_ω's were chosen mutually disjoint, it follows that we can repeat this argument for all $\omega\in\mathbf{c}\Theta_{=2}$ and conclude that $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ arises as a quotient of $\pi_1(\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}})$ by the relations (3) and (22) for $\omega\in\mathbf{c}\Theta_{=2}$. □

The next theorem shows that in many cases the relations from Proposition 2.10 yield a presentation of a *free* group.

Theorem 2.11. *Let $\Theta\subset\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}$ be a closed poset such that either*

(i) *Θ contains all $\omega\in\mathbf{\Omega}_{\langle d]}$ such that $\omega=(\underbrace{1\cdots1}_{i}2\underbrace{1\cdots1}_{j}2\underbrace{1\cdots1}_{\ell})$ for some $j>0$;*

or

(ii) Θ *is such that for* $\omega \in \mathbf{\Omega}_{\langle d]}$ *with* $|\omega|' = 2$ *we have*

$$\omega \in \Theta \;\Leftrightarrow\; \omega = (\underbrace{1\cdots 1}_{i}\,3\,\underbrace{1\cdots 1}_{j})$$

for some $i, j \geq 0$.

Then the fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ *is a free group in* $\mathsf{rank}(\bar{H}_{d-2}(\bar{\mathcal{P}}_d^{\Theta};\mathbb{Z}))$[4] *generators. In particular, in case (ii), we have* $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta}) = \mathbb{Z}$ *for* $d \geq 4$.

Proof.

(i) By Proposition 2.10 and the assumptions on Θ, the fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is a quotient of the free group by arbitrary relations of type (3) and some specific of type (22).

Any relation of type (3) identifies some $\gamma_{i,j+1}$ with $\gamma_{i+1,j}$. Any relation of type (22) that can appear identifies some γ_{i+j+2} with $\gamma_{i,j+\ell+2}$.

In either case the relation and one generator can be dropped without changing the isomorphism type of the group.

Note that the number of generators of the free group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is the rank of $H_1(\mathcal{P}_d^{\mathbf{c}\Theta};\mathbb{Z})$. By the Alexander duality, it is equal the rank of $H^{d-2}(\bar{\mathcal{P}}_d^{\Theta};\mathbb{Z})$.

The assertion follows.

(ii) By Proposition 2.10 and the assumptions on Θ, the fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is a quotient of the free group by *all* possible relations of type (22).

First consider compositions of type $(\underbrace{1,\ldots,1}_{i},2,2,\underbrace{1,\ldots,1}_{\ell})$ of $d' = i+4+\ell \leq d$ where d and d' are of the same parity. The corresponding relation $\gamma_{i,\ell}\,\gamma_{i+2,\ell}\,\gamma_{i,\ell+2}^{-1}\,\gamma_{i,\ell}^{-1} = 1$ identifies $\gamma_{i+2,\ell}$ and $\gamma_{i,\ell+2}$. As a consequence this relation, one generator can be dropped. By iterating this procedure, we identify all $\gamma_{i+2,\ell}$ and $\gamma_{i,\ell+2}$ for $i = 0,\ldots,d'-4$ and even $d-d' \geq 0$. As a consequence, for a fixed d', we have identified $\gamma_{i,\ell}$ and $\gamma_{i',\ell'}$, where $i+\ell = d'-2 = i'+\ell'$ if both i and i' are even or both i and i' are odd. After this identification, the group is generated by $\gamma_{1,d'-3}$ for $3 \leq d' \leq d$.

Let $d' \geq 5$ and consider $\omega = (2,1,2,\underbrace{1\cdots 1}_{d'-5})$. Then the relation

$$\gamma_{1,d'-5}\,\gamma_{3,d'-5}\,\gamma_{0,d'-2}^{-1}\,\gamma_{0,d'-4}^{-1} = 1$$

implies $\gamma_{1,d'-5}\,\gamma_{0,d'-4}^{-1} = \gamma_{3,d'-5}\,\gamma_{0,d'-2}^{-1}$, which identifies $\gamma_{1,d'-5}$ and $\gamma_{3,d'-5}$ and, by transitivity, $\gamma_{1,d'-5}$ and $\gamma_{1,d'-3}$. Making this identification and removing the corresponding relations leaves a set of trivial relations on the unique remaining generator $\gamma_{1,1}$ for d even or $\gamma_{1,2}$ for d odd. Thus we have arrived at a group with one generator and no relation. The assertion now follows.

□

As an immediate consequence, we obtain the following result about certain natural spaces of real polynomials.

[4] In [KSW], we will compute $\mathsf{rank}(\bar{H}_{d-2}(\bar{\mathcal{P}}_d^{\Theta};\mathbb{Z}))$ in pure combinatorial terms.

Corollary 2.12. *The fundamental group of the following spaces is a free group.*

(i) *The space of all degree d polynomials with no two distinct roots of multiplicity ≥ 2.*
(ii) *The space of all degree d polynomials with no root of multiplicity ≥ 3. In this case, the fundamental group is $\mathbb{Z}$.*

Proof. The assertions follow immediately from Theorem 2.11. □

The following illuminating example shows however that, in general, $\mathcal{P}_d^{\mathbf{c}\Theta}$ is not free.

Example 2.13. Let $d = 6$ and consider the smallest closed $\Theta \subseteq \mathbf{\Omega}_{\langle 6]}$ containing

$$\{(3,1),(1,3),(1,3,1,1),(1,1,3,1),(2,2,1,1),(1,2,2,1),(1,1,2,2),(2,1,1,2)\}. \tag{2.3}$$

By Proposition 2.10, we need to consider the group freely generated by

$$\gamma_{1,1}, \gamma_{2,0}, \gamma_{1,3}, \gamma_{2,2}, \gamma_{3,1}, \gamma_{4,0}$$

as well as by the dummy generators $\gamma_{0,2} = \gamma_{0,4} = 1$. We then have to impose the relations of type (3) for the elements

$$(3,1,1,1),\ (1,1,1,3)$$

and relations of type (22) for

$$(2,2),\ (2,1,2,1),\ (1,2,1,2).$$

Figure 6 sketches the 2-complex representing these relations. The type (3) relations identify $\gamma_{0,4}$ and $\gamma_{1,3}$ as well as $\gamma_{3,1}$ and $\gamma_{4,0}$. The type (22) relation, corresponding to $(2,2)$, then identifies $\gamma_{0,2}$ and $\gamma_{2,0}$. After these identifications, the type (22) relations that correspond to $(2,1,2,1)$ and $(1,2,1,2)$ can be transformed as follows:

$$\gamma_{1,1}\,\gamma_{3,1}\,\gamma_{0,4}^{-1}\,\gamma_{0,2}^{-1} \longrightarrow \gamma_{1,1}\,\gamma_{3,1},$$

$$\gamma_{2,0}\,\gamma_{4,0}\,\gamma_{1,3}^{-1}\,\gamma_{1,1}^{-1} \longrightarrow \gamma_{3,1}\,\gamma_{1,1}^{-1}.$$

Thus we are left with the generators $\gamma_{1,1}$ and $\gamma_{3,1}$, subject to $\gamma_{1,1}\,\gamma_{3,1} = 1 = \gamma_{3,1}\,\gamma_{1,1}^{-1}$. After eliminating $\gamma_{3,1}$, we are left with the generator $\gamma_{1,1}$, subject to $\gamma_{1,1}^2 = 1$ which is a presentation of $\mathbb{Z}/2\mathbb{Z}$. Hence, we have shown that $\pi_1(\mathcal{P}_6^{\mathbf{c}\Theta}) \approx \mathbb{Z}/2\mathbb{Z}$ for Θ introduced in (2.3).

Using the same Θ as in (2.3) (Example 2.13), but for $d = 8$, one can check that $\pi_1(\mathcal{P}_8^{\mathbf{c}\Theta})$ is a free group. Guided by this example, one is tempted to speculate that, for any closed Θ and d large enough, $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is free.

We will see that this is, in general, not the case, but first we will show how $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ stabilizes for d large. This stabilization of the fundamental group follows almost immediately from the next lemma, a consequence of Proposition 2.10.

Lemma 2.14. *Let $\Theta \subset \mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}$ be a closed poset and $\Theta' = \Theta \cap \mathbf{\Omega}_{\langle d-2]}$. Assume that no ω with $|\omega| \in \{d, d-2\}$ resides in $\Theta_{=2}$. Then $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta}) \cong \pi_1(\mathcal{P}_{d-2}^{\mathbf{c}\Theta'})$.*

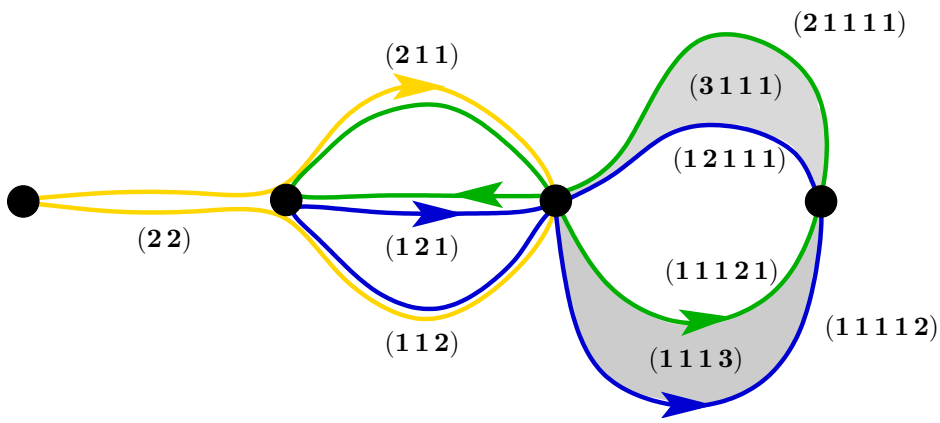

Figure 6. The assembly instructions for the 2-dimensional CW-complex that realizes the fundamental group $\pi_1(\mathcal{P}_6^{\mathbf{c}\Theta})$ for Θ in (2.3). The 2-cells that realize the relations of type (3) are shaded. The three loops, to which the 2-disks that realize relations of type (22) are attached, are shown in three colors (gold for $(2,2)$, green for $(2,1,2,1)$, blue for $(2,1,1,2)$).

Proof. By Proposition 2.10 and the structure of Θ, all the relations of type (3) and (22), that correspond to $\omega \in \mathbf{\Omega}_{\langle d],|\sim|'=2}$ with $|\omega| \in \{d, d-2\}$ and define $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$, are part of a presentation of $\pi_1(\mathcal{P}_{d-2}^{\mathbf{c}\Theta'})$. Indeed, the generators γ_{ij} for $i+j+2=d$ and the relations of type (3) and (22), corresponding to $\omega \in \mathbf{c}\Theta_{=2}$ and $|\omega| = d$, are the only difference to the presentations of $\pi_1(\mathcal{P}_{d-2}^{\mathbf{c}\Theta'})$ and $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$.

The relations of type (3) identify all γ_{ij} for $i+j+2 \in \{d, d-2\}$. Since $\gamma_{0,d-2} = \gamma_{0,d-4} = 1$, this implies that $\gamma_{ij} = 1$ for $i+j+2 \in \{d, d-2\}$. The relations of type (22), corresponding to $\omega \in \mathbf{c}\Theta_{=2}$ with $|\omega| = 2$, then turn into identifications of some γ_{ij} for $i+j+2 = d-2$. But since these elements are already identified with 1, the relations of type (22) imply no independent identities, and hence can be removed.

As a result, we have obtained the generators and relations for $\pi_1(\mathcal{P}_{d-2}^{\mathbf{c}\Theta'})$, and the assertion follows. □

Now we can deduce a stabilization result for the fundamental group.

Theorem 2.15. *Let $\Theta \subseteq \mathbf{\Omega}_{\langle d],|\sim|'\geq 2}$ be a closed poset. For $d' \geq d+2$ such that $d' = d \equiv 2$, let $\Theta_{d'}$ be the smallest closed poset in $\mathbf{\Omega}_{\langle d']}$ containing Θ. Then for $d' \geq d+2$, we have an isomorphism $\pi_1(\mathcal{P}_{d'}^{\mathbf{c}\Theta_{d'}}) \cong \pi_1(\mathcal{P}_{d+2}^{\mathbf{c}\Theta_{d+2}})$.*

Proof. We prove the assertion by induction on d'. For $d' = d+2$, the assertion is trivial, since the relevant polynomial spaces are identical. For $d' \geq d+4$, we have that $\Theta_{=2}$ contains no ω such that $|\omega| \in \{d', d'-2\}$. This implies that $(\Theta_{d'})_{=2}$ contains no ω such that $|\omega| \in \{d', d'-2\}$, and $\Theta_{d'-2} = \Theta_{d'} \cap \mathbf{\Omega}_{\langle d'-2]}$. Hence by Lemma 2.14, $\pi_1(\mathcal{P}_{d'}^{\mathbf{c}\Theta_{d'}}) \cong \pi_1(\mathcal{P}_{d'-2}^{\mathbf{c}\Theta_{d'-2}})$. The assertion now follows from the induction hypothesis. □

Before we can provide a negative answer to the question whether $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is free for large d, we need the following simple lemma, which is an immediate consequence of Proposition 2.10 and the definition of the free product of finitely presented groups.

Lemma 2.16. *Let $\Theta \subset \mathbf{\Omega}_{\langle d], |\sim|' \geq 2}$ be closed. Assume that for some $d' < d$ such that $d' \equiv d \mod 2$, the poset Θ contains all $\omega \in \mathbf{\Omega}_{\langle d], |\sim|' = 2}$ with $|\omega| = d'$.*

Then $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is the free product of $\pi_1(\mathcal{P}_{d'-2}^{\mathbf{c}\Theta})$ and a group presented by relations of type (3) and (22) for $\omega \notin \Theta$, such that $|\omega| > d'$ and $|\omega|' = 2$.

Let $\Theta \subset \mathbf{\Omega}_{\langle 6], |\sim|' = 2}$ be the poset from Example 2.13 (see (2.3)). For $d \geq 10$, let Θ_d be the smallest closed poset in $\mathbf{\Omega}_{\langle d]}$ containing Θ and all $\omega \in \mathbf{\Omega}_{\langle d], |\sim|' = 2}$ with $|\omega| = 8$. For $d \geq 10$, we have that Θ_d satisfies the conditions of Lemma 2.16 with $d' = 8$. Then, for $d \geq 10$, $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta_d})$ is the free product of $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta_6}) \cong \mathbb{Z}/2\mathbb{Z}$ and some group. In particular, $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta_d})$ is not free *stably*, as $d \to \infty$.

Next we formulate some consequences of our results about spaces of polynomials with restrictions on the multiplicities of their *critical points*. For any closed poset $\Theta \subset \mathbf{\Omega}$ and any $d > 0$, we denote by $\mathcal{P}_{d+1}^{\mathsf{crit}\,\mathbf{c}\Theta}$ the space of polynomials of degree $d+1$, whose derivatives belong to $\mathcal{P}_d^{\mathbf{c}\Theta}$. The homotopy statement of the following corollary to Theorem 2.4 and Theorem 2.11 is well known.

Corollary 2.17. *For any closed poset $\Theta \subset \mathbf{\Omega}$ and any $d > 0$, the space $\mathcal{P}_{d+1}^{\mathsf{crit}\,\mathbf{c}\Theta}$ is homotopy equivalent to the space $\mathcal{P}_d^{\mathbf{c}\Theta}$. In particular, $\pi_1(\mathcal{P}_{d+1}^{\mathsf{crit}\,\mathbf{c}\Theta})$ is free for*

(i) $\Theta = \mathbf{\Omega}_{\langle d], |\sim|' \geq 2}$,
(ii) Θ *contains all* $\omega \in \mathbf{\Omega}_{\langle d]}$ *such that* $\omega = (\underbrace{1\cdots1}_{i}2\underbrace{1\cdots1}_{j}2\underbrace{1\cdots1}_{\ell})$ *for some* $j > 0$.
(iii) Θ *is such that for* $\omega \in \mathbf{\Omega}_{\langle d]}$ *with* $|\omega|' = 2$ *we have*

$$\omega \in \Theta \Leftrightarrow \omega = (\underbrace{1\cdots1}_{i}3\underbrace{1\cdots1}_{j})$$

for some $i, j \geq 0$.

Moreover, in case (iii), *we have* $\pi_1(\mathcal{P}_{d+1}^{\mathsf{crit}\,\mathbf{c}\Theta}) = \mathbb{Z}$.

Proof. The homotopy equivalence follows from the simple fact that $\frac{d}{dx} : \mathcal{P}_{d+1}^{\mathsf{crit}\,\mathbf{c}\Theta} \to \mathcal{P}_d^{\mathbf{c}\Theta}$ is a trivial fibration with fiber $\mathbb{R}$. Then (i) follows from Theorem 2.4 and (ii)-(iii) from Theorem 2.11. □

Corollary 2.17 (iii) can be reformulated so that it can be seen as an analogue of Arnold's Theorem A in finite degrees.

Corollary 2.18. *The fundamental group of the space of real monic polynomials of a fixed odd degree $d > 1$ with no real critical points of multiplicity higher than 2 is isomorphic to the group $\mathbb{Z}$.*

3. $\pi_1(\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 2}})$ AND COBORDISMS OF PLANE CURVES WITH RESTRICTED VERTICAL TANGENCIES

Results and constructions in this section are similar to the ones from Arnold's Theorem B. They also provide a glimpse into a new area (that we hope to discover) which will describe new "bordism" and "quasi-isotopy" theories of traversing flows on manifolds with boundary. Crudely, one might think of such theories as classical smooth bordisms $\mathcal{B}_*(\mathcal{P}_d^{\mathbf{c}\Theta})$ or homotopy groups $\pi_*(\mathcal{P}_d^{\mathbf{c}\Theta})$ of spaces like $\mathcal{P}_d^{\mathbf{c}\Theta}$. These future area which we baptized as "Bordisms of traversing flows with constrained tangency to the boundary patterns", will depend crucially on this paper and its sequel [KSW].

Consider a smooth n-manifold Y and an *immersion* $\beta : X \to \mathbb{R} \times Y$ of a smooth closed n-manifold X into the interior of $\mathbb{R} \times Y$. We denote by $\mathcal{L}$ the one-dimensional foliation, defined by the fibers of the projection map $\pi : \mathbb{R} \times Y \to Y$. For each point $\mathsf{x} \in X$, we define the natural number $\mu_\beta(\mathsf{x})$ as the multiplicity of tangency between the x-labeled branch of $\beta(X)$ — the β-image of the vicinity of $\mathsf{x} \in X$— and the leaf of $\mathcal{L}$ through $\beta(\mathsf{x})$. In particular, if the branch is transversal to the leaf, then $\mu_\beta(\mathsf{x}) = 1$.

We fix a natural number d and assume that β is such that each leaf $\mathcal{L}_\mathsf{y}$, $\mathsf{y} \in Y$, hits $\beta(X)$ so that the following inequality holds:

$$m_\beta(\mathsf{y}) := \sum_{\{a \in \mathcal{L}_\mathsf{y} \cap \beta(X)\}} \Big(\sum_{\{\mathsf{x} \in \beta^{-1}(a)\}} \mu_\beta(\mathsf{x}) \Big) \leq d. \tag{3.1}$$

We order the points $\{a_i\}$ of $\mathcal{L}_\mathsf{y} \cap \beta(X)$ by the values of their projections on $\mathbb{R}$ and introduce the combinatorial pattern $\omega^\beta(\mathsf{y})$ of $\mathsf{y} \in Y$ as the sequence of multiplicities $\big\{\omega_i(\mathsf{y}) := \sum_{\{\mathsf{x} \in \beta^{-1}(a_i)\}} \mu_\beta(\mathsf{x})\big\}_i$. We denote by $D^\beta(\mathsf{y})$ the real divisor of the intersection $\mathcal{L}_\mathsf{y} \cap \beta(X)$ with multiplicities $\{\omega_i(\mathsf{y})\}_i$.

Proposition 3.1. *Given manifolds X and Y as above, for any immersion $\beta : X \to \mathbb{R} \times Y$ that satisfies ((3.1)), together with the parity condition $m_\beta(\mathsf{y}) \equiv d \mod 2$, there exists a continuous map $\Phi : Y \to \mathcal{P}_d$ such that*

$$\big\{(x, \mathsf{y}) \in \mathbb{R} \times Y |\ \Phi(\mathsf{y})(x) = 0\big\} = \beta(X).$$

If, for a given closed poset $\Theta \subset \mathbf{\Omega}_{\langle d]}$, the immersion β is such that no $\omega^\beta(\mathsf{y})$ belongs to Θ, then Φ maps Y to $\mathcal{P}_d^{\mathbf{c}\Theta}$.

Proof. We claim that the loci $\beta(X)$ may be viewed as the solution set of the equations $\big\{x^d + \sum_{j=0}^{d-1} a_j(\mathsf{y})\, x^j = 0\big\}_{\mathsf{y} \in Y}$, where $\{a_j : Y \to \mathbb{R}\}_j$ are some smooth functions.

Let us justify this claim. By Lemma 4.1 from [Ka2] and Morin's Theorem [Mor1, Mor2], if a particular branch $\beta(X)_\kappa$ of $\beta(X)$ is tangent to the leaf $\mathcal{L}_{\mathsf{y}_0}$ at a point $b = (\alpha, \mathsf{y}_0)$ with the order of tangency j, then there is a system of local coordinates $(u, \tilde{y}, \tilde{z})$ in the vicinity of b in $\mathbb{R} \times Y$ such that:

(1) $\beta(X)_\kappa$ is given by the equation $\big\{u^j + \sum_{k=0}^{j-2} \tilde{y}_k\, u^k = 0\big\}$;
(2) each nearby leaf $\mathcal{L}_\mathsf{y}$ is given by the equations $\{\tilde{y} = \overrightarrow{const}, \tilde{z} = \overrightarrow{const'}\}$.

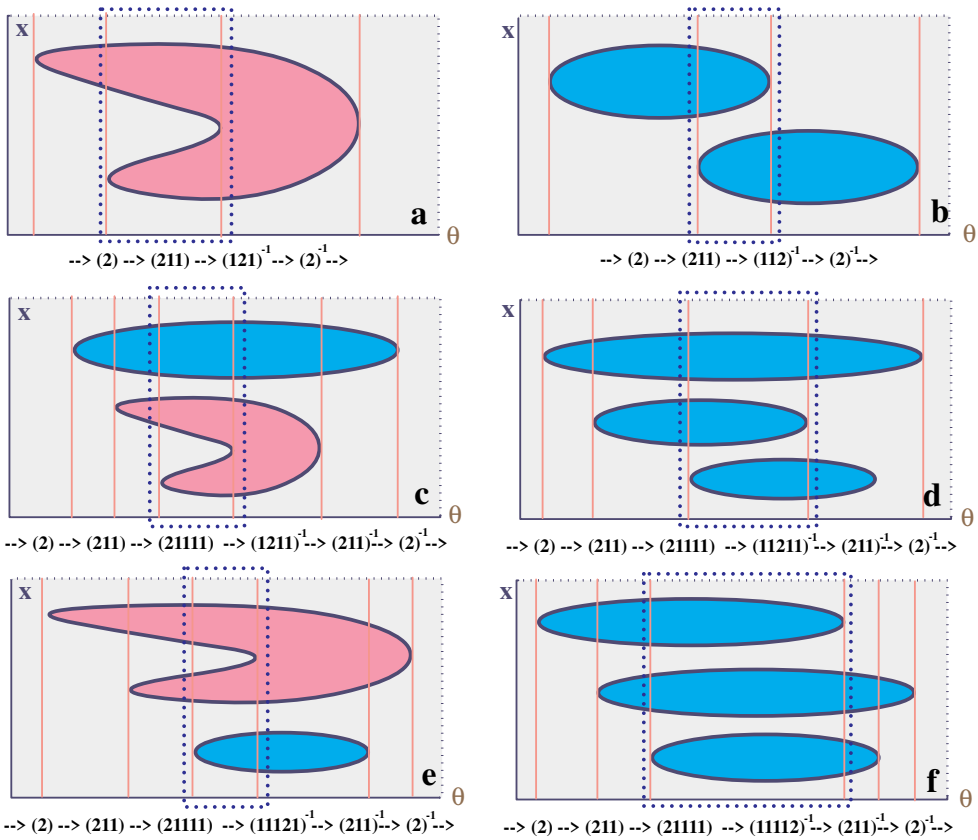

Figure 7. A set $\{a, b, c, d, e, f\}$ of six generators, freely generating the bordism group $\mathcal{B}(S^1 \times \mathbb{R};\ \mathbf{c\Omega}_{\langle 6], |\sim|' \geq 2}) \approx \pi_1(\mathcal{P}_6^{\mathbf{c\Omega}_{\langle 6], |\sim|' \geq 2}})$, is shown as collections of curves in the cylinder with the coordinates $(\psi, x) \in S^1 \times \mathbb{R}$. Each collection of curves is generated by a specific map $\gamma : S^1 \to \mathcal{P}_6^{\mathbf{c\Omega}_{\langle 6], |\sim|' \geq 2}}$ as the set of pairs (ψ, x) with the property $\gamma(\psi)(x) = 0$. Each line $\{\psi = const\}$ is either transversal to the collection, or is quadratically tangent to it. No double tangent lines are permitted. Each collection of curves is equipped with the circular word (written under each of the six diagrams) that reflects the transversal intersections of the loop $\gamma(S^1)$ with the discriminant variety $\mathcal{D}_6 \subset \mathcal{P}_6$.

Setting $u = x - \alpha$ and writing $\tilde{y}_k$'s as smooth functions of y, the same $\beta(X)_\kappa$ can be given by the equation

$$\big\{P_{\alpha,\kappa}(x, \mathsf{y}) := (x - \alpha)^j + \sum_{j=0}^{j-2} a_{\kappa,k}(\mathsf{y})\,(x - \alpha)^k = 0\big\},$$

where $a_{\kappa,k} : Y \to \mathbb{R}$ are smooth functions, vanishing at y_0. Therefore, there exists an open neighborhood U_{y_0} of y_0 in Y such that, in $\mathbb{R} \times U_{\mathsf{y}_0}$, the locus $\beta(X)$ is given by the monic

polynomial equation

$$\Big\{P_{\mathsf{y}_0}(x,\mathsf{y}) := \prod_{(\alpha,\,\mathsf{y}_0)\,\in\,\mathcal{L}_{\mathsf{y}_0}\cap\,\beta(X)} \Big(\prod_{\kappa\in A_\alpha} P_{\alpha,\kappa}(x,\mathsf{y})\Big) = 0\Big\},$$

of degree $m_\beta(\mathsf{y}_0) \le d$ in x. Here the finite set A_α labels the local branches of $\beta(X)$ that contain the point $(\alpha,\mathsf{y}_0) \in \mathcal{L}_{\mathsf{y}_0} \cap \beta(X)$.

By multiplying $P_{\mathsf{y}_0}(x,\mathsf{y})$ with $(x^2+1)^{\frac{d-m_\beta(\mathsf{y}_0)}{2}}$, we get a polynomial $\tilde{P}_{\mathsf{y}_0}(x,\mathsf{y})$ of degree d which, for each $\mathsf{y} \in U_{\mathsf{y}_0}$, shares with $P_{\mathsf{y}_0}(x,\mathsf{y})$ the zero set $\beta(X) \cap (\mathbb{R} \times U_{\mathsf{y}_0})$, as well as the divisors $D^\beta(\mathsf{y})$.

For each $\mathsf{y} \in Y$, consider the space $\mathcal{X}_\beta(\mathsf{y})$ of monic polynomials $\tilde{P}(x)$ of degree d such that their real divisors coincide with the β-induced divisor $D^\beta(\mathsf{y})$. We view $\mathcal{X}_\beta := \coprod_{\mathsf{y}\in Y} \mathcal{X}_\beta(\mathsf{y})$ as a subspace of $Y \times \mathcal{P}_d$. It is equipped with the obvious projection $p : \mathcal{X}_\beta \to Y$. The smooth sections of p are exactly the smooth functions $\tilde{P}(x,\mathsf{y})$ that interest us. Each p-fiber $\mathcal{X}_\beta(\mathsf{y})$ is a convex set. Now, given several smooth sections $\{\sigma_i\}_i$ of p, we conclude that $\sum_i \phi_i \cdot \sigma_i$ is again a section of p, provided that the smooth functions $\phi_i : Y \to [0,1]$ have the property $\sum_i \phi_i \equiv \mathbf{1}$. Note that $\phi_i \cdot \sigma_i \notin \mathcal{P}_d$.

Since X is compact, $\pi(\beta(X)) \subset Y$ is compact as well. So it admits a finite cover by the open sets $\{U_{\mathsf{y}_i}\}_i$ as above. Let $\{\phi_i : Y \to [0,1]\}_i$ be a smooth partition of unity, subordinated to this finite cover. Then the monic x-polynomial

$$\tilde{P}(x,\mathsf{y}) := \sum_i \phi_i(\mathsf{y}) \cdot \tilde{P}_{\mathsf{y}_i}(x,\mathsf{y})$$

of degree d has the desired properties. In particular, its divisor is $D^\beta(\mathsf{y})$ for each $\mathsf{y} \in Y$. Thus, using $\tilde{P}(x,\mathsf{y})$, any immersion $\beta : X \to \mathbb{R} \times Y$, such that no $\omega^\beta(\mathsf{y})$ belongs to Θ, is realized by a smooth map $\Phi : Y \to \mathcal{P}_d^{\mathbf{c}\Theta}$ for which $\beta(X) = \{\Phi(\mathsf{y})(x) = 0\}$. □

We denote by $\mathcal{L}$ the foliation of the cylinder $A = S^1 \times \mathbb{R}$, formed by the fibers $\{\ell_\psi\}_{\psi\in S^1}$ of the obvious projection $q : S^1 \times \mathbb{R} \to S^1$, and by $\mathcal{L}^\bullet$ the 1-dimensional foliation of $A \times [0,1]$, formed by the fibers of the obvious projection $Q : S^1 \times \mathbb{R} \times [0,1] \to S^1 \times [0,1]$. We pick a base point $\psi_\star \in S^1$ and the leaf $\ell_\star$ of $\mathcal{L}$ that corresponds to $\psi_\star$. Similarly, for each $t \in [0,1]$, we fix the base leaf $\ell_\star(t)$ of $\mathcal{L}^\bullet$.

We consider immersions $\beta : M \to A$ of closed smooth 1-dimensional manifolds M such that:

(P1) for each $\psi \in S^1$, the multiplicity from (3.1) satisfies the inequality $m_\beta(\psi) \le d$;
(P2) no leaf ℓ_ψ of $\mathcal{L}$ has the combinatorial tangency pattern $\omega^\beta(\psi) \in \mathbf{\Omega}_{\langle d],\,|\sim|'\ge 2}$;
(P3) the map $q \circ \beta : M \to S^1$ has only Morse type singularities;
(P4) $\beta(M) \cap \ell_\star = \emptyset$.

The next definition lays the foundation for our notion of cobordism, which deviates from the usual cobordism theory.

Definition 3.2. We say that a pair of immersions $\beta_0 : M_0 \to A$, $\beta_1 : M_1 \to A$ is *cobordant*, if there exists a compact smooth orientable surface W with boundary $\partial W = M_1 \coprod (-M_0)$ and an immersion $B : W \to A \times \mathbb{R}$ such that:

- $B|_{M_0} = \beta_0$ and $B|_{M_1} = \beta_1$;
- for each $(\psi, t) \in S^1 \times [0,1]$, the multiplicity $m_B((\psi,t)) \le d$ (see (3.1));
- for each $(\psi, t) \in S^1 \times [0,1]$, the tangency pattern $\omega^B((\psi,t))$ does not belong to $\mathbf{\Omega}_{\langle d], |\sim|' \ge 2}$;
- the composition of $B : W \to S^1 \times [0,1]$ with the obvious map $\Pi : A \times [0,1] \to [0,1]$, is a Morse function with the regular values 0 and 1.

We denote by $\mathcal{B}(A; \mathbf{c\Omega}_{\langle d], |\sim|' \ge 2})$ the set of cobordism classes of such immersions $\beta : M \to A$.

Note that $\mathcal{B}(A; \mathbf{c\Omega}_{\langle d], |\sim|' \ge 2})$ is set of cobordism classes of immersed curves (and not the usual set/group of cobordisms of 1-manifolds with the target A).

In fact, the set $\mathcal{B}(A; \mathbf{c\Omega}_{\langle d], |\sim|' \ge 2})$ also carries a *group* structure, where the group operation $\beta \odot \beta'$ is defined as follows. Since $\beta(M) \cap \ell_\star = \emptyset$ and $\beta'(M') \cap \ell_\star = \emptyset$, we may view $\beta(M)$ as subset of the strip $[0, 2\pi] \times \mathbb{R}$, and $\beta'(M')$ as subset of the strip $[2\pi, 4\pi] \times \mathbb{R}$. Then $\beta(M) \coprod \beta'(M') \subset [0, 4\pi] \times \mathbb{R}$. We use the linear map $\lambda : [0, 4\pi] \to [0, 2\pi]$ to place the locus $\beta(M) \coprod \beta'(M')$ in $[0, 2\pi] \times \mathbb{R}$, and thus in A. Evidently, this operation produces a pattern $\beta(M) \odot \beta'(M')$ satisfying (P1)-(P4).

The next result is similar to Arnold's Theorem B.

Theorem 3.3. *The cobordism group $\mathcal{B}(A; \mathbf{c\Omega}_{\langle d], |\sim|' \ge 2}))$, where $d \equiv 0 \mod 2$, is isomorphic to the fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \ge 2}})$, and thus is a free group in $\frac{d(d-2)}{4}$ generators.*

See Figure 7 for the case $d = 6$. In this case, $\mathcal{B}(A; \mathbf{c\Omega}_{\langle 6], |\sim|' \ge 2})$ is the free group on 6 generators.

Proof. Each continuous loop $\gamma : S^1 \to \mathcal{P}_d$ produces a 1-dimensional locus Ξ_γ (a collection of curves) in the cylinder $A = S^1 \times \mathbb{R}$ by the formula

$$\Xi_\gamma := \{(\psi, x) \in S^1 \times \mathbb{R} \mid \gamma(\psi)(x) = 0\}.$$

If γ is a smooth loop in $\mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \ge 2}}$ that is either transversal to the non-singular part $\mathcal{D}_d^\circ \subset \mathcal{P}_d^{\mathbf{c\Omega}_{\langle d], |\sim|' \ge 2}}$ of the discriminant variety $\mathcal{D}_d \subset \mathcal{P}_d$, or is quadratically tangent to $\mathcal{D}_d^\circ$, then the locus Ξ_γ is the image of a compact 1-manifold M under an immersion $\beta : M \to A$. Indeed, for each $\psi \in S^1$ such that the polynomial $\gamma(\psi)$ has only simple real roots, Ξ_γ is a disjoint union of several smooth arcs over the vicinity of ψ, and the projection $q : \Xi_\gamma \to S^1$ is a local diffeomorphism. So we only need to sort out what happens over the vicinity of $\psi_\star$ such that $\gamma(\psi_\star)$ has a single root $x_\star$ of multiplicity 2. In the vicinity of $(\psi_\star, x_\star)$ in A, the intersections $\{\Xi_\gamma \cap \ell_\psi\}_\psi$ either:

(1) have cardinality 2 for all $\psi \ne \psi_\star$, or
(2) have cardinality 2 for all $\psi > \psi_\star$ and cardinality 0 all $\psi < \psi_\star$, or
(3) have cardinality 2 for all $\psi < \psi_\star$ and cardinality 0 all $\psi > \psi_\star$.

The case (1) arises when γ is quadratically tangent to $\mathcal{D}_d^\circ$ at $\gamma(\psi_\star)$. In such a case, locally, $\gamma(\psi)(x) = \big((x - x_\star)^2 - (\psi - \psi_\star)^2\big) \cdot T(x, \psi)$, where T is an x-polynomial with smooth coefficients (in ψ) and simple roots, so that $T(x_\star, \psi_\star) \ne 0$. Then two local branches of Ξ_γ

intersect at $(\psi_\star, x_\star)$. Both branches are transversal to $\ell_{\psi_\star}$, and the restriction of the map $q : S^1 \times \mathbb{R} \to S^1$ to each branch is a diffeomorphism. So the composition $q \circ \beta$ has no critical points in the vicinity of $\beta^{-1}((\psi_\star, x_\star))$.

When γ is transversal to $\mathcal{D}_d^\circ$ at $\gamma(\psi_\star)$, the cases (2) and (3) are realized. Then the q-images of Ξ_γ, localized to the vicinity of $(\psi_\star, x_\star)$, are semi-open intervals, bounded by $\psi_\star$. In such cases, locally, $\gamma(\psi)(x) = \big((x-x_\star)^2+(\psi-\psi_\star)\big)\cdot T(x,\psi)$, where T is an x-polynomial with smooth coefficients (in ψ) and simple roots, so that $T(x_\star, \psi_\star) \neq 0$. The locus Ξ_γ is quadratically tangent to $\ell_{\psi_\star}$ at $(\psi_\star, x_\star)$, and the Morse function $q \circ \beta$ attains its extremum at the unique point in M whose β-image is $(\psi_\star, x_\star)$.

So the triple self-intersections of Ξ_γ, the double tangencies to the leaves $\{\ell_\psi\}_\psi$, and the cubic tangencies to $\{\ell_\psi\}_\psi$ are forbidden when $\gamma(S^1) \subset \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}$: they correspond to compositions $\omega_\psi \in \mathbf{\Omega}_{\langle d], |\sim|' \geq 2}$.

Thus, if γ is a smooth loop in $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}$ that is either transversal to $\mathcal{D}^\circ$ or quadratically tangent to it, then $\Xi_\gamma \subset A$ satisfies (P1)-(P3).

If the image $\gamma(\star)$ of the base point $\star \in S^1$ belongs to the d-cell $\mathsf{R}_d^{()} \subset \mathcal{P}_d$ that represents polynomials with no real roots, then property (P4) is also satisfied.

Of course, by a small homotopy, we may assume that γ is *transversal* to $\mathcal{D}_d^\circ$ and is a regular embedding when $d > 2$. For such a loop γ, Ξ_γ does not have self-intersections (is a disjoint union of regularly embedded loops). However, even if the cobordism $B : W \to A \times [0,1]$ between two regular embeddings, $\beta_0 : M_0 \to A \times \{0\}$ and $\beta_1 : M_1 \to A \times \{1\}$, itself is a regular embedding, some t-slices of W will develop singularities (in particular, self-intersections). For that reason, there is the need to consider immersions (and not regular embeddings only) in (P1)-(P4).

Conversely, assume $\beta : M \to A$ is an immersion satisfying (P1)-(P4). By Proposition 3.1, we may lift β to a smooth loop $\gamma : S^1 \to \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}$ such that $\Xi_\gamma = \beta(M)$, γ is transversal or quadratically tangent to $\mathcal{D}_d^\circ$, and $\gamma(\star)$ belongs to the cell with no real roots.

So the correspondence $\gamma \rightsquigarrow \Xi_\gamma$ is the candidate for realizing the group isomorphism $\Xi_* : \pi_1(\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}) \to \mathcal{B}(A; \mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2})$.

We have shown already that any immersion $\beta : M \to A$ which satisfies (P1)-(P4) is realizable by a loop $\gamma : S^1 \to \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}$. It remains to prove that:

(1) homotopic loops $\gamma_0, \gamma_1 : S^1 \to \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}$ produce cobordant patters Ξ_{γ_0} Ξ_{γ_1} in A (so that the correspondence Ξ_* is well-defined);
(2) if $\Xi(\gamma)$ is cobordant in A to $\emptyset$, then γ is contractible in $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}$ (i.e., Ξ_* is an injective map).

In order to validate these two claims, we consider the domain

$$\mathcal{E}_d := \big\{(\vec{a}, x) |\ P(x, \vec{a}) \leq 0\big\} \subset \mathcal{P}_d \times \mathbb{R},$$

where $P(x, \vec{a}) := x^d + \sum_{j=0}^{d-1} a_j x^j$. We denote by $\partial\mathcal{E}_d$ the boundary of $\mathcal{E}_d$, a smooth hypersurface. Let $\pi : \mathbb{R} \times \mathcal{P}_d \to \mathcal{P}_d$ denote the obvious projection. Then $\pi^{-1}(\vec{a}) \cap \partial\mathcal{E}$ is the support of the real divisor of the x-polynomial $P(x, \vec{a})$.

By a general position argument, we may assume that the homotopy $\Gamma : S^1 \times [0,1] \to \mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\geq 2}}$ that links γ_0 and γ_1 is smooth and transversal to all the strata $\{\mathsf{R}_d^\omega\}_\omega$ of codimension ≤ 1. Note, that by definition, it misses all the strata of codimension ≥ 2. In other words, we may assume that γ_0, γ_1, and Γ are transversal to the $(d-1)$-cells $\mathring{\mathsf{R}}_d^\omega$, $\omega \in \mathbf{\Omega}_{\langle d],\,|\sim|'=1}$, that form the non-singular portion $\mathcal{D}_d^\circ := \mathcal{D}_d \cap \mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\geq 2}}$ of the discriminant variety $\mathcal{D}_d = \mathcal{P}_d^{\mathbf{\Omega}_{\langle d],\,|\sim|'\geq 1}} \subset \mathcal{P}_d$. For $d > 4$, by a general position argument, we may also assume that Γ is a regular embedding.

Set $G := \Gamma^{-1}(\mathcal{D}_d^\circ)$, a collection of disjoint arcs and loops in the cylinder $S^1 \times [0,1]$. We may perturb the obvious function $t : S^1 \times [0,1] \to [0,1]$ so that the new $\tilde{t} : S^1 \times [0,1] \to [0,1]$ has the following properties:

(1) $\tilde{t}(S^1 \times \{0\}) = t(S^1 \times \{0\}) = 0$;
(2) $\tilde{t}(S^1 \times \{1\}) = t(S^1 \times \{1\}) = 1$;
(3) $\tilde{t}$ has no critical points;
(4) the restriction $\tilde{t} : G \to [0,1]$ is a Morse function with critical points in the interior of G.

From now and on, we retain the old notation "t" for this perturbed $\tilde{t}$. If Γ is transversal to $\mathcal{D}_d^\circ$, then we claim that the map

$$\Lambda := \Gamma \times \mathsf{id}_{\mathbb{R}} : (S^1 \times [0,1]) \times \mathbb{R} \longrightarrow \mathcal{P}_d^{\mathbf{c\Omega}_{\langle d],\,|\sim|'\geq 2}} \times \mathbb{R}$$

is transversal to the hypersurface $\partial\mathcal{E}_d \subset \mathcal{P}_d \times \mathbb{R}$. Indeed, consider a line $\ell_{\psi,t} := \pi^{-1}(\Gamma(\psi,t))$ and a point $\Lambda(\psi,t,x) \in \partial\mathcal{E}_d$. If x is a simple real root of the polynomial $P(\sim, \Gamma(\psi,t))$, then the line $\ell_{\psi,t}$ is transversal to the hypersurface $\partial\mathcal{E}_d$ at $\Lambda(\psi,t,x)$. If x is a real root of multiplicity 2, then $\Gamma(\psi,t) \in \mathcal{D}_d^\circ$, and by the transversality of Γ to $\mathcal{D}_d^\circ$, the map Λ is transversal at the point $\Lambda(\psi,t,x)$ to the boundary $\partial\mathcal{E}$.

The Λ-transversality implies that $W := \Lambda^{-1}(\partial\mathcal{E})$ is a regularly embedded surface in the shell $(S^1 \times [0,1]) \times \mathbb{R} \cong A \times [0,1]$. With the help of $Q : A \times [0,1] \to S^1 \times [0,1]$ that takes each (x,ψ,t) to (ψ,t), the surface W projects on the cylinder $S^1 \times [0,1]$. The surface W interacts with the leaves of $\mathcal{L}^\bullet$ (the Q-fibers) in the ways that are described by the first three bullets in Definition 3.2.

To validate the last bullet from Definition 3.2, we need to check that the composition $t^\bullet : W \xrightarrow{Q} S^1 \times [0,1] \xrightarrow{t} [0,1]$ is a Morse function.

Consider the $(d-1)$-dimensional non-singular affine variety

$$\mathcal{F}_d := \left\{(\vec{a},x) |\ P(x,\vec{a}) = 0,\ \frac{\partial}{\partial x}P(x,\vec{a}) = 0\right\} \subset \partial\mathcal{E}_d \subset \mathcal{P}_d \times \mathbb{R},$$

where $P(x,\vec{a}) := x^d + \sum_{j=0}^{d-1} a_j x^j$.

Note that $\pi(\mathcal{F}_d) = \mathcal{D}_d$, the discriminant variety. The locus $\mathcal{F}_d^\circ := \pi^{-1}(\mathcal{D}_d^\circ)$ is an open and dense subset of $\mathcal{F}_d$, characterized by the inequality $\frac{\partial^2}{\partial x^2}P(x,\vec{a}) \neq 0$. The projection $\pi : \mathcal{F}_d^\circ \to \mathcal{D}_d^\circ$ is a diffeomorphism.

Set $Z := W \cap \mathcal{F}_d$ and put $G := \pi(Z) \subset S^1 \times [0,1]$. Moreover, since $\pi : \mathcal{F}_d^\circ \to \mathcal{D}_d^\circ$ is a diffeomorphism, so is the map $\pi : Z \to G$. Using that Γ is transversal to $\mathcal{D}_d^\circ =$

$\pi(\mathcal{F}_d^\circ)$, we get that W is transversal to $\mathcal{F}_d$. Thus Z is a smooth regular 1-dimensional submanifold of the surface W. By its construction, Z is exactly *the folding locus* of the map $Q : W \to S^1 \times [0,1]$. Therefore, away from Z, the map $Q : W \to S^1 \times [0,1]$ is a local diffeomorphism. Thus, away from Z, its composition $t^\bullet : W \xrightarrow{Q} S^1 \times [0,1] \to [0,1]$ is a non-singular function. As a result, the critical points of $t^\bullet : W \to [0,1]$ are located along Z and are among the critical points of the function $t^\bullet : Z \xrightarrow{\pi} G \xrightarrow{t} [0,1]$. Since Z is the folding locus of $Q : W \to S^1 \times [0,1]$, the critical points of $t^\bullet : W \to [0,1]$ are exactly the critical points of $t^\bullet : Z \to [0,1]$, the later function being Morse by the construction of $t : S^1 \times [0,1] \to [0,1]$ and the fact that $\pi : Z \to G$ is a diffeomorphism. Therefore, all the critical points of $t^\bullet : W \to [0,1]$ are of the Morse type.

Remarkably, changes in the topology of the slices $\{(t^\bullet)^{-1}(t) \cap W\}_{t\in[0,1]}$ and of the slices $\{(t^\bullet)^{-1}(t) \cap Z\}_{t\in[0,1]}$ are *synchronized* in t.

So $W = \Lambda^{-1}(\partial\mathcal{E})$ delivers the desired cobordism between the loop patterns $W \cap (A \times \{0\})$ and $W \cap (A \times \{1\})$. As a result, the map $\Xi_\star$ is well-defined and onto.

To validate (2), we use again Proposition 3.1 to produce a smooth (ψ, t)-parameter family of x-polynomials $\{P(x, \psi, t)\}$, whose roots form the surface W. The immersion (embedding when $d > 4$) $B : W \to A \times [0,1]$ bounds the given immersion $\beta : M \to A \times \{0\}$. For each $t \in [0,1]$, the ψ-family $\{P(x, \psi, t)\}_\psi$ gives rise to a loop $\gamma_t : S^1 \to \mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}$ that depends continuously on t. Since for some $t_\star \in [0,1]$, the $t_\star$-slice of W is empty, the loop $\gamma_{t_\star}$ is a subset of $\mathsf{R}_d^{()}$ and thus is contractible in $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}$. So the loop γ_0 is contractible in $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \geq 2}}$. □

4. Further directions

We conclude the paper by stating a few open problems and hinting to further research directions.

By Proposition 2.10, the fundamental groups of spaces $\mathcal{P}_d^{\mathbf{c}\Theta}$ for closed posets $\Theta \subset \mathbf{\Omega}_{\langle d], |\sim|' \geq 2}$ are completely described in terms of generators and relations. But, besides Theorem 2.11, we know little about the group theoretical properties of these fundamental groups. For instance, there seem to exist many more cases, beyond Theorem 2.11, when $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ is free.

Problem 4.1. For which closed posets $\Theta \subset \mathbf{\Omega}_{\langle d], |\sim|' \geq 2}$ is the fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ a free group?

Note that, when answering this question, one may assume that $\mathbf{\Omega}_{\langle d], |\sim|' \geq 3} \subseteq \Theta$.

More generally, one may ask about group theoretic properties of $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$.

Problem 4.2. Are there any common group theoretic properties for the groups that are presented by the relations of type (3) and (22) from Proposition 2.10?

The stabilization result Theorem 2.15 asks for homotopy theoretical explanation.

Problem 4.3. Let Θ be a closed poset in $\mathbf{\Omega}_{\langle d']}$, and for $d \geq d'$ such that $d \equiv d' \mod 2$ let Θ_d be the smallest closed poset in $\mathbf{\Omega}_{\langle d]}$ containing Θ. Is there a stabilization, as $d \to \infty$, of the spaces $\mathcal{P}_d^{\mathbf{c}\Theta_d}$ in terms of homotopy theory?

This problem is related to the calculation of the homology of $\mathcal{P}_d^{\mathbf{c}\Theta_d}$ in purely combinatorial terms and results on its stabilization from the forthcoming paper [KSW].

In § 3, we show that the fundamental group of $\mathcal{P}_d^{\mathbf{c}\mathbf{\Omega}_{\langle d], |\sim|' \leq 2}}$ can be interpreted as special bordisms of curves, immersed in the cylinder. Since Proposition 2.10 provides us with a presentation of the fundamental group $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ in the general case, this result raises the following question.

Problem 4.4. Given a closed poset $\Theta \subseteq \mathbf{\Omega}_{\langle d], |\sim|' \leq 2}$, is there an interpretation of $\pi_1(\mathcal{P}_d^{\mathbf{c}\Theta})$ in terms of such bordisms? That is, in terms of the bordisms of immersed curves in the cylinder $S^1 \times \mathbb{R}$ with only *some* cubical tangency patterns to the fibers of the map $S^1 \times \mathbb{R} \to S^1$ and *some* double tangent patterns (both types belonging to $\mathbf{c}\Theta$) being permitted?

We suspect that such interpretation is within reach.

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