

FINITE ORBITS AND INVARIANT BEZOUTIANS FOR CORRESPONDENCES ON THE LINE

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ABSTRACT. We study reduced algebraic correspondences on the projective line with finite iteration orbits and the factorisation of their Bezoutians. If sufficiently many points have forward orbits of size at most N , then all orbits are fibres of a rational quotient of degree at most N ; we give an explicit threshold. Over $\mathbb{C}$, uncountably many finite forward orbits suffice. The proof rests on forward stabilisation and the fact that a finite algebraic preorder on a smooth projective curve is an equivalence relation. Monodromy identifies absolute Bezoutian factors with ordered-pair orbits and invariant factors with unordered-pair orbits. For polynomials, this yields the complete invariant reducibility locus. Its normalised composition components are affine spaces; their projective closures are swept by planes and have degrees $S(r + s - 2, r)$. For rational maps, we construct a Grassmann-bundle model, compute its Plücker degree, and determine the composition components when a marked pole fibre has exactly one ramified point. We also describe prime-degree affine monodromy and translation correspondences with polynomial orbit growth.

1. INTRODUCTION

Unless otherwise stated, we work over $\mathbb{C}$. A reduced curve $\Gamma \subset \mathbf{P}^1 \times \mathbf{P}^1$ defines a multivalued map: from z one may move to any w with $(z, w) \in \Gamma$. Throughout we exclude components contained in a fibre of either projection. Write $\mathcal{O}^+(x)$, $\mathcal{O}^-(x)$ and $\mathcal{O}(x)$ for the forward, backward and full orbits, including the initial point. The full orbit consists of points reachable by finite sequences of forward and backward steps in any order. A correspondence is called *finite-orbit* ($\mathcal{FIO}$) when all its full orbits are finite. We use the *projection-degree convention*

$$(n, m) = (\deg(\Gamma \rightarrow \mathbf{P}_z^1), \deg(\Gamma \rightarrow \mathbf{P}_w^1)).$$

Thus an affine equation has degree n in w and degree m in z .

The basic examples come from a rational function $f = P/Q$. Its fibre relation is $P(z)Q(w) - P(w)Q(z) = 0$, and removing the diagonal gives the Bezoutian

$$B_{P,Q}(z, w) = \frac{P(z)Q(w) - P(w)Q(z)}{z - w}. \quad (1.1)$$

Every orbit of a component lies in a fibre of f . Our first result recovers this construction from a one-sided hypothesis.

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Theorem 1.1 (Finite-orbit quotient). *For a reduced correspondence Γ over $\mathbb{C}$ without fibre components, the following conditions are equivalent:*

- (i) *uncountably many points have finite forward orbit;*
- (ii) *every forward orbit is finite;*
- (iii) *every forward and every backward orbit is finite;*
- (iv) *the sizes of all full orbits are uniformly bounded;*
- (v) *a nonconstant rational function is constant on every orbit;*
- (vi) *Γ is a union of components of the fibre relation of a rational function.*

In this case $n = m$, and the forward, backward and full orbits coincide at every point. There is a minimal quotient f , unique up to postcomposition by $\mathrm{PGL}_2(\mathbb{C})$, whose fibres, as sets, are exactly the orbits. Its degree is the generic orbit size, and every rational first integral belongs to $\mathbb{C}(f)$. Each irreducible component satisfies the same conclusions with its own quotient.

The effective statement is valid over every algebraically closed field of characteristic zero. It also gives a shorter proof of the preceding theorem.

Theorem 1.2 (Effective forward-orbit test). *Put $M_N = \sum_{i=0}^N m^i$, where $N \geq 1$. If more than*

$$C(n, m, N) = 2nNM_N$$

points have forward orbit of size at most N , then all orbits are fibres of a quotient of degree at most N . If Γ is symmetric, the bound may be replaced by $2nN(nN + 1)$.

The distinction between forward and full orbits is essential. For the graph $w = z^2$, the points with finite forward orbit are 0 , ∞ and the roots of unity. They form an infinite, Zariski-dense set, but no nonconstant rational first integral exists. There is also no orbit-size bound depending only on (n, m) : finite-order Möbius transformations already give arbitrarily large orbits in projection degree $(1, 1)$.

Relation to earlier work. Bellaïche’s finitary self-correspondence theorem [2, Theorem 2.2.1] gives the equivalence between infinitely many finite complete sets and a nonconstant function satisfying $f \circ \pi_1 = f \circ \pi_2$. It includes the qualitative two-sided quotient statement, also on arbitrary smooth projective curves. The contribution here is the elementary forward-stabilisation argument, its explicit bounded-orbit test, and their connection with the geometry of Bezoutian factorisation.

A core for a correspondence between two curves allows $a \circ \pi_1 = b \circ \pi_2$ with two different functions a, b ; see [7, 16]. This is weaker than a first integral for iteration on one identified line. For example $w = z^2$ has such a core, taking $a(z) = z^2$ and $b(w) = w$, although its iteration orbits are generally infinite. Buchacher’s orbits [5, §3] are defined by repeatedly keeping one coordinate fixed on the correspondence, and should be distinguished from the iteration orbits used here. Bonnet–Hardouin [4] study a related field-intersection construction.

Correspondences also occur in the study of polynomial-root invariant sets: the equations $T[(x - z)^d] = 0$ in [1] turn a differential operator into a multivalued map. Example 3.7 gives a finite-orbit instance. Our emphasis on

explicit low-degree equations and on the geometry of the parameter spaces complements this viewpoint.

Factorisation and parameter spaces. The components of $B_{P,Q} = 0$ correspond to the nontrivial orbitals of $\text{Mon}(f)$ on ordered pairs in a generic fibre. An orbital is called connected when its underlying undirected graph is connected. Irreducible non-diagonal finite-orbit correspondences with generic orbit size N are thus classified by pairs $(f, \mathcal{O})$, where $\deg f = N$, f is taken modulo target Möbius transformations, and $\mathcal{O}$ is a connected nontrivial orbital. An orbital of size Nd gives a component of projection degree (d, d) . Transposition of pairs detects symmetry.

Put $u = z + w$, $v = zw$. The invariant Bezoutian $\overline{B}_{P,Q}(u, v)$ has factors indexed by unordered-pair orbits. Consequently absolute irreducibility means 2-transitivity, whereas invariant irreducibility means 2-homogeneity. Already $B_{t^3} = (z - \omega w)(z - \omega^2 w)$ becomes the irreducible polynomial $u^2 - v$.

Affine (or linear) equivalence of polynomials means independent source and target changes $P = L_1 \circ Q \circ L_2$, with $\deg L_1 = \deg L_2 = 1$. Möbius equivalence of rational maps likewise permits independent changes in source and target. When only postcomposition is allowed, we say so.

For a polynomial of degree $k \geq 3$, Fried's theorem gives the following complete criterion: $\overline{B}_P$ is reducible precisely when P is decomposable or $k \geq 5$ is prime and P is affinely equivalent to a Dickson polynomial $D_k(t, a)$. In the centred monic slice with zero constant term,

$$\tilde{P}_k = \{t^k + a_{k-2}t^{k-2} + \cdots + a_1t\} \simeq \mathbb{A}^{k-2},$$

let $\mathcal{D}_{r,s}$ denote the locus with an outer factor of degree r and an inner factor of degree s , where $rs = k$ and $r, s > 1$.

For a vector bundle E , $\mathbf{P}(E)$ parametrises one-dimensional subspaces, with tautological line bundle $\mathcal{O}_{\mathbf{P}(E)}(-1)$. We use $\text{Gr}_2(E)$ for the bundle of two-dimensional subspaces and $\text{Gr}(2, V)$ for the corresponding Grassmannian of a vector space.

Theorem 1.3 (Composition geometry). *The normalised composition map is a closed embedding $\mathbb{A}^{r-1} \times \mathbb{A}^{s-2} \rightarrow \tilde{\mathcal{P}}_{rs}$. The closure of its image in $\mathbf{P}^{rs-2}$ is the birational image of $\mathbf{P}(E)$, where*

$$E = \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-r) \quad \text{on } \mathbf{P}^{s-2},$$

and has degree $S(r + s - 2, r)$, a Stirling number of the second kind. For rational maps, the closure of the degree- s right-factor locus in $\text{Gr}(2, V_{rs})$, where $V_d = H^0(\mathbf{P}^1, \mathcal{O}(d))$, is the birational image of $\text{Gr}_2(\text{Sym}^r \mathcal{S})$ over $\text{Gr}(2, V_s)$. It has codimension $2(r - 1)(s - 1)$ and degree given by (6.1).

The polynomial components are smooth; their intersections are governed by Ritt theory. We also describe the rational composition components when all finite poles are simple, and the affine-monodromy cases in prime degree. In particular absolute and invariant reducibility differ in prime degree only for degrees 3 and 7. The final appendix records the translation models for symmetric $(2, 2)$ curves and the family $(z, w) = (g^d, (g + c)^d)$, whose generic growth exponent is $\varphi(d)$.

2. FORWARD STABILISATION AND THE QUOTIENT

All algebraic arguments in this section are over an algebraically closed field K of characteristic zero; countability assertions are over $\mathbb{C}$. Write $\Gamma^0 = \Delta$ and use set-theoretic composition of relations. Put

$$F_j = \bigcup_{i=0}^j \Gamma^i, \quad e_j = \deg(F_j \rightarrow \mathbf{P}_z^1), \quad \hat{e}_j = \deg(F_j \rightarrow \mathbf{P}_w^1).$$

Each F_j is a reduced closed curve without fibre components. Indeed, a space of paths of fixed length is an iterated fibre product of finite maps; its endpoint image is closed by properness and finite over either endpoint. These fibre products have pure dimension one, since the original finite maps to a smooth curve are flat. Thus no isolated components are introduced.

Lemma 2.1 (Stabilisation). *The chain $F_0 \subseteq F_1 \subseteq \dots$ has the following properties:*

- (a) *a strict inclusion increases each projection degree by at least one;*
- (b) *$e_{j+1} \leq 1 + ne_j$ and $\hat{e}_j \leq \sum_{i=0}^j m^i$;*
- (c) *if $F_j = F_{j+1}$, the chain is stable thereafter, and its stable value is the full forward relation $\bigcup_{i \geq 0} \Gamma^i$.*

Proof. A new component has positive degree over both coordinates. The identity $F_{j+1} = \Delta \cup (\Gamma \circ F_j)$ gives the degree bounds and shows that equality at one step persists at every later step. $\square$

Lemma 2.2 (Finite algebraic preorders). *Let X be a smooth projective curve and $E \subset X \times X$ a reduced curve without fibre components. If $\Delta \subset E$ and $E \circ E = E$, then E is symmetric, hence an equivalence relation.*

Proof. Let e be the first projection degree. Outside a finite set Z every fibre $E(x)$ has e distinct elements. For x outside $Z \cup E^{-1}(Z)$ and $y \in E(x)$, transitivity gives $E(y) \subseteq E(x)$. Both sets have e elements, so they are equal. Since $x \in E(x)$, we obtain $(y, x) \in E$. These pairs are dense in every component of E ; closedness gives $E^{-1} \subseteq E$, hence equality. $\square$

Lemma 2.3 (Quotient). *A reduced equivalence relation $E \subset \mathbf{P}^1 \times \mathbf{P}^1$ without fibre components is the reduced fibre relation of a rational function f of degree $e = \deg(E \rightarrow \mathbf{P}^1)$. Its field of rational invariants is $K(f)$.*

Proof. On an affine chart the generic class has monic equation $T^e + c_1(z)T^{e-1} + \dots + c_e(z)$. Its coefficients are constant on classes, and not all are constant, since a constant generic class would contradict $\Delta \subset E$. Let $L = K(c_1, \dots, c_e)$. Two general points have the same coefficient vector exactly when they have the same class, so $[K(z) : L] = e$. By Lüroth's theorem $L = K(f)$ with $\deg f = e$. Every rational invariant descends through the map defined by the coefficient vector, and therefore lies in L .

The equality $f(z) = f(w)$ holds generically on every component of E , hence everywhere as an equality of morphisms to $\mathbf{P}^1$. Thus E lies in the reduced fibre relation of f . Both curves have projection degree e and neither has fibre components, so they are equal. This also proves equality of classes and fibres at exceptional points. Two generators of L differ by a Möbius transformation. $\square$

Proof of Theorem 1.2. For a reduced curve D with projection degrees $(e, \hat{e})$ and no fibre components, the binary discriminant in w of a defining form is a nonzero form in z of degree $(2e - 2)\hat{e}$. Hence

$$\#\{x : |D(x)| < e\} \leq (2e - 2)\hat{e}. \quad (2.1)$$

The statement includes $e = 1$, when every fibre consists of one point.

Suppose some $e_j > N$, and choose the first such j . Then $j \leq N$: otherwise either an earlier equality stabilises the chain, or the first N strict inclusions already give $e_N \geq N + 1$. Moreover $e_j \leq nN + 1$ and $\hat{e}_j \leq M_N$. Every point whose forward orbit has at most N elements belongs to the exceptional set in (2.1) for $D = F_j$. There are at most $2nNM_N$ such points, a contradiction. Thus $e_j \leq N$ for all j , and the chain stabilises. Its stable relation is reflexive and transitive, so Lemmas 2.2 and 2.3 apply.

If Γ is symmetric, so is each F_j and $\hat{e}_j = e_j$. At the first crossing the same estimate is therefore at most $2nN(nN + 1)$. $\square$

Proof of Theorem 1.1. The set of points with finite forward orbit is the countable union of the sets with orbit size at most N . If it is uncountable, one such set is infinite and Theorem 1.2 gives a stabilised forward relation E and a rational quotient. Since E is symmetric, forward, backward and full orbits coincide everywhere. The remaining equivalences follow directly: a first integral bounds every orbit by its degree, and an invariant reduced curve is a union of components of its fibre relation.

For a component C of Γ , its two normalised projections satisfy $f \circ \pi_1 = f \circ \pi_2$. Multiplicativity of degrees yields $\deg \pi_1 = \deg \pi_2$. Its own orbits are bounded, so the argument also applies to C . Finally every invariant belongs to $K(f)$ by Lemma 2.3; a nonconstant $g = R \circ f$ has degree $(\deg R)(\deg f)$, proving minimality and uniqueness as stated. $\square$

Corollary 2.4 (Balance). *For a finite-orbit correspondence every forward-invariant set is backward invariant. Each irreducible component has equal projection degrees.*

Remark 2.5. The effective test is algebraic and works over $\overline{\mathbb{Q}}$. The uncountable-point formulation is specific to an uncountable ground field. We make no deduction from unbounded pointwise finite forward orbits over a countable field. A horizontal component must be excluded: the relation $(w - c)(w - z) = 0$ has finite forward orbits, but the backward orbit of c is the whole line.

2.1. An arbitrary smooth projective curve.

Proposition 2.6. *On a smooth projective curve X over $\mathbb{C}$, conditions (i)–(iv) of Theorem 1.1 are equivalent to the existence of a nonconstant morphism $f : X \rightarrow Y$ to a smooth projective curve whose fibres are the orbits. The minimal quotient is unique up to isomorphism of its target, and the two projection degrees of every component of Γ are equal.*

Proof. The path curves and Lemmas 2.1 and 2.2 are unchanged. If their degrees were unbounded, every point outside a countable union of finite exceptional sets would have infinite forward orbit. Thus uncountably many finite forward orbits force stabilisation. For the stable relation, the map

sending a general x to its class defines a nonconstant rational map $X \dashrightarrow \text{Sym}^e X$. Normalise its image to obtain Y . The resulting morphism $f : X \rightarrow Y$ has degree e and its generic fibres are the classes. Equality of the two reduced relations follows from their projection degrees, as in Lemma 2.3. Descent of invariant functions and multiplication of degrees finish the proof. $\square$

Remark 2.7 (An effective bound on X). Let $g = g(X)$, $E_N = nN + 1$ and $M_N = \sum_{i=0}^N m^i$. Over an algebraically closed field of characteristic zero, the sufficient bound

$$C_X = \left\lceil \frac{(E_N + M_N)^2}{2} \right\rceil + \max(2g - 2, 0)M_N$$

follows from the same first-crossing argument. Here is the required control of exceptional fibres, including collisions at singular points. A reduced path curve $D \subset X \times X$ is a Cartier divisor, finite and flat over X . The trace discriminant of $\pi_{1*}\mathcal{O}_D$ is nonzero and has degree

$$-2 \deg(\pi_{1*}\mathcal{O}_D) = D^2 + (2g - 2)\hat{e}.$$

This follows from Riemann–Roch and adjunction. Its zeros contain every fibre with fewer than e distinct points. If H is the sum of the two fibre classes, then $H^2 = 2$, $D \cdot H = e + \hat{e}$, and the Hodge index theorem gives $D^2 \leq (e + \hat{e})^2/2$. At the first crossing, $e \leq E_N$ and $\hat{e} \leq M_N$, which proves the bound. Using only the branch divisor of the normalisation would miss collisions between distinct branches.

Remark 2.8. For symmetric correspondences, infinitely many finite orbits already imply finitariness by Bellaïche’s theorem [2], since these orbits are finite complete sets. This stronger qualitative fact is separate from the explicit bounded-orbit test above.

3. MONODROMY AND THE ORBITAL DICTIONARY

Let $f = P/Q : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be a rational function of degree $N \geq 2$ with P, Q coprime. Projectively, $P(z)Q(w) - P(w)Q(z) = 0$ is the fibre product of f with itself; the diagonal is one component and its residual curve is the projective closure of $B_{P,Q} = 0$.

Proposition 3.1 (Residual fibre curves). *The irreducible components of $B_{P,Q} = 0$ are indexed by the nontrivial orbitals of $G = \text{Mon}(f)$ on ordered pairs in a generic fibre. If an orbital has size Nd , the corresponding component has bidegree (d, d) . Consequently*

$$B_{P,Q} \text{ is irreducible over } \mathbb{C} \iff G \text{ is 2-transitive.}$$

The polynomial $B_{P,Q}$ is always symmetric, being a quotient of two antisymmetric polynomials; written in $u = z + w$, $v = zw$, its irreducible factors in $\mathbb{C}[u, v]$ are indexed by the orbits of G on unordered pairs. Thus the invariant Bezoutian is irreducible if and only if G is 2-homogeneous. Individual components of $B_{P,Q} = 0$ need not be symmetric.

Proof. Remove the branch values of f . Over the remaining base the residual fibre curve is an unramified cover whose fibre is the set of ordered pairs (x, y)

of distinct points in a fibre of f . Its connected components are therefore the orbits of monodromy on those pairs, and their Zariski closures are the irreducible components. The degree over the first coordinate is the size of the orbital divided by N , which gives (d, d) after applying the same argument to the second coordinate.

For the invariant statement, let σ interchange z, w . The residual Bezoutian is squarefree and has no diagonal factor in characteristic zero. A flip-stable irreducible factor can be chosen symmetric: the alternative would be an antisymmetric polynomial, divisible by $z - w$. Since $\mathbb{C}[z, w]^\sigma = \mathbb{C}[u, v]$, such a factor remains irreducible in the invariant ring. If F and F^σ are distinct irreducible factors, their product is irreducible there: by unique factorisation, any invariant divisor containing F must also contain F^σ . Thus invariant factors correspond to flip-orbits of orbitals, equivalently to unordered-pair orbits. $\square$

Theorem 3.2 (Orbital dictionary). *Irreducible non-diagonal finite-orbit correspondences with generic orbit size N are in bijection with pairs $(f, \mathcal{O})$, where $\deg f = N$, the map f is taken modulo target Möbius transformations, and $\mathcal{O}$ is a connected nontrivial orbital of $\text{Mon}(f)$. The generic orbit digraph is the orbital digraph. It is the complete directed graph, with both orientations of every edge, precisely when $\text{Mon}(f)$ is 2-transitive; in that case the curve is the whole residual fibre curve.*

Proof. Given $(f, \mathcal{O})$, Proposition 3.1 produces an irreducible component $\Gamma_{f, \mathcal{O}}$ whose orbits lie in fibres of f . Connectedness of the orbital digraph says that a generic fibre is a single orbit, so its size is $N = \deg f$.

Conversely, let Γ be irreducible and $\mathcal{FIO}$. By Theorem 1.1 its minimal quotient f has degree equal to the generic orbit size and Γ is a component of the residual fibre relation, hence determines a unique orbital by Proposition 3.1. That orbital must be connected: otherwise its connected components form a nontrivial block system, yielding a proper right component of f which is still constant on Γ -orbits, contradicting minimality. The bidegree and symmetry assertions follow from Proposition 3.1; the final statement is the definition of 2-transitivity. $\square$

Remark 3.3 (Relation with Poncelet curves). Mirman–Shukla polynomials encode complete finite orbit graphs and hence the 2-transitive case of Theorem 3.2; see [18]. A Bezoutian representation is an algebraic statement and should not, by itself, be confused with a geometric Poncelet realisation relative to a prescribed conic; additional real or projective hypotheses enter that converse, compare [12].

The dictionary also has a useful geometric form. Let Y be the Galois closure of f , let G be its deck group, and let H be a point stabiliser in the monodromy action; then $Y/G \simeq \mathbf{P}^1$ and $Y/H \simeq \mathbf{P}^1$, and we write $q : Y \rightarrow Y/H$.

Corollary 3.4 (Galois normal form). *A non-diagonal irreducible curve is $\mathcal{FIO}$ if and only if it has a parametrisation*

$$\Gamma = \overline{\{(q(y), q(\gamma y)) : y \in Y\}}$$

for data (Y, G, H, γ) such that G acts faithfully on the smooth projective curve Y , both Y/G and Y/H have genus zero, H is core-free, $\gamma \notin H$, and the double coset $H\gamma H$ generates G together with H . The generic orbit size is $[G : H]$ and

$$\deg_z \Gamma = \deg_w \Gamma = |H\gamma H|/|H|.$$

Transposition replaces $H\gamma H$ by $H\gamma^{-1}H$.

Proof. Apply Theorem 3.2 and the standard correspondence between orbitals of the transitive G -set G/H and double cosets $H \backslash G/H$. The generation condition is equivalent to connectedness of the orbital graph, and the displayed quotient of group orders is its valency. $\square$

There is also a quick obstruction visible directly from the equation. Choose the target coordinate of the minimal quotient so that $f = P/Q$ with $\deg P = N > \deg Q = \ell$. The top homogeneous part of its residual fibre relation is, up to a nonzero scalar,

$$z^\ell w^\ell h_{N-\ell-1}(z, w), \quad h_j(z, w) = \frac{z^{j+1} - w^{j+1}}{z - w}.$$

Corollary 3.5 (Leading-form test). *Let $F(z, w) = 0$ be a finite-orbit correspondence with no diagonal component. Its top homogeneous part divides*

$$z^\ell w^\ell h_{N-\ell-1}(z, w) \quad \text{for some } N > \ell \geq 0.$$

In particular every nonzero finite slope of a linear factor of the leading binary form is a nontrivial $(N - \ell)$ th root of unity, and the leading form is squarefree away from $zw = 0$.

Proof. By Theorem 1.1, F divides the residual fibre relation of its minimal quotient. Taking top homogeneous parts gives the assertion, and the linear factors of $h_{N-\ell-1}$ are distinct. $\square$

Proposition 3.6 (Bidegree $(1, 1)$). *The irreducible $\mathcal{FIO}$ -curves of bidegree $(1, 1)$ are exactly the graphs of finite-order Möbius transformations. Such a graph is symmetric precisely when the transformation satisfies $\phi^2 = \text{id}$ (including the identity).*

Proof. An irreducible curve of bidegree $(1, 1)$ is the graph $w = \phi(z)$ of a Möbius transformation. Its forward and backward orbits are the orbits of $\langle \phi \rangle$, which are all finite if and only if ϕ has finite order. Transposition replaces ϕ by ϕ^{-1} . $\square$

Example 3.7 (An operator-induced finite orbit). Let ω have order $q > 1$ and set $T = (\omega - 1)x d/dx + 1$. In the construction of [1], restricted to linear polynomials, the equation $T[x - z] = \omega x - z = 0$ defines $z \mapsto \omega^{-1}z$. Its generic orbit has q elements and its minimal quotient is z^q . Thus the operator viewpoint leads directly to a finite-orbit correspondence and its Bezoutian fibre relation.

4. RATIONAL PENCILS WITH A MARKED FIBRE

Let $V_d = H^0(\mathbf{P}^1, \mathcal{O}(d))$. A base-point-free pencil $\Lambda = \langle A, B \rangle \in \text{Gr}(2, V_d)$ determines the degree- d map $(A : B)$ up to postcomposition. The map

$$\Lambda \longmapsto [A(z)B(w) - A(w)B(z)] \in |\mathcal{O}(d, d)| \quad (4.1)$$

is the Plücker embedding followed by a linear embedding. Dividing out the fixed diagonal gives the corresponding linear embedding into symmetric forms of bidegree $(d-1, d-1)$. In particular these embeddings preserve degree.

For $k > \ell \geq 0$, let $\mathcal{P}_{k,\ell}$ be the affine space of pairs (P, Q) with P, Q monic of degrees k, ℓ and $[t^\ell]P = 0$. It has dimension $k + \ell - 1$. On the coprime locus, $f = P/Q$ has

$$e_f(\infty) = k - \ell. \quad (4.2)$$

The top homogeneous part of $B_{P,Q}$ is $z^\ell w^\ell h_{k-\ell-1}(z, w)$, with h_j as in Corollary 3.5. The word “separable” below refers to squarefreeness of Q , not to separability of a map in characteristic zero.

Proposition 4.1 (Composition and closedness). *If $f = g \circ h$ with $\deg g = r > 1$, $\deg h = s > 1$, $h = A/B$ and $g = C/D$, write $X = (X_1, X_2)$ and set $\widehat{C}(X) = X_2^r C(X_1/X_2)$, and similarly for $\widehat{D}$. The hat on $B_g = B_{C,D}$ denotes its bihomogenisation of bidegree $(r-1, r-1)$. Then, up to a nonzero scalar,*

$$B_f(z, w) = B_h(z, w) \widehat{B}_g((A(z), B(z)), (A(w), B(w))). \quad (4.3)$$

Both factors are nonconstant and symmetric. On the space of degree- rs base-point-free pencils, having a right factor of degree s is a closed condition. There are finitely many such right factors up to postcomposition.

Proof. Substitute the homogeneous forms for h into

$$\widehat{C}(X)\widehat{D}(Y) - \widehat{C}(Y)\widehat{D}(X) = (X_1Y_2 - X_2Y_1)\widehat{B}_g(X, Y)$$

and divide by $z-w$. The factors have bidegrees $(s-1, s-1)$ and $(rs-s, rs-s)$.

For closedness, let $\mathcal{S}$ be the tautological bundle on $\mathrm{Gr}(2, V_s)$. Multiplication embeds $\mathrm{Sym}^r \mathcal{S}$ into the trivial bundle with fibre V_{rs} : for independent A, B , their ratio is nonconstant, so the powers $B^r, AB^{r-1}, \dots, A^r$ are linearly independent. Thus $\mathrm{Gr}_2(\mathrm{Sym}^r \mathcal{S}) \rightarrow \mathrm{Gr}(2, V_{rs})$ is a proper morphism. If its image pencil is base-point free, so is $\langle A, B \rangle$; otherwise every image form has its common base divisor to the r th power. The outer pencil must also be base-point free, since h is surjective. Its image on this open locus is therefore exactly the decomposition locus. Finiteness of right factors follows from their correspondence with block systems of the finite monodromy group. $\square$

Lemma 4.2 (Wreath products). *The imprimitive action of $S_s \wr S_r$ on rs points has exactly one nontrivial block system, consisting of r blocks of size s .*

Proof. The stabiliser of a point has orbit sizes $1, s-1, s(r-1)$. A block containing that point is a union of these orbits. Besides $1, s, rs$, the only possible size is $1 + s(r-1)$, which is strictly between $rs/2$ and rs and cannot divide rs . $\square$

Lemma 4.3 (Wreath criterion). *Suppose $f = g \circ h$, with $\deg g = r > 1$ and $\deg h = s > 1$, and h is indecomposable. If some critical value of f has exactly one ramified preimage, of local degree two, then $\mathrm{Mon}(f) = S_s \wr \mathrm{Mon}(g)$ in its imprimitive action. In particular, if $\mathrm{Mon}(g) = S_r$, then $\mathrm{Mon}(f) = S_s \wr S_r$.*

Proof. The fibres of h give r blocks, on which the induced group is $\text{Mon}(g)$. Their setwise stabilisers induce $\text{Mon}(h)$, hence act primitively inside each block. The specified branch cycle is a transposition; it must lie inside one block, since moving two blocks moves at least $2s > 2$ points. Its conjugates define a graph whose edges stay inside the blocks. The connected components give an invariant refinement of the block system. Primitivity inside each block and transitivity on blocks force this graph to be connected on every block. The edge transpositions therefore generate S_s^r . This is the full kernel of the ambient wreath product, and the monodromy surjects onto $\text{Mon}(g)$, proving the claim. $\square$

4.1. Simple finite poles.

Theorem 4.4 (Separable-fibre dichotomy). *Suppose $1 \leq \ell \leq k - 2$, P, Q are coprime, and Q is squarefree. Then either $f = P/Q$ is decomposable, or $\text{Mon}(f)$ is 2-transitive. Consequently decomposability, absolute reducibility of B_f and invariant reducibility of $\bar{B}_f$ are equivalent on this locus.*

Proof. Decomposition gives the factors in (4.3). If f is indecomposable, its monodromy is primitive and contains a $(k - \ell)$ -cycle fixing ℓ points. For $\ell \geq 3$, Jones's theorem [13, Corollary 1.3] implies that it contains A_k . For $\ell = 1$ the cycle already makes the stabiliser of the fixed point transitive on its complement. For $\ell = 2$, with fixed points a, b , either the stabiliser of a is transitive on the complement or it fixes b . In the latter case the two stabilisers coincide and fix exactly $\{a, b\}$; their translates form a nontrivial block system, a contradiction. Thus the group is 2-transitive, and Proposition 3.1 applies. $\square$

For the remainder of this subsection assume $1 \leq \ell \leq k - 2$, and set $\mathcal{U}_{k,\ell} = \{(P, Q) \in \mathcal{P}_{k,\ell} : \gcd(P, Q) = 1, Q \text{ squarefree}\}$. For $s \mid k$, $1 < s < k$, let $\mathcal{D}_s$ be its degree- s right-factor locus.

Theorem 4.5 (Composition components). *Assume $1 \leq \ell \leq k - 2$, and put $r = k/s$ and $b = k - \ell$. The locus $\mathcal{D}_s$ is nonempty precisely in one of the following cases:*

case	condition	dimension	codimension
A	$s \geq b$	$2r + 2s - b - 3$	$2(r - 1)(s - 1)$
B	$s \mid \ell, a = b/s$	$2r + s - a - 2$	$k + \ell + 1 - 2r - s + a$

The cases overlap only at $s = b$, where they give the same family. Every nonempty $\mathcal{D}_s$ is closed, irreducible and rational. Its general member has monodromy $S_s \wr S_r$ and exactly two invariant factors, of bidegrees $(s - 1, s - 1)$ and $(k - s, k - s)$. The nonempty $\mathcal{D}_s$ are exactly the irreducible components of the reducibility locus in $\mathcal{U}_{k,\ell}$.

Proof. Let $f = g \circ h$, and move $h(\infty)$ to ∞ by an intermediate Möbius transformation. Local degrees give $e_g(\infty)e_h(\infty) = b$. Every other point in the pole fibre of f is simple. Hence either $e_g(\infty) = 1$ (Case A), or $h^{-1}(\infty) = \{\infty\}$ (Case B). In Case A, $h = A/B$ has numerator and denominator degrees s and $s - b$, while $g = C/D$ has degrees $r, r - 1$. In Case B, $h = A$ is polynomial and g has degrees $r, r - a$, with $\ell = (r - a)s$. These conditions

are sufficient: choose simple finite poles of g away from the critical values of h , and choose B squarefree in Case A. If both conditions hold, $s \mid b$ and $s \geq b$, hence $s = b$.

Normalise A, B to be monic and set $[t^{\deg B}]A = 0$, using the residual affine freedom in the intermediate coordinate. In Case B this means $A(0) = 0$. Take C, D monic. The condition $[t^\ell]P = 0$ determines the coefficient of degree $r - 1$ in C in Case A and that of degree $r - a$ in Case B, with coefficient one. The remaining parameters form a nonempty open subset of an affine space, of dimensions respectively

$$(s + s - b - 1) + (2r - 2), \quad (s - 1) + (2r - a - 1).$$

Its fibres over f are finite, since normalisation fixes a chosen right factor uniquely. This proves irreducibility and the dimension formulas. Closedness is Proposition 4.1.

For generic monodromy, one can choose a degree- d rational map with one specified branch cycle $(b', 1^{d-b'})$ and all other branching simple: factor the inverse b' -cycle into $b' - 1$ transpositions and append $(1\ j), (1\ j)$ for $b' < j \leq d$. The tuple generates S_d , has product one, and its index sum is $2d - 2$. Riemann existence gives a genus-zero cover. Apply this to g and h with the prescribed pole profiles. Move the simple branch values into general position so that some simple critical value v of h is unramified for g and $g(v)$ meets no other ramification of the composition. These are nonempty open conditions on the branch values. Then Lemma 4.3 gives full wreath monodromy. The same branch and noncollision conditions give a dense open set of such pairs.

The two nontrivial unordered-pair orbits of the wreath product are pairs inside one block and pairs in distinct blocks, giving the stated factors. By Lemma 4.2 the generic right factor is unique, so the parameter map is birational and $\mathcal{D}_s$ is rational. A full-wreath member of $\mathcal{D}_s$ lies in no $\mathcal{D}_{s'}$ for $s' \neq s$. Thus none of these closed irreducible loci contains another. The dichotomy proves that their union is the reducibility locus. $\square$

Example 4.6. In type $(k, \ell) = (6, 2)$, only Case B with $s = 2$ occurs. The locus has dimension 4 and codimension 3. An explicit member is

$$f(t) = \frac{t^6}{t^2 - c} = \left(\frac{y^3}{y - c} \right) \circ t^2, \quad c \neq 0,$$

and

$$B_f = (z + w)(z^4 w^2 + z^2 w^4 - c(z^4 + z^2 w^2 + w^4)).$$

For $(k, \ell) = (6, 1)$ neither case occurs, so every coprime pair has absolutely irreducible Bezoutian.

Remark 4.7. The decomposition locus remains closed without the square-freeness condition, but the displayed ramification patterns no longer exhaust it. Nor does decomposition exhaust reducible Bezoutians: indecomposable dihedral and elliptic examples occur. The range $\ell \leq k - 2$ is essential. For instance,

$$P(t) = t^5, \quad Q(t) = \frac{t^5 - (t - 1)^5}{5}$$

is a normalised coprime pair with Q monic and squarefree of degree four. With $h = t/(t - 1)$ one has $P/Q = 5h^5/(h^5 - 1)$, so the monodromy is cyclic

and the invariant Bezoutian is reducible, although prime degree excludes decomposition. We make no complete classification outside the stated range.

4.2. Absolute and invariant factors.

Corollary 4.8 (Kantor's obstruction). *If B_f is reducible and $\overline{B}_f$ is irreducible, then $\deg f = q \equiv 3 \pmod{4}$ is a prime power and $\text{Mon}(f)$ is a 2-homogeneous subgroup of $\text{AGL}_1(\mathbb{F}_q)$. Moreover*

$$B_f(z, w) = c \Phi(z, w) \Phi(w, z), \quad \deg_z \Phi = \deg_w \Phi = (q - 1)/2, \quad (4.4)$$

with the two absolute factors distinct and irreducible.

Proof. Apply Kantor's classification [14] to the monodromy criterion in Proposition 3.1. A 2-homogeneous group which is not 2-transitive has two nontrivial ordered-pair orbits, interchanged by the flip. $\square$

Remark 4.9. In nonprime degree, the classification does not assert containment of the full affine group with square multipliers. Semilinear subgroups require separate treatment. In prime degree there are no field automorphisms and the point stabiliser must be precisely the subgroup of squares; this case is settled in Corollary 8.2.

5. INVARIANT FACTORISATION FOR POLYNOMIAL MAPS

Let $P \in \mathbb{C}[t]$ have degree $k \geq 2$. Since $B_P(z, w)$ is symmetric there is a unique $\overline{B}_P \in \mathbb{C}[u, v]$ with $B_P(z, w) = \overline{B}_P(z + w, zw)$. We assign weights $\text{wt}(u) = 1$, $\text{wt}(v) = 2$, so that the weighted degree of $\overline{B}_P$ is $k - 1$. Its *final invariant factorisation type* is the multiset of weighted degrees of its irreducible factors in $\mathbb{C}[u, v]$. We display this multiset as a tuple; its ordering is immaterial. An unordered-pair orbit may combine two paired orbitals, so invariant and absolute factorisation need not coincide.

5.1. The complete criterion. We use the normalisation of Dickson polynomials determined by $D_n(x + a/x, a) = x^n + (a/x)^n$. We will use the following elementary observation.

Lemma 5.1. *Let $\varphi(u) \in \mathbb{C}[u]$ and $\lambda \in \mathbb{C}^*$. Then $\varphi(u) - \lambda v$ is irreducible in $\mathbb{C}[u, v]$.*

Proof. It has degree one in v , and its content as a polynomial in v over $\mathbb{C}[u]$ is $\gcd(\varphi, \lambda) = 1$. Apply Gauss's lemma. $\square$

Theorem 5.2 (Invariant polynomial criterion). *Let $P \in \mathbb{C}[t]$ have degree $k \geq 3$. The invariant Bezoutian $\overline{B}_P$ is reducible in $\mathbb{C}[u, v]$ if and only if either P is decomposable or $k = p \geq 5$ is prime and P is linearly equivalent to $D_p(t, a)$.*

Proof. If $P = G \circ H$ then

$$B_P(z, w) = B_H(z, w) B_G(H(z), H(w)), \quad (5.1)$$

and both factors are nonconstant and symmetric.

Conversely, invariant reducibility implies absolute reducibility. Fried's classification of reducible polynomial difference quotients [8, 22], stated explicitly in [15, Theorem 6], says that an indecomposable polynomial with

reducible B_P is linearly equivalent to a prime-degree cyclic or dihedral polynomial; in the above normalisation these are included in $D_p(t, a)$, with the cyclic case $a = 0$. The cubic family does not split in the invariant ring: after affine changes $P = t^3 + at$, and $B_P = z^2 + zw + w^2 + a$ becomes $u^2 - v + a$, irreducible by Lemma 5.1. This is the smallest instance of the discrepancy between the two rings, since $B_{t^3} = (z - \omega w)(z - \omega^2 w)$ splits.

For $p \geq 5$ let ζ be a primitive p th root of unity and set $\alpha_j = \zeta^j + \zeta^{-j}$, $\beta_j = \zeta^j - \zeta^{-j}$. The Dickson identity gives

$$\frac{D_p(z, a) - D_p(w, a)}{z - w} = \prod_{j=1}^{(p-1)/2} (z^2 - \alpha_j zw + w^2 + \beta_j^2 a). \quad (5.2)$$

Each factor is symmetric and becomes $u^2 + \beta_j^2 a - (\alpha_j + 2)v$. Here $\alpha_j + 2 \neq 0$, because $\alpha_j = -2$ would force $\zeta^j = -1$, impossible for odd p ; so Lemma 5.1 applies and each factor is irreducible in $\mathbb{C}[u, v]$. The α_j are pairwise distinct for $1 \leq j \leq (p-1)/2$, so the factors are pairwise non-associate, and there are at least two of them when $p \geq 5$. $\square$

For $a = 0$, that is for $P = t^p$, the quadratic factors arise by combining the linear factors $z - \zeta^j w$ in the pairs $\{j, p-j\}$; for $a \neq 0$ the corresponding conics are already irreducible over $\mathbb{C}[z, w]$ and the orbitals are self-paired.

5.2. The normalised slice and the composition loci. Affine changes in the variable act by automorphisms of $\mathbb{C}[u, v]$. It is therefore natural to take the centred monic slice with zero constant term

$$\tilde{\mathcal{P}}_k = \{t^k + a_{k-2}t^{k-2} + \cdots + a_1t\} \simeq \mathbb{A}^{k-2}.$$

This slice removes translations on both sides but not all of linear equivalence: a residual $\mathbb{G}_m$ acts by

$$P(t) \mapsto \alpha^{-k} P(\alpha t), \quad \alpha \in \mathbb{C}^*, \quad (5.3)$$

and every locus considered below is invariant under it. On the Dickson curve it acts by $a \mapsto \alpha a^{-2}$. For $k = rs$ with $r, s > 1$ define $\mathcal{D}_{r,s}$ as the image of

$$H(t) = t^s + c_{s-2}t^{s-2} + \cdots + c_1t, \quad G(t) = t^r + b_{r-1}t^{r-1} + \cdots + b_1t, \quad (5.4)$$

under $(G, H) \mapsto G \circ H$.

Proposition 5.3 (Composition loci). *The normalised composition map is a closed embedding*

$$\iota_{r,s} : \tilde{\mathcal{G}}_r \times \tilde{\mathcal{H}}_s \simeq \mathbb{A}^{r-1} \times \mathbb{A}^{s-2} \hookrightarrow \tilde{\mathcal{P}}_k,$$

where $\tilde{\mathcal{G}}_r$ and $\tilde{\mathcal{H}}_s$ are the spaces of polynomials G and H in (5.4). Consequently

$$\mathcal{D}_{r,s} \simeq \mathbb{A}^{r+s-3}, \quad \text{codim } \mathcal{D}_{r,s} = (r-1)(s-1),$$

and a point of $\mathcal{D}_{r,s}$ has a unique decomposition of the form (5.4).

Proof. The coefficients of G and H are recovered polynomially and triangularly from those of $P = G \circ H$. For $2 \leq j \leq s-1$,

$$[t^{rs-j}]P = r c_{s-j} + F_j(c_{s-2}, \dots, c_{s-j+1}),$$

because every H^q with $q < r$ has degree at most $rs - s$. This successively recovers $c_{s-2}, \dots, c_1$. Once H is known, the coefficients of degrees $s(r -$

$1), s(r-2), \dots, s$ successively recover $b_{r-1}, \dots, b_1$. Applying these recovery formulas to an arbitrary P produces candidates G_P, H_P polynomially in the coefficients of P ; the image is the common zero set of the coefficients of $P - G_P \circ H_P$, so it is closed. The dimension and codimension follow from $\dim \tilde{\mathcal{P}}_k = rs - 2$. $\square$

Theorem 5.4 (Generic factorisation). *There is a dense open subset $\mathcal{D}_{r,s}^\circ \subset \mathcal{D}_{r,s}$ on which $\text{Mon}(P) = S_s \wr S_r$ in its natural action on r blocks of size s . For $P = G \circ H \in \mathcal{D}_{r,s}^\circ$ the two factors in (5.1) are irreducible in $\mathbb{C}[u, v]$, of weighted degrees $s-1$ and $rs-s$.*

Proof. Choose G and H with simple critical points and distinct critical values, and impose the open condition that critical values from the two levels do not collide after composition. Then G and H have monodromy S_r and S_s . In particular H is indecomposable. For a simple critical point v of H , the same noncollision condition ensures that the fibre of $G \circ H$ over $G(H(v))$ contains exactly one ramification point, of local degree two. Thus Lemma 4.3 applies and gives $\text{Mon}(G \circ H) = S_s \wr S_r$. This wreath product has exactly two orbits on unordered pairs: pairs in one block and pairs in distinct blocks. By Proposition 3.1 the invariant Bezoutian has exactly two irreducible factors; the same-block factor is B_H , of degree $s-1$ in either variable, and the complementary factor has degree $rs-s$. $\square$

Corollary 5.5 (Complete normalised locus). *Let $\tilde{\mathcal{R}}_k \subset \tilde{\mathcal{P}}_k$ be the locus on which $\bar{B}_P$ is reducible. Then*

$$\tilde{\mathcal{R}}_k = \bigcup_{\substack{k=rs \\ r,s>1}} \mathcal{D}_{r,s} \cup \mathcal{X}_k, \quad (5.5)$$

where $\mathcal{X}_k = \{D_k(t, a) - D_k(0, a) : a \in \mathbb{C}\}$ if $k \geq 5$ is prime and is empty otherwise. In composite degree the normalised Dickson family $\{D_k(t, a) - D_k(0, a) : a \in \mathbb{C}\}$ is already contained in the composition union. The displayed sets are precisely the irreducible components.

Proof. The set-theoretic assertion is Theorem 5.2. Any decomposition of a centred monic polynomial with zero constant term can be put uniquely in the normalisation (5.4): monicity normalises the right component, the coefficient of t^{rs-1} centres it, and constants are moved from the right component to the left. In prime degree, affine equivalence and the same normalisation reduce the exceptional family to $D_p(t, a)$.

Each $\mathcal{D}_{r,s}$ is closed and irreducible by Proposition 5.3. A generic member has monodromy $S_s \wr S_r$ by Theorem 5.4, whose only nontrivial system of imprimitivity has blocks of size s by Lemma 4.2; since polynomial decompositions correspond to systems of imprimitivity, such a member lies in no other $\mathcal{D}_{r',s'}$, and therefore no $\mathcal{D}_{r,s}$ is contained in a distinct $\mathcal{D}_{r',s'}$. Finally, for $k = rs$, put $c = D_s(0, a)$ and use

$$H(t) = D_s(t, a) - c, \quad G(y) = D_r(y + c, a^s) - D_r(c, a^s).$$

The Dickson composition identity gives $G \circ H = D_{rs}(t, a) - D_{rs}(0, a)$, with both factors in the required normalisation. $\square$

If p is the smallest prime divisor of a composite k , the components of largest dimension correspond to the most unbalanced factorisations and give

$$\dim \tilde{\mathcal{R}}_k = k/p + p - 3, \quad \text{codim } \tilde{\mathcal{R}}_k = (p-1)(k/p-1).$$

5.3. The completely split Dickson family.

Proposition 5.6. *For $k \geq 3$ put $P_{k,a} = D_k(t, a) - D_k(0, a)$. Its final invariant factorisation type is $(2, \dots, 2)$ for k odd and $(1, 2, \dots, 2)$ for k even. More precisely, with ζ a primitive k th root of unity and α_j, β_j as above,*

$$\bar{B}_{P_{k,a}}(u, v) = \begin{cases} \prod_{j=1}^{(k-1)/2} (u^2 - (\alpha_j + 2)v + \beta_j^2 a), & k \text{ odd}, \\ u \prod_{j=1}^{(k-2)/2} (u^2 - (\alpha_j + 2)v + \beta_j^2 a), & k \text{ even}. \end{cases}$$

Proof. This is the Dickson difference identity (5.2); in even degree the self-paired root -1 contributes the additional factor $z + w = u$. Every quadratic displayed is linear in v with nonzero v -coefficient, hence irreducible in $\mathbb{C}[u, v]$. $\square$

6. THE COMPOSITION SCROLL AND ITS DEGREE

The components of $\tilde{\mathcal{R}}_k$ are affine spaces, so it is natural to ask for their degrees in the ambient projective space. We compute them here. Throughout, $k = rs$ with $r, s \geq 2$, and $\mathbf{P}^{k-2}$ is the projective closure of $\tilde{\mathcal{P}}_k \simeq \mathbb{A}^{k-2}$ obtained as follows. Let

$$W = \langle t, t^2, \dots, t^{k-2}, t^k \rangle \subset \mathbb{C}[t], \quad \dim W = k-1,$$

and identify $\mathbf{P}^{k-2} = \mathbf{P}(W)$; then $\tilde{\mathcal{P}}_k$ is the affine chart in which the coefficient of t^k equals one. We write $\bar{\mathcal{D}}_{r,s}$ and $\bar{\mathcal{X}}_p$ for the closures of the composition locus and of the Dickson curve.

6.1. The scroll. Give $\mathbf{P}^{s-2}$ the homogeneous coordinates $(c_0 : c_1 : \dots : c_{s-2})$ and put

$$\hat{H}_c(t) = c_0 t^s + c_{s-2} t^{s-2} + \dots + c_1 t,$$

the homogenisation of the right-hand normalisation (5.4).

Lemma 6.1. *For every $c \in \mathbf{P}^{s-2}$ and every $1 \leq j \leq r$ one has $\hat{H}_c^j \in W$, and the polynomials $\hat{H}_c, \hat{H}_c^2, \dots, \hat{H}_c^r$ are linearly independent. Consequently*

$$E := \mathcal{O}(-1) \oplus \dots \oplus \mathcal{O}(-r) \xrightarrow{\varepsilon} W \otimes \mathcal{O}_{\mathbf{P}^{s-2}},$$

$$\varepsilon(j\text{-th summand}) = \hat{H}_c^j,$$

is a fibrewise injective map of vector bundles, and E is a rank r subbundle of the trivial bundle $W \otimes \mathcal{O}$.

Proof. The powers have zero constant term. For $j < r$ their degrees are at most $k-s \leq k-2$, while $\hat{H}_c^r$ has no t^{k-1} term because $\hat{H}_c$ has no t^{s-1} term. Thus all powers lie in W . They have distinct degrees, since $\hat{H}_c$ is nonconstant, and are therefore independent. Their coefficients are homogeneous of degree j in c , giving the indicated twists and a fibrewise injective bundle map. $\square$

Theorem 6.2 (Polynomial composition degree). *The morphism*

$$\Psi : \mathbf{P}(E) \longrightarrow \mathbf{P}(W) = \mathbf{P}^{k-2}$$

induced by ε is birational onto its image, which is $\overline{\mathcal{D}}_{r,s}$. Thus the closure is swept by $(r-1)$ -planes parametrised by $\mathbf{P}^{s-2}$, and

$$\begin{aligned} \deg \overline{\mathcal{D}}_{r,s} &= \int_{\mathbf{P}^{s-2}} s_{s-2}(E) = [h^{s-2}] \prod_{j=1}^r \frac{1}{1-jh} \\ &= h_{s-2}(1, 2, \dots, r) = S(r+s-2, r), \end{aligned}$$

where h_m is the complete homogeneous symmetric polynomial and $S(\cdot, \cdot)$ is the Stirling number of the second kind.

Proof. Over $c_0 = 1$, write $H = \widehat{H}_c$. The fibre is $\mathbf{P}\langle H, H^2, \dots, H^r \rangle$, whose intersection with the monic degree- k chart is

$$\left\{ H^r + \sum_{j=1}^{r-1} b_j H^j \right\}.$$

Conversely, an image polynomial of degree $k = rs$ must come from $c_0 \neq 0$ and have a nonzero coefficient of H^r . Hence the image in this chart is exactly $\mathcal{D}_{r,s}$. Properness gives the projective closure, and unique normalised decomposition (Proposition 5.3) gives birationality.

By Lemma 6.1, $\mathcal{O}_{\mathbf{P}(E)}(-1) \subset \pi^*E \subset W \otimes \mathcal{O}$, so $\Psi^*\mathcal{O}_{\mathbf{P}^{k-2}}(1) = \mathcal{O}_{\mathbf{P}(E)}(1)$; write ξ for its first Chern class. By the projection formula and birationality,

$$\deg \overline{\mathcal{D}}_{r,s} = \int_{\mathbf{P}(E)} \xi^{r+s-3} = \int_{\mathbf{P}^{s-2}} \pi_* (\xi^{(r-1)+(s-2)}) = \int_{\mathbf{P}^{s-2}} s_{s-2}(E),$$

where $s(E) = c(E)^{-1}$ is the Segre class [9, Ch. 3]. Since $c(E) = \prod_{j=1}^r (1-jh)$ with h the hyperplane class of $\mathbf{P}^{s-2}$, we get $s_{s-2}(E) = [h^{s-2}] \prod_{j=1}^r (1-jh)^{-1} = h_{s-2}(1, \dots, r)$. The identity $\sum_{m \geq 0} S(m+r, r)x^m = \prod_{j=1}^r (1-jx)^{-1}$ is the standard generating function for Stirling numbers of the second kind [21, Ch. 1], which identifies this coefficient with $S(r+s-2, r)$. $\square$

Proposition 6.3. *For prime $p \geq 5$ the closure $\overline{\mathcal{X}}_p \subset \mathbf{P}^{p-2}$ of the Dickson curve is a rational curve of degree $(p-1)/2$.*

Proof. In the normalisation above, $D_p(t, a) = \sum_{i=0}^{(p-1)/2} \frac{p}{p-i} \binom{p-i}{i} (-a)^i t^{p-2i}$, so the coefficient of t^{p-2i} is a nonzero multiple of a^i . The parametrisation $a \mapsto D_p(t, a)$ is therefore given by a base-point-free linear series of forms of degrees $0, 1, \dots, (p-1)/2$ in a , and it is injective because the coefficient of t^{p-2} is $-pa$. Hence the closure is a rational normal curve in its linear span of degree $\max_i i = (p-1)/2$. $\square$

6.2. Consequences and examples.

Corollary 6.4. $\deg \overline{\mathcal{D}}_{r,2} = 1$; $\deg \overline{\mathcal{D}}_{2,s} = 2^{s-1} - 1$; $\deg \overline{\mathcal{D}}_{r,3} = \binom{r+1}{2}$.

Proof. These are $S(r, r)$, $S(s, 2)$ and $S(r+1, r)$. The first also follows directly: for $s = 2$ the right component is t^2 , so $\mathcal{D}_{r,2}$ is a linear subspace. $\square$

Example 6.5. For $(r, s) = (2, 3)$ one has

$$(t^3 + ct)^2 + b(t^3 + ct) = t^6 + 2ct^4 + bt^3 + c^2t^2 + bct.$$

These polynomials form an affine plane embedded nonlinearly in $\tilde{\mathcal{P}}_6$; its projective closure is a cubic ruled surface, since $S(3, 2) = 3$.

6.3. The rational composition locus. Let $k = rs$, with $r, s \geq 2$, and let $\mathcal{S}$ be the tautological rank-two bundle on $\mathrm{Gr}(2, V_s)$. The multiplication map $\mathcal{W} = \mathrm{Sym}^r \mathcal{S} \hookrightarrow V_k \otimes \mathcal{O}$ is fibrewise injective, including at pencils with base points, by the proof of Proposition 4.1. Hence it induces a morphism

$$\Psi : \mathrm{Gr}_2(\mathcal{W}) \longrightarrow \mathrm{Gr}(2, V_k).$$

Theorem 6.6 (Rational composition degree). *Let $\mathcal{C}_s$ be the locus of base-point-free degree- k pencils admitting a right factor of degree s . The morphism Ψ is birational onto its image, and*

$$\overline{\mathcal{C}}_s = \Psi(\mathrm{Gr}_2(\mathcal{W})), \quad \dim \overline{\mathcal{C}}_s = 2(r + s - 2), \quad \mathrm{codim} \overline{\mathcal{C}}_s = 2(r - 1)(s - 1).$$

Put $D = 2(r + s - 2)$, let x, y be formal Chern roots of $\mathcal{S}^*$, and define

$$w_i = (r - i)x + iy, \quad P_{r,s}(x, y) = \sum_{0 \leq i < j \leq r} \frac{(w_i + w_j)^D}{\prod_{l \neq i, j} (w_i - w_l)(w_j - w_l)}.$$

The sum is a symmetric polynomial of degree $2s - 2$. The Plücker degree is

$$\deg \overline{\mathcal{C}}_s = \int_{\mathrm{Gr}_2(\mathcal{W})} c_1(\det \mathcal{T}^*)^D = [x^s y^{s-1}] (x - y) P_{r,s}(x, y), \quad (6.1)$$

where $\mathcal{T}$ is the tautological rank-two subbundle.

Proof. By Proposition 4.1, the preimage of the base-point-free locus of $\mathrm{Gr}(2, V_k)$ maps onto $\mathcal{C}_s$. This preimage is a nonempty open subset of the irreducible Grassmann bundle $\mathrm{Gr}_2(\mathcal{W})$, so the full image is $\overline{\mathcal{C}}_s$. For general rational maps g, h with simple branching, the branch-value noncollision argument in Theorem 5.4 and Lemma 4.3 gives $\mathrm{Mon}(g \circ h) = S_s \wr S_r$. By Lemma 4.2 such a composite has a unique degree- s right factor up to post-composition. Its outer pencil is then uniquely recovered inside $\mathrm{Sym}^r \langle A, B \rangle$. Thus Ψ is generically injective and birational onto its image. The dimension is the sum of the base dimension $2(s - 1)$ and fibre dimension $2(r - 1)$; the codimension follows.

The Plücker bundle pulls back to $\det \mathcal{T}^*$. The Grassmann-bundle Gysin formula [10, Ch. 4] sends $c_1(\det \mathcal{T}^*)^D$ to the displayed symmetric sum $P_{r,s}$; the apparent denominators cancel because this is a characteristic polynomial in the Chern roots. On $\mathrm{Gr}(2, s + 1)$, integration of a symmetric polynomial P of degree $2s - 2$ is $[x^s y^{s-1}](x - y)P$, as follows by expanding in Schur polynomials. The projection formula proves (6.1). $\square$

Remark 6.7 (Recovering the polynomial scroll). Pencils containing $L = t_0^s$ form $\mathbf{P}(V_s / \langle L \rangle) \simeq \mathbf{P}^{s-1}$. Imposing the centred condition selects $\mathbf{P}^{s-2}$, with complementary line represented by the homogenisation of $\hat{H}_c$. On this subspace,

$$\mathcal{S} = \mathcal{O} \oplus \mathcal{O}(-1), \quad \mathrm{Sym}^r \mathcal{S} = \mathcal{O} \oplus E.$$

One must also require the output two-plane to contain L^r . Those two-planes form $\mathbf{P}(E)$, recovering exactly Theorem 6.2. Thus both the centred hyperplane and the output constant line are needed to recover the polynomial model from the rational Grassmann bundle.

The first values of (6.1) are as follows; the centred polynomial model, for comparison, has degrees $S(r + s - 2, r)$, equal to 1, 3, 7, 15 in the first row.

$\deg \overline{\mathcal{C}}_s$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
$r = 2$	21	370	6 125	98 406
$r = 3$	240	18 060	1 099 700	59 295 852
$r = 4$	2 030	444 150	71 640 800	9 604 985 390
$r = 5$	14 490	7 364 280	2 590 487 900	723 573 342 492

Remark 6.8. The $r = 2$ row admits an independent derivation: for $r = 2$ the bundle $\mathcal{W}$ has rank three, so $\mathrm{Gr}_2(\mathcal{W}) \cong \mathbf{P}(\mathcal{W}^*)$ and $\xi = -c_1(\mathcal{W}) + \zeta$ with $p_*(\zeta^j) = s_{j-2}(\mathcal{W}^*)$ a Segre class, which reproduces 21, 370, 6125, 98406. For $(r, s) = (2, 2)$, on $\mathrm{Gr}(2, 3) \simeq \mathbf{P}^2$ one has $c_1(\mathcal{W}^*) = 3\sigma_1$ and $c_2(\mathcal{W}^*) = 6\sigma_1^2$, giving $\binom{4}{2} \cdot 9 - \binom{4}{3} \cdot 9 + 3 = 21$ directly. A closed form for (6.1), and more generally the Chow class of $\overline{\mathcal{C}}_s$ in $\mathrm{Gr}(2, V_k)$, would be the natural sequel; on $\mathrm{Gr}_2(\mathrm{Sym}^r S)$ the latter is a finite Schubert computation.

7. RITT GEOMETRY AND INTERSECTIONS

The components in (5.5) are individually affine spaces. Their intersections record polynomials with two inequivalent decompositions, and Ritt theory gives a conceptual description without coefficient elimination.

For an ordered factorisation $\mathbf{d} = (d_1, \dots, d_\nu)$ of k , with all $d_i \geq 2$, let $\mathcal{C}_{\mathbf{d}}$ be the normalised locus of polynomials $P = F_1 \circ \dots \circ F_\nu$ with monic components with zero constant term and centred innermost component.

Proposition 7.1 (Composition chains). *The normalised composition map is a closed embedding and $\mathcal{C}_{\mathbf{d}} \simeq \mathbb{A}^{d_1 + \dots + d_\nu - \nu - 1}$. Generically its monodromy is the corresponding iterated wreath product, and the final invariant factorisation type is*

$$d_\nu - 1, \quad d_\nu(d_{\nu-1} - 1), \quad \dots, \quad d_\nu \cdots d_2(d_1 - 1). \quad (7.1)$$

Proof. The parameter count is $(d_\nu - 2) + \sum_{i < \nu} (d_i - 1)$. Repeated triangular recovery, as in Proposition 5.3, proves that the map is a closed embedding; at intermediate stages one also recovers the next-to-leading coefficient, without imposing centring. For generic monodromy, apply Lemma 4.3 successively to an outer composite and a new innermost factor with simple, noncolliding critical values. This gives the iterated wreath product by induction. Orbits on unordered pairs are classified by the first level of the nested block chain at which two points separate; iterating (5.1) gives the degrees (7.1). $\square$

The gcd reduction underlying all pairwise intersections is usually quoted from the Ritt–Engstrom theory. For polynomials it has a short self-contained proof, which we give because it also explains why the decomposition lattice is so rigid.

Lemma 7.2 (Divisor lattice). *Let $P \in \mathbb{C}[t]$ have degree k and let $G = \text{Mon}(P)$ act on a generic fibre Ω . Then the G -block systems on Ω correspond to a subset $S \subseteq \text{Div}(k)$ containing 1 and k and closed under gcd and lcm. In particular P has at most one right component of each degree up to affine postcomposition, and any two right components generate, by gcd and lcm, further right components.*

Proof. Since P is totally ramified over ∞ , the branch cycle there is a k -cycle $\sigma \in G$. Identify Ω with $\mathbb{Z}/k$ so that σ acts by $+1$. A G -block system is in particular a $\langle \sigma \rangle$ -block system, hence a partition into cosets of a subgroup of $\mathbb{Z}/k$, and there is exactly one subgroup of each order $s \mid k$. The meet and the join of two G -congruences are G -congruences, and on coset partitions of $\mathbb{Z}/k$ these are the coset partitions of the intersection and of the generated subgroup, that is of orders gcd and lcm. Blocks of size s correspond to right components of degree s by Lüroth's theorem. They can be chosen polynomial, since the unique point above ∞ for P remains unique in every intermediate cover. $\square$

Theorem 7.3 (Pairwise intersections). *Suppose $(r, s) \neq (r', s')$, $rs = r's' = k$, and $P \in \mathcal{D}_{r,s} \cap \mathcal{D}_{r',s'}$. Put $a = \gcd(r, r')$ and $b = \gcd(s, s')$. There are coprime positive integers m, n with*

$$r = am, \quad r' = an, \quad s = bn, \quad s' = bm,$$

and, after normalisation, $P = A \circ Q \circ B$ with $\deg A = a$, $\deg B = b$, where Q has two decompositions with the coprime degree patterns (m, n) and (n, m) . Conversely every such polynomial lies in the intersection.

If $m, n > 1$, Ritt's second theorem says that, up to the linear changes absorbed by A and B , the core Q is of at least one of the following two types:

(i) *a power collision*

$$Q_W(t) = t^{e\ell} W(t^\ell)^\ell, \quad M = d\ell + e, \quad 1 \leq e < \ell, \quad \gcd(e, \ell) = 1,$$

where $\ell = \min(m, n)$, $M = \max(m, n)$ and W is monic of degree d ;

(ii) *a Dickson collision $Q_c(t) = D_{mn}(t, c) - D_{mn}(0, c)$.*

The normalised families meet at t^{mn} , obtained from $c = 0$ and $W(y) = y^d$. For $\ell = 2$ the Dickson family lies in the power family (Proposition 7.5). For $\ell \geq 3$ they meet only at t^{mn} : the coefficient of t^{mn-2} is zero in the power family and $-mnc$ in the Dickson family. If one of m, n equals one, the intersection is instead the composition-chain locus obtained by inserting the quotient of the two comparable degrees.

Proof. First the arithmetic. Write $r = am$ and $r' = an$ with $\gcd(m, n) = 1$. From $rs = r's'$ we get $ms = ns'$, so $n \mid s$; writing $s = bn$ gives $s' = bm$ and $\gcd(s, s') = b\gcd(n, m) = b$, as claimed.

Now the reduction. By Lemma 7.2 the block systems of sizes s and s' generate ones of sizes $\gcd(s, s') = b$ and $\text{lcm}(s, s') = bmn$. The first yields a common right component B of degree b , and the second a common left component A of degree $k/\text{lcm}(s, s') = abmn/bmn = a$. Writing $P = A \circ Q \circ B$, the two given decompositions become two decompositions of Q with the coprime patterns (m, n) and (n, m) . The converse follows by composition.

When both core degrees exceed one, Ritt's second theorem in normalised form gives the power and Dickson alternatives [20, 26, 23, 25]; coprimality of e and ℓ is equivalent to $\gcd(M, \ell) = 1$, that is to $\gcd(m, n) = 1$. When one core degree is one, no collision remains and the two decompositions are the two coarsenings of a three-step chain. $\square$

For example, if $s \mid s'$ and $q = s'/s = r/r'$, then

$$\mathcal{D}_{r,s} \cap \mathcal{D}_{r',s'} = \mathcal{C}_{(r',q,s)}, \quad (7.2)$$

a smooth affine space of dimension $r' + q + s - 4$ with generic type $(sq(r' - 1), s(q - 1), s - 1)$.

Corollary 7.4 (Singular locus). *For $k \geq 3$, give $\tilde{\mathcal{R}}_k$ its reduced structure. In composite degree,*

$$\text{Sing}(\tilde{\mathcal{R}}_k) = \bigcup_{(r,s) \neq (r',s')} \mathcal{D}_{r,s} \cap \mathcal{D}_{r',s'}, \quad rs = r's' = k.$$

The variety $\tilde{\mathcal{R}}_k$ is smooth precisely when k is prime or the square of a prime; the empty degree-three locus is smooth.

Proof. Every component $\mathcal{D}_{r,s}$ is smooth. A point on only one component has a neighbourhood in the reduced union equal to a neighbourhood in that component; a point on two distinct components has a local ring with at least two minimal primes and is singular. For composite k there is exactly one proper ordered factorisation precisely when $k = p^2$, and then $\tilde{\mathcal{R}}_{p^2} = \mathcal{D}_{p,p}$ is an affine space. For $k = p \geq 5$ prime we have $\tilde{\mathcal{R}}_p = \mathcal{X}_p \simeq \mathbb{A}^1$, which is smooth, and $\tilde{\mathcal{R}}_3 = \emptyset$. All components meet at t^k , so for composite $k \neq p^2$ the singular locus is nonempty. $\square$

The most useful explicit incomparable intersection is the following.

Proposition 7.5 (The swapped $(m, 2)$ intersection). *Let $m = 2d + 1 \geq 3$. Then*

$$\mathcal{D}_{m,2} \cap \mathcal{D}_{2,m} = \{P_W(t) = t^2 W(t^2)^2 : W \text{ monic of degree } d\} \simeq \mathbb{A}^d.$$

Writing $H_W(t) = tW(t^2)$, one has

$$B_{P_W}(z, w) = (z + w) \frac{H_W(z) - H_W(w)}{z - w} \frac{H_W(z) + H_W(w)}{z + w}.$$

This intersection contains the one-parameter Dickson subfamily, with the completely split type of Proposition 5.6.

Proof. Membership in $\mathcal{D}_{m,2}$ means that P is even. If also $P = H^2 + bH \in \mathcal{D}_{2,m}$, set $K = H + b/2$. Then $K(-t)^2 = K(t)^2$. Since m is odd, comparison of leading terms forces $K(-t) = -K(t)$; hence $K(0) = b/2 = 0$. Thus $H = tW(t^2)$ and $P = t^2 W(t^2)^2$. The converse is immediate, and triangular recovery from the square shows that this parametrisation is a closed embedding of $\mathbb{A}^d$. The displayed factorisation follows by taking the difference of the two squares. Finally, $D_{2m}(t, c) - D_{2m}(0, c) = D_m(t, c)^2$, and $D_m(t, c)$ is odd, proving the Dickson inclusion directly. $\square$

For $k = 10$ this yields

$$P_{A,B}(t) = t^2(t^4 + At^2 + B)^2 \quad (7.3)$$

and

$$\begin{aligned} \overline{B}_{P_{A,B}}(u, v) &= u[u^4 - 3u^2v + v^2 + A(u^2 - v) + B] \\ &\quad \times [u^4 - 5u^2v + 5v^2 + A(u^2 - 3v) + B]. \end{aligned}$$

Generically the type is $(4, 4, 1)$. The two quartics split simultaneously exactly when $B = A^2/5$; this curve is the Dickson subfamily and has type $(2, 2, 2, 2, 1)$. Indeed, the two quartics are quadratic in v , with discriminants

$$5u^4 + 2Au^2 + A^2 - 4B, \quad 5u^4 + 10Au^2 + 9A^2 - 20B,$$

and either is a square in $\mathbb{C}[u]$ precisely when $B = A^2/5$. Gauss's lemma then gives the assertion, and $A = -5c$, $B = 5c^2$ identifies the curve with the Dickson family.

8. PRIME ORBIT SIZE AND ELLIPTIC CORRESPONDENCES

The orbital dictionary is especially explicit when the generic orbit size is an odd prime p . A transitive group of prime degree is primitive; Burnside's theorem says that it is either 2-transitive or contained in the affine group on $\mathbb{F}_p$, and the genus-zero condition then leaves only Euclidean branch data. For $M \leq \mathbb{F}_p^*$ put $\text{AGL}_1(M) = \{x \mapsto ax + b : a \in M, b \in \mathbb{F}_p\}$.

Theorem 8.1 (Prime-degree residual curves). *Let $f : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ have prime degree $p \geq 3$. If $\text{Mon}(f)$ is not 2-transitive, then $\text{Mon}(f) = \text{AGL}_1(M)$ with $m = |M| \in \{1, 2, 3, 4, 6\}$, and f is Möbius equivalent to one of the following maps:*

m	signature of $Y \rightarrow Y/G$	genus of Y	map
1	(p, p)	0	the power map z^p
2	$(p, 2, 2)$	0	a Dickson map
2	$(2, 2, 2, 2)$	1	a degree- p isogeny on $E/\{\pm 1\}$
3	$(3, 3, 3)$	1	a hexagonal CM isogeny quotient
4	$(2, 4, 4)$	1	a square CM isogeny quotient
6	$(2, 3, 6)$	1	a hexagonal CM isogeny quotient

The table lists the signature of the Galois closure $Y \rightarrow Y/G$; the branch data of f itself is read off from it, a branch cycle of multiplier order d having cycle type $(d, \dots, d, 1)$ and a translation having type (p) . The CM cases exist only when $p \equiv 1 \pmod{m}$. Under the standing non-2-transitivity hypothesis one also has $m < p - 1$; thus the formal possibilities $(p, m) = (3, 2), (5, 4), (7, 6)$ belong instead to the 2-transitive case. There are $(p - 1)/m$ absolute components, all of bidegree (m, m) . A component is symmetric if and only if $-1 \in M$, equivalently if m is even.

Proof. If $G = \text{Mon}(f)$ is not 2-transitive, Burnside's theorem for primitive groups of prime degree identifies the fibre with $\mathbb{F}_p$ and gives $G = \text{AGL}_1(M)$ [24, Theorem 11.7]. Its orbitals are indexed by the nonzero difference $y - x$ modulo M ; hence there are $(p - 1)/m$ orbitals, each with suborbit length m , and they are self-paired exactly when $-1 \in M$. The action is 2-transitive exactly when $M = \mathbb{F}_p^*$, so the present hypothesis is equivalent to $m < p - 1$.

Let σ_i be the branch cycles. A translation has index $p - 1$; an element with multiplier of order $d > 1$ fixes one point and has index $(p - 1)(1 - 1/d)$. Riemann–Hurwitz, divided by $p - 1$, becomes

$$c + \sum_i (1 - 1/d_i) = 2,$$

where c is the number of translation branch cycles. Its solutions are

$$\begin{aligned} c &= 2; \\ c &= 1, \quad (d_i) = (2, 2); \\ c &= 0, \quad (d_i) \in \{(2, 2, 2, 2), (3, 3, 3), (2, 4, 4), (2, 3, 6)\}. \end{aligned}$$

The branch cycles generate G , so their multipliers generate M and $m = \text{lcm}(d_i)$, with the empty lcm equal to one. Thus $m \in \{1, 2, 3, 4, 6\}$. The first two cases have genus-zero Galois closure and give the cyclic and dihedral maps. The remaining Euclidean signatures have elliptic Galois closure; compare the standard description of Lattès maps in [17]. The normal subgroup C_p acts by translation by a p -torsion point, so its quotient is a degree- p isogeny $E \rightarrow E'$. The point stabiliser C_m acts by automorphisms fixing the origin, and passage to $E/C_m \simeq \mathbf{P}^1$ gives the last four families. Orders 3, 4, 6 force $j(E) = 0, 1728, 0$ respectively, and compatibility with a cyclic kernel of order p gives $p \equiv 1 \pmod{m}$. The component counts and bidegrees follow from Proposition 3.1. $\square$

Combining this with Theorem 3.2 gives a concise classification of prime-orbit $\mathcal{FIO}$ -curves. If the monodromy is 2-transitive, the only nontrivial orbital is the full residual relation, a symmetric curve of bidegree $(p-1, p-1)$. Otherwise the possibilities are the six rows above. The only non-symmetric irreducible cases are graphs of Möbius transformations of order p and, for $p \equiv 1 \pmod{3}$, bidegree $(3, 3)$ orbitals of the hexagonal CM family. The same table settles, in prime degree, when absolute and invariant reducibility genuinely differ: by Corollary 8.2 this happens only for $p = 3$ and $p = 7$.

Corollary 8.2 (Prime-degree discrepancy). *Let f have prime degree p . Its absolute Bezoutian is reducible while its invariant Bezoutian is irreducible precisely in the following cases, up to independent source and target Möbius transformations:*

- (a) $p = 3$ and f is the cyclic map z^3 ;
- (b) $p = 7$ and f is induced by a degree-seven isogeny between elliptic curves with $j = 0$, after quotienting by their order-three automorphisms.

In (b) the kernel is invariant under the order-three automorphism, and the monodromy is $C_7 \rtimes C_3$. The assertion describes the family; it does not assert a unique Möbius-equivalence class.

Proof. In prime degree, a 2-homogeneous but not 2-transitive affine group has multiplier subgroup M with $M \cup (-M) = \mathbb{F}_p^*$ and $-1 \notin M$. Hence $|M| = (p - 1)/2$ is odd. The list in Theorem 8.1 leaves $|M| = 1$ or 3, giving $p = 3$ or 7 and the stated maps. Conversely these two groups are transitive on unordered pairs and have two paired nontrivial orbitals. The degree-seven case exists because the order-three automorphism on the seven-torsion

of a $j = 0$ elliptic curve has invariant one-dimensional eigenspaces over $\mathbb{F}_7$. Quotienting by either such kernel gives the required equivariant isogeny. $\square$

The $m = 2$ elliptic family admits a useful equation. Let $E : y^2 = x^3 + Ax + B$ and assume $4A^3 + 27B^2 \neq 0$, and let $C = \{O, \pm T_1, \dots, \pm T_{(p-1)/2}\} \subset E$ be a subgroup of order p . The isogeny $E \rightarrow E/C$ induces a degree p map f on the quotients by ± 1 .

Theorem 8.3 (Elliptic splitting). *Write $x_j = x(T_j)$. In coordinates in which $f = P/Q$,*

$$B_{P,Q}(z, w) = c \prod_{j=1}^{(p-1)/2} \Phi_{x_j}(z, w), \quad c \in \mathbb{C}^*,$$

where

$$\Phi_{x_0}(z, w) = (zw + x_0(z + w) - A)^2 - 4(z + w + x_0)(x_0zw + B). \quad (8.1)$$

Each factor is an irreducible symmetric curve of bidegree $(2, 2)$; its correspondence sends $x(R)$ to $x(R \pm T_j)$, so its generic orbit has p elements.

Proof. The fibre through $x(R)$ is $\{x(R + T) : T \in C\}$. Pairing T_j with $-T_j$ gives $(p-1)/2$ symmetric relations $x(R) \leftrightarrow x(R \pm T_j)$, each of bidegree $(2, 2)$; their total degree in either variable is $p-1$, so they exhaust the reduced residual fibre curve.

It remains to compute one equation. Three points of E with abscissas z, w, x_0 are collinear if and only if

$$x^3 + Ax + B - (\lambda x + \nu)^2 = (x - z)(x - w)(x - x_0)$$

for some λ, ν . Comparing coefficients gives

$$\lambda^2 = z + w + x_0, \quad 2\lambda\nu = A - zw - x_0(z + w), \quad \nu^2 = x_0zw + B,$$

and eliminating λ, ν yields (8.1). Collinearity is equivalent to $R \pm R' \pm T_j = O$, so this is precisely the required relation. The coefficient of z^2 in Φ_{x_0} is $(w - x_0)^2$, so the bidegree really is $(2, 2)$. Irreducibility holds because, under Proposition 3.1, the component corresponds to the orbital consisting of the pairs whose difference lies in the class $\{\pm T_j\}$, and $\text{AGL}_1(\{\pm 1\})$ is transitive on that class. $\square$

These are classical Poncelet–QRT biquadratics [6, 19, 11]. In a nodal degeneration where the cyclic kernel specialises to μ_p in the smooth multiplicative group, the isogeny becomes a power map on $\mathbb{G}_m$. Passing to the quotients by inversion gives the Dickson map and, in the corresponding quotient coordinate, the quadratic factors of Proposition 5.6. This identifies the Dickson family as a multiplicative boundary of the elliptic finite-orbit family.

The derivation of (8.1) applies to any $T \notin E[2]$, without a torsion hypothesis. Its image is irreducible because it is parametrised by E ; its generic projection degree is two because $x(R + T)$ and $x(R - T)$ are distinct. Infinite-order steps and their degenerations are discussed in Section A.

9. SMALL DEGREES

We now specialise the general results; no degree-by-degree elimination is needed. In the table, $\dim/\text{codim}$ is measured in $\tilde{\mathcal{P}}_k \simeq \mathbb{A}^{k-2}$, “type” lists the weighted degrees of the generic final invariant factors on the indicated component, and the last column gives the degree of its projective closure from Theorem 6.2 and Proposition 6.3.

k	component of $\tilde{\mathcal{R}}_k$	$\dim/\text{codim}$	generic type	degree
3	empty	–	–	–
4	$\mathcal{D}_{2,2}$	1/1	(2, 1)	1
5	$\mathcal{X}_5$	1/2	(2, 2)	2
6	$\mathcal{D}_{3,2}$	2/2	(4, 1)	1
	$\mathcal{D}_{2,3}$	2/2	(3, 2)	3
7	$\mathcal{X}_7$	1/4	(2, 2, 2)	3
8	$\mathcal{D}_{4,2}$	3/3	(6, 1)	1
	$\mathcal{D}_{2,4}$	3/3	(4, 3)	7
9	$\mathcal{D}_{3,3}$	3/4	(6, 2)	6
10	$\mathcal{D}_{5,2}$	4/4	(8, 1)	1
	$\mathcal{D}_{2,5}$	4/4	(5, 4)	15
11	$\mathcal{X}_{11}$	1/8	(2, 2, 2, 2, 2)	5
12	$\mathcal{D}_{6,2}$	5/5	(10, 1)	1
	$\mathcal{D}_{4,3}$	4/6	(9, 2)	10
	$\mathcal{D}_{3,4}$	4/6	(8, 3)	25
	$\mathcal{D}_{2,6}$	5/5	(6, 5)	31

All rows follow from Corollary 5.5 and Theorem 5.4: the prime rows are Dickson curves, and the composite rows are composition components with generic type $(rs - s, s - 1)$ and degree $S(r + s - 2, r)$.

For reference, the principal intersection strata in the table are as follows.

k	intersection	dimension and generic type
6	$\mathcal{D}_{3,2} \cap \mathcal{D}_{2,3}$	1; (2, 2, 1)
8	$\mathcal{D}_{4,2} \cap \mathcal{D}_{2,4} = \mathcal{C}_{(2,2,2)}$	2; (4, 2, 1)
10	$\mathcal{D}_{5,2} \cap \mathcal{D}_{2,5}$	2; (4, 4, 1)
12	$\mathcal{D}_{6,2} \cap \mathcal{D}_{3,4} = \mathcal{C}_{(3,2,2)}$	3; (8, 2, 1)
12	$\mathcal{D}_{6,2} \cap \mathcal{D}_{2,6} = \mathcal{C}_{(2,3,2)}$	3; (6, 4, 1)
12	$\mathcal{D}_{4,3} \cap \mathcal{D}_{2,6} = \mathcal{C}_{(2,2,3)}$	3; (6, 3, 2)

The remaining degree-12 intersections are the power/Dickson strata supplied by Theorem 7.3, namely $\mathcal{D}_{6,2} \cap \mathcal{D}_{4,3}$ (with $a = 2, b = 1$, core the swapped (3, 2) family), $\mathcal{D}_{4,3} \cap \mathcal{D}_{3,4}$ (with $a = b = 1$, so the core is all of P), and $\mathcal{D}_{3,4} \cap \mathcal{D}_{2,6}$ (with $a = 1, b = 2$). The middle one illustrates the non-exclusivity noted in Theorem 7.3: it is the union of the power curve $t^3(t^3 + W_0)^3$ and the Dickson curve, which meet only at t^{12} . In every degree the Dickson curve passes through all compatible composition components and gives the maximally split type of Proposition 5.6.

10. FURTHER QUESTIONS

The main remaining factorisation problem is the reducibility locus for general rational maps, especially on the strata with several ramified points in the marked pole fibre. The Grassmann-bundle model also gives a starting point for computing the full Chow classes of the rational composition loci. For polynomial composition closures the degree is already determined by Theorem 6.2; their boundary singularities and intersection multiplicities are separate questions.

The appendix gives translation models with polynomial orbit growth. A classification of non-symmetric $(2, 2)$ correspondences with polynomial growth remains outside the results proved here.

APPENDIX A. TRANSLATION CORRESPONDENCES AND ORBIT GROWTH

Let $B_j(x)$ be the set reachable by at most j steps in Γ and Γ^{-1} . Its cardinality for very general x equals the first projection degree d_j of the reduced path relation. The relations are closed by properness, and $d_{j+k} \leq d_j d_k$. Hence

$$h(\Gamma) = \lim_{j \rightarrow \infty} j^{-1} \log d_j$$

exists by subadditivity. It vanishes precisely for subexponential generic orbit growth. This counts distinct endpoints, rather than paths or multiplicities; compare the different degree-growth invariant in [3]. If an orbit is infinite, then $|B_{j+1}(x)| > |B_j(x)|$ at every step, so its growth is at least linear. An undirected tree of valency $D \geq 3$ has exponential growth, whereas valency two gives linear growth. We use no assertion that a general algebraic correspondence has a tree as its orbit graph.

A.1. Finite quotients of translations. Let G be $\mathbb{G}_a$, $\mathbb{G}_m$ or an elliptic curve, and let $\Sigma \leq \text{Aut}(G, 0)$ be finite with rational quotient. For elliptic G this requires $\Sigma \neq 1$. Choose a coordinate on the quotient, identify it with an open subset of $\mathbf{P}^1$, and let $q : G \rightarrow G/\Sigma$ be the quotient map. For a nonempty finite Σ -stable subset $S \subset G$, put

$$\Gamma_{G, \Sigma, S} = \overline{\{(q(g), q(g+s)) : g \in G, s \in S\}}. \quad (\text{A.1})$$

We use additive notation for all three group laws. In $\mathbb{G}_m$, translations are multiplication and negatives denote inverses.

Proposition A.1. *The correspondence (A.1) is irreducible precisely when S is a single Σ -orbit. It has projection degrees $(|S|, |S|)$, is symmetric precisely when $S = -S$, and its generic orbit growth satisfies*

$$|B_j(x)| \asymp j^{\text{rank } \Lambda}, \quad \Lambda = \langle S \rangle.$$

Proof. For general g , the images $q(g+s)$ are distinct. Two translation graphs have the same image under $q \times q$ precisely when their steps belong to the same Σ -orbit: equality on a dense open set forces the same linear automorphism on both copies of G . Transposition negates the step.

Every path starting at $q(g)$ lifts, after applying an element of Σ when necessary, to successive additions from $S \cup (-S)$; stability of S ensures this can be done at every step. Conversely every such word projects to a path.

The quotient identifies at most $|\Sigma|$ translates, so the orbit ball is comparable to the word ball of the finitely generated abelian group Λ . Such a ball has size $\asymp j^{\text{rank } \Lambda}$. For nonproper G , this argument concerns points whose paths avoid the boundary, which suffices for very general starting points. $\square$

Finite step groups give finite-orbit correspondences. They do not exhaust all finite-orbit correspondences. The multivalued map q^{-1} gives a coherent group-valued translation coordinate; the proof above uses the explicit quotient and its path-lifting property. No growth conclusion is claimed for a merely existential choice of algebraic branches on each edge.

A.2. Symmetric biquadratics.

Theorem A.2. *Let Γ be an irreducible symmetric curve of projection degrees $(2, 2)$. Up to the same Möbius change in both coordinates, it is one of:*

- (a) *the elliptic relation $x(R) \leftrightarrow x(R \pm T)$, with $T \notin E[2]$, given by (8.1);*
- (b) *the multiplicative relation*

$$z^2 - (\lambda + \lambda^{-1})zw + w^2 + a(\lambda - \lambda^{-1})^2 = 0, \quad a \neq 0, \quad \lambda \neq \pm 1; \quad (\text{A.2})$$

- (c) *the additive relation*

$$(w - z - 1)^2 = 4z. \quad (\text{A.3})$$

The first two are finite-orbit correspondences exactly when T is torsion or λ is a root of unity, respectively. Otherwise their generic growth is linear. Every affine orbit in (c) is infinite with linear growth, while ∞ is a singleton orbit. In the infinite-order multiplicative case ∞ is likewise a singleton orbit.

Proof. Let X be the normalisation, $q = \pi_1 : X \rightarrow \mathbf{P}^1$, ι its deck involution, and τ the lifted flip. Put $\rho = \iota\tau$. Successive nonbacktracking edges are obtained by applying ρ ; the other lift of a vertex gives the reverse path, since $\iota\rho = \rho^{-1}$. Consequently the full orbit of $q(P)$ is

$$\{q(\rho^n P) : n \in \mathbb{Z}\}.$$

The genus of X is at most one.

If it is one, Γ is smooth, since its arithmetic genus is one. It intersects the diagonal, so τ has a fixed point. Both τ and ι are therefore reflections $P \mapsto c - P$, and their product is a translation. Choose the origin so that $\iota(P) = -P$; then $q = x$ up to a Möbius change and Theorem 8.3 gives (a). A step in $E[2]$ would make the image a graph of degree $(1, 1)$, and is excluded.

If $X \simeq \mathbf{P}^1$, ρ is a nonidentity Möbius transformation. In the semisimple case write $\rho(s) = \lambda s$. The relation $\iota\rho = \rho^{-1}$ gives $\iota(s) = a/s$ and $q(s) = s + a/s$, except for commuting involutions whose image is a graph; these are excluded by the projection degrees. Eliminating s gives (b). In the parabolic case choose $\rho(s) = s + 1$, $\iota(s) = -s$ and $q(s) = s^2$; this gives (c). These normal forms also describe the fixed boundary orbit and prove all growth assertions. $\square$

Proposition A.3 (Elliptic twists). *Every smooth $(2, 2)$ curve can be written, after choosing an origin on its elliptic normalisation E , as*

$$\Gamma = \{(q(P), \phi(q(P + T))) : P \in E\}, \quad q : E \rightarrow E/\{\pm 1\}, \quad \phi \in \text{PGL}_2(\mathbb{C}).$$

It is symmetric precisely when ϕ has a lift $\alpha(P) = \varepsilon P + e$ with $\varepsilon \in \{1, -1\}$ and $2e = 0$. An arbitrary lift by an automorphism of E is not sufficient.

Proof. The two degree-two projections have deck involutions $P \mapsto c_i - P$. Choose the origin with $c_1 = 0$ and take T with $-2T = c_2$. The second projection and $q \circ t_T$ have the same deck involution, hence differ by a target Möbius map ϕ .

If the curve is symmetric, its flip is an involution τ with a fixed point, as above. From $\phi q t_T = q \tau$ we get the lift $\alpha = \tau t_{-T}$, whose linear part is -1 . Any automorphism lifting a map of $E/\{\pm 1\}$ normalises $\{\pm 1\}$; for $\alpha(P) = \varepsilon P + e$ this forces $2e = 0$. Conversely a lift of the stated form turns the relation into $q(P) \leftrightarrow q(P + T + e)$, which is symmetric. $\square$

Example A.4 (A liftable nonsymmetric twist). On $E : y^2 = x^3 - x$, the map $\phi(z) = -z$ lifts to the order-four automorphism $(x, y) \mapsto (-x, iy)$. Take $x(T) = 2$. The twisted relation has equation $F(z, w) = \Phi_2(z, -w) = 0$, and

$$F(z, w) - F(w, z) = 8(z - w)(zw + 1) \neq 0.$$

The coefficients of $z^2 w^2$ agree, so the two polynomials are not scalar multiples. This smooth irreducible $(2, 2)$ curve is therefore nonsymmetric, despite the existence of a lift.

A.3. Cyclotomic translation examples. Write ζ_d for a primitive d th root of unity and φ for Euler's totient function.

Theorem A.5. *For $d \geq 1$ and $c \neq 0$, the curve $\Gamma_d = \overline{\{(g^d, (g + c)^d) : g \in \mathbb{C}\}}$ is irreducible of projection degrees (d, d) and has generic growth $\asymp j^{\varphi(d)}$. Its affine full orbit through g^d is*

$$\{(g + \lambda)^d : \lambda \in c\mathbb{Z}[\zeta_d]\}.$$

The step group is discrete precisely for $d \in \{1, 2, 3, 4, 6\}$. Exactly for these d there is a nonconstant meromorphic first integral on $\mathbb{C}$. The projective boundary point ∞ is always a singleton orbit.

Proof. The parametrised image is irreducible. For general g^d , the d choices of g give d distinct values $(g + \zeta_d^i c)^d$, and the same holds in the reverse direction. Thus the projection degrees are (d, d) . This is (A.1) with $G = \mathbb{G}_a$, $\Sigma = \mu_d$ and $S = \mu_d c$. Its step group is $c\mathbb{Z}[\zeta_d]$, of rank $\varphi(d)$. The growth assertion is Proposition A.1.

For $d = 1, 2$ the step group is cyclic and discrete; for $d = 3, 4, 6$ it is a lattice. In the other cases its rank exceeds two, so it is nondiscrete in $\mathbb{C}$. In the discrete cases choose a nonconstant meromorphic function on $(\mathbb{C}/c\mathbb{Z}[\zeta_d])/\mu_d$. Pulling it back to the g -plane gives a meromorphic function $F(g)$ invariant under both the step group and μ_d . The latter invariance yields a unique meromorphic function I on the z -plane with $F(g) = I(g^d)$; at $g = 0$ this follows from the Laurent expansion, whose exponents are all divisible by d . Translation invariance makes I a first integral. If the step group is nondiscrete and I were a meromorphic first integral, then $g \mapsto I(g^d)$ would have arbitrarily small nonzero periods. Differentiating along such a sequence on a disc avoiding poles forces it to be constant. Finally a finite g never maps to ∞ , and the projective parametrisation maps $g = \infty$ to (∞, ∞) . $\square$

Example A.6. For $d = 2$, with $\kappa = c^2$, the equation is $(w - z - \kappa)^2 = 4\kappa z$. For $d = 3$, with $\kappa = c^3$, it is

$$(w - z - \kappa)^3 = 27\kappa zw. \quad (\text{A.4})$$

The latter is nonsymmetric, since $-\mu_3 c \neq \mu_3 c$, and has quadratic growth. For the hexagonal lattice $\Lambda = c\mathbb{Z}[\zeta_3]$,

$$I(z) = \wp'_\Lambda(z^{1/3})$$

is a single-valued meromorphic first integral on $\mathbb{C}$: the identity $\wp'_\Lambda(\zeta_3 g) = \wp'_\Lambda(g)$ removes the choice of cube root. On $\mathbb{C}/\Lambda$ the function $\wp'$ has degree three, so its generic fibres are the μ_3 -orbits. Thus the generic level sets of I are exactly the affine full orbits. This integral does not extend meromorphically to the projective line.

Allowing reducible correspondences realises every positive integral growth exponent: use the union of the translation graphs $w = z + c_i$ with $c_1, \dots, c_r$ linearly independent over $\mathbb{Q}$. The step group has rank r . No converse classification of polynomial-growth correspondences is asserted.

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