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ELEMENTS OF PÓLYA-SCHUR THEORY IN THE FINITE DIFFERENCE SETTING

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ABSTRACT. The Pólya-Schur theory describes the class of hyperbolicity preservers, i.e., the class of linear operators acting on univariate polynomials and preserving real-rootedness. We attempt to develop an analog of Pólya-Schur theory in the setting of linear finite difference operators. We study the class of linear finite difference operators preserving the set of real-rooted polynomials whose mesh (i.e., the minimal distance between the roots) is at least one. In particular, we prove a finite difference version of the classical Hermite-Poulain theorem and several results about discrete multiplier sequences.

1. INTRODUCTION

The systematic study of linear operators acting on $\mathbb{R}[x]$ and sending real-rooted polynomials to real-rooted polynomials was initiated in the 1870s by C. Hermite and later continued by E. Laguerre. Its classical period culminated in 1914 with the publication of the outstanding paper [20], where G. Pólya and I. Schur completely characterized all such linear operators acting diagonally on the standard monomial basis $1, x, x^2, \dots$ of $\mathbb{R}[x]$. This article generated a substantial amount of related literature with contributions by N. Obreschkov, S. Karlin, B. Ya. Levin, G. Csordas, T. Craven, A. Iserles, S. P. Nørsett, E. B. Saff, and, recently by the first author together with the late J. Borcea.

Although several variations of the original set-up have been considered over the years (including complex zero decreasing sequences, real-rooted polynomials on finite intervals, stable polynomials, etc.), it seems that its natural finite difference analog discussed below has so far escaped the attention of the specialists in the area. An exception is [11, §§8.7–8.9].

Denote by $\mathcal{HP} \subset \mathbb{R}[x]$ the set of all real-rooted (also referred to as *hyperbolic*) polynomials. (We consider all constant polynomials to be real-rooted.) A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is called a *real-rootedness preserver* or a *hyperbolicity preserver* if it preserves $\mathcal{HP}$. Given a real-rooted polynomial $p(x) \in \mathcal{HP}$, denote by $\operatorname{mesh}(p)$ its mesh, i.e., the minimal distance between its roots. If a real-rooted $p(x)$ has a multiple root, then $\operatorname{mesh}(p) := 0$. Polynomials of degree at most 1 are defined to have mesh equal to $+\infty$. Denote by $\mathcal{HP}_{\geq\alpha} \subset \mathcal{HP}$ the set of all real-rooted polynomials whose mesh is at least $\alpha \geq 0$. Let $\mathcal{HP}^+_{\geq\alpha} \subset \mathcal{HP}_{\geq\alpha}$ be the subset of such polynomials with only non-negative zeros.

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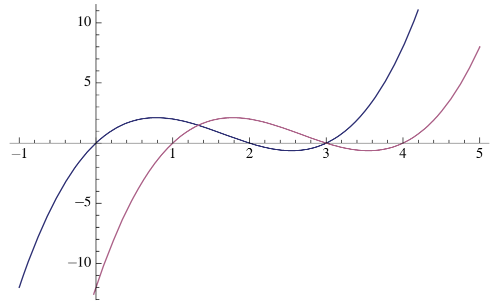

FIGURE 1. Roots of $\Delta(p(x)) = p(x) - p(x-1)$ are the x-coordinates of the intersection points between the graphs of $p(x)$ and $p(x-1)$.

One of rather few known results about linear operators not decreasing the mesh was due originally to M. Riesz and deserves to be better known; see e.g. Corollary 8.48 of [11] and [22].

Theorem 1. *For any hyperbolic polynomial p and any real λ,*

$$\mathrm{mesh}(p - \lambda p') \geq \mathrm{mesh}(p).$$

(To be exact, as was pointed out by one of the anonymous referees, Theorem 1 is stated in [22] without a complete proof. In fact this result was rediscovered and strengthened at least twice; see [24, 27, 28] and also [10].)

Recall that the well-known Hermite-Poulain theorem [17, p. 4] claims that a finite order linear differential operator $T = a_0 + a_1 d/dx + \cdots + a_k d^k/dx^k$ with constant coefficients is hyperbolicity preserving if and only if its *symbol polynomial* $Q_T(t) = a_0 + a_1 t + \cdots + a_k t^k$ is hyperbolic. Thus Theorem 1 combined with the Hermite-Poulain theorem imply the following statement.

Corollary 1. *A hyperbolicity preserving differential operator with constant coefficients does not decrease the mesh of hyperbolic polynomials.*

In the remainder of the introduction we will formulate our main results whose proofs are postponed until §§2-3.

Our first goal is to find an analog of Corollary 1 in the finite difference context. We consider the action on $\mathbb{C}[x]$ of linear finite difference operators T with polynomial coefficients; i.e., operators of the form

$$T(p)(x) = q_0(x)p(x) + q_1(x)p(x-1) + \cdots + q_k(x)p(x-k), \tag{1}$$

where $q_0(x), \ldots, q_k(x)$ are fixed complex- or real-valued polynomials. If $q_k(x) \not\equiv 0$ we say that T has order k. Although no non-trivial T as in (1) preserves $\mathcal{HP}$ (see Lemma 8 below), it can nevertheless preserve $\mathcal{HP}_{\geq 1}$. The simplest example of such an operator is

$$\Delta(p(x)) = p(x) - p(x-1),$$

which is a discrete analog of d/dx; see Figure 1.

Definition 1. A linear finite difference operator (1) is called a **discrete hyperbolicity preserver** if it preserves $\mathcal{HP}_{\geq 1}$.

Obviously, the set of all discrete hyperbolicity preservers is a semigroup with respect to composition.

We start with a finite difference analog of Theorem 1. (A similar result was proved by S. Fisk in [11, Lemma 8.27].)

Theorem 2. *For positive real numbers α and λ, define an operator T by*

$$T(p)(x) = p(x) - \lambda p(x - \alpha).$$

Then for any hyperbolic polynomial $p \in \mathcal{HP}_{\geq\alpha}$,

$$\operatorname{mesh}(T(p)) \geq \operatorname{mesh}(p).$$

Moreover if $\lambda \geq 1$, then T preserves the set $\mathcal{HP}^{+}_{\geq\alpha}$.

Theorem 2 settles Conjecture 2.19 from a recent paper [8]. Our next result is a natural finite difference analog of the Hermite-Poulain theorem.

Theorem 3. *A linear finite difference operator T with constant coefficients of the form*

$$T(p(x)) = a_0 p(x) + a_1 p(x-1) + \cdots + a_k p(x-k) \tag{2}$$

is a discrete hyperbolicity preserver if and only if all zeros of its symbol polynomial $Q_T(t) := a_0 + a_1 t + \cdots + a_k t^k$ are real and non-negative.

As we mentioned above, a famous class of hyperbolicity preservers is the class of multiplier sequences introduced and studied by G. Pólya and I. Schur in [20]. Let us recall this notion and introduce its finite difference analog.

Definition 2. Given a sequence $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ of real numbers, we denote by $T_{\mathcal{A}}$ the linear operator

$$T_{\mathcal{A}}(x^i) = \alpha_i x^i$$

acting diagonally with respect to the monomial basis of $\mathbb{R}[x]$. We refer to $T_{\mathcal{A}}$ as the **diagonal operator** corresponding to the sequence $\mathcal{A}$.

Notice that any diagonal operator T as above can also be written as a formal linear differential operator of (in general) infinite order

$$T = \sum_{i=0}^{\infty} a_i x^i \frac{d^i}{dx^i}.$$

The relation between the sequences $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ and $A = \{a_i\}_{i=0}^{\infty}$ representing the same diagonal operator T is of triangular form and given by

$$\alpha_i = a_0 + i a_1 + i(i-1) a_2 + \cdots + i! a_i,\ i = 0, 1, 2, \ldots .$$

Definition 3. We call a sequence $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ of real numbers a **multiplier sequence** if its diagonal operator $T_{\mathcal{A}}$ preserves $\mathcal{HP}$, i.e., sends an arbitrary hyperbolic polynomial to a hyperbolic polynomial.

The main results of [20] are explicit criteria describing when a given sequence $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ represents a multiplier sequence.

Theorem 4. *Let $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ be a sequence of real numbers, let $T_{\mathcal{A}} : \mathbb{R}[x] \to \mathbb{R}[x]$ be the corresponding linear operator, and define $\Phi(x)$ to be the formal power series*

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k.$$

The following assertions are equivalent:

(a) $\mathcal{A}$ *is a multiplier sequence.*

(b) *The formal power series* $\Phi(x)$ *defines an entire function which is the limit, uniformly on compacts, of polynomials with only real zeros of the same sign.*

(c) *The formal power series* $\Phi(x)$ *or* $\Phi(-x)$ *is an entire function which can be written as*

$$\Phi(x) = Cx^n e^{ax} \prod_{k=1}^{\infty}(1+\beta_k x),$$

where $n \in \mathbb{N}$, $C \in \mathbb{R}$ *and* $a, \beta_k \geq 0$ *for all* $k \in \mathbb{N}$, *and* $\sum_{k=0}^{\infty} \beta_k < \infty$.

(d) *For all non-negative integers* n, *the polynomial* $T[(x-1)^n]$ *is hyperbolic with all zeros of the same sign.*

Let us now describe a finite difference version of multiplier sequences. Natural analogs of monomials in the finite difference setting are the (backwards) Pochhammer polynomials $\{[x]_i\}_{i=0}^{\infty}$ defined by

$$[x]_0 = 1, \quad [x]_i = x(x-1)\cdots(x-i+1),\ i \geq 1. \tag{3}$$

Definition 4. A finite difference operator T as in (1) is called **diagonal** if it acts diagonally with respect to the Pochhammer basis $\{[x]_i\}_{i=0}^{\infty}$.

Analogously to the above case of the usual diagonal operators we can associate to any sequence $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ of real numbers the corresponding diagonal finite difference operator $T_{\mathcal{A}}$ (in general, of infinite order) by assigning

$$T_{\mathcal{A}}([x]_i) = \alpha_i [x]_i,\ i = 0, 1, 2, \dots . \tag{4}$$

Observe that a finite difference analog $x\Delta$ of the Euler operator $x\frac{d}{dx}$ given by

$$x\Delta = x(p(x) - p(x-1))$$

acts diagonally in this basis, namely, $x\Delta([x]_i) = i[x]_i$. Moreover any diagonal finite difference operator T (of finite or infinite order) can be represented as a formal series

$$T = \sum_{i=0}^{\infty} a_i [x]_i \Delta^i.$$

Definition 5. We say that a sequence $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ is a **discrete multiplier sequence** if the corresponding diagonal operator $T_{\mathcal{A}}$ given by (4) preserves $\mathcal{HP}_{\geq 1}^{+}$. In this case $T_{\mathcal{A}}$ itself is called a **discrete multiplier operator**.

Our next result is as follows.

Theorem 5. *An operator* U *given by*

$$U(p(x)) = \alpha p(x) + \beta x\Delta(p(x)) = \alpha p(x) + \beta x(p(x) - p(x-1))$$

is a discrete multiplier operator if α *and* β *are real numbers of the same sign.*

Remark 1. Observe that, in general, the above operator U is **not** mesh-increasing. Therefore, Theorem 5 is not a complete analog of Theorem 2. A simple example of this phenomenon is $U(p(x)) = p(x) + (3/4)x(p(x) - p(x-1))$, i.e., $\alpha = 1,\ \beta = 3/4$. When $p(x) = (x-1)(x-4)(x-7)$, then $U(p(x))$ has three positive roots which are approximately equal to $0.433167, 3.12467, 6.36524$ and its mesh is smaller than 3.

Proposition 6. *If* $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ *is a discrete multiplier sequence, then it is a multiplier sequence in the classical sense.*

Remark 2. Notice that the converse to Proposition 6 fails since the ordinary multiplier sequence $\{\rho^i\}_{i=0}^{\infty}$, where $0 < \rho < 1$, is not a a discrete multiplier sequence (see Proposition 12).

Denote by $\mathcal{L}\text{–}\mathcal{P}_+$ the positive subclass in the Laguerre-Pólya class; i.e., real entire functions which are the uniform limits, on compact subsets of the complex plane, of polynomials with only real non–positive zeros.

Corollary 2. *If $\phi(x) \in \mathcal{L}\text{–}\mathcal{P}_+$, then the sequence $\{\phi(i)\}_{i=0}^{\infty}$ is a discrete multiplier sequence.*

Remark 3. Corollary 2 may be seen as a discrete version of Laguerre's theorem stating that if $\phi(x)$ is a function in the Laguerre-Pólya class and $\phi(x)$ has no positive zero, then the sequence $\{\phi(k)\}_{k=0}^{\infty}$ is a multiplier sequence (and even a complex zero decreasing sequence); see e.g., [7], Theorem 4.1. However Corollary 2 cannot be extended to a complete analog of Laguerre's theorem since if $\phi(x) = e^{-x}$, then the sequence $\{\phi(k)\}_{k=0}^{\infty}$ is not a discrete multiplier sequence by Remark 2 above.

A sequence $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ is said to be *trivial* if $\alpha_i \neq 0$ for at most two indices i. Trivial discrete multiplier sequences are simple to describe.

Proposition 7. *A trivial sequence $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ is a discrete multiplier sequence if and only if there is an integer $m \geq 0$ such that $\alpha_m\alpha_{m+1} \geq 0$ and $\alpha_i = 0$ unless $i \in \{m, m+1\}$.*

We conjecture the following tantalizing characterization of non-trivial discrete multiplier sequences, which would be a discrete analog of the classical result of Pólya and Schur [20].

Conjecture 1. *Let $\mathcal{A} = \{\alpha_i\}_{i=0}^{\infty}$ be a non-trivial sequence such that $\alpha_i > 0$ for some i. Then it is a discrete multiplier sequence if and only if it is a weakly increasing multiplier sequence, that is, $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots$.*

Below we almost prove one direction of Conjecture 1. Namely we show that any discrete multiplier sequence with infinitely many non-zero entries and at least one positive entry is weakly increasing; see Proposition 12.

2. Discrete Hermite-Poulain theorem

The following lemma emphasizes the distinction between ordinary and discrete hyperbolicity preservers.

Lemma 8. *A finite difference operator T given by (1) is hyperbolicity preserving in the classical sense if and only if $q_i(x) \not\equiv 0$ for at most one i, and this $q_i(x)$ is hyperbolic.*

Proof. If T satisfies the conditions of Lemma 8, then T is trivially a hyperbolicity preserver.

To prove the converse, consider the bivariate symbol

$$G_T(x,y) = T(e^{-xy}) = \sum_{j=0}^{k} q_j(x)e^{-(x-j)y} = e^{-xy}\sum_{j=0}^{k} q_j(x)e^{jy}.$$

Note that the image of T is infinite dimensional in $\mathbb{R}[x]$. Hence, if T is a hyperbolicity preserver, then by [1, Theorem 5], $G_T(x,y)$ or $G_T(x,-y)$ is the uniform limit on compact subsets of $\mathbb{C}$ of bivariate polynomials that are non-vanishing whenever $\operatorname{Im} x > 0$ and $\operatorname{Im} y > 0$. It follows that, for each $x_0 \in \mathbb{R}$, the function

$$\sum_{j=0}^{k} q_j(x_0)e^{\pm(x_0-j)y}$$

is in the Laguerre–Pólya class. However this is the case only if $q_j(x_0) \neq 0$ for at most one j as follows: if $q_j(x_0) \neq 0$ for at least two values of j, then the polynomial $\sum_{j=0}^{k} q_j(x_0)z^j$ has a non-zero root of the form $z = e^y$ for some $y \in \mathbb{C}$. But then the numbers $e^{y+2\pi i m}$ for $m \in \mathbb{Z}$ are also roots, and hence the function does not belong to the Laguerre-Pólya class, a contradiction. □

Before we present a proof of Theorem 2, we need to recall some notation and well-known results about hyperbolic polynomials. Let $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ and $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_m$ be the zeros of two hyperbolic polynomials p and q. These zeros *interlace* if either $\gamma_1 \leq \delta_1 \leq \gamma_2 \leq \delta_2 \leq \cdots$ or $\delta_1 \leq \gamma_1 \leq \delta_2 \leq \gamma_2 \leq \cdots$. A pair of hyperbolic polynomials (p, q) are in *proper position*, written $p \ll q$, if their zeros interlace and $W[p,q](x) = p(x)q'(x) - p'(x)q(x) \geq 0$ for all $x \in \mathbb{R}$. Note that if the zeros of two hyperbolic polynomials p and q interlace, then either $p \ll q$ or $q \ll p$. By convention we set $0 \ll p$ and $p \ll 0$ for any hyperbolic polynomial p, and $p \ll q$ whenever p and q have degree at most one and $W[p,q](x) \geq 0$ for all $x \in \mathbb{R}$. Recall that the Hermite–Biehler theorem asserts that $p \ll q$ if and only if either $p \equiv q \equiv 0$ or the polynomial $q + ip$, where $i = \sqrt{-1}$, has no zero with positive imaginary part; see e.g. [21, Theorem 6.3.4].

Lemma 9.

(a) *Let p be a hyperbolic polynomial. Then the sets*

$$\{q \in \mathbb{R}[x] : q \ll p\} \quad \textit{and} \quad \{q \in \mathbb{R}[x] : p \ll q\}$$

are convex cones.

(b) *If $p \ll q$, then $p \ll q + \alpha p$ and $p + \alpha q \ll q$ for all $\alpha \in \mathbb{R}$.*

Proof. Part (a) is [2, Lemma 2.6]. Part (b) follows from (a) by noting that $p \ll \alpha p$ and $\alpha q \ll q$ for all real α. □

Proof of Theorem 2. Set $T(p)(x) = p(x) - \lambda p(x - \alpha)$, where $\alpha, \lambda \geq 0$. We want to prove that, for all $\beta \geq \alpha$, operator T maps $\mathcal{HP}_{\geq\beta}$ to itself. First we show it for $\beta = \alpha$. Note that $p \in \mathcal{HP}_{\geq\alpha}$ if and only if $p(x) \ll p(x - \alpha)$. Lemma 9 (b) implies

$$T(p)(x) \ll p(x) \quad \text{and} \quad T(p)(x) \ll p(x - \alpha),$$

which implies $T(p) \in \mathcal{HP}_{\geq\alpha}$.

Next we prove that if $p, q \in \mathcal{HP}_{\geq\alpha}$ satisfy $p \ll q$, then $T(p) \ll T(q)$. This will prove Theorem 2, since if $p \in \mathcal{HP}_{\geq\beta} \subseteq \mathcal{HP}_{\geq\alpha}$, then $p(x) \ll p(x - \beta)$. Thus $T(p)(x) \ll T(p)(x - \beta)$, which is equivalent to $T(p) \in \mathcal{HP}_{\geq\beta}$. We claim that we may assume that p and q have the same degree. Indeed if $p \ll q$ and $\deg q > \deg p$, then $p + \epsilon q \ll q$ for all real ϵ, by Lemma 9. The claim follows from Hurwitz' theorem on the continuity of zeros [21, Theorem 1.3.8] by letting $\epsilon \to 0$, since then $T(p) + \epsilon T(q) \ll T(q)$ and thus $T(p) + \epsilon T(q) \ll T(q)$. We may also assume that p and q are monic, since if $p(x) \ll q(x)$ and a and b are the leading coefficients of p and q, respectively, then either $a^{-1}p(x) \ll b^{-1}q(x)$ or $b^{-1}q(x) \ll a^{-1}p(x)$, depending on the sign of ab.

To prove that T preserves proper position, we claim that it is enough to show that

$$T((x-a)r) \ll T((x-b)r) \text{ whenever } a \le b \text{ and } r \in \mathcal{HP}_{\ge\alpha}. \tag{5}$$

Proof of the claim. If f and g are hyperbolic polynomials such that $f=(x-a)r$ and $g=(x-b)r$, where $a \le b$ and $r \in \mathcal{HP}_{\ge\alpha}$, we write $f \lhd g$. Now suppose $p,q \in \mathcal{HP}_{\ge\alpha}$ are monic polynomials that have the same degree and satisfy $p \ll q$. Let $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ be the zeros of p and q, respectively. For $0 \le k \le n$, set

$$p_k(x) = (x-\beta_1)(x-\beta_2)\cdots(x-\beta_k)(x-\alpha_{k+1})(x-\alpha_{k+2})\cdots(x-\alpha_n),$$

where $p_0 = p$ and $p_n = q$. For $0 \le k \le n-1$, set

$$q_k(x) = (x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_{k+1})(x-\beta_{k+1})(x-\beta_{k+2})\cdots(x-\beta_n),$$

where $q_0 = q$. By construction,

$$p = p_0 \lhd p_1 \lhd \cdots \lhd p_n = q = q_0 \lhd q_1 \lhd \cdots \lhd q_{n-1} \quad \text{and} \quad p \lhd q_{n-1}.$$

Hence if (5) is true, then

$$T(p) \ll T(p_1) \ll \cdots \ll T(q) \ll T(q_1) \ll \cdots \ll T(q_{n-1}) \quad \text{and} \quad T(p) \ll T(q_{n-1}),$$

which implies $T(p) \ll T(q)$, by e.g. [25, Prop. 3.3]. This proves the claim.

It remains to prove (5). Since $T((x-b)r) = T((x-a)r) - (b-a)T(r)$, it is, by Lemma 9 (a) and invariance under translation, enough to prove that $T(r) \ll T(xr)$ for all $r \in \mathcal{HP}_{\ge\alpha}$. Now

$$T(xr) = (x-\alpha)T(r) + \alpha r.$$

Since $T(r) \ll r$ and $T(r) \ll (x-\alpha)T(r)$, Lemma 9 (b) implies that $T(r) \ll (x-\alpha)T(r) + \alpha r$, as desired.

Finally suppose $p \in \mathcal{HP}^+_{\ge\alpha}$ and $\lambda \ge 1$. Write $p(x) = A\prod_{i=1}^n (x-\theta_i)$, where $\theta_i \ge 0$ for all i. Then for $y \ge 0$,

$$\frac{p(-y)}{p(-y-1)} = \prod_{i=1}^n \frac{y+\theta_i}{y+\theta_i+1} < 1 \le \lambda.$$

Hence $T(p)(-y) \ne 0$, which proves that T preserves $\mathcal{HP}^+_{\ge\alpha}$. □

Proof of Theorem 3. Theorem 2 implies that if the symbol polynomial $Q_T(t) = a_0 + a_1 t + \cdots + a_k t^k$ has only real and non-negative zeros, then the finite difference operator $T(p(x)) = a_0 p(x) + a_1 p(x-1) + \cdots + a_k p(x-k)$ is a discrete hyperbolicity preserver. We need to prove the necessity of the latter condition. Consider the action of T on the Pochhammer polynomial $[x]_i$. Assuming that $i \ge k$, we get

$$T([x]_i) = (x-k)\cdots(x-i+1)R_i(x),$$

where $R_i(x)$ is a hyperbolic polynomial of degree k. Observe that

$$\lim_{i\to\infty} \frac{R_i(ix)}{i^k} = x^k Q_T\left(\frac{x-1}{x}\right), \tag{6}$$

where $Q_T(t)$ is the above symbol polynomial. Hence if T is a discrete hyperbolicity preserver, then $Q_T(t)$ is hyperbolic. We need to show that its zeros are non-negative. Suppose that $Q_T(y) = 0$ for $y < 0$. The assumption $y = (x-1)/x$ implies $0 < x < 1$. By (6) and Hurwitz' theorem on the continuity of zeros it follows that there are real numbers $0 < a < b < 1$ and an integer i_0 such that

$R_i(ix)$ has a zero in the interval (a,b) whenever $i > i_0$. Hence $R_i(x)$ has a zero in (ia, ib) for all $i > i_0$. If we choose $i > i_0$ large enough so that $(ia, ib) \subset (k, i)$, we see that the mesh of $T([x]_i) = (x-k)\cdots(x-i+1)R_i(x)$ is strictly smaller than 1, which contradicts our assumption. Hence all zeros of $Q_T(t)$ are non-negative. $\square$

3. Discrete multiplier sequences

Proof of Theorem 5. It suffices to consider the operator

$$W_\lambda(p) = p(x) + \lambda x \Delta(p(x)) = p(x) + \lambda x\left(p(x) - p(x-1)\right).$$

In other words, we need to show that for each $\lambda \geq 0$,

$$W_\lambda : \mathcal{HP}^+_{\geq 1} \to \mathcal{HP}^+_{\geq 1},$$

i.e., W_λ is a discrete multiplier operator. Take $p \in \mathcal{HP}^+_{\geq 1}$. As in the proof of Theorem 2, we observe that

$$p(x) - p(x-1) \ll p(x) \text{ and } p(x) - p(x-1) \ll p(x-1).$$

Since the degree of $p(x) - p(x-1)$ is one less than that of $p(x)$ and since all the zeros of $p(x) - p(x-1)$ are non-negative (they interlace those of p),

$$\lambda x(p(x) - p(x-1)) \ll p(x) \text{ and } \lambda x(p(x) - p(x-1)) \ll p(x-1).$$

Since $p(x) \ll p(x)$ and $p(x) \ll p(x-1)$, Lemma 9 (a) implies

$$W_\lambda(p)(x) \ll p(x) \text{ and } W_\lambda(p)(x) \ll p(x-1),$$

which in turn implies $W_\lambda(p) \in \mathcal{HP}_{\geq 1}$. Since

$$\mathcal{HP}^+_{\geq 1} \ni x(p(x) - p(x-1)) \ll p(x) \in \mathcal{HP}^+_{\geq 1},$$

these polynomials have the same sign for negative real numbers which implies $W_\lambda(p) \in \mathcal{HP}^+_{\geq 1}$. $\square$

The next result is due to F. Brenti [3]. We provide a proof here for completeness.

Lemma 10. *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by*

$$T(x^i) = [x]_i.$$

If all the zeros of the polynomial $p(x)$ are real and non-negative, then $T(p) \in \mathcal{HP}^+_{\geq 1}$.

Proof. We prove Lemma 10 by induction on n, the degree of p. The cases $n = 0$ and 1 are trivial. We assume that $p(x)$ is a polynomial of degree $n+1 \geq 2$ and write

$$p(x) = (x-\alpha)q(x) = (x-\alpha)\sum_{i=0}^{n} \gamma_i x^i,$$

where $\alpha \geq 0$. By induction we know that $Q(x) = T(q) \in \mathcal{HP}^+_{\geq 1}$. An elementary manipulation shows that

$$T(p) = xQ(x-1) - \alpha Q(x).$$

Since $Q(x) \in \mathcal{HP}^+_{\geq 1}$,

$$xQ(x-1) \ll -Q(x); -\alpha Q(x) \ll -Q(x);$$
$$-xQ(x-1) \ll -Q(x-1); -\alpha Q(x) \ll -Q(x-1).$$

Thus by Lemma 9 (a),

$$T(p) \ll -Q(x) \quad \text{and} \quad T(p) \ll -Q(x-1),$$

which proves $T(p) \in \mathcal{HP}^+_{\geq 1}$. □

Proof of Proposition 6. Suppose that all zeros of a test polynomial $p(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_n x^n$ are real and non-negative. By Lemma 10,

$$\sum_{i=0}^{n} \gamma_i \rho^i [x]_i \in \mathcal{HP}^+_{\geq 1},$$

for all $\rho > 0$. But then $\sum_{i=0}^n \gamma_i \rho^i \alpha_i [x]_i \in \mathcal{HP}^+_{\geq 1}$ and thus $\sum_{i=0}^n \gamma_i \rho^i \alpha_i [x/\rho]_i \in \mathcal{HP}^+_{\geq \rho}$ for all $\rho > 0$. Letting $\rho \to 0$, we see that $\sum_{i=0}^n \gamma_i \alpha_i x^i \in \mathcal{HP}^+_{\geq 0}$, and hence $\{\alpha_i\}_{i=0}^\infty$ is an ordinary multiplier sequence by Theorem 4, since we can choose $p(x) = (x-1)^n$. □

Proof of Corollary 2. Theorem 5 claims that the sequence $\{1 + \lambda i\}_{i=0}^\infty$ is a discrete multiplier sequence for each $\lambda \geq 0$. Since the set of all discrete multiplier sequences is a semigroup under composition, all hyperbolic polynomials with negative zeros give rise to discrete multiplier sequences via $p \mapsto \{p(i)\}_{i=0}^\infty$. The set $\mathcal{L}\text{–}\mathcal{P}_+$ is the closure of such polynomials, from which Corollary 2 follows. □

Lemma 11. *Suppose* $p(x) = \sum_{i=0}^n a_i [x]_i \in \mathcal{HP}^+_{\geq 1}$ *with* $a_n > 0$*. Then* $(-1)^{n-i} a_i \geq 0$ *for all* $0 \leq i \leq n$*.*

Proof. Since $p(x)$ has n non-negative zeros and $p(x) > 0$ for $x > 0$ large enough, we have $(-1)^n p(0) = (-1)^n a_0 \geq 0$. As in the proof of Theorem 5, we see that $\nabla(p) \ll p$ and $\nabla(p) \in \mathcal{HP}^+_{\geq 1}$. Here $\nabla p(x) = p(x+1) - p(x)$ is the forward difference operator. Now

$$\nabla(p) = \sum_{i=0}^{n-1} (i+1) a_{i+1} [x]_i,$$

and Lemma 11 follows by iterating the argument for $i = 0$. (Observe that $\nabla p(x) = \Delta p(x+1)$.) □

An immediate consequence of Lemma 11 is:

Corollary 3. *All non-zero entries of a discrete multiplier sequence have the same sign.*

Next we give the proof of the characterization of trivial discrete multiplier sequences.

Proof of Proposition 7. Suppose $\mathcal{A} = \{\alpha_i\}_{i=0}^\infty$ is a trivial discrete multiplier sequence. Then, by Corollary 3, we may assume that all entries are non-negative. The "only if" direction now follows from the well-known fact that all non-negative and trivial multiplier sequences are of the desired form.

Assume that $\mathcal{A}$ satisfies the conditions in the statement of Proposition 7 with $\alpha_m \alpha_{m+1} \geq 0$. Take $p(x) = \sum_{i=0}^n a_i [x]_i$, and let T be the diagonal finite difference operator associated to $\mathcal{A} = \{\alpha_i\}_{i=0}^\infty$. Then

$$T(p)(x) = \alpha_m a_m [x]_m + \alpha_{m+1} a_{m+1} [x]_{m+1} = -a[x]_m + b[x]_{m+1},$$

where $ab \geq 0$ by Lemma 11. If $b = 0$ we are done, so assume $b > 0$. Then

$$T(p)(x) = b[x]_m(x - m - b/a) \in \mathcal{HP}^+_{\geq 1},$$

as desired. □

Proposition 12. *Let $\mathcal{A} = \{\alpha_i\}_{i=0}^\infty$ be a discrete multiplier sequence. If $\alpha_{m+2} > 0$ for some $m \geq 0$, then $\alpha_m \leq \alpha_{m+1}$.*

Proof. Take $a \geq 0$, and consider

$$\begin{aligned} &T\big([x]_m(x - m - a)(x - 1 - m - a)\big) \\ &\quad = [x]_m\big(\alpha_{m+2}(x - m)(x - m - 1) - 2a\alpha_{m+1}(x - m) + \alpha_m a(a + 1)\big). \end{aligned}$$

Given $A, B, C \geq 0$, a polynomial $Ax(x - 1) - 2Bx + C$ is in $\mathcal{HP}^+_{\geq 1}$ if and only if $AC \leq B^2 + AB$, which yields

$$0 \leq a(\alpha_{m+1}^2 - \alpha_m\alpha_{m+2}) + \alpha_{m+2}(\alpha_{m+1} - \alpha_m), \qquad \text{for all } a \geq 0.$$

In particular $\alpha_{m+1} \geq \alpha_m$. □

Finally, let us present more examples of discrete multiplier sequences.

Example 1. For any non-negative i, the operator $[x]_i\Delta^i$ is a discrete multiplier operator, i.e. $\{(n)_i\}_{n=0}^\infty$ is a discrete multiplier sequence. It follows from the fact that if $p \in \mathcal{HP}^+_{\geq 1}$, then $\Delta p(x) \ll p(x - 1)$, and therefore all zeros of Δp are in $[1, \infty)$.

Example 2. For any non-negative i and any polynomial q with all roots in $(-\infty, i]$, the sequence $\{p(i)\}_{i=0}^\infty$, where

$$p(x) = [x]_i q(x),$$

is a discrete multiplier sequence.

4. Final remarks

The Hermite-Poulain theorem has the following analog in finite degrees.

Proposition 13. *A differential operator $T = a_0 + a_1 d/dx + \cdots + a_k d^k/dx^k$, $a_k \neq 0$, with constant coefficients preserves the set of hyperbolic polynomial of degree at most m if and only if the polynomial $T(x^m)$ is hyperbolic.*

Proposition 13 follows immediately from the algebraic characterization of hyperbolicity preservers; see Theorem 2 of [1].

In the finite difference setting the monomials $\{x^m\}$ should be substituted by the Pochhammer polynomials $\{[x]_m\}$. In particular, Proposition 13 might have the following conjectural analog in the finite difference setting.

Conjecture 2. *A difference operator $T(p(x)) = a_0p(x) + a_1p(x-1) + \cdots + a_kp(x-k)$ with constant coefficients preserves the set of hyperbolic polynomial of degree at most m whose mesh is at least one if and only if the polynomial $T([x]_m)$ is hyperbolic and has mesh at least one.*

We shall now see that there is an alternative formulation of Conjecture 2 which is perhaps more attractive. Let $\nabla p(x) = p(x + 1) - p(x)$ be the forward difference operator, and consider the following product on the space of polynomials of degree at most d:

$$(p \bullet q)(x) = \sum_{k=0}^{d} (\nabla^k p)(0) \cdot (\nabla^{d-k} q)(x).$$

We claim that Conjecture 2 is equivalent to

Conjecture 3. *If p and q are hyperbolic polynomials of degree at most d and of* mesh ≥ 1, *then so is $p \bullet q$.*

Indeed, we may equivalently consider "forward" difference operators of the form

$$T(p(x)) = a_0 p(x) + a_1 p(x+1) + \cdots + a_k p(x+k),$$

which in turn may be written as

$$T = \sum_{j=0}^{\infty} b_j \nabla^j,$$

for some real numbers $b_0, b_1, \ldots$, where we allow infinite sequences (since $\nabla^j f \equiv 0$ if the degree of f is smaller than j). Recall that if f is a polynomial of degree d, then

$$f(x) = \sum_{j=0}^{d} \frac{(\nabla^j f)(0)}{j!} [x]_j.$$

Let

$$p(x) := T([x]_m) = \sum_{j=0}^{m} b_j [m]_j [x]_{m-j}.$$

Then $b_j = (\nabla^{m-j} p)(0)/m!$, and thus

$$T(q) = \sum_{j=0}^{\infty} b_j \nabla^j(q) = \frac{1}{m!} \sum_{j=0}^{m} (\nabla^{m-j} p)(0) \cdot (\nabla^j q) = \frac{(p \bullet q)}{m!}.$$

This proves the equivalence of Conjectures 2 and 3.

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