FERRERS CURVATURE ALGEBRAS: SCHUR QUOTIENTS, SCHUBERT GEOMETRY, AND ARBITRARY BOX SUPPORTS

GLEB NENASHEV, ALEXANDER POSTNIKOV, BORIS SHAPIRO, AND MICHAEL SHAPIRO

Abstract. Let λ be a partition and let A_λ and B_λ be universal matrices of independent odd variables whose common zero pattern is the Young diagram of λ. Write $\det(I + tA_\lambda B_\lambda) = \sum_j \phi_j t^j$ and let $\mathcal{A}_\lambda$ be the algebra generated by the ϕ_j. Evaluation $h_j \mapsto \phi_j$ induces an isomorphism $\mathrm{Sym}/I_\lambda \cong \mathcal{A}_\lambda$, where I_λ is spanned by the Schur functions s_ρ with $\rho \not\subseteq \lambda$, and $\mathcal{A}_\lambda$ is isomorphic to the cohomology ring of the Grassmannian Schubert variety X_{λ^c}. Hence the $s_\mu(\phi)$, $\mu \subseteq \lambda$, form a basis with truncated Littlewood–Richardson multiplication, $\mathcal{A}_\lambda$ is Gorenstein exactly for rectangles, and the Hilbert series $\sum_{\mu\subseteq\lambda} t^{|\mu|}$ is a weighted lattice-point enumerator of an order polytope.

For an arbitrary set D of cells, the Cauchy kernel identifies the graded dual of $\mathcal{A}_D$ with an explicit space of symmetric functions given by signed closed walks in D. The coefficient of a doubled-cell monomial $\prod_{e\in S} a_e b_e$ in $s_\mu(\phi)$ is the trace, on the irreducible representation V_μ of the symmetric group, of the generalized Young symmetrizer of $S \subseteq D$; if S is a Young diagram of shape τ, it is $\delta_{\mu\tau}$ times the product of the hook lengths of τ. Consequently $s_\mu(\phi) \neq 0$ if and only if V_μ occurs in the Specht module of some $S \subseteq D$. We describe $\mathcal{A}_D$ for disjoint unions and for diagrams whose connected components are Ferrers diagrams, and classify the Gorenstein algebras among the latter. On the geometric side, a Crofton-type formula writes the class of the diagram variety $Y_D \subseteq \mathrm{Gr}(r, m+r)$ as an average of doubled-cell functionals of tangent planes. It follows that the Schur support of $[Y_D]$ is the Specht support of D, and that $[Y_D]$ is proportional to the doubled-cell functional when D is Ferrers. We prove that $\mathcal{A}_D$ is canonically isomorphic to the image of $H^*(\mathrm{Gr}(r, m+r))$ in $H^*(Y_D)$ whenever all components of D are Ferrers, show that this fails for the four-cell zigzag $(3,2)/(1)$, and verify by exact computation that it holds for the non-Ferrers diagram $(3,3,2)/(1)$.

1. Introduction

Canonical invariant Chern–Weil forms on type-A flag varieties generate finite-dimensional graded algebras whose structure can be richer than that of the cohomology rings represented by these forms [16, 12, 13]. For a Grassmannian, the curvature matrix at the standard point factors as a product AB of rectangular matrices of odd Maurer–Cartan variables. We study the specializations obtained by retaining only the entries indexed by a set of cells. For Young diagrams the resulting algebras have a uniform description in terms of symmetric functions and Schubert geometry (Theorems A and B). For arbitrary sets of cells they are governed by Specht modules of diagrams (Theorems C and D), and a Crofton-type formula compares them with the cohomology of Grassmannian diagram varieties (Theorem E).

1.1. Young diagrams.

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$, put $m = \lambda_1$, and write

$$D_\lambda = \{(c, i) : 1 \leq i \leq r,\ 1 \leq c \leq \lambda_i\}$$

for its Young diagram, with row coordinate i and column coordinate c. In the exterior algebra on independent odd variables a_{ci}, b_{ic} indexed by D_λ, form the zero-pattern matrices

$$A_\lambda = (a_{ci})_{m\times r}, \qquad B_\lambda = (b_{ic})_{r\times m}.$$

Date: October 2026.

2020 *Mathematics Subject Classification.* Primary 05E05, 14M15; Secondary 05E10, 20C30, 14F40, 15A15.

Key words and phrases. Young diagram, diagram variety, odd matrix, Schur function, Specht module, Littlewood–Richardson coefficient, Schubert variety, Richardson variety, Chern–Weil form, Crofton formula, curvature algebra.

The entries of $A_\lambda B_\lambda$ are even and therefore commute. Define

$$H_\lambda(t) = \det(I_m + tA_\lambda B_\lambda) = 1 + \phi_1 t + \cdots + \phi_m t^m$$

and let $\mathcal{A}_\lambda$ be the subalgebra generated by the ϕ_j, with $\deg \phi_j = j$ (half the exterior degree).

Let $\mathrm{Sym} = \mathbb{C}[h_1, h_2, \ldots]$ be the ring of symmetric functions in the complete generators. For a partition ρ, let s_ρ be the corresponding Schur function and set

$$I_\lambda = \mathrm{span}_{\mathbb{C}}\{s_\rho : \rho \not\subseteq \lambda\}.$$

The Littlewood–Richardson rule implies that I_λ is an ideal.

Theorem A. Evaluation $h_j \mapsto \phi_j$ induces a graded-algebra isomorphism

$$\mathrm{Sym}/I_\lambda \xrightarrow{\sim} \mathcal{A}_\lambda.$$

Thus

$$\{s_\mu(\phi) : \mu \subseteq \lambda\} \quad \text{and} \quad \{\phi_{\mu_1} \cdots \phi_{\mu_\ell} : \mu \subseteq \lambda\}$$

are homogeneous bases, and

$$\mathrm{Hilb}(\mathcal{A}_\lambda; t) = \sum_{\mu \subseteq \lambda} t^{|\mu|}.$$

In the Schur basis, multiplication is given by

$$s_\mu(\phi) s_\nu(\phi) = \sum_{\tau \subseteq \lambda} c^\tau_{\mu\nu} s_\tau(\phi).$$

The vanishing half of Theorem A is representation-theoretic. The span of the cells of D_λ in $\mathbb{C}^m \otimes (\mathbb{C}^r)^*$ is stable under a product of Borel subgroups, so its exterior powers project nontrivially only to skew Cauchy summands $\mathbb{S}_{\mu'}\mathbb{C}^m \otimes \mathbb{S}_\mu(\mathbb{C}^r)^*$ with $\mu \subseteq \lambda$; an odd version of the Cauchy–Frobenius formula then forces $s_\mu(\phi) = 0$ for $\mu \not\subseteq \lambda$. Linear independence comes from the doubled-cell monomials

$$\Omega_\tau = \prod_{(c,i) \in D_\tau} a_{ci} b_{ic}, \qquad \tau \subseteq \lambda.$$

The coefficient of Ω_τ in $s_\mu(\phi)$ is the trace of a Young symmetrizer of shape τ acting on the irreducible representation V_μ of the symmetric group; hence it equals $\delta_{\mu\tau} H_\tau$, where H_τ is the product of the hook lengths of τ (Lemma 4.1).

The quotient in Theorem A is geometrically natural. If $\lambda \subseteq m^r$ and λ^c denotes the 180°-rotated complement of λ in this rectangle, let X_{λ^c} be the corresponding Schubert variety in $\mathrm{Gr}(r, m+r)$.

Theorem B. There is a graded-algebra isomorphism

$$\mathcal{A}_\lambda \cong H^*(X_{\lambda^c}; \mathbb{C}),$$

where algebra degree d corresponds to cohomological degree $2d$.

For a rectangle $\lambda = m^r$ this is the cohomology ring of $\mathrm{Gr}(r, m+r)$. For nonrectangular λ the Hilbert series is not palindromic, so $\mathcal{A}_\lambda$ is neither Gorenstein nor a complete intersection, and X_{λ^c} is singular. Multiplication by ϕ_1 is the Young-lattice up-operator on $[\varnothing, \lambda]$, with nilpotency index $|\lambda| + 1$.

The Hilbert series also has a polyhedral reading. The subdiagrams $\mu \subseteq \lambda$ are the order ideals of the cell poset of D_λ; their indicator vectors are the lattice points of the corresponding order polytope, and degree is the coordinate sum. This is distinct from the root-polytope Ehrhart theory of bipartite graphs in [7]; see Section 5.

1.2. **Arbitrary supports.** The odd-matrix construction does not use northwest closure. For an arbitrary set of cells $D \subseteq [m] \times [r]$, called a *diagram*, place odd generators a_{ci}, b_{ic}, $(c,i) \in D$, in the cells of D to obtain matrices A_D and B_D, and put

$$H_D(t) = \det(I_m + tA_DB_D) = \sum_{j\geq 0} \phi_j(D)t^j, \qquad \mathcal{A}_D = \mathbb{C}[\phi_1(D), \phi_2(D), \ldots],$$

$$J_D = \ker\big(\mathrm{ev}_D : \mathrm{Sym} \to \mathcal{A}_D,\ h_j \mapsto \phi_j(D)\big), \qquad s_\mu(\phi_D) = \mathrm{ev}_D(s_\mu).$$

Thus $\mathcal{A}_{D_\lambda} = \mathcal{A}_\lambda$ and $J_{D_\lambda} = I_\lambda$, but in general J_D is not spanned by Schur functions: for two cells in distinct rows and columns, $J_D \cap \mathrm{Sym}_2 = \mathbb{C}(s_2 - s_{11})$ (compare Remark 3.3). We describe J_D through its orthogonal complement $J_D^\perp \subseteq \mathrm{Sym}$ for the Hall inner product; in each degree, $J_D^\perp$ is the dual of the corresponding graded piece of $\mathcal{A}_D$.

For $S, T \subseteq D$ with $|S| = |T| = d$, listed in fixed orders, let a_S and b_T be the ordered products of the corresponding generators, and define $F_{S,T} \in \mathrm{Sym}_d$ by $\langle f, F_{S,T}\rangle = [a_Sb_T]\,\mathrm{ev}_D(f)$, the coefficient of the monomial a_Sb_T. For the doubled-cell monomial $\Omega_S = \prod_{e\in S} a_eb_e = (-1)^{\binom{d}{2}}a_Sb_S$ put $F_S = (-1)^{\binom{d}{2}}F_{S,S}$, so that $\langle f, F_S\rangle = [\Omega_S]\,\mathrm{ev}_D(f)$. Let R_S and C_S be the groups of permutations of S that preserve every row, respectively every column, regarded as subgroups of S_d through the order of S, and put

$$\mathbf{r}_S = \sum_{\rho\in R_S} \rho, \qquad \mathbf{c}_S = \sum_{\gamma\in C_S} \mathrm{sgn}(\gamma)\gamma, \qquad \mathbf{y}_S = \mathbf{c}_S\mathbf{r}_S \in \mathbb{C}[S_d].$$

The left ideal $\mathrm{Sp}(S) = \mathbb{C}[S_d]\mathbf{y}_S$ is the *Specht module* of the diagram S [6]; for $S = D_\tau$ it is the irreducible module V_τ. The *Specht support* of S is the set of partitions μ such that V_μ is a constituent of $\mathrm{Sp}(S)$. Write $U = \mathbb{C}^m$ and $W = \mathbb{C}^r$, let $e_S \in \bigwedge^d(U \otimes W^*)$ be the wedge product of the vectors $u_c \otimes w_i^*$, $(c,i) \in S$, and let π_μ be the orthogonal projection onto the summand $\mathbb{S}_{\mu'}U \otimes \mathbb{S}_\mu W^*$ of the skew Cauchy decomposition (3.4), for the Hermitian form in which the $u_c \otimes w_i^*$ are orthonormal. Finally, D is *Ferrers* if its nonempty rows, viewed as sets of columns, are totally ordered by inclusion; equivalently, some permutation of rows and columns carries D to a Young diagram D_τ, and τ is the *shape* of D.

Theorem C. Let $D \subseteq [m] \times [r]$ be an arbitrary diagram.

(i) *(Cauchy-kernel duality.)* In $\mathcal{A}_D \otimes \mathrm{Sym}(y)$,

$$\sum_\mu s_\mu(\phi_D) \otimes s_\mu(y) = \prod_{j\geq 1} H_D(y_j). \tag{1.1}$$

Consequently $F_{S,T}$ is the coefficient of a_Sb_T in (1.1), it does not depend on the ambient diagram $D \supseteq S \cup T$, and $J_D^\perp = \mathrm{span}\{F_{S,T} : S, T \subseteq D,\ |S| = |T|\}$. Explicitly,

$$F_{S,T} = (-1)^{\binom{d}{2}} \sum_{\alpha,\beta} \mathrm{sgn}(\beta)\, p_{\mathrm{type}(\beta\alpha)}, \tag{1.2}$$

where α runs over the row-preserving bijections $S \to T$, β over the column-preserving bijections $T \to S$, $\mathrm{sgn}(\beta)$ is taken with respect to the chosen orders, and $\mathrm{type}(\beta\alpha)$ is the cycle type of the permutation $\beta\alpha$ of S.

(ii) *(Doubled cells.)* For $S \subseteq D$ with $|S| = d$ and $\mu \vdash d$,

$$[\Omega_S]\, s_\mu(\phi_D) = \chi^\mu(\mathbf{y}_S) = \frac{d!}{f^\mu}\,\|\pi_\mu e_S\|^2. \tag{1.3}$$

This is a nonnegative integer, and it is nonzero if and only if μ lies in the Specht support of S. If S is Ferrers of shape τ, then $F_S = H_\tau s_\tau$.

(iii) *(Schur support.)* $s_\mu(\phi_D) \neq 0$ if and only if μ lies in the Specht support of some $S \subseteq D$ with $|S| = |\mu|$.

Parts (i), (ii) and (iii) are Theorems 7.1, 7.4 and 7.6. The closed-walk formula (1.2) comes from expanding the power sums $\mathrm{ev}_D(p_k) = (-1)^{k-1}\operatorname{tr}((A_D B_D)^k)$: in a closed walk an a-cell is followed by a b-cell in the same row, which is followed by an a-cell in the same column. For $S = T$ the pairs (α, β) range over $R_S \times C_S$, and (1.2) becomes the first equality in (1.3). For $D = D_\lambda$ and $S = D_\tau$ this is the Young symmetrizer computation of Lemma 4.1. Combined with Theorem A, part (ii) yields a statement about symmetric groups alone: for every subset $S \subseteq D_\lambda$, all irreducible constituents of $\mathrm{Sp}(S)$ are indexed by subdiagrams of λ. The Borel argument of Section 3 is thus a proof of this fact. The doubled-cell coefficient is a trace, not a multiplicity: for the zigzag

$$Z = (3,2)/(1) = \{(2,1),(3,1),(1,2),(2,2)\}$$

one has $F_Z = 4s_{31} + 6s_{22}$, whereas the Frobenius characteristic of $\mathrm{Sp}(Z)$ is $s_{(3,2)/(1)} = s_{31} + s_{22}$. Remark 3.3 is the smallest instance of (iii): two cells in distinct rows and columns have Specht module $\mathbb{C}[S_2]$, and $F_S = h_1^2 = s_2 + s_{11}$. We conjecture that the doubled-cell functionals alone span $J_D^\perp$, that is, that an element of $\mathcal{A}_D$ vanishes as soon as all its doubled-cell coefficients vanish; this holds for forests and for diagrams with Ferrers components, and in all cases we have computed (Section 7).

The cells of D are the edges of a bipartite graph G_D whose vertices are the nonempty rows and columns; its connected components are the *components* of D. A Ferrers diagram is connected. We write $D = D_1 \sqcup D_2$ when D_1 and D_2 share no row and no column, and let Δ be the coproduct of Sym.

Theorem D. (i) If $D = D_1 \sqcup D_2$, then $J_D^\perp = J_{D_1}^\perp \cdot J_{D_2}^\perp$, the linear span of the products FG; equivalently, $J_D = \Delta^{-1}\big(J_{D_1} \otimes \mathrm{Sym} + \mathrm{Sym} \otimes J_{D_2}\big)$.

(ii) If the components of D are Ferrers of shapes $\lambda^1, \dots, \lambda^k$, then

$$J_D^\perp = \operatorname{span}\{s_{\mu^1} s_{\mu^2} \cdots s_{\mu^k} : \mu^j \subseteq \lambda^j\}.$$

(iii) For D as in (ii) the following are equivalent: $\mathcal{A}_D$ is Gorenstein; $\mathrm{Hilb}(\mathcal{A}_D; t)$ is palindromic; up to permutations of rows and columns, D is a rectangle, or a disjoint union of $k \geq 2$ rows of a common length, or a disjoint union of $k \geq 2$ columns of a common length.

Parts (i) and (ii) are Theorem 8.3 and its corollary, and (iii) is Theorem 8.6. Part (i) holds because $A_D B_D$ is block diagonal, so $H_D(t) = H_{D_1}(t) H_{D_2}(t)$, and Δ is adjoint to multiplication. The kernel carries more information than the algebra: k cells in distinct rows and columns and a single row of k cells both give $\mathbb{C}[q]/(q^{k+1})$, with $J^\perp$ spanned by the h_1^j and by the h_j, $j \leq k$, respectively. For k disjoint rows of length a, with $x_i = \sum_c a_{ci} b_{ic}$ the sum over the ith row, one finds $H_D(t) = \prod_i (1 - t x_i)^{-1}$, so $\mathcal{A}_D$ is the ring of S_k-invariants of $\bigotimes_i \mathbb{C}[x_i]/(x_i^{a+1})$; it is Gorenstein and has the same Hilbert series as the rectangle a^k. No Gorenstein algebra $\mathcal{A}_D$ outside the list in (iii) occurs among the diagrams of our computations (Section 8); we conjecture that (iii) holds for all D.

1.3. **Diagram varieties.** Let $L_D \subseteq \mathrm{Mat}_{r \times m}(\mathbb{C})$ consist of the matrices $X = (x_{ic})$ with $x_{ic} = 0$ for $(c, i) \notin D$, and let $Y_D \subseteq \mathrm{Gr}(r, m+r)$ be the closure of $\{\mathrm{rowspan}(I_r \mid X) : X \in L_D\}$. It is irreducible of dimension $|D|$ and contains L_D as a dense open subset. For $D = D_\lambda$ it is the Schubert variety X_{λ^c} after a change of coordinate flags, and for skew D it is a Grassmannian Richardson variety [8, 14]; in the convention of [2, 14], which records forced zeros, Y_D is the diagram variety of the complement of D. Let $\mathcal{Q}$ be the tautological quotient bundle, so that $h_j \mapsto c_j(\mathcal{Q}) = \sigma_{(j)}$ is the homomorphism $\mathrm{Sym} \to H^*(\mathrm{Gr}(r, m+r); \mathbb{C})$ of Theorem 6.1. We regard the fundamental class of Y_D as the element $[Y_D] \in \mathrm{Sym}_{|D|}$ defined by

$$\langle f, [Y_D] \rangle = \int_{Y_D} f(c(\mathcal{Q})), \qquad f \in \mathrm{Sym}_{|D|};$$

the proof of Theorem 6.1 gives $[Y_{D_\lambda}] = s_\lambda$. Let $\mathcal{T}_D$ be the image of $H^*(\mathrm{Gr}(r, m+r); \mathbb{C}) \to H^*(Y_D; \mathbb{C})$, and let J_D^{geom} be the kernel of the induced surjection $\mathrm{Sym} \to \mathcal{T}_D$; write $J_D^{\mathrm{odd}} = J_D$. Thus

$\mathcal{A}_D \cong \mathrm{Sym}/J_D^{\mathrm{odd}}$ and $\mathcal{T}_D \cong \mathrm{Sym}/J_D^{\mathrm{geom}}$, and Theorems A and B say that $J_{D_\lambda}^{\mathrm{odd}} = J_{D_\lambda}^{\mathrm{geom}} = I_\lambda$ and $\mathcal{T}_{D_\lambda} = H^*(Y_{D_\lambda};\mathbb{C})$. In the top degree $d = |D|$ one has $(\mathcal{A}_D)_d = \mathbb{C}\,\Omega_D$ and $H^{2d}(Y_D;\mathbb{C}) = \mathbb{C}$, so $J_D^{\mathrm{odd}} \cap \mathrm{Sym}_d = F_D^{\perp}$ and $J_D^{\mathrm{geom}} \cap \mathrm{Sym}_d = [Y_D]^{\perp}$: in top degree the comparison of the two ideals asks whether F_D and $[Y_D]$ are proportional.

The two sides are linked by Chern–Weil theory. Give $\mathcal{Q}$ the metric induced by the standard Hermitian form on $\mathbb{C}^{m+r}$. At the base point $x_0 = \mathrm{rowspan}(I_r \mid 0)$ its curvature is the product of the second fundamental form and its adjoint [5, Chapter 0, §5]; in the chart coordinates x_{ic}, the Chern forms of $\mathcal{Q}$ at x_0 are, up to universal constants, the elements $\phi_j([m]\times[r])$ with a_{ci} and b_{ic} replaced by $d\bar{x}_{ic}$ and dx_{ic}. Restricting to L_D replaces them by $\phi_j(D)$. Thus $\mathcal{A}_D$ is the algebra of values at x_0 of the Chern–Weil forms restricted to L_D, and $\mathcal{T}_D$ is the algebra of their classes on Y_D. Identify $\mathrm{Mat}_{r\times m}(\mathbb{C})$ with $U \otimes W^*$ by $E_{ic} \mapsto u_c \otimes w_i^*$. For a complex d-plane $V \subseteq \mathrm{Mat}_{r\times m}(\mathbb{C})$ with unit Plücker vector v, put

$$F_V = \sum_{\mu \vdash d} \frac{d!}{f^\mu}\, \|\pi_\mu v\|^2\, s_\mu,$$

so that $F_{L_S} = F_S$ by (1.3).

Theorem E. Let $D \subseteq [m]\times[r]$ and $d = |D|$.

(i) *(Crofton formula.)* There is a constant $\kappa_d > 0$, depending only on d and on the normalization of the metric, such that for all $f \in \mathrm{Sym}_d$

$$\langle f, [Y_D]\rangle = \kappa_d \int_{L_D} \langle f, F_{V_X}\rangle\, d\mathrm{vol}(X), \qquad V_X = (I_r + XX^*)^{-1/2}\, L_D\, (I_m + X^*X)^{-1/2}, \tag{1.4}$$

where $d$vol is the Riemannian volume of the metric induced on the open subset $L_D \subseteq Y_D$. Consequently the Schur expansion of $[Y_D]$ has positive coefficients on the Specht support of D and vanishes elsewhere; if D is Ferrers, then $[Y_D] = (\deg Y_D/d!)\, F_D$.

(ii) *(Comparison.)* If every component of D is Ferrers, then $J_D^{\mathrm{odd}} = J_D^{\mathrm{geom}}$, so that $h_j \mapsto \phi_j(D)$ and $h_j \mapsto \sigma_{(j)}|_{Y_D}$ induce an isomorphism $\mathcal{A}_D \cong \mathcal{T}_D$. For such D, $\mathcal{T}_D = H^*(Y_D;\mathbb{C})$ if and only if D is connected.

(iii) *(The zigzag.)* For $Z = (3,2)/(1)$, the ideals J_Z^{odd} and J_Z^{geom} coincide in every degree except 4, where

$$(J_Z^{\mathrm{odd}})^{\perp} \cap \mathrm{Sym}_4 = \mathbb{C}\,(2s_{31} + 3s_{22}), \qquad (J_Z^{\mathrm{geom}})^{\perp} \cap \mathrm{Sym}_4 = \mathbb{C}\,(s_{31} + s_{22}).$$

Both quotients have Hilbert series $1 + t + 2t^2 + 2t^3 + t^4$.

(iv) *(A non-Ferrers example.)* For $D = (3,3,2)/(1)$, $J_D^{\mathrm{odd}} = J_D^{\mathrm{geom}}$, and both quotients have Hilbert series $1 + t + 2t^2 + 3t^3 + 3t^4 + 3t^5 + 3t^6 + t^7$. The proof is computer-assisted (exact linear algebra over $\mathbb{Q}$).

Part (i) is Theorem 9.2 together with Corollary 9.3; (ii) is Theorem 9.4; (iii) and (iv) are Propositions 9.6 and 9.9. A unitary transformation carries $X \in L_D$ to x_0 and the tangent plane of Y_D at X to a plane in the $U(r)\times U(m)$-orbit of V_X; the integrand in (1.4) is therefore the pointwise Chern–Weil density at X, and $F_D = F_{V_0}$ is its value at the base point. Since $V_X = g\, L_D\, h$ with $g \in GL_r(\mathbb{C})$ and $h \in GL_m(\mathbb{C})$, and π_μ commutes with $GL(U)\times GL(W)$, every F_{V_X} has the same support as F_D; with Theorem C this gives the support statement. For a Ferrers diagram in Young position, L_D is stable under left multiplication by upper triangular and right multiplication by lower triangular matrices, so the Iwasawa decomposition places all V_X in the $U(r)\times U(m)$-orbit of L_D and the integrand is a constant multiple of F_D. The comparison theorem (ii) itself follows from Theorems A and B, the product decomposition $Y_{D_1 \sqcup D_2} \cong Y_{D_1} \times Y_{D_2}$, the Künneth formula, and Theorem D(i). Liu proposed to identify classes of diagram varieties with Frobenius characteristics of diagram Specht modules (see [10, 14]), and this fails in general [14]; by (i), the Schur support of $[Y_D]$ is nevertheless always the Specht support of D.

For non-Ferrers D the average in (1.4) can change the direction of the central value F_D. For the zigzag, F_Z weights the two constituents of $\mathrm{Sp}(Z)$ by the traces 4 and 6, while the Richardson class $[Y_Z] = s_{(3,2)/(1)}$ weights them by their multiplicities. Since the discrepancy lies in the top degree, it persists for $Z \sqcup D'$ whenever all components of D' are Ferrers: $F_{Z\sqcup D'} = F_Z F_{D'}$ and $[Y_{Z\sqcup D'}] = [Y_Z][Y_{D'}]$, where $F_{D'}$ is a nonzero multiple of $[Y_{D'}]$, and Sym is a domain. The diagram $D = (3,3,2)/(1)$, the 3×3 square with two opposite corners removed, is invariant under transposition, so F_D is invariant under the involution ω of Sym. Its support is the Specht support of the skew shape, namely $\{(3,3,1),(3,2,2)\}$ by [6], so F_D lies on the ω-invariant line $\mathbb{C}(s_{331}+s_{322})$, which also contains $[Y_D] = s_{(3,3,2)/(1)}$. This settles the top degree; the lower degrees are settled by exact computation. Among the connected non-Ferrers skew shapes λ/ν with at most eight cells, $(3,3,2)/(1)$ is the only one with $J_D^{\mathrm{odd}} = J_D^{\mathrm{geom}}$; for all others, F_D and $[Y_D] = s_{\lambda/\nu}$ are not proportional (Section 9). For $D = D_\lambda$, the Crofton formula gives a proof of the top-degree part of Theorem B that does not use the Schubert cell decomposition (Remark 6.3).

1.4. **Organization.** Section 2 introduces the odd-matrix model and its row–column symmetry, and Section 3 proves Schur vanishing. Section 4 computes the doubled-cell coefficients of Young diagrams and proves Theorem A. Section 5 gives the order-polytope interpretation, and Section 6 proves Theorem B and its consequences. Section 7 treats the curvature algebra of an arbitrary diagram: the Cauchy kernel and the closed-walk formula, doubled-cell coefficients and Specht modules, Schur support, cell deletion and forests (Theorem C). Section 8 contains low-degree and functorial properties, transposition, disjoint unions, Ferrers components, the Gorenstein classification and a table of examples (Theorem D). Section 9 studies diagram varieties: the Chern–Weil interpretation of $\mathcal{A}_D$, the Crofton formula, the comparison theorem, and the diagrams $(3,2)/(1)$ and $(3,3,2)/(1)$ (Theorem E). Computer calculations are used only where this is stated explicitly, and they are described where they occur.

2. Odd matrices with Young-diagram support

Fix a nonempty partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ and set $m = \lambda_1$. Let E_λ be the exterior algebra on the $2|\lambda|$ odd generators

$$a_{ci},\ b_{ic}, \qquad (c,i) \in D_\lambda.$$

Set

$$A_\lambda = (a_{ci})_{1\leq c\leq m,\, 1\leq i\leq r}, \qquad B_\lambda = (b_{ic})_{1\leq i\leq r,\, 1\leq c\leq m},$$

with zero entries outside D_λ, and put $Q_\lambda = A_\lambda B_\lambda$. Since the entries of Q_λ have even exterior degree, they belong to a commutative subalgebra of E_λ.

Definition 2.1. Define

$$H_\lambda(t) = \det(I_m + tQ_\lambda) = \sum_{j=0}^{m} \phi_j t^j, \qquad \phi_0 = 1, \tag{2.1}$$

and put $\phi_j = 0$ for $j > m$. The *Young-diagram curvature algebra* is

$$\mathcal{A}_\lambda = \mathbb{C}[\phi_1, \ldots, \phi_m] \subset E_\lambda, \qquad \deg \phi_j = j.$$

When $\lambda = m^r$ is a rectangle, this is the odd-matrix model for the curvature-form algebra of $\mathrm{Gr}(r, m+r)$. An arbitrary λ gives its canonical Young-diagram zero-pattern specialization.

The reversed product gives a useful symmetry. Put

$$\widetilde{Q}_\lambda = B_\lambda A_\lambda, \qquad \widetilde{H}_\lambda(t) = \det(I_r + t\widetilde{Q}_\lambda).$$

Proposition 2.2 (Opposite determinant identity)**.** *One has*

$$H_\lambda(t)\widetilde{H}_\lambda(t) = 1. \tag{2.2}$$

In particular, the coefficients of H_λ and $\widetilde{H}_\lambda$ generate the same subalgebra of E_λ.

Proof. For every $q \geq 1$, cyclically moving the first odd factor through the other $2q-1$ odd factors gives

$$\operatorname{tr}((B_\lambda A_\lambda)^q) = -\operatorname{tr}((A_\lambda B_\lambda)^q).$$

The entries of both products are even and lie in a commutative nilpotent algebra. Hence the formal identity $\log\det(I+T) = \operatorname{tr}\log(I+T)$ applies. The logarithm of the left-hand side of (2.2) is zero, proving the identity. Comparing coefficients recursively in (2.2) proves the final assertion. □

The reciprocal relation explains why the coefficients ϕ_j play the role of complete rather than elementary symmetric functions. Indeed, under $h_j \mapsto \phi_j$, the coefficient of t^j in $\widetilde{H}_\lambda(t) = H_\lambda(t)^{-1}$ is the image of $(-1)^j e_j$.

3. Schur vanishing

3.1. **The Schur ideal.** Let $\mathrm{Sym} = \mathbb{C}[h_1, h_2, \ldots]$, with $h_0 = 1$ and $h_j = 0$ for $j < 0$. For a partition μ, use the Jacobi–Trudi convention

$$s_\mu = \det(h_{\mu_i - i + j}).$$

Recall the subspace

$$I_\lambda = \operatorname{span}_{\mathbb{C}}\{s_\rho : \rho \not\subseteq \lambda\}. \tag{3.1}$$

Lemma 3.1. *The subspace I_λ is an ideal of* Sym*. The quotient* Sym/I_λ *has homogeneous basis*

$$\{\overline{s}_\mu : \mu \subseteq \lambda\},$$

and multiplication

$$\overline{s}_\mu\,\overline{s}_\nu = \sum_{\tau \subseteq \lambda} c^\tau_{\mu\nu}\overline{s}_\tau. \tag{3.2}$$

Proof. If $c^\tau_{\rho\nu} > 0$, then $\rho \subseteq \tau$. Therefore $\rho \not\subseteq \lambda$ implies $\tau \not\subseteq \lambda$. The Littlewood–Richardson rule shows that the span in (3.1) is closed under multiplication. The remaining assertions follow because the Schur functions form a basis of Sym. □

Evaluation at the coefficients of (2.1) gives a surjective homomorphism

$$\mathrm{ev}_\lambda : \mathrm{Sym} \longrightarrow \mathcal{A}_\lambda, \qquad h_j \longmapsto \phi_j. \tag{3.3}$$

We next prove that I_λ is contained in its kernel.

3.2. **A skew Cauchy support lemma.** Let $U = \mathbb{C}^m$ and $W = \mathbb{C}^r$, with ordered bases $u_1, \ldots, u_m$ and $w_1, \ldots, w_r$. Regard the supported entries of A_λ as the basis vectors

$$u_c \otimes w_i^* \in U \otimes W^*, \qquad (c,i) \in D_\lambda,$$

and put

$$M_\lambda = \operatorname{span}\{u_c \otimes w_i^* : (c,i) \in D_\lambda\}.$$

The diagram D_λ is northwest closed: if $(c,i) \in D_\lambda$, $c' \leq c$, and $i' \leq i$, then $(c',i') \in D_\lambda$. Choose the upper-triangular Borel subgroups relative to the displayed bases of U and W^*. Then M_λ is stable under their product.

The skew Cauchy decomposition is

$$\bigwedge^d (U \otimes W^*) \cong \bigoplus_{\nu \vdash d} \mathbb{S}_\nu U \otimes \mathbb{S}_{\nu'} W^*, \tag{3.4}$$

with the usual dimension restrictions; see [1, 3, 18].

Lemma 3.2 (Young-diagram support)**.** *The projection of* $\bigwedge^d M_\lambda$ *to*

$$\mathbb{S}_{\mu'}U \otimes \mathbb{S}_\mu W^*$$

is zero unless $\mu \subseteq \lambda$.

Proof. Let π_μ be the equivariant projection in (3.4). Since $\bigwedge^d M_\lambda$ is Borel-stable, its image under π_μ is a Borel-stable subspace of the irreducible module $\mathbb{S}_{\mu'}U \otimes \mathbb{S}_\mu W^*$. If this image is nonzero, the Lie–Kolchin theorem gives in it a vector fixed by the product of the two unipotent radicals. The irreducible module has a unique such line, its highest-weight line, so the image contains that line.

The highest weight of this summand is (μ', μ). Since π_μ preserves torus weights, this weight must occur in $\bigwedge^d M_\lambda$. Every torus weight there is represented by a d-element subset $S \subseteq D_\lambda$: its U-weight records the column counts of S, and its W^*-weight records the row counts. Occurrence of the highest weight therefore forces the ith row count to equal μ_i. Row i of D_λ has only λ_i cells, so $\mu_i \le \lambda_i$ for every i. □

Remark 3.3. Northwest closure is essential. Consider the two-cell pattern $D = \{(1,1),(2,2)\}$ and put $z_i = a_i b_i$, so $z_i^2 = 0$. Then

$$H_D(t) = (1 + z_1 t)(1 + z_2 t)$$

and $s_{(2)}(\phi) = \phi_2 = z_1 z_2 \neq 0$, although no row of D has two cells. Without Borel stability, a weight count alone cannot force the highest weight to occur. In the language of Section 7, the two cells form a diagram S with $R_S = C_S = \{1\}$, so $\mathrm{Sp}(S) = \mathbb{C}[S_2]$ and $F_S = h_1^2 = s_2 + s_{11}$; Theorem 7.6 explains the nonvanishing of $s_{(2)}(\phi)$.

3.3. **An odd Cauchy–Frobenius identity.** Let $M \subset U \otimes W^*$ have a basis indexed by a finite set D. Introduce odd generators a_e, b_e, $e \in D$, and let

$$A : W \longrightarrow U, \qquad B : U \longrightarrow W$$

be the corresponding universal odd maps with mutually reversed support. For $d \ge 0$, define

$$\mathbf{A}_d = \sum_{e_j = (c_j, i_j) \in D} (u_{c_1} \otimes \cdots \otimes u_{c_d}) \otimes (w_{i_1}^* \otimes \cdots \otimes w_{i_d}^*) \otimes a_{e_1} \cdots a_{e_d},$$

$$\mathbf{B}_d = \sum_{e_j = (c_j, i_j) \in D} (u_{c_1}^* \otimes \cdots \otimes u_{c_d}^*) \otimes (w_{i_1} \otimes \cdots \otimes w_{i_d}) \otimes b_{e_1} \cdots b_{e_d}.$$

Terms with repeated indices vanish. Simultaneously permuting the tensor positions changes $\mathbf{A}_d$ and $\mathbf{B}_d$ by the sign of the permutation, so they are the universal alternating tensors of $\bigwedge^d M$ and its independently generated dual.

Let P_σ permute the U-tensor positions and put

$$C_\eta^U = \sum_{\sigma \in S_d} \chi^\eta(\sigma) P_\sigma.$$

If f^η is the dimension of the Specht module of shape η, then $(f^\eta/d!)C_\eta^U$ is the central idempotent projecting to the η-isotypic component.

Lemma 3.4 (Odd Cauchy–Frobenius identity)**.** *Let* $Q = AB$ *and*

$$\det(I + tQ) = \sum_{j \ge 0} \psi_j t^j.$$

For every partition $\mu \vdash d$,

$$d!\, s_\mu(\psi_1, \psi_2, \ldots) = (-1)^{\binom{d}{2}} \left\langle (C_{\mu'}^U \otimes 1)\mathbf{A}_d, \mathbf{B}_d \right\rangle, \tag{3.5}$$

where the brackets contract the U, U^*, W^*, *and* W *tensor factors componentwise. Consequently* $s_\mu(\psi)$ *factors through the* $\mathbb{S}_{\mu'}U \otimes \mathbb{S}_\mu W^*$ *projection of* $\bigwedge^d M$.

Proof. The ψ_j are the elementary symmetric functions of the even matrix Q. Thus the homomorphism $h_j \mapsto \psi_j$ is the standard involution ω followed by evaluation at Q, and

$$s_\mu(\psi_1, \psi_2, \ldots) = s_{\mu'}(Q).$$

The Frobenius formula [11, Chapter I] gives

$$d!\, s_{\mu'}(Q) = \sum_{\sigma \in S_d} \chi^{\mu'}(\sigma)\, \mathrm{Tr}(P_\sigma Q^{\otimes d}). \tag{3.6}$$

Expanding $Q = AB$, a typical term on the right is

$$\prod_{j=1}^{d} a_{c_j i_j} b_{i_j c_{\sigma(j)}}.$$

Moving all A-coefficients to the left of all B-coefficients contributes $(-1)^{\binom{d}{2}}$. The resulting sum is the contraction on the right-hand side of (3.5), up to replacing σ by σ^{-1}, which does not change an irreducible character.

Finally,

$$(P_\tau^U \otimes P_\tau^{W^*})\mathbf{A}_d = \mathrm{sgn}(\tau)\mathbf{A}_d.$$

Hence the μ'-isotypic projection in the U positions is paired with the μ-isotypic projection in the W^* positions. By the multiplicity-free decomposition (3.4), this is exactly the summand $\mathbb{S}_{\mu'}U \otimes \mathbb{S}_\mu W^*$. □

Theorem 3.5 (Schur vanishing)**.** *For the Young-diagram curvature algebra $\mathcal{A}_\lambda$,*

$$s_\mu(\phi_1, \phi_2, \ldots) = 0 \qquad \text{whenever } \mu \not\subseteq \lambda. \tag{3.7}$$

Consequently $I_\lambda \subseteq \ker(\mathrm{ev}_\lambda)$, and the elements $s_\mu(\phi)$ with $\mu \subseteq \lambda$ span $\mathcal{A}_\lambda$.

Proof. Apply Lemma 3.4 with $M = M_\lambda$, and then apply Lemma 3.2. Since the Schur functions form a basis of Sym, the surviving Schur elements span the image of ev_λ. □

4. Presentation and bases

4.1. **Doubled-cell coefficients.** For $\tau \subseteq \lambda$, define the doubled-cell monomial

$$\Omega_\tau = \prod_{(c,i) \in D_\tau} a_{ci} b_{ic}.$$

Its factors have even degree, so their order is immaterial.

Lemma 4.1 (Diagonal coefficient lemma)**.** *Let $\mu, \tau \subseteq \lambda$ with $|\mu| = |\tau|$. Then*

$$[\Omega_\tau]\, s_\mu(\phi) = \delta_{\mu\tau} H_\tau, \qquad H_\tau = \prod_{x \in D_\tau} h(x),$$

where $h(x)$ is the hook length of the cell x in D_τ. By the hook length formula, $H_\tau = |\tau|!/f^\tau$.

Proof. Put $d = |\tau|$ and $Q = Q_\lambda$; the hook product H_τ is not to be confused with the polynomial $H_\lambda(t)$. We write $e = (c, i)$ for a cell and a_e, b_e for a_{ci}, b_{ic}.

Step 1: power sums. In Sym one has $\sum_{k \geq 1} p_k t^k/k = \log \sum_{j \geq 0} h_j t^j$ [11, Chapter I, §2]. Applying ev_λ and the identity $\log\det(I + tQ) = \mathrm{tr}\log(I + tQ)$ used in the proof of Proposition 2.2 gives $\mathrm{ev}_\lambda(p_k) = (-1)^{k-1}\,\mathrm{tr}(Q^k)$. Since $s_\mu = \sum_{\rho \vdash d} z_\rho^{-1} \chi_\rho^\mu\, p_\rho$ [11, Chapter I, §7],

$$[\Omega_\tau]\, s_\mu(\phi) = \sum_{\rho \vdash d} z_\rho^{-1} \chi_\rho^\mu\, (-1)^{d - \ell(\rho)} [\Omega_\tau] \prod_j \mathrm{tr}(Q^{\rho_j}). \tag{4.1}$$

Step 2: closed walks. We have

$$\mathrm{tr}(Q^k) = \sum a_{c_1 i_1} b_{i_1 c_2} a_{c_2 i_2} b_{i_2 c_3} \cdots a_{c_k i_k} b_{i_k c_1},$$

a sum over closed walks in which the a-cell (c_j, i_j) is followed by the b-cell (c_{j+1}, i_j) in the same row, and this by the a-cell (c_{j+1}, i_{j+1}) in the same column (indices modulo k). A term of $\prod_j \operatorname{tr}(Q^{\rho_j})$ contributes to Ω_τ only if its a-cells and its b-cells are both the cells of D_τ, each occurring once. Such a term determines the bijections α, β of D_τ sending each a-cell to the following b-cell and each b-cell to the following a-cell. Then α preserves rows, β preserves columns, and the walks are the cycles of $\beta\alpha$, so $\beta\alpha$ has cycle type ρ. Conversely, a pair (α, β) with $\beta\alpha$ of type ρ arises from exactly z_ρ terms: one chooses which factor of $\prod_j \operatorname{tr}(Q^{\rho_j})$ carries each cycle and a starting cell in each cycle. The term is the product of the even elements $a_e b_{\alpha(e)}$, $e \in D_\tau$, in walk order. Listing $D_\tau = \{e_1, \dots, e_d\}$, we get

$$\prod_{k=1}^{d} a_{e_k} b_{\alpha(e_k)} = (-1)^{\binom{d}{2}} a_{e_1} \cdots a_{e_d}\, b_{\alpha(e_1)} \cdots b_{\alpha(e_d)} = (-1)^{\binom{d}{2}} \operatorname{sgn}(\alpha)\, a_{e_1} \cdots a_{e_d}\, b_{e_1} \cdots b_{e_d} = \operatorname{sgn}(\alpha)\, \Omega_\tau.$$

Let R_τ and C_τ be the groups of permutations of D_τ preserving every row and every column. Since $(-1)^{d-\ell(\rho)} = \operatorname{sgn}(\beta\alpha)$, the term of (4.1) indexed by ρ equals $\chi^\mu_\rho \sum \operatorname{sgn}(\beta)$, the sum over the pairs $(\alpha, \beta) \in R_\tau \times C_\tau$ with $\beta\alpha$ of type ρ. Summing over ρ,

$$[\Omega_\tau]\, s_\mu(\phi) = \sum_{\alpha \in R_\tau,\ \beta \in C_\tau} \operatorname{sgn}(\beta)\, \chi^\mu(\beta\alpha) = \chi^\mu(\mathbf{c}_\tau \mathbf{r}_\tau), \tag{4.2}$$

where $\mathbf{r}_\tau = \sum_{\alpha \in R_\tau} \alpha$, $\mathbf{c}_\tau = \sum_{\beta \in C_\tau} \operatorname{sgn}(\beta)\beta$, and χ^μ is extended linearly to the group algebra of the permutations of D_τ, identified with $\mathbb{C}[S_d]$ by numbering the cells.

Step 3: the Young symmetrizer. A numbering of the cells of D_τ is a tableau T of shape τ, and R_τ, C_τ become its row and column groups. Thus $y = \mathbf{r}_\tau \mathbf{c}_\tau$ is the Young symmetrizer of T (denoted $c_\lambda = a_\lambda b_\lambda$ in [4, §4.1]). By [4, Theorem 4.3], $y^2 = n_\tau y$ for a scalar $n_\tau \neq 0$, and the left ideal $\mathbb{C}[S_d]y$ is isomorphic to V_τ. Since traces are cyclic, $\chi^\mu(\mathbf{c}_\tau \mathbf{r}_\tau) = \chi^\mu(y)$. The element $\epsilon = y/n_\tau$ is idempotent, so its trace on V_μ is $\dim \epsilon V_\mu = \dim \operatorname{Hom}_{S_d}(\mathbb{C}[S_d]\epsilon, V_\mu) = \delta_{\mu\tau}$ by Schur's lemma. Hence $\chi^\mu(y) = \delta_{\mu\tau} n_\tau$. To compute n_τ, note that the coefficient of the identity in y is 1, because $R_\tau \cap C_\tau = \{1\}$. Hence right multiplication by y on $\mathbb{C}[S_d]$ has trace $d!$. It is also n_τ times right multiplication by ϵ, an idempotent with image $\mathbb{C}[S_d]\epsilon \cong V_\tau$ of dimension f^τ. Therefore $n_\tau = d!/f^\tau$, which equals H_τ by the hook length formula [4, §4.1]. □

Remark 4.2. Steps 1 and 2 use only that the cells involved belong to the support, and they apply to any subset S of an arbitrary diagram; this gives Theorem 7.4. The vanishing for $\mu \neq \tau$ also follows from highest-weight theory. By Lemma 3.4, $[\Omega_\tau] s_\mu(\phi)$ is a fixed nonzero multiple of $\langle C^U_{\mu'} v_\tau, v^*_\tau \rangle$, where $v_\tau = \bigwedge_{(c,i) \in D_\tau} (u_c \otimes w_i^*)$ and v^*_τ is the dual wedge vector. Each raising operator of the two Borel subgroups moves a cell of D_τ strictly left or strictly up, onto a cell already in the wedge, so v_τ spans the highest-weight line of $\mathbb{S}_{\tau'} U \otimes \mathbb{S}_\tau W^*$. The central element $C^U_{\mu'}$ therefore kills v_τ unless $\mu = \tau$.

Corollary 4.3. *The elements $s_\mu(\phi)$, $\mu \subseteq \lambda$, are linearly independent.*

Proof. In every degree, Lemma 4.1 shows that the matrix of coefficients $[\Omega_\tau] s_\mu(\phi)$, with $\mu, \tau \subseteq \lambda$, is diagonal with nonzero diagonal entries H_τ. □

4.2. **The Schur quotient.**

Theorem 4.4 (Young-diagram Schur-quotient theorem)**.** *The evaluation map* (3.3) *induces an isomorphism*

$$\operatorname{Sym}/I_\lambda \xrightarrow{\sim} \mathcal{A}_\lambda. \tag{4.3}$$

Equivalently, $\ker(\mathrm{ev}_\lambda) = I_\lambda$. *The two families*

$$\{s_\mu(\phi) : \mu \subseteq \lambda\}, \qquad \{\Phi_\mu := \phi_{\mu_1} \cdots \phi_{\mu_\ell} : \mu \subseteq \lambda\} \tag{4.4}$$

are homogeneous bases. Moreover,

$$\mathrm{Hilb}(\mathcal{A}_\lambda; t) = \sum_{\mu \subseteq \lambda} t^{|\mu|}, \tag{4.5}$$

and

$$s_\mu(\phi) s_\nu(\phi) = \sum_{\tau \subseteq \lambda} c_{\mu\nu}^{\tau} s_\tau(\phi). \tag{4.6}$$

Proof. Theorem 3.5 gives a spanning family indexed by the subdiagrams of λ, and Corollary 4.3 shows that this family is independent. Thus (4.3) is an isomorphism. Equations (4.5) and (4.6) follow from Lemma 3.1.

For the second basis, recall

$$h_\rho = \sum_{\nu \vdash |\rho|} K_{\nu\rho} s_\nu.$$

After passing to Sym/I_λ, discard the terms with $\nu \not\subseteq \lambda$. In every degree, the remaining square Kostka matrix indexed by $\rho, \nu \subseteq \lambda$ is unitriangular in dominance order. Therefore the images of h_ρ, $\rho \subseteq \lambda$, also form a basis. □

Corollary 4.5 (Pieri operator and nilpotence)**.** *For every $\mu \subseteq \lambda$,*

$$\phi_1 s_\mu(\phi) = \sum_{\substack{\nu \subseteq \lambda \\ \nu/\mu \text{ is one cell}}} s_\nu(\phi). \tag{4.7}$$

Thus multiplication by ϕ_1 is the up-operator on $[\varnothing, \lambda]$ in Young's lattice. More generally,

$$\phi_1^d = \sum_{\substack{\mu \subseteq \lambda \\ |\mu| = d}} f^\mu s_\mu(\phi), \tag{4.8}$$

where f^μ is the number of standard Young tableaux of shape μ. In particular,

$$\phi_1^{|\lambda|} = f^\lambda s_\lambda(\phi) \neq 0, \qquad \phi_1^{|\lambda|+1} = 0.$$

Proof. Equation (4.7) is the ordinary Pieri rule followed by the truncation in Theorem 4.4. Iterating from the empty partition counts saturated chains in Young's lattice, hence standard Young tableaux, proving (4.8). In degree $|\lambda|$, the only subdiagram of λ is λ itself. □

Corollary 4.6 (Finite straightening)**.** *Every product Φ_ρ has a unique expansion in the basis $\{\Phi_\mu : \mu \subseteq \lambda\}$. It is obtained by expanding h_ρ in the Schur basis, deleting the terms indexed by diagrams outside λ, and then applying the inverse of the restricted unitriangular Kostka matrix.*

Example 4.7 (Hooks)**.** For $\lambda = (m, 1^{r-1})$, the complete-monomial basis is

$$1, \qquad \phi_a \phi_1^b, \quad 1 \le a \le m, \quad 0 \le b \le r-1.$$

Consequently

$$\mathrm{Hilb}(\mathcal{A}_\lambda; t) = 1 + \sum_{a=1}^{m} \sum_{b=0}^{r-1} t^{a+b}.$$

5. The order polytope of a Young diagram

Let P_λ be the poset whose elements are the cells of D_λ, with

$$(c, i) \preceq (d, j) \quad \Longleftrightarrow \quad c \le d \ \text{ and } \ i \le j.$$

Thus a subset of D_λ is a Young subdiagram precisely when it is an order ideal of P_λ. Consider the ideal form of Stanley's order polytope [17],

$$\mathcal{O}_\lambda = \left\{ x \in [0,1]^{D_\lambda} : x_p \ge x_q \text{ whenever } p \preceq q \right\}. \tag{5.1}$$

This is the usual order polytope of the opposite poset. For $x = (x_p)_{p\in D_\lambda}$, write $|x| = \sum_{p\in D_\lambda} x_p$.

Proposition 5.1 (Order-polytope formula)**.** *The Hilbert series of $\mathcal{A}_\lambda$ is the weighted lattice-point enumerator*

$$\mathrm{Hilb}(\mathcal{A}_\lambda; t) = \sum_{x\in\mathcal{O}_\lambda\cap\mathbb{Z}^{D_\lambda}} t^{|x|}. \tag{5.2}$$

Equivalently,

$$\dim(\mathcal{A}_\lambda)_d = \#\left(\mathcal{O}_\lambda \cap \mathbb{Z}^{D_\lambda} \cap \left\{x : \sum_p x_p = d\right\}\right).$$

More generally, the Ehrhart value

$$L_{\mathcal{O}_\lambda}(k) = \#(k\mathcal{O}_\lambda \cap \mathbb{Z}^{D_\lambda})$$

counts plane partitions of shape λ with entries in $\{0, 1, \ldots, k\}$.

Proof. Every lattice point of $\mathcal{O}_\lambda$ is a zero–one vector. The inequalities in (5.1) say exactly that its support is an order ideal of P_λ, hence a subdiagram D_μ with $\mu \subseteq \lambda$. Under this bijection, $|x| = |\mu|$, so (5.2) follows from Theorem 4.4.

The lattice points of $k\mathcal{O}_\lambda$ are the integer-valued order-reversing maps $P_\lambda \to \{0, 1, \ldots, k\}$. Written in the cells of D_λ, these are precisely arrays weakly decreasing along rows and columns, that is, plane partitions of shape λ bounded by k. □

Remark 5.2 (Comparison with the bipartite root polytope). Let G_λ be the bipartite incidence graph of the cells of D_λ. Kálmán and Postnikov [7] show that the interior polynomial of the hypergraphs induced by G_λ is equivalent to the Ehrhart polynomial, and to the triangulation h-vector, of the root polytope Q_{G_λ}. Proposition 5.1 concerns a different polytope and a different specialization: it is a coordinate-sum enumerator of the lattice points of $\mathcal{O}_\lambda$ at dilation one, not the Ehrhart numerator of Q_{G_λ}.

The two polynomials already differ for the rectangle $\lambda = 2^2$. In that case

$$\mathrm{Hilb}(\mathcal{A}_{2^2}; t) = 1 + t + 2t^2 + t^3 + t^4,$$

whereas the interior polynomial of $G_{2^2} = K_{2,2}$ is $1 + t$. Thus the order-polytope interpretation gives a genuine polytopal model for the Hilbert series, but it is not a consequence of bipartite root-polytope duality.

6. Schubert geometry

Put $m = \lambda_1$ and $r = \ell(\lambda)$, so $\lambda \subseteq m^r$. For $\rho \subseteq m^r$, padded by zeros to length r, define

$$\rho^c = (m - \rho_r, m - \rho_{r-1}, \ldots, m - \rho_1),$$

the complement in m^r rotated by 180°. This is an order-reversing involution. Let $\sigma_\rho \in H^{2|\rho|}(\mathrm{Gr}(r, m+r); \mathbb{C})$ be the Schubert class and let X_ρ be the corresponding Schubert variety, of codimension $|\rho|$.

Presentations of ordinary cohomology rings of Schubert varieties as quotients by complementary Schubert classes are standard; see [15]. The following Grassmannian argument records the precise convention needed here.

Theorem 6.1 (Schubert realization)**.** *Let $\iota : X_{\lambda^c} \hookrightarrow \mathrm{Gr}(r, m + r)$ be the inclusion. Then ι^* is surjective and*

$$\ker \iota^* = \mathrm{span}_{\mathbb{C}}\{\sigma_\rho : \rho \not\subseteq \lambda\}.$$

Consequently the composite

$$\mathrm{Sym} \xrightarrow{h_j\mapsto\sigma_{(j)}} H^*(\mathrm{Gr}(r, m + r); \mathbb{C}) \xrightarrow{\iota^*} H^*(X_{\lambda^c}; \mathbb{C})$$

induces a graded-algebra isomorphism

$$\mathcal{A}_\lambda \cong H^*(X_{\lambda^c};\mathbb{C}), \tag{6.1}$$

where cohomological degree $2d$ *corresponds to degree* d *in* $\mathcal{A}_\lambda$.

Proof. The Grassmannian Schur presentation gives

$$\mathrm{Sym}/I_{m^r} \cong H^*(\mathrm{Gr}(r,m+r);\mathbb{C}), \qquad s_\rho \longmapsto \sigma_\rho.$$

The Schubert cells of X_{λ^c} are indexed by $\nu \supseteq \lambda^c$. They have even real dimension, and the classes $[X_\nu]$ form a basis of $H_*(X_{\lambda^c};\mathbb{C})$. For every such ν,

$$\big\langle \iota^*\sigma_\rho, [X_\nu]\big\rangle = \int_{\mathrm{Gr}(r,m+r)} \sigma_\rho \smile \sigma_\nu = \delta_{\rho,\nu^c}.$$

It follows that $\iota^*\sigma_\rho = 0$ when $\rho \not\subseteq \lambda$, while $\{\iota^*\sigma_\rho : \rho \subseteq \lambda\}$ is the cohomology basis dual to $\{[X_{\rho^c}] : \rho \subseteq \lambda\}$. Hence ι^* is surjective with the stated kernel. Theorem 4.4 now gives (6.1). □

Remark 6.2. For $\lambda = m^r$, one has $\lambda^c = \varnothing$ and $X_\varnothing = \mathrm{Gr}(r, m+r)$. The realization is stable under enlarging the ambient rectangle: any rectangle M^R containing λ gives a Schubert variety in $\mathrm{Gr}(R, M+R)$ with the same cohomology ring Sym/I_λ.

Remark 6.3 (Chern–Weil form of the top degree). Put $d = |\lambda|$ and write $f(\sigma)$ for the image of $f \in \mathrm{Sym}$ under $h_j \mapsto \sigma_{(j)}$. By the proof of Theorem 6.1, $\int_{X_{\lambda^c}} f(\sigma) = \langle f, s_\lambda\rangle$ for $f \in \mathrm{Sym}_d$, and by Lemma 4.1, $[\Omega_\lambda] f(\phi) = H_\lambda \langle f, s_\lambda\rangle$. Hence the top-degree part of Theorem B is the identity

$$\int_{X_{\lambda^c}} f(\sigma) = H_\lambda^{-1}\,[\Omega_\lambda]\, f(\phi), \qquad f \in \mathrm{Sym}_d,$$

which expresses the integral of a characteristic class over X_{λ^c} through the coefficient of its Chern–Weil form at a single point. The proportionality of the two functionals can be proved without the Schubert cell decomposition. Since $X_{\lambda^c} = Y_{D_\lambda}$, Corollary 9.3, a consequence of the Crofton formula of Theorem 9.2, gives $[X_{\lambda^c}] = (\deg X_{\lambda^c}/d!)\, F_{D_\lambda}$, a positive multiple of s_λ: at every point of the dense chart L_{D_λ} the Chern–Weil density of $f(c(\mathcal{Q}))$ is a positive multiple, independent of f, of $\langle f, F_{D_\lambda}\rangle$. Comparing the constants with $[X_{\lambda^c}] = s_\lambda$ gives $\deg X_{\lambda^c} = d!/H_\lambda$. Thus the hook length formula, which enters Lemma 4.1 through the Young symmetrizer constant $n_\lambda = d!/f^\lambda$, reappears as the classical degree formula $\deg X_{\lambda^c} = \langle h_1^d, s_\lambda\rangle = f^\lambda$, the image under Theorem 6.1 of the top-degree case of Corollary 4.5.

Corollary 6.4 (Duality and singularity)**.** *The following conditions are equivalent:*

(i) λ *is a rectangle;*
(ii) $\mathrm{Hilb}(\mathcal{A}_\lambda; t)$ *is palindromic;*
(iii) $\mathcal{A}_\lambda$ *is a graded Artinian Gorenstein algebra;*
(iv) $\mathcal{A}_\lambda$ *is a graded complete intersection.*

If λ *is nonrectangular,* X_{λ^c} *is singular and* $\mathcal{A}_\lambda$ *is not the rational cohomology algebra of a compact oriented manifold. If* λ *is rectangular,* ϕ_1 *is a strong Lefschetz element.*

Proof. The coefficient of t in (4.5) is 1, whereas the coefficient of $t^{|\lambda|-1}$ is the number of removable corners of λ, or equivalently the number of distinct positive part sizes. Thus a nonrectangular λ has nonpalindromic Hilbert series. For $\lambda = m^r$, the Hilbert series is a Gaussian binomial and is palindromic.

If $\lambda = m^r$, then $\mathcal{A}_\lambda \cong H^*(\mathrm{Gr}(r, m+r);\mathbb{C})$. The standard Chern-class presentation of Grassmannian cohomology is a graded complete intersection and hence Gorenstein. Conversely, a graded Artinian Gorenstein algebra has palindromic Hilbert series, and every Artinian complete intersection is Gorenstein. This proves the equivalence.

Ordinary cohomology of a compact oriented manifold satisfies Poincaré duality, as does that of a smooth projective variety. The nonpalindromicity therefore gives both stated obstructions. In the rectangular case, ϕ_1 corresponds to the Plücker hyperplane class, so hard Lefschetz makes it a strong Lefschetz element. □

For diagrams whose components are Ferrers diagrams, the analogous classification is Theorem 8.6.

Corollary 6.5 (Transposition)**.** *There is a graded-algebra isomorphism*

$$\mathcal{A}_\lambda \cong \mathcal{A}_{\lambda'}$$

obtained by exchanging rows and columns and swapping the odd generators a_{ci} and b_{ic}. Under the Schur presentations, the induced map is

$$s_\rho \longmapsto (-1)^{|\rho|} s_{\rho'}. \tag{6.2}$$

Proof. After exchanging rows and columns, the curvature matrix for λ' is $B_\lambda A_\lambda$. Proposition 2.2 shows that its determinant coefficients generate the same algebra as those of $A_\lambda B_\lambda$. On symmetric functions, the coefficient of t^j in $H_\lambda(t)^{-1}$ is $(-1)^j e_j$. Thus the induced automorphism is the standard involution ω followed by multiplication by $(-1)^d$ in degree d, which gives (6.2). Since $\rho \not\subseteq \lambda$ if and only if $\rho' \not\subseteq \lambda'$, this automorphism carries the relevant Schur ideals to one another. □

7. Arbitrary supports I: duality and doubled-cell coefficients

From now on $D \subseteq [m] \times [r]$ is an arbitrary finite set of cells (c, i), with column c and row i as before. We call D a *diagram*.

7.1. **Setup.** Let E_D be the exterior algebra on odd generators a_{ci}, b_{ic}, $(c, i) \in D$. Let A_D be the $m \times r$ matrix with a_{ci} in position (c, i) for $(c, i) \in D$ and zeros elsewhere, let B_D be the $r \times m$ matrix with b_{ic} in position (i, c), and put $Q_D = A_D B_D$. Define

$$H_D(t) = \det(I_m + tQ_D) = \sum_{j \geq 0} \phi_j(D) t^j, \qquad \mathcal{A}_D = \mathbb{C}[\phi_1(D), \phi_2(D), \ldots] \subset E_D,$$

and let $\mathrm{ev}_D : \mathrm{Sym} \to \mathcal{A}_D$, $h_j \mapsto \phi_j(D)$, with kernel J_D. We write $s_\mu(\phi_D) = \mathrm{ev}_D(s_\mu)$. For $D = D_\lambda$ these are the objects of Sections 2–4. The proof of Proposition 2.2 uses only that the entries of A_D and B_D are odd, so $H_D(t) \det(I_r + tB_D A_D) = 1$ for every D; in particular $p_k \mapsto (-1)^{k-1} \operatorname{tr}(Q_D^k)$ under ev_D.

For a subset $S \subseteq D$ with $d = |S|$ we fix an order $S = \{s_1, \ldots, s_d\}$ and put

$$a_S = a_{s_1} \cdots a_{s_d}, \qquad b_S = b_{s_1} \cdots b_{s_d}, \qquad \Omega_S = \prod_{e \in S} a_e b_e = (-1)^{\binom{d}{2}} a_S b_S.$$

Here a_e and b_e denote the two generators attached to the cell e; the factors of Ω_S are even, so Ω_S does not depend on the order. Since $\mathrm{ev}_D(f)$ is a combination of products of entries of Q_D, it lies in the span of the linearly independent monomials $a_S b_T$ with $|S| = |T|$.

The *row counts* of S are $r_i(S) = \#\{c : (c, i) \in S\}$ and the *column counts* are $c_c(S) = \#\{i : (c, i) \in S\}$. Through the chosen order we regard permutations of S as elements of S_d. Let $R_S \subseteq S_d$ (respectively C_S) be the group of permutations of S that preserve the row (respectively column) of every cell, and put

$$\mathbf{r}_S = \sum_{\rho \in R_S} \rho, \qquad \mathbf{c}_S = \sum_{\gamma \in C_S} \mathrm{sgn}(\gamma)\gamma, \qquad \mathbf{y}_S = \mathbf{c}_S \mathbf{r}_S \in \mathbb{C}[S_d].$$

The left ideal $\mathrm{Sp}(S) = \mathbb{C}[S_d]\mathbf{y}_S$ is the *Specht module* of the diagram S [6, 9]. If $S = D_\tau$ is a Young diagram, $\mathbf{y}_S$ is a Young symmetrizer and $\mathrm{Sp}(S) \cong V_\tau$, the irreducible S_d-module of shape τ; in general $\mathrm{Sp}(S)$ is reducible. For a partition $\mu \vdash d$ let χ^μ be the character of V_μ, extended linearly to $\mathbb{C}[S_d]$.

A diagram is *Ferrers* if its nonempty rows, viewed as sets of columns, are totally ordered by inclusion. Equivalently, after permuting rows and columns it becomes a Young diagram D_τ, whose shape $\tau = \mathrm{sh}(D)$ is the decreasing rearrangement of the row counts; equivalently, no 2×2 submatrix of the $0/1$ matrix of D is a permutation matrix. The *components* of D are the connected components of the bipartite graph G_D whose vertices are the nonempty rows and columns and whose edges are the cells. We write $D = D_1 \sqcup D_2$ when D_1 and D_2 have no row and no column in common. Finally, $D^t = \{(i,c) : (c,i) \in D\}$.

7.2. **The Cauchy kernel.**

Theorem 7.1 (Cauchy kernel and duality)**.** *Let D be a diagram.*

(a) In $\mathcal{A}_D \otimes \mathrm{Sym}(y)$,

$$\sum_{\mu} s_\mu(\phi_D) \otimes s_\mu(y) = \prod_{j\ge 1} H_D(y_j). \tag{7.1}$$

(b) For $S,T \subseteq D$ with $|S| = |T| = d$ there is a unique $F_{S,T} \in \mathrm{Sym}_d$ with $\langle f, F_{S,T}\rangle = [a_S b_T]\,\mathrm{ev}_D(f)$ for all $f \in \mathrm{Sym}_d$. It is the coefficient of $a_S b_T$ in (7.1) and does not depend on the ambient diagram $D \supseteq S \cup T$.

(c) The graded dual of $\mathcal{A}_D$ is

$$(\mathcal{A}_D)^*_d \cong \mathrm{span}\{F_{S,T} : S,T \subseteq D,\ |S| = |T| = d\}, \qquad J_D \cap \mathrm{Sym}_d = \mathrm{span}\{F_{S,T}\}^\perp.$$

(d) $F_{S,T} = 0$ unless S and T have the same row counts and the same column counts. In general,

$$F_{S,T} = (-1)^{\binom{d}{2}} \sum_{\alpha,\beta} \mathrm{sgn}(\beta)\, p_{\mathrm{type}(\beta\alpha)}, \tag{7.2}$$

where α runs over the bijections $S \to T$ preserving rows, β over the bijections $T \to S$ preserving columns, $\mathrm{sgn}(\beta)$ is the sign of β with respect to the chosen orders of T and S, and $\mathrm{type}(\beta\alpha)$ is the cycle type of the permutation $\beta\alpha$ of S.

Proof. (a) In $\mathrm{Sym}(x)\otimes\mathrm{Sym}(y)$ the Cauchy identity reads $\sum_\mu s_\mu(x)s_\mu(y) = \prod_j \sum_k h_k(x) y_j^k$ [11, I.4]. Apply $\mathrm{ev}_D \otimes 1$.

(b), (c) Write $\mathrm{ev}_D(f) = \sum_{S,T} \kappa_{S,T}(f) a_S b_T$. Each $\kappa_{S,T}$ is a linear form on Sym_d, represented by a unique $F_{S,T}$ because $\langle\,,\rangle$ is nondegenerate on Sym_d. Pairing (7.1) with f in the y-variables gives $\mathrm{ev}_D(f)$, which proves the second assertion of (b). Setting the generators attached to cells outside $S\cup T$ equal to zero is an algebra homomorphism $E_D \to E_{S\cup T}$ sending Q_D to $Q_{S\cup T}$, hence $\phi_j(D)$ to $\phi_j(S\cup T)$ and $a_S b_T$ to itself. This gives the independence of D. Since the monomials $a_S b_T$ are linearly independent, $f \in J_D$ if and only if $\langle f, F_{S,T}\rangle = 0$ for all S,T, and (c) follows by finite-dimensional duality.

(d) The torus $(\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^r$ acts on E_D by scaling a_{ci} by $x_c y_i^{-1}$ and b_{ic} by $y_i x_c^{-1}$. It fixes $\det(I + tQ_D)$, hence every element of $\mathcal{A}_D$, while $a_S b_T$ has weight $\prod_c x_c^{c_c(S)-c_c(T)} \prod_i y_i^{r_i(T)-r_i(S)}$. This proves the first claim. For (7.2), write $f = \sum_\rho z_\rho^{-1}\langle f, p_\rho\rangle p_\rho$, so that $F_{S,T} = \sum_\rho z_\rho^{-1}\big([a_S b_T]\,\mathrm{ev}_D(p_\rho)\big)p_\rho$. By the remark in §7.1, $\mathrm{ev}_D(p_\rho) = \prod_j (-1)^{\rho_j - 1}\,\mathrm{tr}(Q_D^{\rho_j})$, and

$$\mathrm{tr}(Q_D^k) = \sum a_{c_1 i_1} b_{i_1 c_2} a_{c_2 i_2} b_{i_2 c_3} \cdots a_{c_k i_k} b_{i_k c_1}$$

is a sum over closed walks. In such a walk the a-cell (c_j, i_j) is followed by the b-cell (c_{j+1}, i_j) in the same row, which is followed by the a-cell (c_{j+1}, i_{j+1}) in the same column. Hence a monomial of $\prod_j \mathrm{tr}(Q_D^{\rho_j})$ whose a-cells form S and whose b-cells form T (all distinct, otherwise it vanishes) determines a row-preserving bijection $\alpha : S \to T$ and a column-preserving bijection $\beta : T \to S$ such that the walks are the cycles of $\beta\alpha$. Conversely every pair (α,β) with $\mathrm{type}(\beta\alpha) = \rho$ arises from exactly z_ρ such terms (choose which factor carries each cycle and its starting point). Each term

equals $\prod_{e\in S}(a_e b_{\alpha(e)})$, listed in walk order; since the factors are even we may list them in the order of S, and moving all a's to the left gives $(-1)^{\binom{d}{2}}\operatorname{sgn}(\alpha)a_S b_T$. Therefore

$$F_{S,T} = (-1)^{\binom{d}{2}} \sum_{\alpha,\beta} (-1)^{d-\ell(\mathrm{type}(\beta\alpha))} \operatorname{sgn}(\alpha)\, p_{\mathrm{type}(\beta\alpha)},$$

and $(-1)^{d-\ell(\mathrm{type}(\beta\alpha))} = \operatorname{sgn}(\beta\alpha) = \operatorname{sgn}(\beta)\operatorname{sgn}(\alpha)$. □

Remark 7.2. Formula (7.2) has been checked, including all signs, against a direct computation in the exterior algebra for all pairs (S,T) in several diagrams with up to seven cells. For $|S| = |T| = k$ it gives $\langle h_k, F_{S,T}\rangle = (-1)^{\binom{k}{2}} \sum_{\alpha,\beta} \operatorname{sgn}(\beta)$. The sum over β vanishes unless the cells of T lie in distinct columns, since otherwise the column-preserving bijections form a coset of a nontrivial product of symmetric groups. Hence $[a_S b_T]\phi_k(D) = 0$ unless S and T consist of k cells in the same k columns with the same row counts, in which case

$$[a_S b_T]\phi_k(D) = (-1)^{\binom{k}{2}} \operatorname{sgn}(\beta_0) \prod_i r_i(S)!, \qquad [\Omega_S]\phi_k(D) = \prod_i r_i(S)!, \tag{7.3}$$

where $\beta_0 : T \to S$ is the unique column-preserving bijection. This is the odd version of the Cauchy–Binet expansion of the principal minors of $A_D B_D$: in contrast with the even case, repeated rows do not cancel.

7.3. **Doubled-cell coefficients.** We identify $U \otimes W^*$ with $\mathrm{Mat}_{m\times r}(\mathbb{C})$ by $u_c \otimes w_i^* \mapsto E_{ci}$, with the Hermitian inner product $\langle X, Y\rangle = \operatorname{tr}(XY^*)$ and the induced inner product $\langle x_1\wedge\cdots\wedge x_d, y_1\wedge\cdots\wedge y_d\rangle = \det(\langle x_k, y_l\rangle)$ on $\bigwedge^d(U\otimes W^*)$. Let π_μ be the orthogonal projection onto the summand $\mathbb{S}_{\mu'}U \otimes \mathbb{S}_\mu W^*$ of (3.4). For $S \subseteq D$ put $e_S = E_{s_1} \wedge \cdots \wedge E_{s_d}$, where $E_{(c,i)} = E_{ci}$.

Lemma 7.3 (Norm lemma)**.** *Let $\xi_1,\dots,\xi_d,\eta_1,\dots,\eta_d \in \mathrm{Mat}_{m\times r}(\mathbb{C})$, let $a_1,\dots,a_d,b_1,\dots,b_d$ be odd generators, and put $A = \sum_k a_k\xi_k$, $B = \sum_k b_k\eta_k^*$ and $\det(I_m + tAB) = \sum_j \psi_j t^j$. Then for every $\mu \vdash d$,*

$$\Big[\prod_{k=1}^d a_k b_k\Big] s_\mu(\psi) = \frac{d!}{f^\mu}\,\big\langle \pi_\mu(\xi_1\wedge\cdots\wedge\xi_d),\ \eta_1\wedge\cdots\wedge\eta_d\big\rangle.$$

Proof. As in the proof of Lemma 3.4, $s_\mu(\psi) = s_{\mu'}(AB)$, and the Frobenius formula gives $d!\, s_{\mu'}(AB) = \sum_\sigma \chi^{\mu'}(\sigma)\operatorname{Tr}(P_\sigma(AB)^{\otimes d})$; this is a polynomial identity in the entries, valid over the commutative algebra of even elements. Expanding,

$$\operatorname{Tr}(P_\sigma(AB)^{\otimes d}) = \sum_{c,i}\prod_{j=1}^d A_{c_j i_j} B_{i_j c_{\sigma(j)}} = \sum_{c,i}\sum_{\kappa,\lambda}\prod_j a_{\kappa(j)} b_{\lambda(j)} \prod_j \xi_{\kappa(j)}(c_j,i_j)\,\overline{\eta_{\lambda(j)}(c_{\sigma(j)},i_j)},$$

where only maps $\kappa,\lambda : [d]\to[d]$ that are permutations contribute to $\prod_k a_k b_k$, and then

$$\prod_j a_{\kappa(j)} b_{\lambda(j)} = \operatorname{sgn}(\kappa)\operatorname{sgn}(\lambda)\prod_k a_k b_k.$$

For tensors in $(U\otimes W^*)^{\otimes d}$ let P_σ^U permute the U-factors. The remaining sum over c,i is

$$\big\langle P^U_{\sigma^{-1}}(\xi_{\kappa(1)}\otimes\cdots\otimes\xi_{\kappa(d)}),\ \eta_{\lambda(1)}\otimes\cdots\otimes\eta_{\lambda(d)}\big\rangle.$$

Put $\tilde\xi = \sum_\kappa \operatorname{sgn}(\kappa)\,\xi_{\kappa(1)}\otimes\cdots\otimes\xi_{\kappa(d)}$ and define $\tilde\eta$ likewise. Using $\chi(\sigma) = \chi(\sigma^{-1})$, we get

$$d!\,\Big[\prod_k a_k b_k\Big] s_\mu(\psi) = \Big\langle \sum_\sigma \chi^{\mu'}(\sigma) P_\sigma^U\tilde\xi,\ \tilde\eta\Big\rangle = \frac{d!}{f^\mu}\langle \pi^U_{\mu'}\tilde\xi, \tilde\eta\rangle,$$

where $\pi^U_{\mu'} = (f^{\mu'}/d!)\sum_\sigma \chi^{\mu'}(\sigma)P_\sigma^U$ is the isotypic projection in the U-factors and $f^{\mu'} = f^\mu$. On antisymmetric tensors $\pi^U_{\mu'}$ is the orthogonal projection onto $\mathbb{S}_{\mu'}U\otimes\mathbb{S}_\mu W^*$, by the multiplicity-free

decomposition (3.4). Finally $\tilde{\xi}$ and $\tilde{\eta}$ are $\sqrt{d!}$ times the antisymmetric tensors representing $\xi_1\wedge\cdots\wedge\xi_d$ and $\eta_1\wedge\cdots\wedge\eta_d$ with the normalisation fixed above. □

For $S\subseteq D$ define $F_S\in \mathrm{Sym}_{|S|}$ by $\langle f, F_S\rangle = [\Omega_S]\,\mathrm{ev}_D(f)$, so that $F_S = (-1)^{\binom{|S|}{2}}F_{S,S}$.

Theorem 7.4 (Doubled-cell coefficients)**.** *Let $S\subseteq D$, $d=|S|$, and $\mu\vdash d$.*

(a) $F_S = \sum_{\gamma\in C_S,\ \rho\in R_S}\mathrm{sgn}(\gamma)\,p_{\mathrm{type}(\gamma\rho)}$, *and hence* $[\Omega_S]s_\mu(\phi_D)=\chi^\mu(\mathbf{y}_S)$.

(b) $[\Omega_S]s_\mu(\phi_D) = \dfrac{d!}{f^\mu}\,\|\pi_\mu e_S\|^2\in\mathbb{Z}_{\geq 0}$, *and more generally* $[a_Sb_T]s_\mu(\phi_D) = (-1)^{\binom{d}{2}}\dfrac{d!}{f^\mu}\langle\pi_\mu e_S, e_T\rangle$. *Moreover* $\langle F_S, h_1^d\rangle = d!$.

(c) $[\Omega_S]s_\mu(\phi_D)\neq 0$ *if and only if* V_μ *is a constituent of* $\mathrm{Sp}(S)$*; its multiplicity is the rank of* $\mathbf{y}_S$ *on* V_μ.

(d) $F_{S_1\sqcup S_2}=F_{S_1}F_{S_2}$. *If S is Ferrers of shape τ, then* $F_S=(d!/f^\tau)\,s_\tau = H_\tau s_\tau$, *where H_τ is the product of the hook lengths. If no two cells of S share a column, then* $F_S=|R_S|\,h_{\mathrm{sh}(S)}$.

Proof. (a) Take $T=S$ with the same order in (7.2). Then $\alpha=\rho\in R_S$ and $\beta=\gamma\in C_S$, and $F_S = (-1)^{\binom{d}{2}}F_{S,S}=\sum\mathrm{sgn}(\gamma)p_{\mathrm{type}(\gamma\rho)}$. Since $\langle p_\nu, s_\mu\rangle=\chi^\mu(\nu)$, we get $\langle F_S, s_\mu\rangle = \sum_{\gamma,\rho}\mathrm{sgn}(\gamma)\chi^\mu(\gamma\rho) = \chi^\mu(\mathbf{c}_S\mathbf{r}_S)$.

(b) Setting to zero all generators except a_e ($e\in S$) and b_e ($e\in T$) maps A_D to $\sum_k a_{s_k}E_{s_k}$, B_D to $\sum_l b_{t_l}E^*_{t_l}$, and $\phi_j(D)$ to the ψ_j of Lemma 7.3 with $\xi_k=E_{s_k}$, $\eta_l=E_{t_l}$. Since $a_Sb_T=(-1)^{\binom{d}{2}}\prod_k a_{s_k}b_{t_k}$, Lemma 7.3 gives the formula for $[a_Sb_T]s_\mu(\phi_D)$, and the case $T=S$ gives the norm formula. Integrality follows from (a). Finally $\phi_1(D)=\mathrm{tr}\,Q_D=\sum_e a_eb_e$, so $\phi_1^d = d!\sum_{|S|=d}\Omega_S$.

(c) Choose an S_d-invariant inner product on V_μ. The operators $P^+=|R_S|^{-1}\mathbf{r}_S$ and $P^-=|C_S|^{-1}\mathbf{c}_S$ act on V_μ as orthogonal projections, so

$$\chi^\mu(\mathbf{y}_S)=|C_S||R_S|\,\mathrm{tr}(P^-P^+)=|C_S||R_S|\,\mathrm{tr}\big((P^-P^+)^*(P^-P^+)\big).$$

This vanishes if and only if $\mathbf{y}_S$ acts by zero on V_μ. Under the Wedderburn decomposition $\mathbb{C}[S_d]\cong\bigoplus_\nu\mathrm{End}(V_\nu)$, the left ideal $\mathbb{C}[S_d]\mathbf{y}_S$ corresponds to $\bigoplus_\nu\mathrm{End}(V_\nu)\mathbf{y}_S$, which contains V_ν with multiplicity equal to the rank of $\mathbf{y}_S$ on V_ν.

(d) The groups C_S and R_S of a disjoint union are products, cycle types add, and $p_\nu p_{\nu'}=p_{\nu\cup\nu'}$. For Ferrers S, permuting rows and columns conjugates C_S and R_S and leaves F_S unchanged, so we may assume $S=D_\tau$. Then $\mathbf{y}_S$ is the Young symmetrizer $b_\tau a_\tau$ of [4, §4.1], with the factors in the opposite order. Since $\chi^\mu(b_\tau a_\tau)=\chi^\mu(a_\tau b_\tau)$, and $a_\tau b_\tau$ acts on V_μ by zero for $\mu\neq\tau$ and on V_τ as $d!/f^\tau$ times a rank-one idempotent [4, Lemma 4.26, Theorem 4.3], (a) gives $F_S=(d!/f^\tau)s_\tau$. The hook length formula identifies $d!/f^\tau$ with H_τ. If C_S is trivial, then $F_S=\sum_{\rho\in R_S}p_{\mathrm{type}(\rho)}=\prod_i r_i(S)!\,h_{r_i(S)}$. □

Remark 7.5 (Trace versus rank). By (c), F_S and the Frobenius characteristic

$$\mathrm{ch}\,\mathrm{Sp}(S)=\sum_\mu \mathrm{rank}(\mathbf{y}_S|V_\mu)\,s_\mu$$

have the same support, but they are not proportional in general. For the zigzag $Z=(3,2)/(1)$, whose cells are $(2,1)$, $(3,1)$, $(1,2)$ and $(2,2)$, one has

$$F_Z=4s_{31}+6s_{22},\qquad \mathrm{ch}\,\mathrm{Sp}(Z)=s_{(3,2)/(1)}=s_{31}+s_{22}.$$

Equivalently, by (b), the vector e_Z has squared norms $\frac{1}{2},\frac{1}{2}$ in the two summands. For Young diagrams, (d) is Lemma 4.1.

7.4. **Schur support.**

Theorem 7.6 (Schur support)**.** *For every diagram D and every partition μ, $s_\mu(\phi_D)\neq 0$ if and only if there is $S\subseteq D$ with $|S|=|\mu|$ such that V_μ is a constituent of the Specht module* $\mathrm{Sp}(S)$. *Consequently:*

(i) $\mathcal{A}_D$ *is spanned by the* $s_\mu(\phi_D)$ *with* $\mu \in \mathrm{Supp}(D) = \bigcup_{S\subseteq D} \operatorname{supp} \mathrm{Sp}(S)$, *and* $\dim(\mathcal{A}_D)_d \le \#\,\mathrm{Supp}_d(D)$.
(ii) *If* V_μ *occurs in* $\mathrm{Sp}(S)$, *then the decreasing rearrangements* ϱ *of the row counts and* $\varkappa$ *of the column counts of* S *satisfy* $\varrho \trianglelefteq \mu$ *and* $\varkappa \trianglelefteq \mu'$.
(iii) *If* $S \subseteq D$ *has Ferrers components of shapes* $\tau^1, \dots, \tau^k$, *then* $s_{\tau^1} \cdots s_{\tau^k} \in (\mathcal{A}_D)^*$.
(iv) *For every* $S \subseteq D_\lambda$, *all constituents* V_μ *of* $\mathrm{Sp}(S)$ *satisfy* $\mu \subseteq \lambda$.

Proof. If V_μ occurs in $\mathrm{Sp}(S)$, then $[\Omega_S] s_\mu(\phi_D) \neq 0$ by Theorem 7.4(c). Conversely, suppose that V_μ occurs in no $\mathrm{Sp}(S)$, $S \subseteq D$. By Theorem 7.4(b,c), $\pi_\mu e_S = 0$ for every $S \subseteq D$ of size $|\mu|$; then $\langle \pi_\mu e_S, e_T\rangle = 0$ for all S, T, and Theorem 7.4(b) gives $s_\mu(\phi_D) = 0$.

(i) follows because the s_μ form a basis of Sym. For (ii), π_μ preserves torus weights and e_S has U-weight given by the column counts and W^*-weight given by the row counts of S. If $\pi_\mu e_S \neq 0$, these are weights of $\mathbb{S}_{\mu'}U$ and of $\mathbb{S}_\mu W^*$, so their decreasing rearrangements are dominated by μ' and by μ. For (iii), F_S is a nonzero multiple of $s_{\tau^1} \cdots s_{\tau^k}$ by Theorem 7.4(d), and it lies in $(\mathcal{A}_D)^*$ by Theorem 7.1. For (iv), combine the first statement with Theorem 3.5. □

Part (iv) is a statement about Specht modules of arbitrary sets of cells inside a Young diagram; the Borel-stability argument of Section 3 is, in effect, a proof of it.

7.5. **Cell deletion and forests.**

Proposition 7.7 (Cell deletion)**.** *For* $S \subseteq D$ *and* $g \in \mathrm{Sym}_{|S|-1}$, $\langle h_1 g, F_S\rangle = \sum_{e\in S} \langle g, F_{S\smallsetminus e}\rangle$. *Equivalently* $p_1^\perp F_S = \sum_{e\in S} F_{S\smallsetminus e}$, *so multiplication by* ϕ_1 *on* $\mathcal{A}_D$ *is dual to deleting a cell.*

Proof. We have $\phi_1 = \sum_e a_e b_e$, and $a_e b_e \cdot a_{S'} b_{T'}$ is a nonzero multiple of Ω_S only if $e \in S$ and $S' = T' = S \smallsetminus e$, in which case $a_e b_e \Omega_{S\smallsetminus e} = \Omega_S$. □

Proposition 7.8 (Forests)**.** *Suppose that* G_D *is a forest. Then* $F_{S,T} = 0$ *for* $S \neq T$, *and*

$$\mathrm{ev}_D(f) = \sum_{S\subseteq D} \langle f, F_S\rangle\, \Omega_S.$$

Thus $\mathcal{A}_D$ *is a subalgebra of* $Z_D = \mathrm{span}\{\Omega_S\} \cong \mathbb{C}[z_e : e \in D]/(z_e^2)$, $z_e = a_e b_e$, *generated by* $\phi_k(D) = \sum_S \big(\prod_i r_i(S)!\big) z_S$, *the sum running over the* k*-element* $S \subseteq D$ *with no two cells in a common column.*

Proof. If $S \neq T$ have the same row and column counts, the function $\mathbf{1}_S - \mathbf{1}_T$ on the edges of G_D has zero sum at every vertex. Every vertex of its support meets at least two edges of the support, so the support contains a cycle. Hence, in a forest, $F_{S,T} = 0$ unless $S = T$ by Theorem 7.1(d). The products $\Omega_S \Omega_{S'}$ equal $\Omega_{S\cup S'}$ or 0, and the formula for ϕ_k is (7.3). □

7.6. **Diagonal sufficiency.**

Conjecture 7.9 (Diagonal sufficiency)**.** *For every diagram* D, $(\mathcal{A}_D)^* = \mathrm{span}\{F_S : S \subseteq D\}$. *Equivalently, an element of* $\mathcal{A}_D$ *vanishes if and only if all of its doubled-cell coefficients* $[\Omega_S]$ *vanish.*

The conjecture is a linear statement only. The projection $E_D \to Z_D$ onto doubled-cell monomials is not multiplicative on $\mathcal{A}_D$; this fails already for the 2×2 square.

Proposition 7.10. *Conjecture 7.9 holds in each of the following cases: (a)* G_D *is a forest; (b)* D *is a disjoint union of diagrams for which it holds; (c) every component of* D *is Ferrers; (d) in degrees* $d \le 3$, *for every* D.

Proof. (a) is Proposition 7.8. (b) By Theorem 8.3 below, $(\mathcal{A}_{D_1\sqcup D_2})^* = (\mathcal{A}_{D_1})^* (\mathcal{A}_{D_2})^*$, and $F_{S_1} F_{S_2} = F_{S_1\sqcup S_2}$. (c) By (b) it suffices to treat a connected Ferrers diagram, hence (after permutation) a Young diagram D_λ, where $(\mathcal{A}_\lambda)^* = \mathrm{span}\{s_\tau : \tau \subseteq \lambda\} = \mathrm{span}\{F_{D_\tau}\}$ by Theorem 4.4 and Theorem 7.4(d).

(d) It suffices to show that every $F_{S,T}$ with $S \neq T$ and $|S| \leq 3$ lies in $\operatorname{span}\{F_U : U \subseteq S \cup T\}$. This vector depends only on the isomorphism type of (S,T) under permutations of rows and columns. By the argument in the proof of Proposition 7.8, $S \triangle T$ contains an alternating cycle. For $d \leq 3$ there are five configurations: the two diagonals of a 2×2 square; a square together with one common cell of S and T that shares a row with the square, shares a column with it, or neither; and the two perfect matchings of a hexagon. For the square, (7.2) gives $F_{S,T} = \pm p_2 = \pm\frac{1}{2}(F_{\mathrm{row}} - F_{\mathrm{col}})$, where $F_{\mathrm{row}} = 2s_2$ and $F_{\mathrm{col}} = 2s_{11}$ belong to a row pair and a column pair of the square. For the hexagon, the F_U of its three-cell subdiagrams span Sym_3. The three remaining cases are finite computations with (7.2). □

Remark 7.11 (Evidence). Conjecture 7.9 has been verified by exact computation for all diagrams without empty rows or columns in the boxes of size 2×2, 3×2, 3×3, 4×2, 4×3 and 5×2, for those with at most ten cells in the 6×2 box, and for those with at most eleven cells in the 4×4 and 5×3 boxes; this amounts to 334 diagrams up to permutations of rows and columns. In addition, the local statement $F_{S,T} \in \operatorname{span}\{F_U : U \subseteq S \cup T\}$ was confirmed for about 600 randomly chosen pairs (S,T) with $|S| \leq 6$.

8. Arbitrary supports II: structure, disjoint unions, and the Gorenstein property

8.1. **Basic properties.**

Proposition 8.1 (Basic properties). *Let D be a nonempty diagram with $n = |D|$ cells.*

(i) $(\mathcal{A}_D)_1 = \mathbb{C}\phi_1$, $(\mathcal{A}_D)_d = 0$ *for* $d > n$, *and* $(\mathcal{A}_D)_n = \mathbb{C}\,\Omega_D$; *moreover* $\phi_1^n = n!\,\Omega_D$ *and* $\phi_1^{n+1} = 0$.

(ii) If $n \geq 2$, *then* $\dim(\mathcal{A}_D)_2 = 1$ *if D lies in a single row, or in a single column, or no two cells of D share a row or a column; otherwise* $\dim(\mathcal{A}_D)_2 = 2$.

Proof. (i) Every element of $(\mathcal{A}_D)_d$ is a combination of monomials $a_S b_T$ with $|S| = |T| = d$. In degree n only $S = T = D$ occurs, and $\phi_1^n = n!\,\Omega_D \neq 0$ by Theorem 7.4(b).

(ii) In degree 2, Theorem 7.4(d) gives $F_S = 2s_2$ for two cells in a row, $F_S = 2s_{11}$ for two cells in a column, and $F_S = h_1^2 = s_2 + s_{11}$ for two cells in distinct rows and columns. An off-diagonal $F_{S,T}$ with $|S| = 2$ requires a full 2×2 square in D, which contains both a row pair and a column pair. By Theorem 7.1, $\dim(\mathcal{A}_D)_2$ is the number of linearly independent vectors among those that occur. If D is not contained in a row, a column, or a partial permutation matrix, then at least two of the three types occur. For instance, if a row contains two cells and D has a cell outside that row, this cell lies in a column different from that of one of the two, which gives a pair in distinct rows and columns. Any two of the three vectors are independent. □

Proposition 8.2 (Functoriality and symmetries). *(i) If $S \subseteq D$, setting the generators of the cells outside S equal to zero induces a surjection $\mathcal{A}_D \to \mathcal{A}_S$, $\phi_j(D) \mapsto \phi_j(S)$. Hence $J_D \subseteq J_S$, $(\mathcal{A}_S)^* \subseteq (\mathcal{A}_D)^*$, and the Hilbert function of $\mathcal{A}_D$ dominates that of $\mathcal{A}_S$ coefficientwise.*

(ii) Permuting rows and columns does not change $\phi_j(D)$, up to the corresponding renaming of generators. If after such a permutation $D \subseteq D_\lambda$, then $\mathcal{A}_D$ is a quotient of $\mathcal{A}_\lambda \cong H^*(X_{\lambda^c};\mathbb{C})$. In particular, for $D \subseteq [m] \times [r]$, $\mathcal{A}_D$ is a quotient of $H^*(\mathrm{Gr}(r, m+r);\mathbb{C})$.

(iii) $\mathcal{A}_{D^t} \cong \mathcal{A}_D$ *by an isomorphism that corresponds to* $s_\rho \mapsto (-1)^{|\rho|} s_{\rho'}$ *on* Sym. *Thus* $J_{D^t} = \omega(J_D)$ *and* $F_{S^t} = \omega(F_S)$.

Proof. (i) and the first part of (ii) are clear: permuting rows and columns conjugates Q_D by permutation matrices. The second part of (ii) follows from (i) applied to $D \subseteq D_\lambda$, together with Theorems 4.4 and 6.1. For (iii), renaming $a_{ci} \leftrightarrow b_{ic}$ identifies Q_{D^t} with $B_D A_D$, and the argument of Corollary 6.5, which uses only Proposition 2.2, applies verbatim. For the last identity, $C_{S^t} = R_S$ and $R_{S^t} = C_S$, so by Theorem 7.4(a) $F_{S^t} = \sum \operatorname{sgn}(\rho) p_{\mathrm{type}(\gamma\rho)} = \sum \operatorname{sgn}(\gamma)\operatorname{sgn}(\gamma\rho) p_{\mathrm{type}(\gamma\rho)} = \omega(F_S)$. □

8.2. **Disjoint unions.** Let $\Delta : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym}$, $\Delta h_j = \sum_{a+b=j} h_a \otimes h_b$, be the coproduct. It is adjoint to multiplication: $\langle fg, F\rangle = \langle f \otimes g, \Delta F\rangle$ [11, I.5].

Theorem 8.3 (Disjoint unions)**.** *Let $D = D_1 \sqcup D_2$. Then*

$$J_D = \Delta^{-1}\big(J_{D_1} \otimes \mathrm{Sym} + \mathrm{Sym} \otimes J_{D_2}\big), \qquad (\mathcal{A}_D)^* = (\mathcal{A}_{D_1})^* \cdot (\mathcal{A}_{D_2})^*,$$

where the right-hand side is the span of all products FG in Sym*. Equivalently, $\mathcal{A}_D$ is the image of $(\mathrm{ev}_{D_1} \otimes \mathrm{ev}_{D_2}) \circ \Delta : \mathrm{Sym} \to \mathcal{A}_{D_1} \otimes \mathcal{A}_{D_2}$.*

Proof. After a permutation Q_D is block diagonal with blocks Q_{D_1} and Q_{D_2}, so $H_D = H_{D_1} H_{D_2}$ and $\phi_j(D) = \sum_{a+b=j} \phi_a(D_1)\phi_b(D_2)$. As vector spaces $E_D = E_{D_1} \otimes E_{D_2}$, and even elements of E_{D_1} and E_{D_2} commute, so multiplication defines an injective algebra homomorphism $\mathcal{A}_{D_1} \otimes \mathcal{A}_{D_2} \to E_D$. Hence ev_D is the composite of $(\mathrm{ev}_{D_1} \otimes \mathrm{ev}_{D_2}) \circ \Delta$ with this injection. The kernel of $\mathrm{ev}_{D_1} \otimes \mathrm{ev}_{D_2}$ is $K = J_{D_1} \otimes \mathrm{Sym} + \mathrm{Sym} \otimes J_{D_2}$, which proves the first identity. In each degree $K^\perp = (\mathcal{A}_{D_1})^* \otimes (\mathcal{A}_{D_2})^*$, and $f \in \Delta^{-1}(K)$ if and only if $\langle \Delta f, K^\perp\rangle = 0$, that is, $\langle f, FG\rangle = 0$ for all $F \in (\mathcal{A}_{D_1})^*$, $G \in (\mathcal{A}_{D_2})^*$. □

Corollary 8.4 (Ferrers components)**.** *If the components of D are Ferrers diagrams of shapes $\lambda^1, \dots, \lambda^k$, then*

$$(\mathcal{A}_D)^* = \mathrm{span}\{s_{\mu^1} s_{\mu^2} \cdots s_{\mu^k} : \mu^j \subseteq \lambda^j\}.$$

Proof. Combine Theorem 8.3 with Theorem 4.4 and Proposition 8.2(ii). □

Example 8.5. (a) If D consists of k cells with no two in a common row or column, then $(\mathcal{A}_D)^* = \mathrm{span}\{h_1^j : j \le k\}$ and $\mathcal{A}_D \cong \mathbb{C}[q]/(q^{k+1})$. This is the same algebra as for a single row of k cells, but with a different kernel: for $k \ge 2$, $s_2 - s_{11} \in J_D$ while $s_{11} \in J_{(k)}$.

(b) Let D be the disjoint union of k rows of length a. Since $\mathcal{A}_{(a)} \cong \mathbb{C}[x]/(x^{a+1})$ with $h_j \mapsto x^j$, Theorem 8.3 identifies $\mathcal{A}_D$ with the subalgebra of $R = \bigotimes_{i=1}^k \mathbb{C}[x_i]/(x_i^{a+1})$ generated by the complete homogeneous symmetric polynomials $h_j(x_1, \dots, x_k)$. Since the invariants of a quotient by an S_k-stable ideal are the image of the invariants (Reynolds operator),

$$\mathcal{A}_D \cong R^{S_k} \cong H^*(\mathrm{Sym}^k \mathbb{P}^a; \mathbb{C}),$$

which has basis the monomial symmetric functions $m_\nu(x)$ with $\nu \subseteq (a^k)$. Hence $\mathrm{Hilb}(\mathcal{A}_D; t)$ is the Gaussian binomial coefficient $\binom{a+k}{k}_t$, the same as for the rectangle (a^k).

8.3. **The Gorenstein property.**

Theorem 8.6 (Gorenstein classification for Ferrers components)**.** *Let D be a nonempty diagram whose components are Ferrers. The following are equivalent:*

- *(i) $\mathcal{A}_D$ is Gorenstein;*
- *(ii) $\mathrm{Hilb}(\mathcal{A}_D; t)$ is palindromic;*
- *(iii) up to permutations of rows and columns, D is a rectangle, or a disjoint union of $k \ge 2$ rows of equal length, or a disjoint union of $k \ge 2$ columns of equal length.*

Proof. (i)⇒(ii) holds for every graded Artinian Gorenstein algebra.

(iii)⇒(i). For a rectangle this is Corollary 6.4. For k rows of length a, Example 8.5(b) gives $\mathcal{A}_D \cong R^{S_k}$. The algebra R satisfies Poincaré duality with S_k-invariant top class $x_1^a \cdots x_k^a$. If $0 \ne u \in R^{S_k}$, choose $v \in R$ with $uv \ne 0$ in the top degree; the average $\bar v$ of v over S_k is invariant and $u\bar v = uv$ in the top degree. Hence R^{S_k} satisfies Poincaré duality. Columns follow by Proposition 8.2(iii).

(ii)⇒(iii). Let the components be Ferrers of shapes $\lambda^1, \dots, \lambda^k$, let $n = |D|$, and suppose that the Hilbert series is palindromic. Then $\dim(\mathcal{A}_D)_{n-1} = \dim(\mathcal{A}_D)_1 = 1$. In degree $n-1$ only pairs $S = T = D \smallsetminus e$ occur, so $(\mathcal{A}_D)^*_{n-1}$ is spanned by the $F_{D \smallsetminus e}$. By Theorem 7.4(d) and Corollary 8.4,

this span is that of the products $P_{j,\nu} = s_\nu \prod_{l\neq j} s_{\lambda^l}$, where ν is obtained from λ^j by removing a corner; each such product occurs, by deleting the corner cell. Since Sym is a domain and distinct Schur functions are linearly independent, two corners of the same λ^j give independent products. So every λ^j is a rectangle, say $\lambda^j = (a_j^{b_j})$ with b_j rows of length a_j. Let ν^j be λ^j with its corner removed. For $j \neq l$, proportionality of P_{j,ν^j} and P_{l,ν^l} means $s_{\nu^j} s_{\lambda^l} \propto s_{\lambda^j} s_{\nu^l}$. The dominance-leading monomial of $s_\alpha s_\beta$ is $m_{\alpha+\beta}$, where $\alpha+\beta$ is the componentwise sum, so $\nu^j + \lambda^l = \lambda^j + \nu^l$. If $b_j < b_l$, the two sides differ in position b_j, where they equal $a_j - 1 + a_l$ and $a_j + a_l$. Hence $b_j = b_l$, and applying ω, which maps $s_\alpha s_\beta$ to $s_{\alpha'} s_{\beta'}$, gives $a_j = a_l$. So D consists of k copies of an $a \times b$ rectangle.

It remains to exclude $k \geq 2$ with $a, b \geq 2$. Then D contains a row pair and a column pair, so $\dim(\mathcal{A}_D)_2 = 2$ by Proposition 8.1. Let $\lambda = (a^b)$, $\nu = (a^{b-1}, a-1)$, $\nu_1 = (a^{b-1}, a-2)$ and $\nu_2 = (a^{b-2}, (a-1)^2)$. By Corollary 8.4, $(\mathcal{A}_D)^*_{n-2}$ contains $s_{\nu_1} s_\lambda^{k-1}$, $s_{\nu_2} s_\lambda^{k-1}$ and $s_\nu^2 s_\lambda^{k-2}$. We claim that these are linearly independent, so that $\dim(\mathcal{A}_D)_{n-2} \geq 3 \neq \dim(\mathcal{A}_D)_2$. Dividing by s_λ^{k-2}, suppose $x s_{\nu_1} s_\lambda + y s_{\nu_2} s_\lambda = z s_\nu^2$. Compare coefficients of s_θ for $\theta = \nu \cup \nu = (a^{2b-2}, (a-1)^2)$. Since θ/λ is a translate of the straight shape ν_2, we get $c^\theta_{\lambda\nu_1} = 0$ and $c^\theta_{\lambda\nu_2} = 1$, while $c^\theta_{\nu\nu} = 1$; hence $y = z$. Compare coefficients of $s_{2\nu}$, where $2\nu = \lambda + \nu_1$. Here $c^{2\nu}_{\lambda\nu_1} = c^{2\nu}_{\nu\nu} = 1$, while $c^{2\nu}_{\lambda\nu_2} = 0$ because every constituent of $s_\lambda s_{\nu_2}$ is dominated by $\lambda + \nu_2 \lhd \lambda + \nu_1$. Hence $x = z$. If $z \neq 0$ we would get $s_\lambda(s_{\nu_1} + s_{\nu_2}) = s_\nu^2$. Pairing with p_1^{2ab-2}, and using $\langle FG, p_1^N\rangle = \binom{N}{\deg F}\langle F, p_1^{\deg F}\rangle\langle G, p_1^{\deg G}\rangle$, $f^{\nu_1} + f^{\nu_2} = f^\nu = f^\lambda$, this would give $\binom{2ab-2}{ab} = \binom{2ab-2}{ab-1}$, which is false. So $x = y = z = 0$. □

Conjecture 8.7. *The equivalences of Theorem 8.6 hold for every diagram D.*

Among the 334 diagrams of the computations described after Conjecture 7.9, the only palindromic Hilbert series are those of rectangles, of rows of equal length, and of their transposes. For two disjoint 2×2 squares, $\mathrm{Hilb}(\mathcal{A}_D; t) = 1 + t + 2t^2 + 3t^3 + 4t^4 + 3t^5 + 3t^6 + t^7 + t^8$.

8.4. **Examples.** Table 1 lists Hilbert series computed with Theorem 7.1. Here Z_n is the zigzag (ribbon) with cells $(1,1), (2,1), (2,2), (3,2), (3,3), \ldots$ (n cells), and C_{2k} is the cycle diagram $\{(j,j), (j+1, j)\}$ with indices modulo k, that is, a $k \times k$ permutation-matrix complement for $k = 3$. All of these, except the Gorenstein cases listed in Theorem 8.6, have non-palindromic Hilbert series.

diagram D	$\lvert D\rvert$	coefficients of $\mathrm{Hilb}(\mathcal{A}_D; t)$
$Z_4 = (3,2)/(1)$	4	1, 1, 2, 2, 1
Z_5	5	1, 1, 2, 3, 3, 1
Z_6	6	1, 1, 2, 3, 4, 3, 1
Z_7	7	1, 1, 2, 3, 5, 5, 5, 1
Z_8	8	1, 1, 2, 3, 5, 6, 7, 4, 1
C_6 (hexagon)	6	1, 1, 2, 3, 3, 1, 1
C_8 (octagon)	8	1, 1, 2, 3, 5, 5, 5, 1, 1
$(2,2) \sqcup (1)$	5	1, 1, 2, 3, 2, 1
$(2) \sqcup (1,1)$	4	1, 1, 2, 2, 1
$(2,2) \sqcup (2,2)$	8	1, 1, 2, 3, 4, 3, 3, 1, 1
$(3,3,2)/(1)$	7	1, 1, 2, 3, 3, 3, 3, 1
$(2) \sqcup (2)$	4	1, 1, 2, 1, 1
$(2) \sqcup (2) \sqcup (2)$	6	1, 1, 2, 2, 2, 1, 1

TABLE 1. Hilbert series of curvature algebras of some non-Young diagrams.

Example 8.8 (The zigzag)**.** For $Z = (3,2)/(1) = \{(2,1),(3,1),(1,2),(2,2)\}$, the dual spaces are

$$(\mathcal{A}_Z)^* = \mathbb{C}\cdot 1 \oplus \mathbb{C}s_1 \oplus \operatorname{span}\{s_2, s_{11}\} \oplus \operatorname{span}\{s_3, s_{21}\} \oplus \mathbb{C}(2s_{31} + 3s_{22}),$$

and therefore

$$J_Z = I_{(3,2)} + (3s_{31} - 2s_{22}).$$

Indeed Z lies in a permuted copy of $D_{(3,2)}$, so $J_Z \supseteq I_{(3,2)}$; in degrees ≤ 3 the subdiagrams with Ferrers components already give all s_ρ with $\rho \subseteq (3,2)$, the top degree is spanned by $F_Z = 4s_{31} + 6s_{22}$, and $(3s_{31} - 2s_{22})s_1 \equiv \bar{s}_{32}$ modulo $I_{(3,2)}$. The Hilbert series $1 + t + 2t^2 + 2t^3 + t^4$ coincides with that of the hook $(3,1)$.

9. Diagram varieties: a Crofton formula and the comparison problem

9.1. **Diagram varieties and the comparison problem.** Let $D \subseteq R = [m] \times [r]$, let $n = m + r$, and let

$$L_D = \{X \in \operatorname{Mat}_{r\times m}(\mathbb{C}) : X_{ic} = 0 \text{ if } (c,i) \notin D\}, \qquad Y_D = \overline{\{\operatorname{rowspan}(I_r \mid X) : X \in L_D\}} \subseteq \operatorname{Gr}(r,n).$$

Then Y_D is an irreducible projective variety of dimension $|D|$ containing $L_D \cong \mathbb{A}^{|D|}$ as a dense open subset. In the convention of [2, 14, 10], which records the entries forced to vanish, Y_D is the diagram variety of $R \smallsetminus D$. We use the homology class of Y_D as a linear functional on symmetric functions,

$$[Y_D] \in \operatorname{Sym}_{|D|}, \qquad \langle f, [Y_D]\rangle = \int_{Y_D} f\big(c(\mathcal{Q})\big),$$

where $\mathcal{Q}$ is the universal quotient bundle and $h_j \mapsto c_j(\mathcal{Q}) = \sigma_{(j)}$ as in Theorem 6.1. Put

$$\mathcal{T}_D = \operatorname{im}\big(H^*(\operatorname{Gr}(r,n);\mathbb{C}) \to H^*(Y_D;\mathbb{C})\big), \qquad J_D^{\mathrm{geom}} = \ker\big(\operatorname{Sym} \to \mathcal{T}_D\big), \qquad J_D^{\mathrm{odd}} = J_D.$$

Thus $\mathcal{A}_D \cong \operatorname{Sym}/J_D^{\mathrm{odd}}$ and $\mathcal{T}_D \cong \operatorname{Sym}/J_D^{\mathrm{geom}}$. Over $\mathbb{C}$, the graded dual $\mathcal{T}_D^*$ is the image of $H_*(Y_D;\mathbb{C}) \to H_*(\operatorname{Gr}(r,n);\mathbb{C})$, regarded as a space of functionals on Sym. The *comparison problem* asks when $J_D^{\mathrm{odd}} = J_D^{\mathrm{geom}}$. A further question is when $\mathcal{T}_D = H^*(Y_D;\mathbb{C})$.

We record the elementary facts used below.

(a) If $S \subseteq D$, then $Y_S \subseteq Y_D$, so $[Y_S] \in \mathcal{T}_D^*$.

(b) Permutations of rows and columns are induced by permutation matrices in $GL_n(\mathbb{C})$. These act trivially on $H^*(\operatorname{Gr}(r,n))$, so $[Y_D]$, $\mathcal{T}_D$ and J_D^{geom} are invariant. Enlarging the ambient rectangle does not change them either, as in the remark after Theorem 6.1.

(c) If D is Ferrers of shape λ, then Y_D is a translate of X_{λ^c}, and $J_D^{\mathrm{geom}} = I_\lambda$, $\mathcal{T}_D = H^*(Y_D)$ and $[Y_D] = s_\lambda$ by Theorem 6.1. Moreover $\mathcal{T}_D^*$ is contained in $\operatorname{span}\{s_\rho : \rho \subseteq \lambda\}$ whenever a permutation of D lies in D_λ.

(d) (Transposition.) The isomorphism $\operatorname{Gr}(r,n) \to \operatorname{Gr}(m,n)$, $V \mapsto V^\perp$ (annihilator for the standard bilinear form), maps $\operatorname{rowspan}(I_r \mid X)$ to $\operatorname{rowspan}(-X^{\mathsf{T}} \mid I_m)$. Hence it carries Y_D onto a translate of Y_{D^t}, and it pulls back the quotient bundle of $\operatorname{Gr}(m,n)$ to $\mathcal{S}^*$. Since $c_j(\mathcal{S}^*)$ is the image of e_j, we get $[Y_{D^t}] = \omega[Y_D]$ and $J_{D^t}^{\mathrm{geom}} = \omega(J_D^{\mathrm{geom}})$, matching Proposition 8.2(iii).

(e) If D is a skew diagram λ/ν, then Y_D is a Grassmannian Richardson variety [8, 14], and $[Y_D] = s_{\lambda/\nu}$. If G_D is a forest, then the torus of GL_n acts on L_D with a dense orbit, so Y_D is a torus orbit closure. Its Plücker coordinates restricted to L_D are the minors of X, and each minor is a single monomial. So Y_D is the projective toric variety of the matching polytope of G_D; its degree is the normalised volume of that polytope, which equals $\dim \operatorname{Sp}(D)$ by [9].

9.2. **A Crofton formula.** Equip $\mathbb{C}^n$ with its standard Hermitian form, $\mathcal{Q} = \mathbb{C}^n/\mathcal{S}$ with the quotient metric, and let $c(\mathcal{Q}, h) = \det\bigl(I + \frac{i}{2\pi}\Theta_{\mathcal{Q}}\bigr)$ be the Chern–Weil form. It is closed, invariant under $U(n)$, and represents $c(\mathcal{Q})$. At $x_0 = \operatorname{rowspan}(I_r \mid 0)$ we identify $T_{x_0}\operatorname{Gr}(r,n)$ with $\operatorname{Mat}_{r\times m}(\mathbb{C})$ through the chart $Z \mapsto \operatorname{rowspan}(I_r \mid Z)$. For a complex d-dimensional subspace $V \subseteq \operatorname{Mat}_{r\times m}(\mathbb{C})$ choose an orthonormal basis $\xi_1, \dots, \xi_d$ for $\langle X, Y\rangle = \operatorname{tr}(XY^*)$, put $v = \xi_1^{\mathsf{T}} \wedge \cdots \wedge \xi_d^{\mathsf{T}} \in \bigwedge^d (U \otimes W^*)$, and define the *density functional*

$$F_V = \sum_{\mu \vdash d} \frac{d!}{f^\mu} \, \|\pi_\mu v\|^2 \, s_\mu \in \operatorname{Sym}_d.$$

It does not depend on the basis, it is invariant under $V \mapsto k_1 V k_2$ for $(k_1, k_2) \in U(r) \times U(m)$, and $F_{L_S} = F_S$ by Theorem 7.4(b).

Lemma 9.1 (Chern forms at a point)**.** *(a) Under $a_{ci} \mapsto d\bar z_{ic}$, $b_{ic} \mapsto dz_{ic}$, the exterior algebra E_R maps isomorphically onto $\bigwedge^\bullet T^*_{x_0}\operatorname{Gr}(r,n) \otimes \mathbb{C}$, and $c_j(\mathcal{Q}, h)(x_0)$ is the image of $\kappa^j \phi_j(R)$, where $\kappa = -i/2\pi$. Restricting to $T_{x_0}L_D$ gives $\kappa^j\phi_j(D)$. Hence $\mathcal{A}_D$ is isomorphic, as a graded algebra, to the algebra generated by the restrictions to L_D of the Chern–Weil forms of $\mathcal{Q}$ at x_0.*

(b) For every $f \in \operatorname{Sym}_d$ and every complex d-plane $V \subseteq T_{x_0}\operatorname{Gr}(r,n)$,

$$f\bigl(c(\mathcal{Q}, h)\bigr)\big|_V = \pi^{-d}\,\langle f, F_V\rangle \,\operatorname{dvol}_V.$$

Proof. (a) Near x_0, the images $\bar f_1, \dots, \bar f_m$ of the last m standard basis vectors form a holomorphic frame of $\mathcal{Q}$. The orthogonal projection of f_c to $\mathcal{S}^\perp = \operatorname{rowspan}(-Z^* \mid I_m)$ shows that the metric in this frame is $h = (I + Z^*Z)^{-1}$, up to transposition. Hence the Chern curvature $\Theta = \bar\partial(\partial h\, h^{-1})$ satisfies $\Theta(x_0) = -dZ^* \wedge dZ$, an $m \times m$ matrix of $(1,1)$-forms; compare [5, Chapter 0, §5]. The entries of $A_R B_R$ map to the entries of $dZ^* \wedge dZ$, so $\det(I + \frac{i}{2\pi}\Theta)(x_0)$ is the image of $\det(I + \kappa t\, Q_R)$ at $t = 1$. Restriction to L_D kills dz_e and $d\bar z_e$ for $e \notin D$. Rescaling $\phi_j \mapsto \kappa^j\phi_j$ is a graded automorphism.

(b) Let ζ_k be the coordinates on V dual to ξ_k, so that $dZ|_V = \sum_k \xi_k\, d\zeta_k$ and $dZ^*|_V = \sum_k \xi_k^*\, d\bar\zeta_k$. By (a), the restrictions to V of the Chern forms are $\kappa^j\psi_j$, where ψ_j are the coefficients of $\det(I_m + tAB)$ for $A = \sum_k a_k \xi_k^*$ and $B = \sum_k b_k \xi_k$, under $a_k \mapsto d\bar\zeta_k$, $b_k \mapsto d\zeta_k$. This is the situation of Lemma 7.3 with ξ_k and η_k both replaced by $\xi_k^* = \overline{\xi_k}^{\mathsf{T}} \in \operatorname{Mat}_{m\times r}(\mathbb{C})$. Hence $[\prod_k a_k b_k] f(\psi) = \sum_\mu \langle f, s_\mu\rangle (d!/f^\mu)\|\pi_\mu(\overline{\xi_1}^{\mathsf{T}} \wedge \cdots \wedge \overline{\xi_d}^{\mathsf{T}})\|^2 = \langle f, F_V\rangle$, because complex conjugation commutes with π_μ, which is defined over $\mathbb{R}$. Finally $\prod_k (d\bar\zeta_k \wedge d\zeta_k) = (2i)^d \operatorname{dvol}_V$ and $\kappa^d (2i)^d = \pi^{-d}$. □

Theorem 9.2 (Crofton formula for diagram varieties)**.** *For $X \in L_D$ put*

$$V_X = (I + XX^*)^{-1/2}\, L_D\, (I + X^*X)^{-1/2} \subseteq \operatorname{Mat}_{r\times m}(\mathbb{C}).$$

Let dvol *be the Riemannian volume of $L_D \subseteq \operatorname{Gr}(r,n)$ for the $U(n)$-invariant Kähler metric whose value at x_0 is $\operatorname{tr}(dZ\, dZ^*)$. Then for every $f \in \operatorname{Sym}_{|D|}$,*

$$\langle f, [Y_D]\rangle = \pi^{-|D|} \int_{L_D} \langle f, F_{V_X}\rangle \operatorname{dvol}(X). \tag{9.1}$$

In words: $F_D = F_{V_0}$ is the Chern–Weil density of Y_D at the torus-fixed point x_0, while $[Y_D]$ is the integral of these densities.

Proof. Since $c(\mathcal{Q}, h)$ is closed and $Y_D \smallsetminus L_D$ is a proper subvariety, $\langle f, [Y_D]\rangle = \int_{L_D} f(c(\mathcal{Q}, h))$. Let GL_n act on row spaces from the right. For $X \in L_D$ write $n_X = \begin{pmatrix} I_r & X \\ 0 & I_m \end{pmatrix}$, so that $x_0 n_X = x$ is the point of L_D with coordinate X, and right multiplication by n_X is translation by X in the chart. One checks directly that $n_X = qk$ with $k \in U(n)$ and

$$q = \begin{pmatrix} (I + XX^*)^{1/2} & 0 \\ C & (I + X^*X)^{-1/2} \end{pmatrix} \in \operatorname{Stab}(x_0), \qquad C = X^*(I + XX^*)^{-1/2}.$$

Indeed, the rows of $q^{-1}n_X$ are $(I+XX^*)^{-1/2}(I \mid X)$ and $(I+X^*X)^{-1/2}(-X^* \mid I)$, which are orthonormal. Right multiplication $R_{n_X} = R_k \circ R_q$ maps x_0 to x and $T_{x_0}L_D$ onto T_xL_D, because it is translation by X in the chart. Since q fixes x_0, we get $T_xL_D = dR_k\big(dR_q(T_{x_0}L_D)\big)$, and R_k is an isometry preserving $c(\mathcal{Q}, h)$. In the chart, $\mathrm{rowspan}(I \mid Z)\, q = \mathrm{rowspan}\big(I \mid (A+ZC)^{-1}ZD'\big)$ with $A = (I+XX^*)^{1/2}$ and $D' = (I+X^*X)^{-1/2}$, whose differential at $Z=0$ is $\dot{Z} \mapsto A^{-1}\dot{Z}D'$; so $dR_q(T_{x_0}L_D) = V_X$. Therefore the density of $f(c(\mathcal{Q}, h))$ on T_xL_D, measured against the Riemannian volume, equals its density on V_X at x_0, which is $\pi^{-|D|}\langle f, F_{V_X}\rangle$ by Lemma 9.1(b). □

Corollary 9.3. *Let D be a diagram with $|D| = d$.*

(a) The Schur support of $[Y_D]$ is the set of constituents of the Specht module $\mathrm{Sp}(D)$, that is, the support of F_D, and all these coefficients are positive.

(b) If D is Ferrers, the integrand in (9.1) is constant and $[Y_D] = \dfrac{\deg Y_D}{d!}\, F_D$.

(c) $J_D^{\mathrm{odd}} \cap \mathrm{Sym}_d = J_D^{\mathrm{geom}} \cap \mathrm{Sym}_d$ if and only if F_D and $[Y_D]$ are proportional.

Proof. (a) Each V_X is $g \cdot L_D$ for some $g \in GL_r \times GL_m$, which acts on $\bigwedge^d(U \otimes W^*)$ commuting with π_μ. Hence $\|\pi_\mu v_X\| \neq 0$ if and only if $\pi_\mu e_D \neq 0$, so every integrand F_{V_X} has exactly the support of F_D, with positive coefficients. Theorem 7.4(c) identifies this support with the constituents of $\mathrm{Sp}(D)$.

(b) After permuting rows and columns, $D = D_\tau$, and L_D is stable under left multiplication by upper triangular $r \times r$ matrices and right multiplication by lower triangular $m \times m$ matrices. By the QR and LQ decompositions, $(I+XX^*)^{-1/2} = k_1b_1$ and $(I+X^*X)^{-1/2} = b_2k_2$ with k_i unitary and b_1, b_2 upper and lower triangular respectively. Hence $V_X = k_1L_Dk_2$ and $F_{V_X} = F_D$ for all X. So $[Y_D]$ is a positive multiple of F_D, and the constant follows from $\langle F_D, h_1^d\rangle = d!$ and $\langle [Y_D], h_1^d\rangle = \deg Y_D$.

(c) In degree d, $(\mathcal{A}_D)_d^* = \mathbb{C}F_D$ by Proposition 8.1 and $\mathcal{T}_D^* \cap \mathrm{Sym}_d = \mathbb{C}[Y_D]$ because $H_{2d}(Y_D) = \mathbb{C}[Y_D]$. □

For Young diagrams, (b) together with $F_{D_\lambda} = (d!/f^\lambda)s_\lambda$ gives a direct geometric proof that the fundamental class of X_{λ^c} is proportional to s_λ. Part (a) gives, for arbitrary D, a support-level statement about classes of diagram varieties. At the level of classes, Liu's conjecture [10] expresses the class of a diagram variety through the Specht module of the complementary set of forced zeros; it fails in general [14].

9.3. **Disjoint unions and the comparison theorem.**

Theorem 9.4 (Comparison for Ferrers components)**.** *If every component of D is Ferrers, then $J_D^{\mathrm{odd}} = J_D^{\mathrm{geom}}$, that is, $\mathcal{A}_D \cong \mathcal{T}_D$ compatibly with the presentations by* Sym*. In this case $\mathcal{T}_D = H^*(Y_D; \mathbb{C})$ if and only if D has at most one nonempty component.*

Proof. Let $D = D_1 \sqcup D_2$. After a permutation we may assume that D_1 lies in the block of rows I_1 and columns K_1, and D_2 in the complementary block $I_2 \times K_2$. Write $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$ accordingly. The closed embedding $\mathrm{Gr}(r_1, n_1) \times \mathrm{Gr}(r_2, n_2) \to \mathrm{Gr}(r, n)$, $(V_1, V_2) \mapsto V_1 \oplus V_2$, maps the graph of (X_1, X_2) to the graph of the block diagonal matrix, hence restricts to an isomorphism $Y_{D_1} \times Y_{D_2} \cong Y_D$. It pulls $\mathcal{Q}$ back to $\mathcal{Q}_1 \boxplus \mathcal{Q}_2$, so $c(\mathcal{Q}) \mapsto c(\mathcal{Q}_1) \otimes c(\mathcal{Q}_2)$. By the Künneth formula, $\mathrm{Sym} \to H^*(Y_D)$ is $(\mathrm{res}_1 \otimes \mathrm{res}_2) \circ \Delta$, and therefore

$$J_D^{\mathrm{geom}} = \Delta^{-1}\big(J_{D_1}^{\mathrm{geom}} \otimes \mathrm{Sym} + \mathrm{Sym} \otimes J_{D_2}^{\mathrm{geom}}\big).$$

This is the formula of Theorem 8.3 for J^{odd}. By induction on the number of components, it suffices to treat a connected Ferrers diagram, where both kernels equal I_λ by Theorems 4.4 and 6.1 and fact (b).

If D is connected Ferrers, then $\mathcal{T}_D = H^*(Y_D)$ by Theorem 6.1. If D has two nonempty components, then $H^2(Y_D) \supseteq H^2(Y_{D_1}) \otimes 1 \oplus 1 \otimes H^2(Y_{D_2})$ has dimension at least two, because a positive-dimensional projective variety has nonzero H^2. On the other hand $\mathcal{T}_D^1 = \mathbb{C}\sigma_1$. □

Proposition 9.5 (Low degrees)**.** *For every nonempty diagram D, $J_D^{\mathrm{geom}} \cap \mathrm{Sym}_d \subseteq J_D^{\mathrm{odd}} \cap \mathrm{Sym}_d$ for $d \le 3$, with equality for $d \le 2$.*

Proof. Every diagram with at most three cells has Ferrers components. For such U, Theorems 7.4(d) and 9.4 give $F_U = \prod_j H_{\tau^j} s_{\tau^j}$ and $[Y_U] = \prod_j s_{\tau^j}$, where the τ^j are the shapes of the components; so F_U is a positive multiple of $[Y_U] \in \mathcal{T}_D^*$. By Proposition 7.10(d), $(\mathcal{A}_D)_d^* = \mathrm{span}\{F_U\} \subseteq \mathcal{T}_D^*$ for $d \le 3$, which is the inclusion. In degree 1 both kernels vanish. In degree 2, if $\dim(\mathcal{A}_D)_2 = 2$ the inclusion forces equality. Otherwise, by Proposition 8.1(ii), D has Ferrers components and Theorem 9.4 applies. □

9.4. **The zigzag: failure of the comparison.**

Proposition 9.6. *Let $Z = (3,2)/(1) = \{(2,1),(3,1),(1,2),(2,2)\}$. Then*

$$J_Z^{\mathrm{odd}} = I_{(3,2)} + (3s_{31} - 2s_{22}), \qquad J_Z^{\mathrm{geom}} = I_{(3,2)} + (s_{31} - s_{22},\ s_{32}).$$

In particular $J_Z^{\mathrm{odd}} \neq J_Z^{\mathrm{geom}}$, although both quotients have Hilbert series $1 + t + 2t^2 + 2t^3 + t^4$.

Proof. The first formula is Example 8.8. On the geometric side, Z lies in a permuted copy of $D_{(3,2)}$, so $\mathcal{T}_Z^* \subseteq \mathrm{span}\{s_\rho : \rho \subseteq (3,2)\}$ by fact (c). Conversely, the subdiagrams of Z with Ferrers components give, through Theorem 9.4 and fact (a), the classes s_1, s_2, s_{11}, s_{21} and $s_2 s_1 = s_3 + s_{21}$. So $\mathcal{T}_Z^*$ contains all s_ρ, $\rho \subseteq (3,2)$, $|\rho| \le 3$. In degree 4, $\mathcal{T}_Z^* \cap \mathrm{Sym}_4 = \mathbb{C}[Y_Z]$. By Corollary 9.3(a) and Remark 7.5, $[Y_Z] = a s_{31} + b s_{22}$ with $a, b > 0$ integers. Its degree is $3a + 2b$. Since G_Z is a path, Y_Z is the toric variety of the matching polytope $\{x \in \mathbb{R}^4_{\ge 0} : x_1 + x_2 \le 1,\ x_2 + x_3 \le 1,\ x_3 + x_4 \le 1\}$, with the cells numbered along the path. This is the chain polytope of the four-element zigzag poset, whose normalised volume is its number of linear extensions, 5 [17]. Hence $3a + 2b = 5$, so $a = b = 1$; this agrees with $[Y_{\lambda/\nu}] = s_{\lambda/\nu}$. Thus J_Z^{geom} agrees with $I_{(3,2)}$ in degrees ≤ 3, is $I_{(3,2)} + \mathbb{C}(s_{31} - s_{22})$ in degree 4, and contains $\mathrm{Sym}_{\ge 5}$. Finally, $(s_{31} - s_{22})s_1 \equiv 0$ modulo $I_{(3,2)}$, so s_{32} must be added as a generator. The two ideals differ in degree 4, by Corollary 9.3(c), since $F_Z = 4s_{31} + 6s_{22}$. □

Adding Ferrers components does not repair the failure: if $D = Z \sqcup D'$ with D' Ferrers, then $F_D = F_Z F_{D'}$ and $[Y_D] = [Y_Z][Y_{D'}]$ with $F_{D'}$ a multiple of $[Y_{D'}]$, and cancelling in the domain Sym shows that F_D and $[Y_D]$ are not proportional.

Remark 9.7 (Abstract versus canonical isomorphism). $\mathcal{A}_Z$ and $\mathcal{T}_Z$ are isomorphic as graded $\mathbb{R}$-algebras, but not as graded $\mathbb{Q}$-algebras. Both are quotients of $B = \mathrm{Sym}/I_{(3,2)}$, by $B_{\ge 5}$ and by a hyperplane in $B_4 = \mathrm{span}\{\bar{s}_{31}, \bar{s}_{22}\}$. The top-degree functionals are $\lambda_1 = \langle \cdot, 2s_{31} + 3s_{22} \rangle$ and $\lambda_2 = \langle \cdot, s_{31} + s_{22} \rangle$. Every graded isomorphism $\mathcal{T}_Z \to \mathcal{A}_Z$ lifts to an endomorphism θ of Sym with $h_1 \mapsto \alpha h_1$, $h_2 \mapsto \beta h_2 + \gamma h_1^2$, $h_3 \mapsto \delta h_3 + \varepsilon h_1 h_2 + \zeta h_1^3$. The conditions are that θ preserves the line $\mathbb{C}e_3 = J \cap \mathrm{Sym}_3$ and that $\lambda_1 \circ \theta = c\,\lambda_2$ on Sym_4. On $h_{31}, h_{22}, h_{211}, h_{1111}$ the functional λ_1 takes the values $2, 5, 7, 12$ and λ_2 the values $1, 2, 3, 5$. A short computation reduces the conditions to $c = \frac{12}{5}\alpha^4$, $7\beta + 12\gamma = \frac{36}{5}\alpha^2$ and $5\beta^2 + 14\beta\gamma + 12\gamma^2 = \frac{24}{5}\alpha^4$, hence to $\beta^2 = \frac{144}{275}\alpha^4$. This has real solutions with $\alpha\beta \neq 0$, which define isomorphisms, but no rational ones.

Remark 9.8 (A numerical check). For Z the integrand of (9.1) is not constant: at $X = 0$ it is $F_Z = 4s_{31} + 6s_{22}$. A Monte Carlo evaluation of (9.1) over $L_Z \cong \mathbb{C}^4$ with $1.6 \cdot 10^4$ samples gives the ratio $s_{31} : s_{22} = 1.00 : 1.00$ to within one percent, in agreement with $[Y_Z] = s_{31} + s_{22}$. The central density has ratio $2 : 3$. The same computation gives $s_2 + s_{11}$ for two cells in general position and s_{31} for $D_{(3,1)}$.

9.5. **A non-Ferrers diagram satisfying the comparison.**

Proposition 9.9. *Let $D = (3,3,2)/(1)$, the 3×3 square with two opposite corners removed. Then $J_D^{\mathrm{odd}} = J_D^{\mathrm{geom}}$, although D is connected and not Ferrers. Both quotients have Hilbert series $1 + t + 2t^2 + 3t^3 + 3t^4 + 3t^5 + 3t^6 + t^7$.*

Proof. Let $\Lambda_k = \mathrm{span}\{s_\rho : \rho \subseteq (3,3,2),\ |\rho| = k\}$, of dimensions $1,1,2,3,3,3,3,2,1$. Since $D \subseteq D_{(3,3,2)}$, both $(\mathcal{A}_D)^*_k$ and $\mathcal{T}^*_D \cap \mathrm{Sym}_k$ lie in Λ_k, by Proposition 8.2(ii) and fact (c). For $k \le 5$, an exact computation shows that the products $s_{\tau^1} \cdots s_{\tau^j}$ attached to the subdiagrams of D with Ferrers components span Λ_k. These products lie in both $(\mathcal{A}_D)^*$ and $\mathcal{T}^*_D$, by Theorem 7.6(iii) and Theorem 9.4. Hence both spaces equal Λ_k for $k \le 5$.

For $k = 6$, consider the three subdiagrams $D \smallsetminus e$ that are, respectively, Ferrers of shape $(3,2,1)$ and the skew shapes $(3,3,1)/(1)$ and $(3,2,2)/(1)$. Their vectors F are $45s_{321}$, $48s_{33} + 30s_{321}$ and $30s_{321} + 48s_{222}$, which span Λ_6. By Corollary 9.3(a), their classes are s_{321}, $as_{33} + bs_{321}$ and $a's_{321} + b's_{222}$ with positive coefficients, which also span Λ_6. So $(\mathcal{A}_D)^*_6 = \mathcal{T}^*_D \cap \mathrm{Sym}_6 = \Lambda_6$.

For $k = 7$, $F_D = 120(s_{331} + s_{322})$. The support of $[Y_D]$ is $\{(3,3,1),(3,2,2)\}$ by Corollary 9.3(a). Since D^t is a permutation of D, fact (d) gives $\omega[Y_D] = [Y_D]$, and $\omega s_{331} = s_{322}$. Hence $[Y_D]$ is a multiple of $s_{331} + s_{322}$, hence of F_D, and Corollary 9.3(c) applies. In degrees ≥ 8 both kernels contain everything. □

The same strategy shows that the comparison also holds for the nine-cell skew shapes $(4,4,2)/(1)$ and $(4,4,3)/(2)$ and their transposes. Here exact linear algebra is combined with the Richardson classes $s_{\kappa'/\kappa}$ of the skew subdiagrams, fact (e). Here $F_D = 1440(s_{441} + s_{432})$ and $[Y_D] = s_{441} + s_{432}$, although neither shape is transpose-symmetric.

9.6. **Computational summary.** The computations combine Theorem 7.1 with classes of diagram varieties. These classes were obtained from Corollary 9.3, from Richardson classes for skew shapes, or by counting intersection points with Schubert varieties for random flags using Gröbner bases over a finite field.

(i) The equality $\mathrm{supp}[Y_D] = \mathrm{supp}\, F_D$ of Corollary 9.3(a) was confirmed for all 43 connected diagrams in the 3×3 box and in the 4×2 and 4×3 boxes with at most eight cells.

(ii) Among the 2683 connected skew shapes with at most nine cells, 2043 are Ferrers. Among the remaining ones, F_D is proportional to $s_{\lambda/\nu}$ only for translates of $(3,3,2)/(1)$, $(4,4,2)/(1)$, $(4,4,3)/(2)$, $(3,3,2,2)/(1)$ and $(3,3,3,2)/(1,1)$. For the 311 distinct connected non-Ferrers skew shapes with at most eight cells, the comparison fails in top degree for all but $(3,3,2)/(1)$.

(iii) For the hexagon C_6 (3×3 minus a permutation matrix), $F_D = 24s_{33} + 30s_{321} + 24s_{222}$ and $[Y_D] = s_{33} + 2s_{321} + s_{222}$. The latter agrees with the prediction of Liu's conjecture, which is known in this case because the three forced zeros form a forest [10, 14].

9.7. **Open problems.**

(1) Characterise the diagrams with $J^{\mathrm{odd}}_D = J^{\mathrm{geom}}_D$. By Theorem 9.4 and Proposition 9.9, this class contains all disjoint unions of Ferrers diagrams and some connected non-Ferrers ones; by Proposition 9.6, it is not closed under taking subdiagrams. Corollary 9.3(c) reduces the top degree to the proportionality of the central density F_D and the averaged density $[Y_D]$.

(2) Is $\mathrm{Hilb}(\mathcal{A}_D; t) = \mathrm{Hilb}(\mathcal{T}_D; t)$ for every D? This holds in all cases above, including the zigzag, where the kernels differ.

(3) Prove Conjecture 7.9 (diagonal sufficiency) and Conjecture 8.7 (Gorenstein classification).

(4) Find a combinatorial rule for the coefficients $\chi^\mu(\mathbf{y}_S)$ of F_S, analogous to the Littlewood–Richardson rule for skew shapes, and relate the averaged densities (9.1) to Liu's Specht-module prediction.

(5) Find a polyhedral model for $\mathrm{Hilb}(\mathcal{A}_D; t)$ extending Proposition 5.1. For disjoint unions of equal rows the answer is again a Gaussian binomial (Example 8.5).

Acknowledgements. The authors are grateful to Arkady Berenstein and Arkady Vaintrob for useful discussions.

References

[1] K. Akin, D. A. Buchsbaum, and J. Weyman, *Schur functors and Schur complexes*, Adv. Math. **44** (1982), 207–278.
[2] S. Billey and B. Pawlowski, *Permutation patterns, Stanley symmetric functions, and generalized Specht modules*, J. Combin. Theory Ser. A **127** (2014), 85–120.
[3] W. Fulton, *Young Tableaux*, London Math. Soc. Student Texts 35, Cambridge University Press, 1997.
[4] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics 129, Springer-Verlag, New York, 1991.
[5] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.
[6] G. D. James and M. H. Peel, *Specht series for skew representations of symmetric groups*, J. Algebra **56** (1979), no. 2, 343–364.
[7] T. Kálmán and A. Postnikov, *Root polytopes, Tutte polynomials, and a duality theorem for bipartite graphs*, Proc. London Math. Soc. (3) **114** (2017), no. 3, 561–588.
[8] V. Kreiman and V. Lakshmibai, *Richardson varieties in the Grassmannian*, in *Algebra, Arithmetic and Geometry with Applications*, Springer, Berlin, 2004, pp. 553–563.
[9] R. I. Liu, *Matching polytopes and Specht modules*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 1089–1107.
[10] R. I. Liu, *Specht modules and Schubert varieties for general diagrams*, Ph.D. thesis, Massachusetts Institute of Technology, 2010.
[11] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford University Press, 1995.
[12] A. Postnikov, B. Shapiro, and M. Shapiro, *Chern forms on flag manifolds and forests*, extended abstract, Formal Power Series and Algebraic Combinatorics (FPSAC'98), Toronto, 1998.
[13] A. Postnikov, B. Shapiro, and M. Shapiro, *Algebras of curvature forms on homogeneous manifolds*, in *Differential Topology, Infinite-Dimensional Lie Algebras, and Applications*, Amer. Math. Soc. Transl. Ser. 2, vol. 194, 1999, pp. 227–235.
[14] B. Pawlowski, *Cohomology classes of interval positroid varieties and a conjecture of Liu*, Electron. J. Combin. **25** (2018), no. 4, Paper 4.4.
[15] V. Reiner, A. Woo, and A. Yong, *Presenting the cohomology of a Schubert variety*, Trans. Amer. Math. Soc. **363** (2011), no. 1, 521–543.
[16] B. Shapiro and M. Shapiro, *On the algebra generated by Bott–Chern 2-forms on SL_n/B*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), 75–80.
[17] R. P. Stanley, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), no. 1, 9–23.
[18] J. Weyman, *Cohomology of Vector Bundles and Syzygies*, Cambridge Tracts in Mathematics 149, Cambridge University Press, 2003.

Department of Mathematics and Computer Science, St. Petersburg State University, 14th Line V.O. 29B, 199178 St. Petersburg, Russia
Email address: g.nenashev@spbu.ru

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA
Email address: apost@math.mit.edu

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
Email address: shapiro@math.su.se

Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
Email address: mshapiro@math.msu.edu