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Topology of intersections of Schubert cells and Hecke algebra

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Abstract

We consider intersections of Schubert cells $\mathscr{B}\alpha \cdot \mathscr{B}$ and $\sigma\mathscr{B}\sigma^{-}\beta \cdot \mathscr{B}$ in the space of complete flags $F = SL/\mathscr{B}$, where $\mathscr{B}$ denotes the Borel subgroup of upper triangular matrices, while α, β and σ belong to the Weyl group W (coinciding with the symmetric group). We obtain a special decomposition of F which subdivides all $\mathscr{B}\alpha \cdot \mathscr{B} \cap \sigma\mathscr{B}\sigma^{-}\beta \cdot \mathscr{B}$ into strata of a simple form. It enables us to establish a new geometrical interpretation of the structure constants for the corresponding Hecke algebra and in particular of the so-called R-polynomials used in Kazhdan-Lusztig theory. Structure constants of the Hecke algebra appear to be the alternating sums of the Hodge numbers for the mixed Hodge structure in the cohomology with compact supports of the above intersections. We derive a new efficient combinatorial algorithm calculating the R-polynomials and structure constants in general.

1. Preliminaries

Intersections of pairs and, more generally, of k-tuples of Schubert cells each belonging to its own Schubert cell decompositions of a flag space appeared in many articles, see e.g. [1,9,16,22,23]. Topological properties of such intersections are of particular importance in representation theory. Intersections of some special interesting arrangements of Schubert cells are directly related to the problem of representability of matroids [16]. Most likely, for somewhat general class of arrangements of Schubert cells their intersections and complements to such intersections do not have nice topological properties. Even the problem of nonemptiness of such intersections in the complex flag varieties is hard. However, in the important case of pairs of

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Schubert cells topology of their intersections appears to be more accessible. Namely, one can obtain a special decomposition of such intersections (generalizing the standard Schubert cell decomposition) into products of algebraic tori and linear subspaces of different dimensions. These strata in their turn are intersections of more than two Schubert cells originating from the initial pair. Such a decomposition enables us to calculate (algorithmically) additive topological characteristics of considered intersections, namely their Euler $E^{p,q}$-characteristics [7]. Generally speaking, the considered decomposition does not stratify all pairwise intersections of Schubert cells, i.e. the closure of a stratum is not always the union of strata of lower dimensions but still it implies interesting topological restrictions. We describe below the construction of this decomposition and formulate combinatorial and topological consequences.

2. Refined double decomposition, main algorithm and modifications

In order to formulate the main results, let us recall and introduce some notions.

Let $F_n = SL_n/B_n$ denote the space of complete flags in $\boldsymbol{C}^n$. Each compete flag f can be interpreted both as a Borel subgroup and as a sequence of enclosed subspaces of all dimensions from 0 to n. The Schubert cell decomposition D_f of F_n relative to f consists of cells formed by all flags having some given set of dimensions of intersections with subspaces of f. Thus we have a family of Schubert cell decompositions parameterized by F_n. Cells of any such decomposition are in 1–1 correspondence with permutations on n elements as follows.

Given a system of coordinates in $\boldsymbol{C}^n$, a flag is called *coordinate* if all its subspaces are spanned by coordinate vectors. Each coordinate flag is obviously identified with a permutation of coordinates. The coordinate flag is called *standard* if it is identified with the unit permutation, i.e. for all i its i-dimensional subspace is spanned by the first i coordinate vectors. For any two flags f and g in F_n one can always choose a system of coordinates such that f will be the standard coordinate flag and g will be some coordinate flag given by a permutation σ.

1. Definition. Pairs (f,g) such that in a suitable system of coordinates f is the standard flag and g is the coordinate flag identified with σ are called *pairs in the relative position* σ.

2. Definition. For any k-tuple of flags $f,g,h,\ldots$ in F_n we introduce *the k-tuple Schubert decomposition* $D_{f,g,h,\ldots}$ consisting of all nonempty intersections of k-tuples of cells one taken from each decomposition D_f, $D_g,\ldots$, etc.

Generally speaking, a k-tuple Schubert decomposition is not a stratification of the space of complete flags and its strata can have very complicated topology.

3. Definition. Given some set of linear subspaces, denote by their *+-completion* the set of sums of all possible subsets of these subspaces. By a +-completion of a pair of flags denote the +-completion of the set of all subspaces constituting these flags.

4. Important definition. *A refined double decomposition* $\bar{D}_{f,g}$ of the space of complete flags relative to a given pair of flags (f,g) is a decomposition into strata formed by all flags with some fixed dimensions of intersections with all subspaces from the +-completion of (f,g). Strata of this decomposition will be called refined double strata.

5. Remark. Refined double decomposition coincides with some special k-tuple decomposition.

6. Remark. Refined double decomposition $\bar{D}_{f,g}$ subdivides the standard decompositions D_f, D_g and the double decomposition $D_{f,g}$, i.e. each Schubert cell with respect to f or g as well as their pairwise intersections consist of some number of refined double strata.

Theorem A. *Each refined double stratum is biholomorphically equivalent to a product of a complex torus by a linear space.*

Now we enumerate (algorithmically) all strata of $\bar{D}_{f,g}$ using the permutation σ. By a decreasing subsequence in a permutation $\sigma = (i_1, \ldots, i_n)$ we understand a subsequence $i_{j_1}, i_{j_2}, \ldots, i_{j_k}$ such that $j_1 < j_2 < \cdots < j_k$ and $i_{j_1} > i_{j_2} > \cdots > i_{j_k}$.

7. Definition. *A cyclic shift of* σ *about a decreasing subsequence* $i_{j_1}, i_{j_2}, \ldots, i_{j_k}$ is a transformation sending i_{j_1} onto i_{j_k}, i_{j_2} onto $i_{j_1}, \ldots$, and i_{j_k} onto $i_{j_{k-1}}$ and preserving the rest of the elements. (If subsequence consists of just one element then the transformation is identical.)

We also assign two numbers to a cyclic shift of σ about some decreasing subsequence.

8. Definition. *The reduced length of a decreasing subsequence* is equal to the number of the elements in decreasing subsequence minus one. *The domination* of a decreasing subsequence is equal to the number of elements in σ for which there exists at least one element from decreasing subsequence which stands further in σ and is strictly bigger.

Example. Consider $\sigma = 6723451$ and its decreasing subsequence 731. Then the cyclic shift of σ about 731 is 6321457. The reduced length is 2 and the number of dominated elements is also 2, namely, the element 6 is dominated by 7 from decreasing subsequence and the element 2 is dominated by 3.

Fixing some permutation σ on n elements let us apply to it the following n-step procedure.

Main algorithm. The algorithm consists of the following steps.

The 1st step. Find all decreasing subsequences in σ. Applying to σ cyclic shifts about each of these decreasing subsequences obtain the set of resulting permutations. In each of these permutations block the first (and the biggest) element in the applied decreasing subsequence. (To block just means that this element will be ignored in all subsequent steps of the algorithm.)

The ith step. For each permutation obtained on the previous step apply the same procedure that was applied to σ in the 1st step. Namely, find all its decreasing subsequences (disregarding all blocked elements). Make cyclic shifts about each of these decreasing subsequences and finally in each of the obtained permutations block the 1st element of the decreasing subsequence applied in this step.

9. Remark. The algorithm stops exactly after n steps since in each permutation obtained after i steps we get exactly i elements blocked (i.e. in the $(i+1)$th step of algorithm we actually work with permutations on $n-i$ elements). Moreover, each permutation with at least one unblocked element contains at least one decreasing subsequence (probably consisting of just one element).

The whole procedure for $\sigma = 321$ is illustrated in Fig. 1. Bold numbers are non-blocked while underlined numbers are blocked and numbers on edges present decreasing subsequences.

10. Definition. By *a chain of permutations* we mean a sequence of $n+1$ permutations starting with σ and such that each consequent permutation is obtained from the preceeding one as the result of a cyclic shift on the corresponding step of the above procedure.

For example, there are 20 chains shown above in Fig. 1. They are just paths in the represented tree starting from the top element σ and going down to the bottom.

11. Definition. Assign to each chain two numbers, namely its *total length* equal to the sum of reduced lengths of all permutations involved and its *total domination* equal to the sum of all dominations. (When we calculate domination in a permutation with blocked elements we just disregard them completely.)

Theorem B. *If a pair of flags are in the relative position σ then strata of the refined double decomposition relative to this pair are in* 1–1 *correspondence with the chains in the above procedure starting with σ. The stratum corresponding to a given chain is a product of a complex torus of dimension equal to the total length of this chain by a linear space of dimension equal to its total domination.*

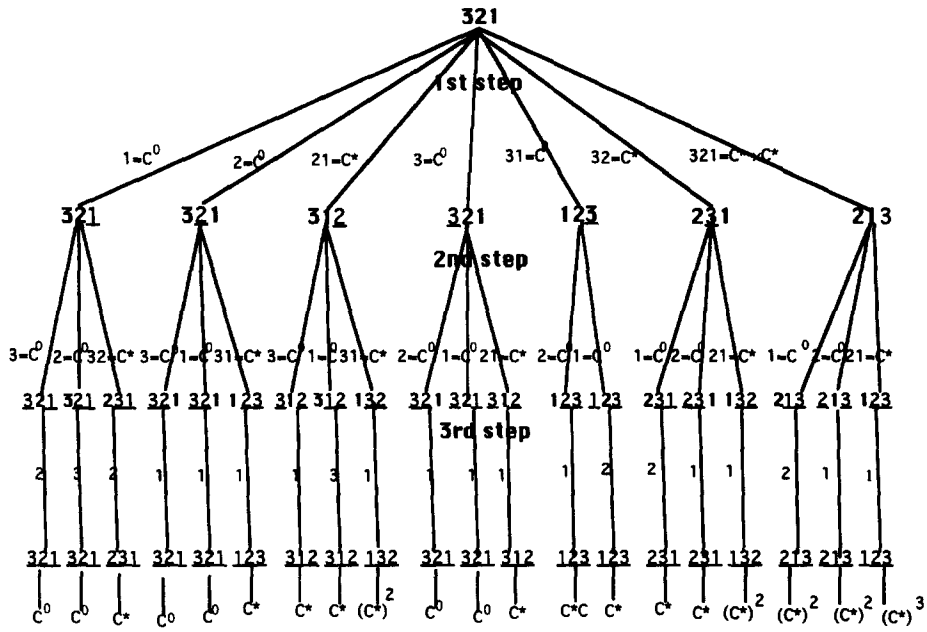

Fig. 1. Illustration of the main algorithm in the case $\sigma = 321$.

Example. The structure of each refined double stratum is given in the bottom line of Fig. 1. For example, $\mathbb{C}^0$ means that the corresponding chain is topologically just a point and $\mathbb{C}^*\mathbb{C}$ means that it is biholomorphically equivalent to a Cartesian product of a punctured line $\mathbb{C}^*$ and $\mathbb{C}$.

From now on we assume that an arbitrary pair of flags (f, g) is already transformed into the standard coordinate pair $(1, \sigma)$ (the element 1 denotes the standard coordinate flag and σ the flag corresponding to the permutation σ).

For any two permutations α and β, denote by $C_{1,\alpha}$ the Schubert cell consisting of all flags which are in relative position α with respect to 1 and by $C_{\sigma,\beta}$ the Schubert cell of all flags in relative position β with respect to σ.

By definition, the cell $C_{1,\alpha}$ belongs to decomposition D_1, the cell $C_{\sigma,\beta}$ belongs to decomposition D_σ and their intersection to the double decomposition $D_{1,\sigma}$.

Since, by definition, the refined double decomposition $\bar{D}_{1,\sigma}$ subdivides decompositions D_1, D_σ and $D_{1,\sigma}$ we now describe all refined double strata included in the Schubert cells $C_{1,\alpha}$, $C_{\sigma,\beta}$ and their intersection.

Let us assign to a chain of permutations the following two new permutations.

12. Definition. *The first permutation of a chain* is a sequence of successively blocked elements, i.e. its ith entry is the element blocked on ith step of procedure. *The second permutation of a chain* is the sequence of positions on which the successively blocked elements stand, i.e. its ith element is the number of position on which the ith blocked element stands.

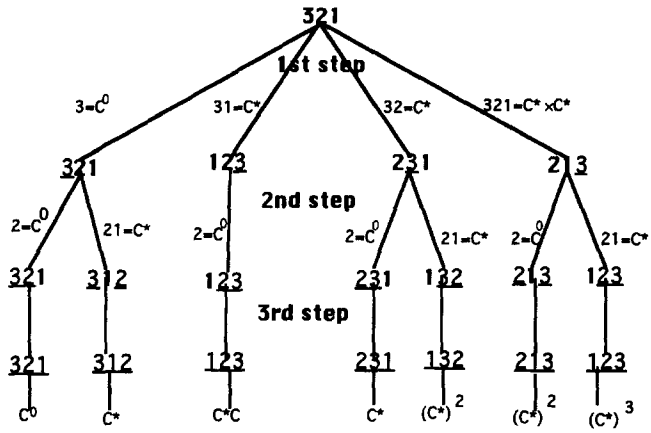

Fig. 2. Modified algorithm in the case $\sigma = \alpha = 321$.

Example. See Fig. 1. For the chain **321**$\to$**12**$\underline{3}$ $\to$**1**$\underline{23}$ $\to$ $\underline{123}$ the first permutation is 321 and the second permutation is 321. For the chain **321**$\to$**31**$\underline{2}$ $\to$**1**$\underline{32}$ $\to$ $\underline{132}$ the first permutation is 231 and the second permutation is 321. For the chain **321**$\to$ $\underline{3}$**21**$\to$ $\underline{31}$**2** $\to$ $\underline{312}$ the first permutation is 321 and the second permutation is 132.

Theorem C. *A stratum from $\bar{D}_{1,\sigma}$ belongs to the Schubert cell $C_{1,\alpha}$ if and only if the first permutation of its chain coincides with α; a stratum from $\bar{D}_{1,\sigma}$ belongs to the Schubert cell $C_{\sigma,\beta}$ if and only if the second permutation of its chain coincides with β. Therefore, a stratum belongs to the intersection $C_{1,\alpha} \cap C_{\sigma,\beta}$ if and only if its first permutation is α and its second permutation is β.*

This theorem enables to modify the described algorithm in order to obtain refined double decompositions of Schubert cells $C_{1,\alpha}$, $C_{\sigma,\beta}$ or their intersection.

Modifications of main algorithm. The following are the modifications.

The 1st modification. In order to obtain decomposition of $C_{1,\alpha}$ one must consider in the ith step for $i = 1,\ldots,n$ only decreasing subsequences starting at the ith element of the permutation α.

The 2nd modification. In order to obtain decomposition of $C_{\sigma,\beta}$ one must consider on the ith step for $i = 1,\ldots,n$ only decreasing subsequences ending at the position whose number is equal to the ith element in the permutation β.

The 3rd modification. Finally, in order to obtain decomposition of $C_{1,\alpha} \cap C_{\sigma,\beta}$ one must consider in the ith step for $i = 1,\ldots,n$ only decreasing subsequences starting at the ith element of permutation α and ending at the position whose number is equal to the ith element in the permutation β. See corresponding three examples on Figs. 2 and 3.

A similar process was proposed by Brenti [3] in the case $\sigma = \omega_0 = (n, n-1, \ldots, 1)$.

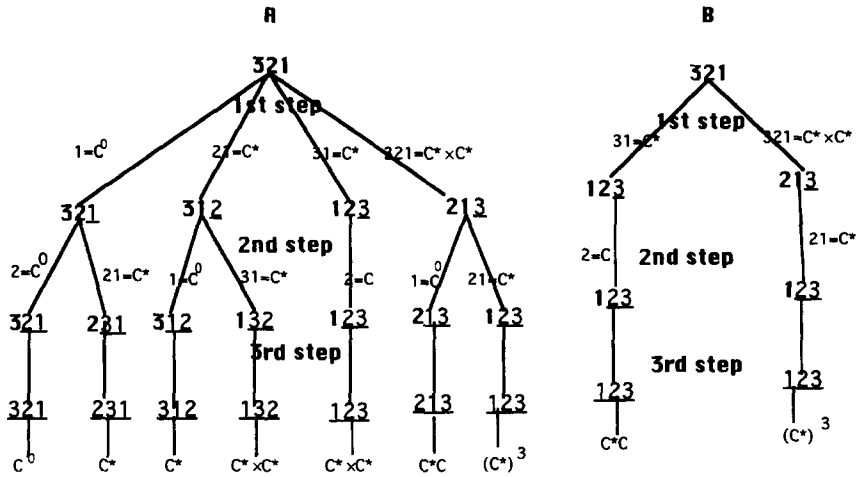

Fig. 3. Modified algorithm in the cases $\sigma = \beta = 321$ and $\sigma = \alpha = \beta = 321$.

The following remark is valid for all versions of the main algorithnm, i.e. for the refined double decomposition of the whole space of complete flags, some Schubert cell or some pairwise decomposition of Schubert cells.

13. Remark. Each step of the above algorithm is geometrically interpreted as projection of flags from the considered set on a linear subspace of the corresponding codimension. This means in particular that sets of restricted chains, i.e. those chains which start at a permutation with blocked elements obtained after some ith step present decomposition of F_{n-i}, (some Schubert cell in F_{n-i} or some pairwise intersection of Schubert cells in F_{n-i} depending on the corresponding modification of the algorithm). This feature will enable us later to obtain inductive formulas for structure constants in Hecke algebras different from the standard ones (cf. [3,11,12,23]). To be more precise we introduce the following operation on permutations with blocked elements. Consider some permutation on n elements with i blocked entries.

14. Definition. *The reduction of the blocked part* from a given permutation is the operation which forms the new permutation on $n-i$ elements in the following way. We exclude all blocked elements and subtract from each nonblocked element the number of all blocked elements which are less.

Example. The reduction of the blocked part from the permutation $75\underline{6}3\underline{2}4\underline{1}$ gives **4312**.

15. Proposition. (I) *The set of all restricted chains, i.e. chains starting at some permutation $\tilde{\sigma}$ obtained after i steps of the algorithm (and thus containing i blocked elements) geometrically presents*:

(1) *for the main algorithm — a refined double decomposition of F_{n-i} relative to the pair $(1, \tilde{\sigma}')$, where $\tilde{\sigma}'$ is obtained from $\tilde{\sigma}$ by reduction of all blocked elements*;

(2) *for the* 1*st modification — a refined double decomposition of the Schubert cell* $C_{1,\alpha'}$ *in* F_{n-i} *where* α' *is obtained by reduction of the first i elements from* α;

(3) *for the* 2*nd modification — a refined double decomposition of the Schubert cell* $C_{\tilde{\sigma}',\beta'}$ *in* F_{n-i}, *where* $\tilde{\sigma}'$ *is the same as above and* β' *is obtained by reduction of the first i elements from* β;

(4) *for the* 3*rd modification of the algorithm a refined double decomposition of* $C_{1,\alpha'} \cap C_{\tilde{\sigma}',\beta'}$.

(II) *Moreover, the geometrical meaning of the ith step of any variant of the algorithm is the decomposition of the initial object* (F_n, $C_{1,\alpha}$, $C_{\sigma,\beta}$ *or* $C_{1,\alpha} \cap C_{\sigma,\beta}$) *into a disjoint union of products of analogous objects in* F_{n-i} *enumerated by the set of all permutations* $\tilde{\sigma}$ *obtained after the ith step by* $(\boldsymbol{C}^*)^{l(\tilde{\sigma})} \times \boldsymbol{C}^{d(\tilde{\sigma})}$. *Here* $l(\tilde{\sigma})$ *is equal to the sum of reduced lengths of all permutations in the chain starting from* σ *and ending at* $\tilde{\sigma}$ *and* $d(\tilde{\sigma})$ *is the sum of all dominations in this chain.*

Example. The set of all reduced chains passing through any permutation obtained after the first step except **123** in Fig. 1 presents refined decomposition of the space of complete flags $F_2 = \mathbf{CP}^1$ relative to $\sigma = 21$ into three strata, namely two points and $\boldsymbol{C}^*$. Chains passing through **123** (or $\sigma = 12$) present the standard Schubert cell decomposition of F_2 into a point and $\boldsymbol{C}$.

3. Combinatorial and topological consequences

Now we list some special combinatorial and topological properties of strata included in some pairwise intersection of Schubert cells.

Theorem D. *Refined double strata included in any nonempty intersection of Schubert cells enjoy the following additional properties.*

(a) *The sum of the* (*complex*) *dimension of a torus and doubled dimension of a linear space equals* $lng(\alpha) + lng(\beta) - lng(\sigma)$ (*where lng is the usual length of a permutation*), *independently on a choice of nonempty stratum in a given* $C_{1,\alpha} \cap C_{\sigma,\beta}$, *in particular, all strata of the same dimension have the same form.*

(b) *If at least one of the permutations* α, β *or* σ *is the longest element* w_0 *then there are no gaps in* (*complex*) *dimension, i.e. there exist strata of all intermediate* (*complex*) *dimensions between minimal and maximal.*

Example. The only nontrivial intersection $C_{1,w_0} \cap C_{w_0,w_0}$ (where $w_0 = 321$) consists of two refined double strata $(\boldsymbol{C}^*)^3$ and $\boldsymbol{C}^* \times \boldsymbol{C}$, see Fig. 3(B).

The next result describes 'adjacency' of strata in $C_{1,\alpha} \cap C_{\sigma,\beta}$, i.e. gives the combinatorial conditions on strata in $C_{1,\alpha} \cap C_{\sigma,\beta}$ which can have nonempty intersection with the closure of some given stratum. Since refined double decomposition is in general

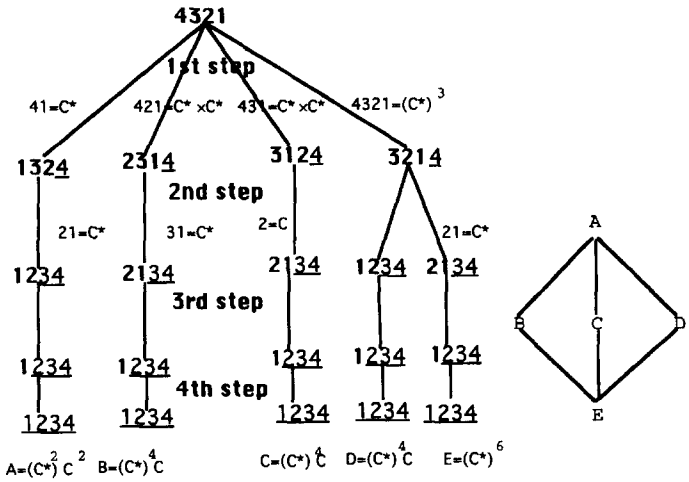

Fig. 4. Modified algorithm in the case $\sigma = \alpha = \beta = 4321$ and 'adjancency' of strata in decomposition.

not a stratification, this is not the standard notion of adjacency. Consider two ordered k-tuples of permutations.

16. Definition. The second k-tuple is called *less than or equal to* the first k-tuple, if each permutation of the second k-tuple is less than or equal to in the Bruhat order, the corresponding permutation of the first k-tuple, see the notion of the Bruhat order in e.g. [19]. The above partial order on the set of all refined double strata (or their chains of permutations) in $C_{1,\alpha} \cap C_{\sigma,\beta}$ is called *the adjacency partial order.*

By Theorem C each nonempty refined double stratum from $\bar{D}_{1,\sigma}$ is contained in the only pairwise intersection $C_{1,\alpha} \cap C_{\sigma,\beta}$, i.e. α equals to the first and β equals to the second permutation of the chain of the considered stratum.

Theorem E. *The closure of a given refined double stratum in its corresponding pairwise intersection of Schubert cells belongs to (but in general does not coincide with) the union of all refined double strata included in the same pairwise intersection such that their chains of permutations are less than or equal to, in the adjacency partial order, the chain of the considered stratum.*

See an example of decomposition and 'adjacency' on Fig. 4.

This theorem enables one to construct, rather easily examples of $C_{1,\alpha} \cap C_{\sigma,\beta}$ refined double decompositions of which are not stratifications. In particular, refined double decomposition of $C_{1234,4231} \cap C_{4231,4231}$ consists of three strata, namely $\boldsymbol{C}^*(\boldsymbol{C})^2$ and two copies of $(\boldsymbol{C}^*)^3\boldsymbol{C}$. The closure of one stratum $C = (\boldsymbol{C}^*)^3\boldsymbol{C}$ is nonempty in the other stratum $B = (\boldsymbol{C}^*)^3\boldsymbol{C}$, moreover the closure of $C = (\boldsymbol{C}^*)^3\boldsymbol{C}$ does not contain the whole stratum $A = \boldsymbol{C}^*(\boldsymbol{C})^2$.

17. Remark. The above results hold in complete generality for the spaces of complete flags over any algebraically closed field or $\boldsymbol{R}$. In this case $\boldsymbol{C}^*$ must be substituted by the multiplicative and $\boldsymbol{C}$ by the additive group of the field (cf. [6]).

Let us briefly recall several notions from complex algebraic geometry. We assume that readers are familiar with the Hodge decomposition in the cohomology of smooth compact Kaehlerian varieties (see e.g. [17]), further referred to as the *pure Hodge structure*.

According to [8–10], in the cohomology (with compact supports) of an arbitrary complex quasiprojective variety V there exists (and is uniquely defined) a so-called *mixed Hodge structure*. It consists of two filtrations W and F in $H^*(V,\mathbb{C})$ (in $H_c^*(V,\mathbb{C})$, respectively); W is an increasing rational filtration determined by the topology of V, while F is a decreasing filtration defined by the analytic structure of V which defines the pure Hodge structure on any quotient W_i/W_{i-1}; both filtrations are respected by arbitrary algebraic maps.

Using the notion of the mixed Hodge structure in, say, $H_c^*(V,\mathbb{C})$, one can define the corresponding *Hodge numbers* h_k^{pq} by the following formula:

$$h_k^{pq} = \dim \frac{F_p(W_{p+q}(H_c^k(V,\mathbb{C}))/W_{p+q-1}(H_c^k(V,\mathbb{C})))}{F_{p-1}(W_{p+q}(H_c^k(V,\mathbb{C}))/W_{p+q-1}(H_c^k(V,\mathbb{C})))}.$$

Corollary F (of Theorems B and E). *The Hodge numbers $h_k^{p,q}$ of any intersection $C_{1,\alpha} \cap C_{\sigma,\beta}$ can be positive only if $p = q$.*

Similar to the notion of the usual Euler characteristics of V, one can define the series of generalized Euler characteristics depending on p and q by

$$\chi^{pq} = \sum_k (-1)^k h_k^{pq},$$

and form their generating function, called the *$E^{p,q}$-polynomial* or just the *E-polynomial* of V,

$$E_V(u,v) = \sum_{p,q} \chi^{pq} u^p v^q.$$

The following crucial property of E_V follows from the additivity of generalized Euler characteristics for the cohomology with compact supports.

18. Lemma (see e.g. Durfee [13]). *If a quasiprojective complex variety V is represented as the disjoint union of quasiprojective subvarieties, $V = \bigcup_i V_i$, then*

$$E_V(u,v) = \sum_i E_{V_i}(u,v).$$

Corollary of Theorem B.

$$E_{\sigma}^{\alpha,\beta} = E_{C_{1,\alpha}\cap C_{\sigma,\beta}}(u,v) = \sum_{ch\in CH} z^{d(ch)}(z-1)^{l(ch)},$$

where $z = uv$; CH denotes the set of chains of all strata included in $C_{1,\alpha}\cap C_{\sigma,\beta}$, $d(ch)$ and $l(ch)$ are the total domination and the total length of a given chain respectively (cf. [11]).

The same expression can be rewritten as an inductive formula using the above remark on the geometrical meaning of our algorithm. More precisely, let SUB denote the set of all decreasing subsequences in σ starting at the element α_1 and ending at the position with the number β_1 (i.e. those subsequences which are used on ther first step of construction the refined double decomposition of $C_{1,\alpha}\cap C_{\sigma,\beta}$). For any decreasing subsequence $sub \in SUB$ let $l(sub)$ and $d(sub)$ denote its reduced length and domination, respectively. Finally, let α' and β' denote the results of reduction of the first elements α_1 and β_1 from α and β, respectively; $\sigma(sub)$ denote the result of the cyclic shift of σ about sub and reduction of the first element of sub, (see description of the algorithm and its modifications above). Then, by Proposition 15,

$$E_{\sigma}^{\alpha,\beta} = E_{C_{1,\alpha}\cap C_{\sigma,\beta}}(u,v) = \sum_{sub\in SUB} z^{d(sub)}(z-1)^{l(sub)}E_{C_{1,\alpha'}\cap C_{\sigma(sub),\beta'}}.$$

For example, $E_{(C_{1234,\,4321}\cap C_{4321,\,4321})} = (z-1)E_{(C_{123,\,321}\cap C_{132,\,321})} + (z-1)^2E_{(C_{123,\,321}\cap C_{231,\,321})} + (z-1)^2E_{(C_{123,\,321}\cap C_{312,\,321})} + (z-1)^3E_{(C_{123,\,321}\cap C_{321,\,321})}$.

Consider now the Leray spectral sequence converging to the cohomology of $C_{1,\alpha}\cap C_{\sigma,\beta}$ with compact supports which is associated with the refined double decomposition of the pairwise intersection of Shubert cells or more precisely with the corresponding filtration of $C_{1,\alpha}\cap C_{\sigma,\beta}$ by union of all strata of dimension less than or equal to some given value. Recall that its first page contains the cohomology with compact supports of the differences between consequent terms of filtration and the differential d_1 is induced by the long exact sequence of triples. In the case when the initial decomposition is a stratification the cohomology with compact supports of the differences between the ith and $(i-1)$th terms of filtration coincides with the direct sum of cohomology with compact supports of all strata of dimension i.

19. Definition. We call a pairwise intersection of Schubert cells $C_{1,\alpha}\cap C_{\sigma,\beta}$ *nice* if the refined double decomposition $RD_{1,\sigma}$ gives its stratification and *almost nice* if for any pair $St_1 < St_2$ of refined double strata (where $<$ denotes the relation in the adjacency partial order) $\dim St_1 \leqslant \dim St_2$. The rest of $C_{1,\alpha}\cap C_{\sigma,\beta}$ will be called hard.

Remark. Filtration by dimensions is the filtration by closed subsets for any (almost) nice $C_{1,\alpha}\cap C_{\sigma,\beta}$.

Theorem G. *The Leray spectral sequence associated with the filtration by dimensions of the refined double decomposition of any (almost) nice pairwise intersection degenerates at the second page.*

Remark. Unfortunately, a pure combinatorial description of d_1 is unavailable at the present moment and apparently is rather complicated.

Conjecture H. If $\boldsymbol{R}C_{1,\alpha} \cap \boldsymbol{R}C_{\sigma,\beta}$ is a nonempty intersection of Schubert cells over $\mathbb{R}$ and $C_{1,\alpha} \cap C_{\sigma,\beta}$ is its complexification then the actions of differentials in the corresponding Leray spectral sequences are concordant, i.e. respected by complex conjugation.

Hypothetical corollary I. *The intersection $\boldsymbol{R}C_{1,\alpha} \cap \boldsymbol{R}C_{\sigma,\beta}$ enjoys the so-called M-property, i.e. the sum of the Betti numbers with $\boldsymbol{Z}/2\boldsymbol{Z}$-coefficients of $\boldsymbol{R}C_{1,\alpha} \cap \boldsymbol{R}C_{\sigma,\beta}$ coincides with that of $C_{1,\alpha} \cap C_{\sigma,\beta}$, (cf. e.g.* [25]).

Finally, let us recall the notion of Hecke algebra H in its simplest version as a $\boldsymbol{C}$-algebra depending on a complex parameter q and given by a standard set of generators and relations.

The basis of H consists of elements T_ω, ω is any permutation on n elements, and multiplication rules are as follows:

(1) $T_\omega T_{\omega'} = T_{\omega\omega'}$ if $lng(\omega\omega') = lng(\omega) + lng(\omega')$, where $lng(\omega)$ is the length function equal to the number of inversions in ω;

(2) $T_{s_i} T_{s_i} = (z-1)T_{s_i} + zT_1$.

If z is a power of some prime number then H can be interpreted as the algebra of functions on GL_n over the finite field of z elements which are constant on double cosets $\mathscr{B}x\mathscr{B}$ with multiplication given by convolution [4,5]. (Analogously for generic z the algebra H can be identified with the set of sheaves constant on Schubert cells with multiplication given by sheaf convolution, see e.g. [28].)

Denote by $c^{w_3}_{w_1,w_2}$ the structure constant of Hecke algebra in the expansion $T_{w_1}T_{w_2} = \sum_{w_3 \in S_n} c^{w_3}_{w_1,w_2}(z)T_{w_3}$.

The structure constants of H are polynomials in z counting the number of points in the intersection of Schubert cells over finite fields if z is a power of prime [4].

Theorem J. *The set of structure constants for Hecke algebra coincides with the set of E-polynomials for the pairwise intersections of cells of the two Schubert decompositions of F_n. Namely,*

$$E^{\alpha,\beta_\sigma}(z) \equiv c^{\sigma}_{\alpha,\beta^{-1}}(z)$$

for any $\alpha, \beta, \sigma \in S_n$.

The above inductive formula of the E-polynomials of $C_{1,\alpha} \cap C_{\sigma,\beta}$ can obviously be rewritten as the inductive formula for the structure constants.

An analogous family of decompositions was introduced 10 years earlier in the case of intersections $C_{1,\alpha} \cap C_{w_0,\beta}$, where w_0 denotes the longest element in an arbitrary finite Coxeter system by Deodhar [11,12] and was extended to all intersections $C_{1,\alpha} \cap C_{\sigma,\beta}$ by Curtis in [6]. These decompositions depend on a reduced expression of the element α as a product of simple reflections and different choices of such expressions lead to different decompositions. Combinatorial data coding strata in the approach of Deodhar–Curtis is similar to chains of permutations corresponding to refined double strata but more lengthy. We have found the correspondence between strata of these two decompositions and proved that the refined double decomposition coincides with one of decompositions suggested by Deodhar for some particular choice of reduced expression.

Namely, let us define for any $\alpha \in S_n$ its reduced expression $\alpha = s_1 s_2 \ldots s_k$ of a special form. Present α in a form $\alpha = ((\alpha^{-1}(1), \alpha^{-1}(1)-1)(\alpha^{-1}(1)-1, \alpha^{-1}(1)-2)\ldots(2,1))\tilde{\alpha}$, where $\tilde{\alpha}$ belongs to S_{n-1}. This enables one to define inductively a reduced expression which will be called the standard expression.

Proposition K. *The refined double decomposition of the Schubert cell $B\alpha \cdot B$ coincides with the decomposition suggested by Deodhar if one chooses the standard reduced expression of α.*

Moreover, refined double decomposition is not the only geometrical way to refine double decompositions. These other decompositions of geometrical origin probably coincide with Deodhar's decompositions corresponding to other choices of reduced expressions.

One should probably mention that Theorem E together with the above proposition answers the question posed in [6] about the closure pattern of strata.

The starting point of this study was an attempt to calculate the cohomology of pairwise intersections of Schubert cells of the maximal dimension [26]. Brenti has attracted our attention to the fact that the properties of the E-polynomials calculated in [26] resemble those of the R-polynomials, and we managed to deduce the coincidence of these polynomials using our refined double decomposition and the results of Deodhar [11]. Later Brenti [3] has extended our combinatorial construction to a more general case, and succeeded in proving the coincidence of the E- and the R-polynomials in a pure combinatorial way. Here we prove analogous results in a more general setting by using geometrical arguments.

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