ENIGMA OF OF TWISTED TOEPLITZ MATRICES

DARIO GIANDINOTO AND BORIS SHAPIRO

ABSTRACT. In this paper we discuss spectral asymptotics of sequences of banded complex-valued Kac-Murdock-Szegö matrices and their perturbations. We conjecture that small banded perturbations of such sequences typically result in a drastic change of the asymptotic spectral measure. In the 3-diagonal case the perturbed measure is the push-forward of the uniform measure on a 2-dimensional cylinder under the symbol of the sequence. We prove this conjecture for special types of perturbations.

1. INTRODUCTION

Definition 1. Consider a Laurent series $a(x, z)$ of two variables given by

$$a(x, z) = \sum_{k \in \mathbb{Z}} a_k(x) z^k$$

where $x \in [0, 1]$, $z \in \mathbb{C}$ and $a_k(x)$ are piecewise continuous complex-valued functions on $[0, 1]$. (In many articles one uses $z = e^{it}$ and (if convergent) $a(x, t)$ is then defined on a cylinder $[0, 1] \times S^1$.) Following M. Kac, W. Murdock and G. Szegö [10], we associate to $a(x, z)$ the sequence of matrices of the form

$$T_n(a) = \left[a_{j-i} \left(\frac{i + j}{2n + 2} \right) \right]_{i.j=0}^{n}, \ n = 1, 2, \ldots . \tag{1.1}$$

Thus $T_n(a)$ is an $(n + 1) \times (n + 1)$-matrix with indices of its entries running from 0 to n which is well-defined independently of the convergence of the Laurent series $a(x, z)$ which is called the *symbol* of the sequence $\{T_n(a)\}$.

In different sources such matrices are called *locally Toeplitz, generalized Toeplitz, Toeplitz-like, twisted Toeplitz, variable coefficient Toeplitz matrices;* instead of being constant as in the classical Toeplitz case, their entries along the $(j - i)$-th diagonal are modelled by piecewise continuous functions $a_{j-i}(x)$. Following [2], we call matrices of this form *Kac-Murdock-Szegö* matrices (KMS-matrices for short) which, in our opinion, is a more suitable name for this interesting class.[1]

It is an easy observation that matrices $T_n(a)$ will be Hermitian if and only if the symbol $a(x, t)$ is real-valued. The main results of [10] called *the first and the second Szegö theorems* are mostly dealing with this special case. Much later

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[1]The term 'Kac-Murdock-Szegö matrix' is sometimes used for special symmetric matrices with matrix elements $A_{jk} = \rho^{|j-k|}$ which were introduced in the same paper [10]. Usually parameter ρ has absolute value smaller than 1. This terminology is somewhat misleading since these matrices are, in fact, Toeplitz.

and independently special cases of such results have been rediscovered in [11]. In particular, both publications contain formulas for the asymptotic spectral measure in the Hermitian case. These results have been extended to the case when the sequence $\{T_n(a)\}$ consists of normal matrices. However it seems to be unknown for which symbols $a(x, z)$ such situation occurs.

For a general symbol one still has a formula for the asymptotic distribution of singular values of $\{T_n(a)\}$. But, to the best of our knowledge, the asymptotic spectral distribution is only known in very few cases. Namely, (banded) Toeplitz matrices have been extensively studied starting from the 1960's in the papers of P. Schmidt and F. Spitzer [16], J. Ullman [25], I. Hirschman [9], H. Widom [26, 27], P.Tilli [20, 21, 22] and several other authors. This material is summarized in e.g. [3]. In addition to this, only some sporadic cases of symbols $a(x, z)$ where additional tools such as e.g. integral presentation of the characteristic polynomials of $\{T_n(a)\}$ are available have been treated in the literature, see e.g. [13, 12, 1, 18, 17]. In recent years the notion of GLT-sequences of matrices (partially) generalizing that of KMS-matrices has attracted substantial attention of the specialists in linear algebra due to its importance in discretization of linear ODE and PDE, see e.g. [5, 6]. Many results have been extended from the Toeplitz and KMS-cases to the GLT-case.

Finally, one should mention a related area of asymptotic spectral theory of real and complex infinite Jacobi matrices going back to the 1920's. It has been developed by a number of famous mathematicians including I. M. Gelfand, Yu. M. Berezanskii, B. Simon, M. Kac, N. Levinson, B. Levitan, P. van Moerbeke, etc. In particular, recent developments in this area have been stimulated by the PT-symmetric quantum mechanics, see e.g. [4]. One origin of such matrices is the spectral theory of quasi-exactly solvable potentials for the Schrödinger equations.

Two major open questions for sequences of KMS-matrices are as follows.

Problem 1. *What conditions on the symbol $a(x, z)$ guarantee the weak convergence of the sequence $\{\mu_n(a)\}$ where $\mu_n(a) = \frac{1}{n+1}\sum_{i=1}^{n+1} \delta_{\lambda_{i,n}}$ is the spectral measure of $T_n(a)$? (Here δ is the Dirac delta and $\lambda_{i,n}$ are the eigenvalues of $T_n(a)$).*

Problem 2 (comp. Problem 3 of [2])**.** *If the above asymptotic spectral measure $\mu(a) = \lim_{n\to\infty} \mu_n(a)$ exists, describe its support and density in terms of the symbol $a(x, z)$. (The limit is understood in the sense of weak convergence of measures.)*

To our surprise, we have found no discussions of Problem 1 in the literature. Numerical experiments strongly suggest that $\mu(a)$ exists if the functions $a_k(x)$ are piecewise analytic on $[0, 1]$ and probably under much weaker assumptions. We state this guess as Conjecture 1 below and in what follows assume its validity hoping to return to this question in a future publication.

As we already mentioned, it seems that for the non-Hermitian/non-normal sequences $\{T_n(a)\}$ there are no known answers to Problem 2, except for the case of (block) Toeplitz matrices and several particular examples.

Our first goal is to suggest a conjectural description of $\mu(a)$ for banded KMS-sequences as a so-called mother body of another explicitly defined measure $\mathcal{M}(a)$ associated to the symbol $a(x, z)$, see Definition 5. Confirmation of our guess comes partially from a known result about the coincidence of the harmonic moments of these two measures (see Proposition 1 and Theorem 2.2 of [2]) and partially from our computer experiments.

The notion of a mother body seems to be rather unknown to the specialists in asymptotic matrix analysis although it can traced back to the notion of balayage of a measure defined by H. Poincare around the turn of the previous century, see [14, 7]. On the physical level of rigour, mother bodies were introduced in the 1960's by a Bulgarian geophysicist D. Zidarov [28] and later mathematically developed by B. Gustafsson [8]. During the last decades mother bodies of solid domains or, more generally, of positive Borel measures were discussed both in geophysics and mathematics, see e.g. [19], [15], [8], [28].

Although a number of interesting results about mother bodies was obtained in several special cases, [15], [8], [28] there is still no full consensus about a general definition and we use below one of possibilities. No general existence and/or uniqueness results for mother bodies are known at present. Therefore in order to rigorously justify our guess one needs first to substantially develop the existing theory. In addition to that, one can suspect that the support of $\mu(a)$ could be related to Stokes lines of linear difference equations, but no such connection is known at present.

Our second goal is to show that the asymptotic spectral measure $\mu(a)$ (which we assume exists) is highly unstable under perturbations. We conjecture that in the case of banded sequence $\{T_n(a)\}$ the spectrum of its generic banded perturbation has the asymptotic density given by the above measure $\mathcal{M}(a)$ which we prove in a number of cases and illustrate numerically. In a sense, the support of $\mathcal{M}(a)$ is an analog of pseudospectrum for banded deformations of banded KMS matrices.

To simplify our considerations about perturbations we restrict them to symmetric symbols of the form

$$a(x,z) = c(x)z^{-1} + b(x) + c(x)z \tag{1.2}$$

with b, c being continuous complex-valued functions on $[0,1]$. Already this case has all essential features present in more general situation and is closely related to the theory of Jacobi matrices and discrete Schroedinger equations with complex potential.

Remark 1. The observed instability of $\mu(a)$ is similar to the one discussed in [**?**, **?**], but both our scenario and our answers are different from that of loc.cit.

The main results of this paper are as follows... The structure of the paper is as follows .....

2. Existence and mother body conjectures

Supported by numerous computer experiments we guess the following.

Conjecture 1. *For a banded symbol* $a(x,z) = \sum_{k=-q}^{p} a_k(x)z^k$ *where each* $a_k(x)$ *is a piecewise analytic function on* $[0,1]$*, the asymptotic eigenvalue measure* $\mu(a)$ *exists and its support is a finite union of semi-analytic curves and points in* $\mathbb{C}$*, see Fig. 1.*

To provide a description of $\mu(a)$, we need to introduce another measure related to the symbol $a(x,z)$. We start with the Toeplitz case.

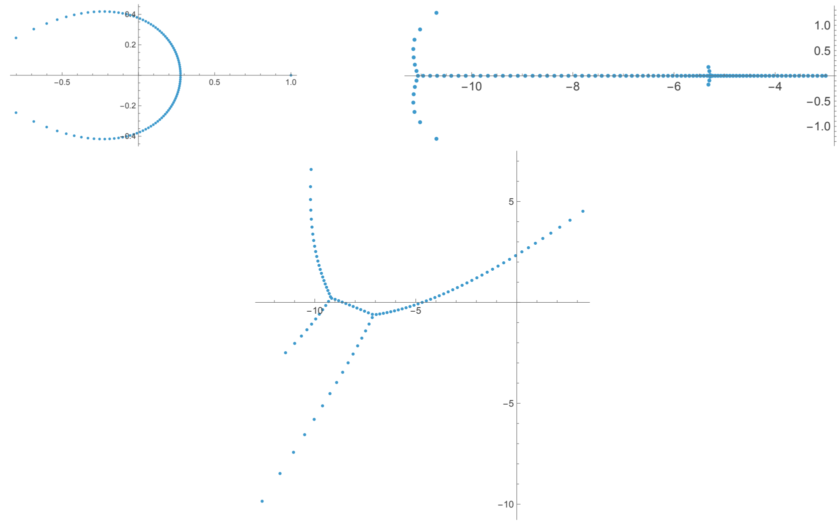

FIGURE 1. Empirical measure $\mu(a)$ (blue dots) in the cases:
(i) $b(x) = 1 - 2x$ and $c(x) = i\sqrt{x(1-x)}$ (left);
(ii) $b(x) = -3 - 5x - 4x^2 - x^3 - x^4 + x^5 - 4x^6 - 3x^7 + 3x^8 + 5x^9 + 3x^{10} - 2x^{11} - 3x^{13} + 4x^{14}$ and $c(c) = i(4x - 4x^2 - x^3)$ (right);
(iii) $b(x) = (-4+i) - 2x - (3+i)x^2 - (3+2i)x^3$ and $c(x) = (-4-2i) + (5+5i)x - (4+3i)x^2 + (4+5i)x^3$ (down).

Definition 2. *Given an infiniite banded Toeplitz matrix T, denote by μ_T its asymptotic eigenvalue measure, i.e. the asymptotic root-counting measure of the sequence of characteristic polynomials of its principal minors.*

The existence and description of μ_T are the main results of [16]. In particular, the support of μ_T has the following description.

Let

$$a(z) = \sum_{k=-q}^{p} a_k z^k \qquad (a_{-q} \neq 0,\ a_p \neq 0)$$

be a Laurent polynomial. The associated Toeplitz matrices are $(p+q+1)$-banded. Let $T_n(a(z))$ denote the $(n+1)\times(n+1)$ Toeplitz matrix with entries

$$(T_n(a(z)))_{ij} = a_{i-j}, \qquad i, j = 0, 1, \ldots, n.$$

Denote its spectrum by

$$\sigma_n = \operatorname{spec}(T_n(a(z))).$$

Define the *asymptotic spectrum* (limit set of eigenvalues) by

$$B = \{\lambda \in \mathbb{C} :\ \exists\, n_m \to \infty,\ \lambda_m \in \sigma_{n_m},\ \lambda_m \to \lambda\}.$$

Equivalently,

$$B = \limsup_{n\to\infty} \sigma_n.$$

For each $\lambda \in \mathbb{C}$, define

$$Q(\lambda; z) = z^q\,(b(z) - \lambda),$$

which is a polynomial in z of degree $p+q$. Let

$$\alpha_1(\lambda) \le \alpha_2(\lambda) \le \cdots \le \alpha_{p+q}(\lambda)$$

be the moduli of the zeros of $Q(\lambda; z)$, listed with multiplicity and in nondecreasing order. Define

$$C = \{\lambda \in \mathbb{C} : \ \alpha_q(\lambda) = \alpha_{q+1}(\lambda)\}.$$

Theorem A (Schmidt–Spitzer, 1960)**.** *With b, $T_n(b)$, and B as above,*

$$B = C,$$

that is, the set of limit points of eigenvalues of the finite banded Toeplitz matrices $T_n(b)$ is exactly

$$B = \big\{\lambda \in \mathbb{C} : \ \alpha_q(\lambda) = \alpha_{q+1}(\lambda)\big\}.$$

Equivalently, if $z_1(\lambda), \dots, z_{p+q}(\lambda)$ are the zeros of $Q(\lambda; z)$ ordered so that

$$|z_1(\lambda)| \le \cdots \le |z_{p+q}(\lambda)|,$$

then

$$B = \{\lambda \in \mathbb{C} : \ |z_q(\lambda)| = |z_{q+1}(\lambda)|\}.$$

Now let us define the second measure associated with a banded symbol $a(x, z)$.

Definition 3. *For a symbol $a(x, z) = \sum_{k=-q}^{p} a_k(x) z^k$ with piecewise analytic functions $a_k(x)$, denote by $T(x) := T_a(x)$, $x \in [0, 1]$ the 1-parameter family of infinite banded Toeplitz matrices obtained by fixing x, i.e., such that the diagonal entries of $T_a(x)$ on the diagonals $-q, -q+1, \dots, p-1, p$ are given by $a_{-q}(x), a_{-q+1}(x), \dots, a_{p-1}(x), a_p(x)$ respectively. Let $\mu_{T_a(x)}$ denote the asymptotic spectral measure of $T_a(x)$. Now define the* **standard measure** *$\mathcal{M}(a)$ of the symbol $a(x, z)$ as given by*

$$\mathcal{M}_a := \int_0^1 \mu_{T_a(x)} dx. \tag{2.1}$$

Although for a non-real symbol $a(x, z)$ the measures $\mu(a)$ and $\mathcal{M}(a)$ are non-equal they still enjoy the following important property.

Definition 4. *Given two compactly supported in $\mathbb{C}$ probability measures we say that they are* **equipotential** *if their logarithmic potentials coincide near ∞.*

Recall that the logarithmic potential $\mathcal{L}_\nu(z)$ of a Borel measure ν in the complex plane is given by

$$\mathcal{L}_\nu(z) = \int_{\mathbb{C}} \log |z - \xi| d\nu(\xi). \tag{2.2}$$

The next result was mentioned to the second author several years ago by A. Kuijlaars.

Proposition 1. *In the above notation, the measures $\mu(a)$ and $\mathcal{M}(a)$ are equipotential.*

To prove Proposition 1 we need the following generalization of the 1st Szegö theorem to the case of complex-valued KMS matrices.

Theorem B (see Theorem 2.2 of [2])**.** *Let $\{T_n(a)\}$ be a sequence of KMS matrices that satisfies condition*

$$\Upsilon_a := \sum_k ||a_k(x)||_\infty < \infty, \tag{2.3}$$

where $||.||_\infty$ is the L_∞-norm on $[0,1]$.

Set $D_a = \{z \in \mathbb{C} : |z| \leq \Upsilon_a\}$. Then for any function ϕ holomorphic in D_a one has

$$\lim_{n\to\infty} \frac{Tr[\phi(T_n(a))]}{n+1} = \frac{1}{2\pi}\int_0^1 \int_0^{2\pi} \phi(a(x,t))dtdx \tag{2.4}$$

where $a(x,t) = a(x, e^{it})$ and Tr stands for the trace of a matrix.

Proof of Proposition 1. Let us apply Theorem B to $\phi_k = z^k$. Then it says that for $k = 0, 1, 2, \dots,$

$$\frac{1}{n+1} Tr[T_n^k(a)] = \frac{1}{n+1}\sum_{j=0}^n \lambda_{j,n}^k(a) = \frac{1}{2\pi}\int_0^1 \int_0^{2\pi} a^k(x,t)dtdx + \epsilon(n),$$

where $\lambda_{0,n}(a), \dots \lambda_{n,n}(a)$ are the eigenvalues of $T_n(a)$ and $\lim_{n\to\infty} \epsilon(n) = 0$. Under the assumption that meaure $\mu(a)$ exists one has

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{j=0}^n \lambda_{j,n}^k(a) = \int_{\mathbb{C}} z^k d\mu(a)dz$$

which is the k-th harmonic moment $m_k(\mu(a))$ of $\mu(a)$. Furthermore the kth harmonic moment of $\mathcal{M}(a)$ is given by the same formula. Indeed for each fixed $x \in [0,1]$, the usual 1st Szegö limit theorem applies to the Toeplitz matrix $T_a(x)$ and gives that the k-th harmonic moment of $\mu_{T_a(x)}$ equals $\frac{1}{2\pi}\int_0^{2\pi} a^k(x,t)dt$. Using Definition (2.1) and integrating the latter expression with respect to $x \in [0,1]$ we get the coincidence of k-th harmonic moments of $\mu(a)$ and $\mathcal{M}(a)$.

Finally, observe that the coincidence of all harmonic moments of two compactly supported measures implies that they have coinciding Cauchy transforms $\mathcal{C}(z)$ near ∞. Recall that the Cauchy transform $\mathcal{C}_\nu(z)$ of a measure ν is given by

$$\mathcal{C}_\nu(z) = \int_{\mathbb{C}} \frac{d\nu(\xi)}{z-\xi}.$$

Moreover, for a compactly supported measure ν, its Cauchy transform has the following expansion at ∞:

$$\mathcal{C}_\nu(z) = \frac{m_0(\nu)}{z} + \frac{m_1(\nu)}{z^2} + \frac{m_2(\nu)}{z^2} + \dots$$

Finally, coincidence of Cauchy transforms of two measures near ∞ implies coincidence of their logarithmic potentials near ∞ and therefore equipotentiality of measures. □

To formulate our main conjecture about $\mu(a)$ we need another definition. Consider a finite positive Borel measure Ξ supported on a bounded domain $\Omega \subset \mathbb{C}$ with a piecewise smooth boundary and whose density is a piecewise analytic function in Ω.

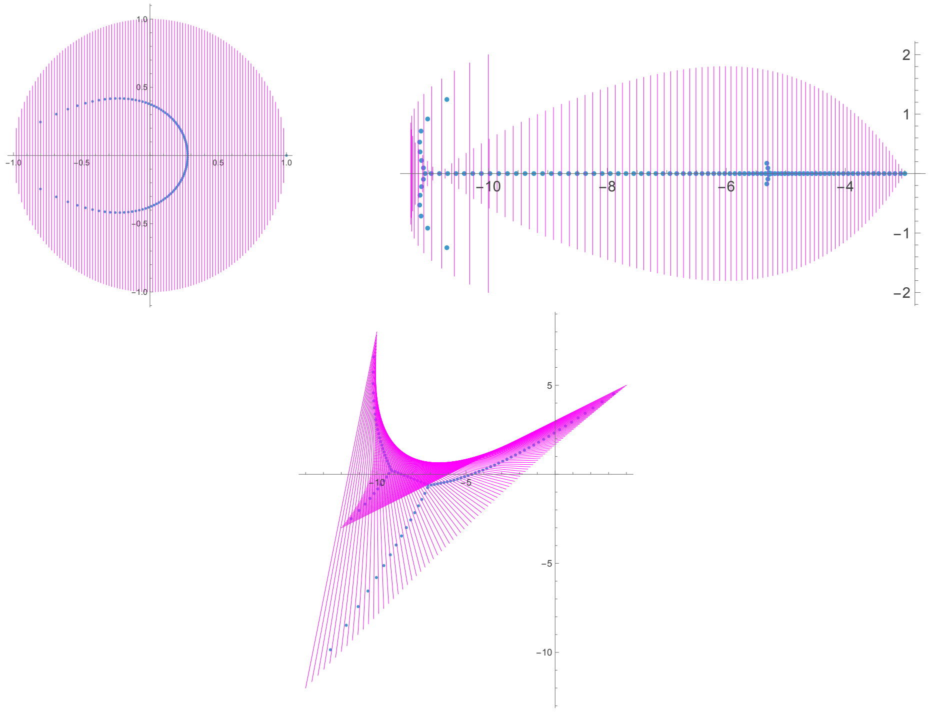

FIGURE 2. Measures $\mathcal{M}(a)$ (pink lines) and $\mu(a)$ (blue dots) in the cases:
(i) $b(x) = 1 - 2x$ and $c(x) = i\sqrt{x(1-x)}$ (left);
(ii) $b(x) = -3 - 5x - 4x^2 - x^3 - x^4 + x^5 - 4x^6 - 3x^7 + 3x^8 + 5x^9 + 3x^{10} - 2x^{11} - 3x^{13} + 4x^{14}$ and $c(c) = i(4x - 4x^2 - x^3)$ (right);
(iii) $b(x) = (-4+i) - 2x - (3+i)x^2 - (3+2i)x^3$ and $c(x) = (-4-2i) + (5+5i)x - (4+3i)x^2 + (4+5i)x^3$ (down).

Definition 5. By a *mother body measure* μ_Ξ of Ξ, we mean a positive measure such that
(i) its support $S := \operatorname{supp} \mu_\Xi$ belongs to Ω and consists of finitely many compact real-analytic curves and finite many points.
(ii) the logarithmic potential of μ_Ξ coincides with that of Ξ in the complement $\mathbb{C} \setminus \Omega$.

Conjecture 2. *In the above notation, measure $\mu(a)$ is a motherbody measure for $\mathcal{M}(a)$, see Fig. 2.*

Remark 2. In many of our numerical experiments the complement $\mathbb{C} \setminus S$ to the support of $\mu(a)$ is path–connected. However in case (i) in Fig. 1 the support of $\mu(a)$ seems to coincide with the famous Szegö curve and is a closed loop.

Remark 3. Observe that the support of a motherbody of a measure Ξ with analytic density and a 2-dimensional support must pass through every singularity of the boundary of the support of Ξ which we observe in Fig. 2–3.

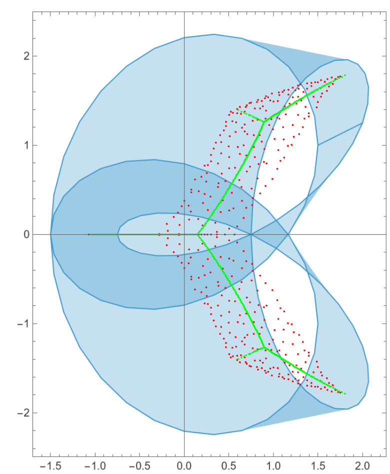

FIGURE 3. Containment of supports of $\mu(a)$ and $\mathcal{M}(a)$. The blue domain is the essential range of the symbol, the red dots are approximating the support of $\mathcal{M}(a)$ and the green dots are approximating the support of $\mu(a)$.

Remark 4. By Proposition 1 the measures $\mu(a)$ and $\mathcal{M}(a)$ are equipotential. For $\mu(a)$ to be a mother body of $\mathcal{M}(a)$ one needs to show is that $\mu(a)$ has a 1-dimensional support belonging to that of $\mathcal{M}(a)$ which we are currently unable to prove, see Fig. 3. Observe that as a consequence of Gershgorin's circle Theorem and condition (2.3), the spectrum of every $T_n(a)$ lies inside the disk D_a.

A much more interesting localization result which however holds with probability one is Theorem 2.11 of [2], see Fig. 3. Its formulation is as follows.

We define the *essential range* $\mathcal{R}(a)$ of the symbol $a(x,t) = a(x, e^{it})$ by :

$$\mathcal{R}(a) := \{z \in \mathbb{C} : \text{meas}\{a^{-1}(D_r(z))\} > 0 \; \forall r > 0.$$

That is, $\mathcal{R}(a)$ is the set of those points $z \in \mathbb{C}$ with the property that the Lebesgue measure of the inverse image of any open set containing z is positive. $D_r(z)$ denotes an open disk in the complex plane with radius r centered at z.

For symbols $a(x,t)$ satisfying the decay condition (2.3), $\mathcal{R}(a)$ is a compact set; hence its complement has just one unbounded connected component, and we can write

$$\mathbb{C} \setminus \mathcal{R}(a) := \bigcup_{j=1}^{\infty} U_j, \quad U_i \cap U_J = \emptyset \text{ for } i \neq j. \tag{2.5}$$

Here each $U_j,\ j \geq 1$, is a connected bounded open set, and U_0 is an unbounded connected open set. Using (2.5) we define the *extended range* of the symbol $a(x,t)$ as

$$\mathcal{ER}(a) := \mathbb{C} \setminus U_0.$$

Hence, the extended range $\mathcal{ER}(a)$ is the union of the range and all the bounded components of its complement.

Theorem C (Theorem 2.11 of [2]). *Let $a(x,t)$ be as above. Then, the extended range $\mathcal{ER}(a)$ is a cluster of the eigenvalues of $T_n(a)$. That is, for any open set for V containing $\mathcal{ER}(a)$ one has*

$$\lim_{n\to\infty} \frac{N(V,n)}{n+1},$$

where $N(V,n)$ is the number of eigenvalues of $T_n(a)$ that lie inside V. In other words, any ϵ-neighborhood of $\mathcal{ER}(a)$ contains all of the eigenvalues of T_n except at most $o(n)$ of them.

Remark 5. Observe that Fig. 3 illustrates that the support of $\mu(a)$ does not lie in the essential range $\mathcal{R}(a)$, but only in the extended range $\mathcal{ER}(a)$. However in our numerical experiments the support of $\mu(a)$ always belongs to that of $\mathcal{M}(a)$.

Remark 6. An idea to prove that the support of $\mu(a)$ belongs to the support of $\mathcal{M}(a)$ is to show that the latter support contains the spectra of all small perturbations of the original KMS-sequence.

3. A WKB-based approach to Problem 1

In this section we record a possible route to Problem 1 in the tridiagonal case. The result below should be understood as a conditional existence theorem: it reduces the weak convergence of the eigenvalue measures to a standard exact WKB statement for the associated second order difference equation. We do not attempt here to give a complete proof of this exact WKB statement.

We consider symbols of the form

$$a(x,z) = c(x)z^{-1} + b(x) + c(x)z, \tag{3.1}$$

where b, c are complex-valued functions analytic in a neighbourhood of $[0,1]$, and $c(x) \neq 0$ there. The corresponding sequence $T_n(a)$ is tridiagonal. We write

$$p_n(\lambda) := \det(\lambda I - T_n(a)).$$

The normalized eigenvalue counting measure is

$$\mu_n(a) = \frac{1}{n+1} \sum_{\lambda \in \operatorname{spec}(T_n(a))} \delta_\lambda,$$

where the eigenvalues are counted with their algebraic multiplicities.

Let

$$u_n(\lambda) := \frac{1}{n+1} \log |p_n(\lambda)|.$$

Then

$$\frac{1}{2\pi} \Delta u_n = \mu_n(a) \tag{3.2}$$

in the sense of distributions. Consequently, weak convergence of $\mu_n(a)$ follows from convergence of u_n in $L^1_{\mathrm{loc}}(\mathbb{C})$.

The characteristic polynomials of the principal minors satisfy a three-term recurrence. Up to an inessential shift in the endpoint convention coming from Definition 1, it has the form

$$p_{k+1,n}(\lambda) = \big(\lambda - b(x_k)\big) p_{k,n}(\lambda) - c(x_k)^2 p_{k-1,n}(\lambda), \qquad x_k = \frac{k}{n+1}. \tag{3.3}$$

Equivalently,

$$\begin{pmatrix} p_{k+1,n} \\ p_{k,n} \end{pmatrix} = A(x_k, \lambda) \begin{pmatrix} p_{k,n} \\ p_{k-1,n} \end{pmatrix}, \qquad A(x, \lambda) = \begin{pmatrix} \lambda - b(x) & -c(x)^2 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

Thus

$$p_n(\lambda) = (1, 0) A(x_n, \lambda) \cdots A(x_0, \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvalues of the local transfer matrix are

$$\rho_\pm(x, \lambda) = \frac{\lambda - b(x) \pm \sqrt{(\lambda - b(x))^2 - 4c(x)^2}}{2}. \tag{3.5}$$

They satisfy

$$\rho_+(x, \lambda) \rho_-(x, \lambda) = c(x)^2.$$

The turning points are the solutions of

$$(\lambda - b(x))^2 - 4c(x)^2 = 0, \qquad \text{equivalently} \qquad \lambda = b(x) \pm 2c(x). \tag{3.6}$$

Away from turning points one can choose analytic branches of $\rho_\pm$, and the local WKB phases are

$$\Phi_\pm(x, \lambda) = \int^x \log \rho_\pm(s, \lambda)\, ds.$$

The following hypothesis is the exact WKB input needed in the sequel.

Hypothesis 1 (Finite exact WKB representation). For every compact set $K \Subset \mathbb{C}$, there exists an exceptional set $E_K \subset K$, contained in a finite union of real-analytic arcs and isolated points, such that on every simply connected component D of $K \setminus E_K$ the characteristic polynomial has a representation

$$p_n(\lambda) = \sum_{\alpha=1}^{m} A_\alpha(\lambda) \exp\{(n+1)\Phi_\alpha(\lambda)\} \big(1 + o(1)\big) \tag{3.7}$$

locally uniformly in $\lambda \in D$. Here $m < \infty$, the functions Φ_α and A_α are holomorphic on D, not all A_α's vanish identically, and the representation is stable under analytic continuation across adjacent WKB charts. We also assume that no exponential cancellation occurs on a set of positive planar Lebesgue measure.

Remark 7. *For analytic b, c, the preceding hypothesis is the usual form of the complex WKB statement for the second order difference equation* (3.3)*. Away from turning points it follows from adiabatic diagonalization of the transfer matrices* (3.4)*. Near a simple turning point one expects a discrete Airy parametrix and the corresponding connection formulae. For polynomial or rational b, c without poles near $[0, 1]$, the turning point equation* (3.6) *is algebraic, and the associated Stokes graph is finite away from algebraic exceptional parameter values. Nevertheless, the uniform WKB connection analysis is a substantive ingredient and is not proved here.*

We next isolate the elementary potential-theoretic step which turns (3.7) into convergence of logarithmic potentials.

Lemma 2. *Let*

$$f_n(\lambda) = \sum_{\alpha=1}^{m} A_\alpha(\lambda) \exp\{n\Phi_\alpha(\lambda)\}(1 + o(1))$$

locally uniformly on a domain D, where the functions A_α and Φ_α are holomorphic and not all A_α's vanish identically. Assume moreover that no exponential cancellation occurs on a set of positive area. Then

$$\frac{1}{n} \log |f_n(\lambda)| \longrightarrow \max_\alpha \operatorname{Re} \Phi_\alpha(\lambda)$$

in $L^1_{\mathrm{loc}}(D)$, after combining terms whose phases differ by a purely imaginary constant.

Proof. Let

$$V(\lambda) := \max_\alpha \operatorname{Re} \Phi_\alpha(\lambda).$$

The upper bound follows from the triangle inequality:

$$|f_n(\lambda)| \le C_D e^{nV(\lambda)}(1 + o(1))$$

on compact subsets of D. On every open set where one phase has strictly larger real part than all the others, the corresponding term dominates and gives locally uniform convergence to V, except at isolated zeros of its amplitude.

The phase-equality set is contained in a finite union of real-analytic arcs, unless two phases have identical real parts on an open set. In the latter case their difference is a purely imaginary constant, and the corresponding terms can be combined into a single amplitude; if this amplitude vanishes identically, the term is removed. The stated non-cancellation assumption excludes cancellation on a set of positive area. Thus pointwise convergence holds almost everywhere.

The functions $n^{-1} \log |f_n|$ are subharmonic up to a locally uniformly vanishing error and are locally uniformly bounded above. Standard compactness for subharmonic functions then upgrades almost everywhere convergence to L^1_{loc}-convergence. □

Theorem 3. *Let $a(x, z)$ be of the form* (3.1)*, where b, c are analytic in a neighbourhood of $[0, 1]$, and $c \neq 0$. Assume Hypothesis 1. Then the sequence $\{\mu_n(a)\}$ converges weakly to a compactly supported probability measure $\mu(a)$.*

Proof. By Gershgorin's theorem, all eigenvalues of $T_n(a)$ belong to a fixed compact subset of $\mathbb{C}$. Hence the measures $\mu_n(a)$ are tight.

The functions

$$u_n(\lambda) = \frac{1}{n+1} \log |p_n(\lambda)|$$

are subharmonic and locally uniformly bounded above. Since p_n is monic and all its zeros stay in a fixed compact set, no subsequential limit of u_n can be identically $-\infty$.

By Hypothesis 1 and Lemma 2, u_n converges in L^1_{loc} on every component of $K \setminus E_K$, for every compact $K \Subset \mathbb{C}$. The exceptional set E_K has planar Lebesgue measure zero. Since the functions u_n are subharmonic and locally uniformly bounded above, any two L^1_{loc}-subsequential limits are uniquely determined by their almost everywhere values off E_K. Thus the full sequence u_n converges in $L^1_{\mathrm{loc}}(\mathbb{C})$ to a subharmonic function u.

Using (3.2), we have

$$\mu_n(a) = \frac{1}{2\pi} \Delta u_n.$$

Taking distributional Laplacians in the L^1_{loc}-limit gives

$$\mu_n(a) \Rightarrow \frac{1}{2\pi} \Delta u.$$

We define

$$\mu(a) := \frac{1}{2\pi} \Delta u.$$

This proves weak convergence. □

Remark 8. *Locally, the limiting logarithmic potential is of the form*

$$u(\lambda) = \max_{\alpha} \operatorname{Re} \Phi_\alpha(\lambda).$$

Hence the support of $\mu(a) = (2\pi)^{-1}\Delta u$ is expected to be contained in the set where two or more WKB actions have equal real part, together with possible contributions from zeros of amplitudes. This explains why numerical experiments in the non-Hermitian tridiagonal case typically produce one-dimensional spectral supports rather than the two-dimensional range of the symbol.

Remark 9. *Theorem 3 should not be interpreted as an unconditional proof of Conjecture 1. Rather, it gives a rigorous reduction of Problem 1 in the analytic tridiagonal case to the finite exact WKB representation in Hypothesis 1.*

3.1. **An unconditional result in the hyperbolic regime.** In this subsection we give a fully rigorous result under a uniform hyperbolicity assumption. Unlike the WKB-based Theorem 3, the statement below does not rely on turning-point analysis.

Theorem 4. *Let $a(x, z) = c(x)z^{-1} + b(x) + c(x)z$, where b, c are analytic in a neighbourhood of $[0, 1]$ and $c(x) \neq 0$ there.*

Let $\Omega \subset \mathbb{C}$ be a domain such that for every compact set $K \Subset \Omega$ there exists $\eta > 0$ and a labeling of the roots

$$\rho_\pm(x, \lambda) = \frac{\lambda - b(x) \pm \sqrt{(\lambda - b(x))^2 - 4c(x)^2}}{2}$$

satisfying

$$|\rho_+(x,\lambda)| \geq (1+\eta)|\rho_-(x,\lambda)| \tag{3.8}$$

for all $x \in [0,1]$ *and* $\lambda \in K$.

Then

$$\frac{1}{n+1}\log|\det(\lambda I - T_n(a))| \longrightarrow \int_0^1 \log|\rho_+(x,\lambda)|\,dx \tag{3.9}$$

in $L^1_{\mathrm{loc}}(\Omega)$.

In particular, on Ω *the logarithmic potentials converge, and hence*

$$\mu_n(a)\big|_\Omega \;\Rightarrow\; \frac{1}{2\pi}\Delta\left(\int_0^1 \log|\rho_+(x,\lambda)|\,dx\right).$$

Proof. We use the transfer matrix representation

$$\begin{pmatrix} p_{k+1,n} \\ p_{k,n} \end{pmatrix} = A(x_k,\lambda)\begin{pmatrix} p_{k,n} \\ p_{k-1,n} \end{pmatrix}, \qquad A(x,\lambda) = \begin{pmatrix} \lambda - b(x) & -c(x)^2 \\ 1 & 0 \end{pmatrix},$$

with $x_k = \frac{k}{n+1}$.

Since the eigenvalues $\rho_\pm(x,\lambda)$ are distinct and satisfy the uniform separation (3.8), the matrix $A(x,\lambda)$ admits an analytic diagonalization

$$A(x,\lambda) = S(x,\lambda)\begin{pmatrix} \rho_+(x,\lambda) & 0 \\ 0 & \rho_-(x,\lambda) \end{pmatrix} S(x,\lambda)^{-1},$$

with S and S^{-1} uniformly bounded on $[0,1]\times K$.

Writing $x_k = k/(n+1)$, one has

$$S(x_k,\lambda)^{-1}S(x_{k-1},\lambda) = I + \frac{1}{n+1}R(x_k,\lambda) + O(n^{-2})$$

uniformly on K. Thus the full transfer product is a diagonal product perturbed by near-identity factors.

By iterating this expansion one obtains

$$A(x_n,\lambda)\cdots A(x_0,\lambda) = S(1,\lambda)\begin{pmatrix} \prod_{k=0}^n \rho_+(x_k,\lambda) & 0 \\ 0 & \prod_{k=0}^n \rho_-(x_k,\lambda) \end{pmatrix} C_n(\lambda) S(0,\lambda)^{-1},$$

where $C_n(\lambda)$ is uniformly bounded and invertible on K.

Due to (3.8), the second diagonal entry is exponentially smaller than the first. Hence

$$\log\|A(x_n,\lambda)\cdots A(x_0,\lambda)\| = \sum_{k=0}^n \log|\rho_+(x_k,\lambda)| + O(1),$$

uniformly on K.

Using

$$p_n(\lambda) = (1,0)\,A(x_n,\lambda)\cdots A(x_0,\lambda)\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the fact that the projection onto the dominant channel is nonzero except at isolated λ, one obtains

$$\log|p_n(\lambda)| = \sum_{k=0}^n \log|\rho_+(x_k,\lambda)| + O(1),$$

locally uniformly outside a discrete set. Dividing by $n+1$ and passing to the limit using Riemann sums gives (3.9). The convergence in $L^1_{\mathrm{loc}}(\Omega)$ follows from subharmonicity. □

Remark 10. *The region Ω corresponds to a uniformly hyperbolic regime where one WKB branch dominates everywhere on $[0,1]$. The complement of such regions is precisely where turning points and Stokes phenomena occur, and where the limiting spectral measure is expected to concentrate.*

4. Two examples

4.1. The Szegö curve.

Consider the tridiagonal KMS symbol

$$a(x,z) = c(x)z^{-1} + b(x) + c(x)z,$$

where

$$b(x) = 1-2x, \qquad c(x) = \mathrm{i}\sqrt{x(1-x)}, \qquad 0 \le x \le 1.$$

The corresponding second order recurrence has local transfer matrix

$$A(x,\lambda) = \begin{pmatrix} \lambda - b(x) & -c(x)^2 \\ 1 & 0 \end{pmatrix}.$$

The two local characteristic roots are

$$\rho_\pm(x,\lambda) = \frac{\lambda - b(x) \pm \sqrt{(\lambda-b(x))^2 - 4c(x)^2}}{2}.$$

For the present choice of b, c, this becomes

$$\rho_\pm(x,\lambda) = \frac{\lambda - 1 + 2x \pm \sqrt{(\lambda-1)^2 + 4\lambda x}}{2}.$$

4.1.1. *Turning points.* The turning points are defined by

$$(\lambda - b(x))^2 - 4c(x)^2 = 0.$$

Substituting $b(x) = 1-2x$ and $c(x) = \mathrm{i}\sqrt{x(1-x)}$, we obtain

$$(\lambda - 1 + 2x)^2 + 4x(1-x) = 0.$$

Expanding and simplifying gives

$$(\lambda-1)^2 + 4\lambda x = 0.$$

Thus the turning point is

$$x_*(\lambda) = \frac{2 - \lambda - \lambda^{-1}}{4}.$$

The turning point curves in the λ-plane are

$$\lambda = b(x) \pm 2c(x) = 1 - 2x \pm 2\mathrm{i}\sqrt{x(1-x)}.$$

Writing $s = 1-2x$, one has

$$x(1-x) = \frac{1-s^2}{4},$$

and therefore

$$\lambda = s \pm \mathrm{i}\sqrt{1-s^2}.$$

Hence the turning point curves form the unit circle

$$|\lambda| = 1.$$

4.1.2. *The action integral.* The Stokes condition is obtained by comparing the two WKB branches. The action difference is

$$\mathcal{S}(\lambda) = \int_{\gamma(\lambda)} \log \frac{\rho_+(x,\lambda)}{\rho_-(x,\lambda)}\, dx,$$

where $\gamma(\lambda)$ is the relevant path from the turning point to the endpoint in the complex x-plane. In this example the turning point is unique, so the relevant Stokes condition is obtained from the endpoint connection between $x = 0$ and the turning point $x_*(\lambda)$.

Set

$$D(x,\lambda) = (\lambda - 1)^2 + 4\lambda x, \qquad w = \sqrt{D(x,\lambda)}.$$

Then

$$x = \frac{w^2 - (\lambda - 1)^2}{4\lambda}, \qquad dx = \frac{w}{2\lambda}\, dw.$$

Moreover,

$$\rho_\pm = \frac{\lambda - 1 + 2x \pm w}{2}.$$

Substituting $x = (w^2 - (\lambda - 1)^2)/(4\lambda)$, one finds

$$\rho_+ = \frac{(w + \lambda)^2 - 1}{4\lambda}, \qquad \rho_- = \frac{(w - \lambda)^2 - 1}{4\lambda}.$$

Therefore

$$\log \frac{\rho_+}{\rho_-} = \log \frac{(w + \lambda)^2 - 1}{(w - \lambda)^2 - 1}.$$

The endpoint $x = 0$ corresponds to $w = \lambda - 1$, while the turning point corresponds to $w = 0$. Hence

$$\mathcal{S}(\lambda) = \int_{\lambda - 1}^{0} \frac{w}{2\lambda} \log \frac{(w + \lambda)^2 - 1}{(w - \lambda)^2 - 1}\, dw.$$

An elementary integration by parts gives

$$\mathcal{S}(\lambda) = \lambda - 1 - \log \lambda,$$

up to an additive purely imaginary constant determined by the choice of branches of the logarithm. Since the Stokes condition only involves the real part of the action, this constant is irrelevant.

Thus the Stokes condition is

$$\operatorname{Re} \mathcal{S}(\lambda) = 0,$$

or equivalently

$$\operatorname{Re}(\lambda - 1 - \log \lambda) = 0.$$

5. The Szegö curve

Since

$$\operatorname{Re}(\lambda - 1 - \log \lambda) = \operatorname{Re} \lambda - 1 - \log |\lambda|,$$

the Stokes condition is

$$\operatorname{Re} \lambda - 1 - \log |\lambda| = 0.$$

Equivalently,

$$\log |\lambda| + 1 - \operatorname{Re} \lambda = 0,$$

which is the same as

$$|\lambda e^{1-\lambda}| = 1.$$

Therefore the candidate limiting spectral curve is

$$\boxed{|\lambda e^{1-\lambda}| = 1, \qquad |\lambda| \le 1.}$$

This is the classical Szegö curve.

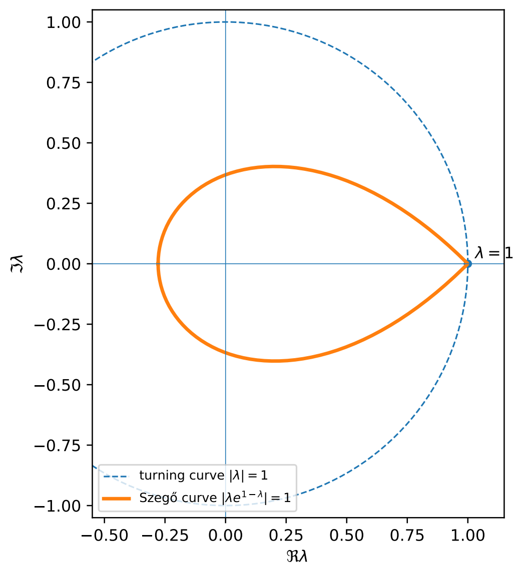

FIGURE 4. The dashed circle is the turning point curve $|\lambda| = 1$. The solid curve is the Szegö curve $|\lambda e^{1-\lambda}| = 1,\ |\lambda| \le 1$, obtained from the Stokes condition.

5.1. **The Lommel example and its WKB interpretation.** We now discuss another explicitly solvable tridiagonal example, namely

$$b(x) = 1 - 2x, \qquad c(x) = \frac{\mathrm{i}}{2}. \tag{5.1}$$

Thus

$$a(x,z) = \frac{\mathrm{i}}{2}z^{-1} + 1 - 2x + \frac{\mathrm{i}}{2}z.$$

This example is important because it exhibits the multi-branch nature of the Stokes phenomenon. Numerically, the limiting support is not a single level curve, but a system of arcs together with straight line segments forming an envelope inside the rectangle

$$[-1,1] + \mathrm{i}[-1,1].$$

The example is closely related to the asymptotic zero distribution of Lommel polynomials studied by Blaschke and Štampach in [1]. In their notation the parameter is $\alpha = \frac{1}{2}$. Their result gives a rigorous description of the limiting zero distribution. Below we explain how the same curves arise from the WKB/Stokes point of view.

The characteristic polynomials satisfy, up to the harmless endpoint convention already mentioned above,

$$p_{k+1,n}(\lambda) = (\lambda - 1 + 2x_k)p_{k,n}(\lambda) + \frac{1}{4}p_{k-1,n}(\lambda), \qquad x_k = \frac{k}{n+1}. \tag{5.2}$$

The local transfer roots are

$$\rho_\pm(x,\lambda) = \frac{\lambda - 1 + 2x \pm \sqrt{(\lambda - 1 + 2x)^2 + 1}}{2}. \tag{5.3}$$

The turning points are determined by

$$(\lambda - 1 + 2x)^2 + 1 = 0,$$

and hence

$$x_\pm(\lambda) = \frac{1 - \lambda \pm \mathrm{i}}{2}. \tag{5.4}$$

Equivalently, their images in the λ-plane are the two horizontal segments

$$[-1,1] + \mathrm{i}, \qquad [-1,1] - \mathrm{i}.$$

A naive Stokes condition comparing only the two WKB branches along the whole interval $[0,1]$ does not give the correct support. The reason is that (5.4) gives two moving turning points, and the endpoint conditions produce several competing exponential contributions. In this case the characteristic polynomial has a multi-term WKB expansion

$$p_n(\lambda) \sim \sum_j A_j(\lambda) \exp\{(n+1)\Phi_j(\lambda)\}.$$

The limiting support is the tropical Stokes set

$$\max_j \operatorname{Re}\Phi_j(\lambda) \quad \text{is attained by at least two indices } j. \tag{5.5}$$

For the Lommel example the relevant actions can be written explicitly. Following the notation of [1], define for $\alpha > 0$

$$\chi_\alpha(w) := \frac{1}{2}\sqrt{4\alpha^2 + (1+w)^2} + \frac{1+w}{2}\log\frac{2\alpha}{1 + w + \sqrt{4\alpha^2 + (1+w)^2}}. \tag{5.6}$$

The function χ_α is precisely the WKB action obtained after the standard change of variables reducing (5.2) to the Lommel recurrence. In the first quadrant

$$\Omega^{(\alpha)} = (0,1) + 2\mathrm{i}\alpha(0,1),$$

the equality of the two dominant real parts is

$$\operatorname{Re}\chi_\alpha(-\lambda) = \frac{\pi}{4}\operatorname{Im}\lambda. \tag{5.7}$$

For the present example $\alpha = 1/2$, and therefore the curved part of the support in the first quadrant is

$$\operatorname{Re}\chi_{1/2}(-\lambda) = \frac{\pi}{4}\operatorname{Im}\lambda, \qquad \lambda \in (0,1) + \mathrm{i}(0,1). \tag{5.8}$$

This curve has one endpoint at $1 + \mathrm{i}$. The second endpoint lies on the imaginary axis and is denoted by ξ. It is determined by

$$\operatorname{Re}\chi_{1/2}(-\xi) = \frac{\pi}{4}\operatorname{Im}\xi, \qquad \xi \in \mathrm{i}(0,1). \tag{5.9}$$

Numerically,

$$\xi \approx 0.3699257664\,\mathrm{i}.$$

The support in the first quadrant consists of the curve (5.8) connecting ξ to $1+\mathrm{i}$, together with the straight segment $[0, \xi]$ on the imaginary axis. The full support is obtained by the symmetries with respect to the coordinate axes. Thus the limiting support is a multi-branch Stokes graph rather than a single Stokes line.

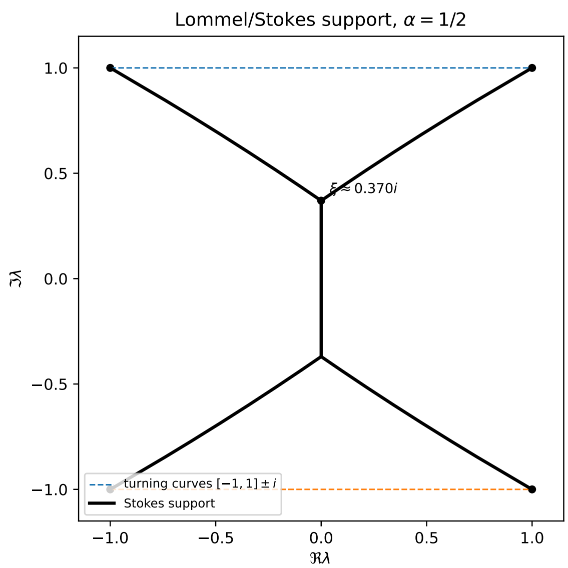

Figure 5. The WKB/Stokes support for the Lommel example $b(x) = 1 - 2x$, $c(x) = \mathrm{i}/2$. The dashed horizontal lines are the turning curves $[-1, 1] \pm \mathrm{i}$. The solid set is the support described by (5.8), its reflections, and the central straight segment on the imaginary axis.

This example explains why the single-action contour is insufficient in general. The limiting spectral support is produced by the competition of several WKB actions. In the Lommel case these actions can be evaluated explicitly, and the result agrees with the rigorous asymptotic zero distribution obtained in [1].

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Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
Email address: dario.giandinoto@math.su.se

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
Email address: shapiro@math.su.se