# Pólya's shire theorem for a class of functions with essential singularities

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**Abstract**

We study the zero asymptotics of successive derivatives of

$$f(z) = \frac{P(z)}{Q(z)} \exp\left(\frac{S(z)}{T(z)}\right),$$

where $P, Q, S, T \in \mathbb{C}[z]$, $\gcd(P, Q) = \gcd(S, T) = 1$, and $T$ is nonconstant. The $n$th derivative carries a polynomial factor $B_n$, and our main result gives uniform asymptotics for $B_n$ on compact subsets of each open Voronoi cell of the singular set $\mathcal{Z}(T) \cup \mathcal{Z}(Q)$: classical Darboux asymptotics on cells attached to poles of $P/Q$, and a parameter-uniform Wright expansion with $m+1$ saddle contributions on cells attached to a pole of $S/T$ of order $m$. These local results yield an $L^1$ convergence theorem for the normalized zero-counting measures, whose limit is supported on the Voronoi diagram together with atoms at the essential singularities. We also study the reduced local model at an essential singularity: for simple poles it gives generalized Laguerre polynomials and the Marchenko–Pastur law, while for higher-order poles it gives a Laguerre-type Sheffer sequence that is $m$-orthogonal.

# 1 Introduction

## 1.1 Short historical account

In 1922, G. Pólya published the paper "Über die Nullstellen sukzessiver Derivierten" [14] in *Mathematische Zeitschrift*. It contains two main results, together with several questions that shaped much of the later development of the subject.

**Definition 1.** Let $f$ be meromorphic in $\mathbb{C}$. The *final set* of $f$ is

$$L(f) := \{z \in \mathbb{C} : \text{ every neighborhood of } z \text{ contains zeros of } f^{(n)} \text{ for infinitely many } n\}.$$

*Theorem* A. Let $f$ be meromorphic in $\mathbb{C}$ and suppose that $f$ has at least one pole. For each pole $A$, let its *shire* be the set of points in $\mathbb{C}$ closer to $A$ than to any other pole. Then the final set $L(f)$ is precisely the union of the boundaries of these shires.

Pólya himself described the picture by saying that the poles repel the zeros, which therefore have no choice but to gather in between. Theorem A is now known as *Pólya's shire theorem*. It gives a complete description of $L(f)$ for meromorphic functions in the plane that are not entire.

Figure 1 illustrates this result.

In the same paper Pólya also determined $L(f)$ for a class of entire functions of the form $f(z) = P(z)e^{Q(z)}$, where $P$ is a polynomial and $Q(z) = cz^q + dz^{q-1} + \dots$ is a nonconstant polynomial, so $q = \deg Q \geq 1$.

*Theorem* B. In the above notation, $L(f)$ consists of the $q$ rays emanating from the point $-\frac{d}{qc}$, parallel to the directions determined by the solutions of $cz^q + 1 = 0$. In particular, these rays are equally spaced.

Pólya returned to successive derivatives several times: in his AMS address [15], in [16], and finally in [17], published 54 years after [14].

His results were later extended in various directions; see [1, 2, 4, 9, 10, 11, 13, 18, 19, 21, 26] and the references therein.

In the entire case, Edrei and MacLane showed in [7] that if $E \subset \mathbb{P}^1(\mathbb{C})$ is any closed set containing $\infty$, then there exists an entire function $F$ of any prescribed order $0 \leq \lambda \leq \infty$ whose final set is exactly

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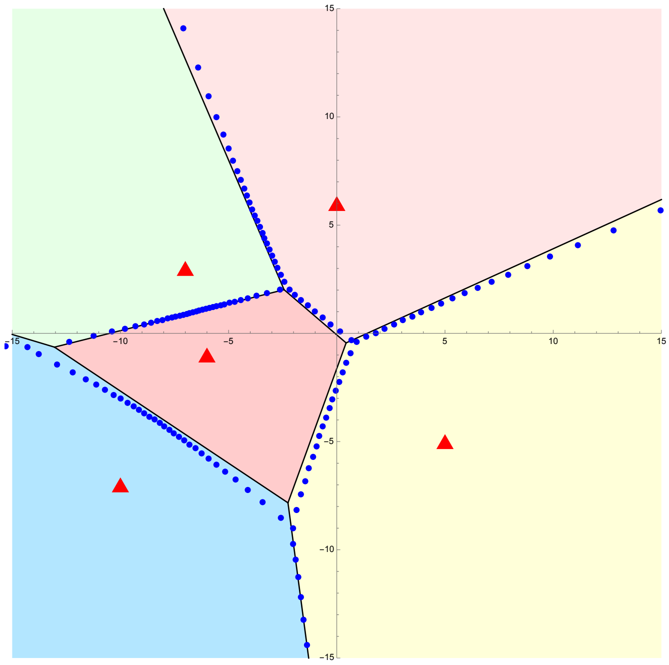


Figure 1: The Voronoi diagram determined by the five poles (red triangles) of some rational function $f$, together with the zeros of $f^{(20)}$ (blue dots).

$E$. By contrast, if $f$ is meromorphic in $\mathbb{C}$ but not entire, then Pólya's theorem from [14] completely determines $L(f)$.

Thus, for single-valued functions meromorphic on the complement of a finite subset of $\mathbb{C}$, the main unresolved case is that of essential singularities. Pólya already considered the example $f(z) = z^{-1}e^{-1/z}$ in § 4 of [15] and showed that $L(f)$ coincides with the set of all nonnegative real numbers. This observation motivated the papers [6, 4], where the authors studied functions of the form $f(z) = g(\frac{1}{z})$ with $g$ entire and having only real zeros. In that setting all zeros of all derivatives are real, and the asymptotic distribution can be studied on $\mathbb{R}$.

At the fixed-scale level, Bøgvad and Hägg [1] proved that for rational functions the normalized zero-counting measures of successive derivatives converge to an explicit measure supported on the Voronoi diagram of the poles. Hägg [11] obtained the corresponding result for $R(z)e^{p(z)}$.

To conclude this short survey, let us mention that analogues of Pólya's results for certain classes of multivalued functions, that is, functions on Riemann surfaces, were considered in [2, 19].

### 1.2 Our setup

The main goal of this paper is to study the zero asymptotics of successive derivatives of single-valued functions with finitely many essential singularities. We do this for the class of functions

$$f(z) = \frac{P(z)}{Q(z)} \exp\left(\frac{S(z)}{T(z)}\right), \tag{1}$$

for which $f'/f \in \mathbb{C}(z)$; in the algorithmic literature such functions are called *hyperexponential* over $\mathbb{C}(z)$, see [12]. On the complement of a finite singular set, the single-valued members of this class are precisely the functions $cR(z)e^{H(z)}$ with $R, H \in \mathbb{C}(z)$ and $c \in \mathbb{C}^{\times}$; Appendix A records this characterization. For the broader Liouvillian setting, see [22, 23, 3].

If one drops the standing assumption that $T$ be nonconstant, the class (1) also contains the entire functions appearing in Theorem B; Figure 2 illustrates the type of behavior we study.

Hyperexponential functions form the natural next class to consider: they have both ordinary poles and finite essential singularities, and these two kinds of singularities interact in the same nearest-singularity geometry. Their main technical advantage is that one can isolate a polynomial factor $B_n$ that carries the zeros of $f^{(n)}$. More precisely,

$$f^{(n)}(z) = \frac{P_T(z)\, B_n(z)}{Q(z)\, W(z)^n} e^{E(z)}. \tag{2}$$

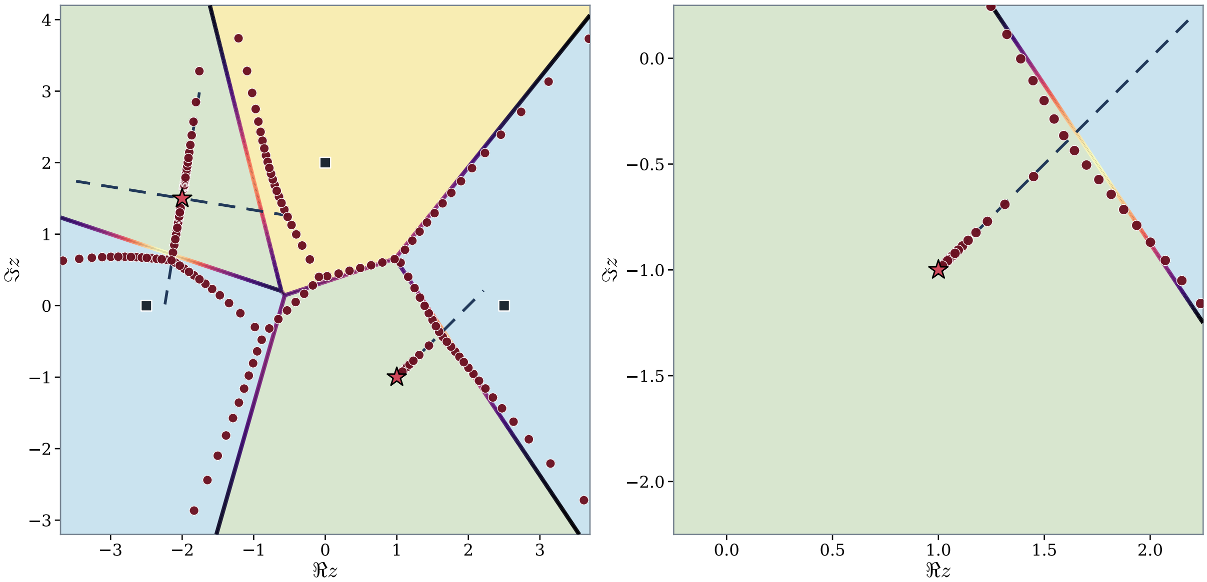


Figure 2: Numerical illustration for $f(z) = \frac{1}{(z+5/2)(z-2i)(z-5/2)} \exp\left(\frac{(3-i)/2}{(z+2-3i/2)^2} + \frac{-1-i}{z-1+i}\right)$. Here $S/T$ has a double pole at $z = -2+3i/2$ and a simple pole at $z = 1-i$. The dark points are the finite zeros of $f^{(30)}$. The left panel shows the global fixed-scale picture, while the right panel zooms into the neighborhood of $z = 1 - i$. The pastel background colors merely distinguish the Voronoi cells of $\mathcal{Z}(T) \cup \mathcal{Z}(Q)$. Solid black curves are the Voronoi edges, and the thin warm-colored band drawn on those edges indicates the limiting edge density $\frac{1}{2\pi\kappa} \frac{|a_i - a_j|}{|z-a_i|\,|z-a_j|}$, with lighter colors corresponding to larger density. The dashed rays are the leading local Stokes rays: four from the order-2 pole of $S/T$ at $z = -2 + 3i/2$, and one from the simple pole at $z = 1 - i$.

Our results have three main parts. First, Theorem 9 gives a complete asymptotic description of $B_n(z)/n!$ on compact subsets of every Voronoi interior. On cells attached to poles of $P/Q$, the analysis reduces to the classical algebraic singularity case. On cells attached to poles of $S/T$, we obtain a parameter-uniform Wright expansion with finitely many saddle contributions, explicit holomorphic amplitudes, and explicit phases. This is the main technical theorem of the paper.

Second, Section 5 studies the reduced microscopic model obtained by retaining only the algebraic factor and the highest-order singular term at an essential singularity. For simple poles this gives generalized Laguerre polynomials and the Marchenko–Pastur law. For higher-order poles it yields a Laguerre-type Sheffer sequence and an explicit algebraic microscopic limit; when the exponent of the local algebraic factor is negative, this sequence is $m$-orthogonal, that is, orthogonal with respect to an $m$-tuple of linear functionals. Corollary 22 makes the connection with Theorem 21 explicit: the atom at an essential singularity represents a cluster of $m_j n + o(n)$ zeros, while the reduced local model resolves that cluster on the scale $n^{-1/m_j}$. The full microscopic law for the complete local factor remains open.

Third, the global zero law on the original $z$-scale in Theorem 21 is a consequence of the local results rather than the starting point of the argument. It follows from Theorem 9, Corollary 11, and subharmonic compactness. On essential cells one cannot rely on pointwise lower bounds, because comparable saddle contributions may cancel along Stokes sets, so the argument proceeds through local $L^1$ estimates. Away from the Stokes set a single saddle dominates, while on the Stokes set one keeps the full multi-saddle expansion; Jensen–Poisson estimates then provide the required $L^1$ bound.

Since $d = \sum_{j=1}^{\check{t}} m_j + N$, the numerator $d - 1$ of the finite-plane mass $(d-1)/\kappa$ splits as $(N - 1) + \sum_{j=1}^{\check{t}} m_j$: the first term is the classical Voronoi contribution of the $N$ finite singular sites, and the second comes from zeros collapsing into the essential singularities. Section 5 identifies the microscopic mechanism behind this second term.

## Acknowledgements

We are grateful to Rikard Bøgvad and Per Alexandersson for valuable discussions.

# 2 Notation and preliminaries

## 2.1 Polynomials, zeros, and orders

For a nonzero polynomial $R \in \mathbb{C}[z]$, we write $\deg R$ for its degree, $\mathcal{Z}(R)$ for its set of distinct finite zeros, $\operatorname{rad}(R)$ for its *squarefree part* (the monic polynomial whose zeros are exactly the elements of $\mathcal{Z}(R)$, each with multiplicity one), and $\operatorname{lc}(R)$ for its leading coefficient. For a meromorphic function $F$ and a point $a \in \mathbb{C}$, we write $\operatorname{ord}_a F$ for the order of $F$ at $a$ (positive for a zero, negative for a pole, and zero otherwise).

## 2.2 Voronoi diagrams

Let $\Sigma = \{a_1, \ldots, a_N\} \subset \mathbb{C}$ be a finite set of points called *sites*. The *Voronoi cell* of $a_i$ is

$$\mathcal{V}_i := \big\{z \in \mathbb{C} : |z - a_i| \le |z - a_j| \text{ for all } 1 \le j \le N\big\},$$

and $\mathcal{V}_i^\circ$ denotes its interior, consisting of all points for which $a_i$ is the unique nearest site. For $i \neq j$, define

$$E_{ij} := \begin{cases} \mathcal{V}_i \cap \mathcal{V}_j, & \text{if } \mathcal{V}_i \cap \mathcal{V}_j \text{ is one-dimensional,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

Each nonempty $E_{ij}$ lies on the perpendicular bisector of $[a_i, a_j]$. The *Voronoi diagram* of $\Sigma$ is the union of the edges $E_{ij}$ and the Voronoi vertices. We write $\mathrm{d}\ell$ for arclength measure on a curve.

## 2.3 Logarithmic potentials and zero-counting measures

A subharmonic function $u : \mathbb{C} \to [-\infty, \infty)$ has a distributional Laplacian $\Delta u$, and in our normalization $(2\pi)^{-1}\Delta u$ is a nonnegative Radon measure; see Ransford [20]. For a nonzero polynomial $R$ of degree $n \ge 1$, the function $\frac{1}{n}\log|R|$ is subharmonic and its distributional Laplacian is $2\pi$ times the *normalized zero-counting measure*

$$\mu_R := \frac{1}{n} \sum_{\zeta:\, R(\zeta)=0} \delta_\zeta,$$

where each zero is counted with multiplicity and $\delta_\zeta$ denotes the unit point mass at $\zeta$. We say that a sequence of measures $\mu_n$ converges *vaguely* on $\mathbb{C}$ to $\mu$ if $\int \varphi \,\mathrm{d}\mu_n \to \int \varphi \,\mathrm{d}\mu$ for every continuous function $\varphi$ with compact support. We say that $\mu_n$ converges *weakly* on $\mathbb{C}$ to $\mu$ if this holds for every bounded continuous function $\varphi : \mathbb{C} \to \mathbb{R}$, and *weakly* on $\widehat{\mathbb{C}}$ if it holds for every continuous function $\varphi : \widehat{\mathbb{C}} \to \mathbb{R}$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.

Here and below, $D \Subset U$ means that $D$ is a relatively compact subset of $U$, and a subscript on an $O$-term indicates the permitted dependence of the implied constant.

For a finite Borel measure $\mu$ on $\mathbb{C}$ for which the integral below converges absolutely, we write

$$C_\mu(z) := \int_{\mathbb{C}} \frac{\mathrm{d}\mu(\zeta)}{z - \zeta}, \qquad z \in \mathbb{C} \setminus \operatorname{supp}\mu,$$

for its Cauchy transform. If $\phi : X \to Y$ is measurable and $\mu$ is a measure on $X$, then $\phi_\# \mu$ denotes the pushforward measure on $Y$.

## 2.4 Notation for hyperexponential functions

Let $P, Q, S, T \in \mathbb{C}[z]$ with $P, Q \neq 0$, $\gcd(P, Q) = \gcd(S, T) = 1$, and $T$ nonconstant. Consider

$$f(z) := \frac{P(z)}{Q(z)} \exp\!\left(\frac{S(z)}{T(z)}\right). \tag{3}$$

Then $f$ is holomorphic on $\mathbb{C} \setminus \big(\mathcal{Z}(T) \cup \mathcal{Z}(Q)\big)$. Each point of $\mathcal{Z}(T)$ is an essential singularity of $f$, while each point of $\mathcal{Z}(Q) \setminus \mathcal{Z}(T)$ is a pole. To isolate the finite-plane singularities, we normalize the constant-prefactor case: if $P/Q \equiv \alpha \neq 0$, choose $c \in \mathbb{C}$ such that $e^c = \alpha$ and replace $(P, Q, S)$ by $(1, 1, S + cT)$. Thus, after this normalization,

$$\frac{P}{Q} \equiv 1 \qquad \text{or} \qquad \frac{P}{Q} \not\equiv \text{const.} \tag{4}$$

Set

$$p := \deg P, \qquad q := \deg Q.$$

After common rescalings of $P, Q$ and of $S, T$, we assume that $T$ is monic and that $Q = 1$ if $q = 0$, whereas $Q$ is monic if $q > 0$. Write

$$E(z) := \frac{S(z)}{T(z)} = H(z) + \frac{M(z)}{T(z)}, \qquad \deg M < \deg T, \tag{5}$$

where $H \in \mathbb{C}[z]$ is the polynomial part and $M \in \mathbb{C}[z]$ is the remainder. Since $M = S - HT$ and $\gcd(S, T) = 1$, one has $\gcd(M, T) = 1$; in particular $M \neq 0$. Define

$$h := \begin{cases} \deg H, & \text{if } H \text{ is nonconstant,} \\ 0, & \text{if } H \text{ is constant.} \end{cases} \tag{6}$$

Write $\check{t} := |\mathcal{Z}(T)|$ for the number of distinct zeros of $T$, and let $c_1, \ldots, c_{\check{t}}$ be those zeros with multiplicities $m_1, \ldots, m_{\check{t}}$. Put

$$T_0 := \operatorname{rad}(T) = \prod_{j=1}^{\check{t}} (z - c_j).$$

Write $\check{q} := |\mathcal{Z}(Q) \setminus \mathcal{Z}(T)|$, and let $b_1, \ldots, b_{\check{q}}$ be the distinct zeros of $Q$ that do not belong to $\mathcal{Z}(T)$, with multiplicities $\ell_1, \ldots, \ell_{\check{q}}$. Set

$$Q_*(z) := \prod_{k=1}^{\check{q}} (z - b_k).$$

Set

$$P_T(z) := \prod_{j=1}^{\check{t}} (z - c_j)^{p_j}, \qquad p_j := \operatorname{ord}_{c_j} P, \qquad \nu_j := \operatorname{ord}_{c_j} Q,$$

and put

$$P_\sharp := \frac{P}{P_T}. \tag{7}$$

Then $P_\sharp$ is coprime to $T$, so $P_\sharp(c_j) \neq 0$ for all $j$. The factor $P_T$ records the zeros of $P$ at $\mathcal{Z}(T)$, which contribute fixed local orders at the essential singularities; zeros of $P$ away from $\mathcal{Z}(T)$ are accounted for separately in the polynomial factor of $f^{(n)}$ (see (15) below).

Define

$$W(z) := T(z)\, T_0(z)\, Q_*(z) = \prod_{j=1}^{\check{t}} (z - c_j)^{m_j + 1} \prod_{k=1}^{\check{q}} (z - b_k). \tag{8}$$

It is monic, of degree

$$d := \deg W = \deg T + \check{t} + \check{q}. \tag{9}$$

The finite singular set of $f$ is

$$\Sigma := \mathcal{Z}(T) \cup \mathcal{Z}(Q) = \{a_1, \ldots, a_N\}, \qquad N := \check{t} + \check{q}. \tag{10}$$

We write $\rho(z) := \operatorname{dist}(z, \Sigma) = \min_{1 \le i \le N} |z - a_i|$ for the Euclidean distance from $z$ to $\Sigma$.

Set

$$\Lambda(z) := E'(z) + \frac{P_T'(z)}{P_T(z)} - \frac{Q'(z)}{Q(z)}, \tag{11}$$

and

$$U(z) := W(z)\, \Lambda(z). \tag{12}$$

By construction, $U$ is a polynomial: at each $c_j \in \mathcal{Z}(T)$ the factor $(z - c_j)^{m_j+1}$ in $W$ cancels the pole of $E'$ of order $m_j + 1$, while $P_T'/P_T$ and $Q'/Q$ have at most simple poles there; at each $b_k \in \mathcal{Z}(Q) \setminus \mathcal{Z}(T)$ the factor $z - b_k$ cancels the simple pole of $Q'/Q$.

Finally, set

$$\kappa := d + h - 1, \tag{13}$$

and

$$\sigma := \begin{cases} 0, & \text{if } h = 0, \\ \log|h\tau_h|, & \text{if } h > 0 \text{ and } H(z) = \tau_h z^h + \cdots. \end{cases} \tag{14}$$

The integer $h$ governs the growth of $e^{E(z)}$ at infinity: when $h = 0$, no positive proportion of the zeros escapes to $\infty$; when $h \geq 1$, the polynomial part of the exponent sends a fraction $h/\kappa$ of the normalized zero mass to $\infty$ in the spherical limit described later.

Since $d = \deg T + \check{t} + \check{q} \geq 2$, one has $\kappa \geq 1$.

# 3 Properties of the polynomial factor $B_n$

## 3.1 Recurrence for $B_n$

**Proposition 2** (recurrence for $B_n$)**.** *For $f(z)$ given by (3), there is a unique polynomial sequence $\{B_n\}_{n\geq 0}$ such that*

$$f^{(n)}(z) = \frac{P_T(z)\, B_n(z)}{Q(z)\, W(z)^n}\, e^{E(z)}, \qquad B_0 = P_\sharp, \tag{15}$$

*and satisfying the recurrence*

$$B_{n+1} = W\, B_n' + (U - nW')\, B_n. \tag{16}$$

*Moreover, $W$ is the unique monic polynomial of least degree such that $W\Lambda \in \mathbb{C}[z]$.*

(See definitions of $P_\sharp$ in (7) and $E(z)$ in (5).)

*Remark* 3. Since $f = P_T P_\sharp Q^{-1} e^E$, the representation (15) yields the *logarithmic-derivative representation*

$$A_n(z) := \frac{f^{(n)}(z)}{f(z)} = \frac{B_n(z)}{P_\sharp(z)\, W(z)^n} \tag{17}$$

on $\mathbb{C} \setminus (\Sigma \cup \mathcal{Z}(P_\sharp))$. By (15), the factor $\frac{P_T(z)}{Q(z)\, W(z)^n} e^{E(z)}$ is holomorphic and nonvanishing on $\mathbb{C} \setminus \Sigma$, so the zeros of $f^{(n)}$ away from $\Sigma$ are exactly the zeros of $B_n$.

*Proof of Proposition 2.* Write

$$f(z) = \frac{P_T(z) P_\sharp(z)}{Q(z)}\, e^{E(z)}.$$

The poles of $\Lambda$ are easy to read off locally. At a zero $c_j$ of $T$ of multiplicity $m_j$, the function $E'$ has a pole of order $m_j + 1$, while $P_T'/P_T$ and $Q'/Q$ have at most simple poles there. At a zero $b_k$ of $Q$ outside $\mathcal{Z}(T)$, the only pole of $\Lambda$ is the simple pole of $-Q'/Q$. There are no other poles. Therefore the unique monic polynomial of least degree clearing all poles of $\Lambda$ is precisely $W$ from (8), so $U = W\Lambda \in \mathbb{C}[z]$.

Now define $B_0 := P_\sharp$ and $B_{n+1}$ by (16). Assume (15) holds for some $n$. Differentiating and using (11) gives

$$\begin{aligned} f^{(n+1)} &= \left(\frac{P_T B_n}{QW^n}\right)' e^E + E' \frac{P_T B_n}{QW^n}\, e^E \\ &= \frac{P_T}{QW^{n+1}} \Big(WB_n' + W\Big(E' + \tfrac{P_T'}{P_T} - \tfrac{Q'}{Q}\Big) B_n - nW' B_n\Big) e^E \\ &= \frac{P_T}{QW^{n+1}} (WB_n' + (U - nW')B_n) e^E \\ &= \frac{P_T B_{n+1}}{QW^{n+1}}\, e^E. \end{aligned}$$

By induction, (15) holds for all $n$. Since (16) with the initial value $B_0 = P_\sharp$ determines the sequence uniquely, this proves the proposition. ☐

### 3.2 Local factorizations of $B_n$ at finite singular points

**Proposition 4** (exact local factorizations). *1. For every zero $c_j$ of $T$ and every $n \geq 0$,*

$$B_n(c_j) = U(c_j)^n P_\sharp(c_j) \neq 0.$$

*Equivalently, $f^{(n)}e^{-E}$ is meromorphic near $c_j$ with $\mathrm{ord}_{c_j}\big(f^{(n)}e^{-E}\big) = p_j - \nu_j - n(m_j+1)$; explicitly, there exists a holomorphic function $\phi_{j,n}$ near $c_j$ with $\phi_{j,n}(c_j) \neq 0$ such that*

$$f^{(n)}(z) = (z - c_j)^{p_j - \nu_j - n(m_j+1)} \phi_{j,n}(z)\, e^{E(z)}. \tag{18}$$

*2. For every pole $b_k$ of $P/Q$ outside $\mathcal{Z}(T)$ and every $n \geq 0$,*

$$B_{n+1}(b_k) = -(\ell_k + n)\, W'(b_k)\, B_n(b_k),$$

*so $B_n(b_k) \neq 0$ for all $n$. Equivalently, $f^{(n)}$ is meromorphic near $b_k$ with $\mathrm{ord}_{b_k} f^{(n)} = -\ell_k - n$; explicitly, there exists a holomorphic function $\psi_{k,n}$ near $b_k$ with $\psi_{k,n}(b_k) \neq 0$ such that*

$$f^{(n)}(z) = (z - b_k)^{-\ell_k - n} \psi_{k,n}(z). \tag{19}$$

*Proof of Proposition 4.* Fix $j$. Since $W$ has a zero of order $m_j + 1 \geq 2$ at $c_j$, both $W(c_j)$ and $W'(c_j)$ vanish. Evaluating (16) at $c_j$ gives

$$B_{n+1}(c_j) = U(c_j)\, B_n(c_j).$$

Write

$$W(z) = (z - c_j)^{m_j+1} \widetilde{W}_j(z), \qquad \widetilde{W}_j(c_j) \neq 0.$$

Because $\gcd(S, T) = 1$, the Laurent expansion of $E$ at $c_j$ begins with a nonzero principal term

$$E(z) = \lambda_{j,m_j}(z - c_j)^{-m_j} + O\big((z - c_j)^{-m_j+1}\big), \qquad \lambda_{j,m_j} \neq 0,$$

so

$$E'(z) = -m_j \lambda_{j,m_j}(z - c_j)^{-m_j-1} + O\big((z - c_j)^{-m_j}\big).$$

The other two terms in $\Lambda$ have at most simple poles at $c_j$; multiplying by $W$ annihilates them at $c_j$. Hence

$$U(c_j) = -m_j \lambda_{j,m_j} \widetilde{W}_j(c_j) \neq 0.$$

Since $B_0 = P_\sharp$ and $P_\sharp(c_j) \neq 0$, the first claim follows by induction. The local form (18) is just (15) rewritten with the exact orders of $P_T$, $Q$, and $W$ at $c_j$.

Now fix $k$. Since $b_k \notin \mathcal{Z}(T)$, the function $E' + P_T'/P_T$ is holomorphic at $b_k$, whereas

$$-\frac{Q'(z)}{Q(z)} = -\frac{\ell_k}{z - b_k} + O(1).$$

Write

$$W(z) = (z - b_k)\widehat{W}_k(z), \qquad \widehat{W}_k(b_k) = W'(b_k) \neq 0.$$

Then $U(b_k) = -\ell_k W'(b_k)$, and evaluating (16) at $b_k$ gives

$$B_{n+1}(b_k) = \big(U(b_k) - nW'(b_k)\big) B_n(b_k) = -(\ell_k + n) W'(b_k)\, B_n(b_k).$$

Since $b_k \notin \mathcal{Z}(P)$ by $\gcd(P, Q) = 1$ and $b_k \notin \mathcal{Z}(T)$, one has $B_0(b_k) = P_\sharp(b_k) \neq 0$. The nonvanishing follows by induction. Since $E$ is holomorphic at $b_k$, (15) yields (19) after absorbing $e^E$ into the holomorphic unit. □

### 3.3 Degree growth and leading coefficients of $B_n$

To state the result in the case $h = 0$, we need an auxiliary function. Since $H$ is constant when $h = 0$, the function $f$ equals $e^H$ times the function $G$ obtained by removing $H$ from the exponent:

$$G(z) := \frac{P(z)}{Q(z)} \exp\left(\frac{M(z)}{T(z)}\right), \tag{20}$$

where $M$ is the remainder in (5). Then $G$ has a Laurent expansion at infinity

$$G(z) = \sum_{\nu=-\infty}^{p-q} G_\nu z^\nu, \qquad G_{p-q} \neq 0.$$

When $p \geq q$, let $J \geq 1$ be the smallest integer such that $G_{-J} \neq 0$. Such $J$ exists: if $G_{-J} = 0$ for all $J \geq 1$, then $G$ agrees near $\infty$ with a polynomial $R$. Hence

$$e^{M/T} = \frac{RQ}{P}$$

on a nonempty open subset of $\mathbb{C} \setminus \big(\mathcal{Z}(P) \cup \mathcal{Z}(T)\big)$, and by the identity theorem on that connected domain $e^{M/T}$ would be rational. This is impossible because $\gcd(M, T) = 1$, so $M/T$ has a pole at every zero of $T$, and $e^{M/T}$ has an essential singularity at each such point.

**Proposition 5** (degree growth)**.** *Let $\gamma_n := \mathrm{lc}(B_n)$.*

1. *If $h > 0$ and $H(z) = \tau_h z^h + \cdots$, then*

$$\deg B_n = \deg B_0 + n\kappa, \qquad \gamma_n = (h\tau_h)^n \gamma_0.$$

2. *If $h = 0$ and $p < q$, then for every $n \geq 0$,*

$$\deg B_n = \deg B_0 + n(d-1), \qquad \gamma_n = \gamma_0 (-1)^n \frac{\Gamma(n+q-p)}{\Gamma(q-p)}.$$

   *Equivalently, $\gamma_{n+1} = (p - q - n)\gamma_n$ for all $n \geq 0$.*

3. *If $h = 0$ and $p \geq q$, then for $0 \leq n \leq p - q$,*

$$\deg B_n = \deg B_0 + n(d-1), \qquad \gamma_n = \gamma_0 \frac{(p-q)!}{(p-q-n)!},$$

   *whereas for $n \geq p - q + 1$,*

$$\deg B_n = q - \deg P_T - J + n(d-1), \qquad \gamma_n = G_{-J} (-1)^n \frac{\Gamma(n+J)}{\Gamma(J)}.$$

*In particular, when $h = 0$,*

$$\deg B_n = (d-1)n + O(1), \qquad \log|\gamma_n| = \log n! + O(\log n).$$

*Proof.* If $h > 0$, then $E'(z) = H'(z) + O(z^{-2})$, so the highest-degree term of $U = W\Lambda$ comes from $WH'$. Therefore

$$\deg U = d + h - 1 = \kappa, \qquad \mathrm{lc}(U) = h\tau_h.$$

In (16) the term $UB_n$ has strictly larger degree than $WB_n'$ and $-nW'B_n$, so

$$\deg B_{n+1} = \deg B_n + \kappa, \qquad \gamma_{n+1} = (h\tau_h)\gamma_n.$$

This proves part 1.

Assume $h = 0$. Since $H$ is constant, $f = e^H G$, and (15) gives

$$B_n(z) = \frac{Q(z)W(z)^n}{P_T(z)} e^{-M(z)/T(z)} G^{(n)}(z).$$

As $z \to \infty$,

$$\frac{Q(z)W(z)^n}{P_T(z)} = z^{q+nd-\deg P_T}\big(1 + O(z^{-1})\big),$$

while

$$e^{-M(z)/T(z)} = 1 + O(z^{-1}).$$

Thus the leading Laurent exponent of $B_n$ is obtained by adding $q + nd - \deg P_T$ to the leading Laurent exponent of $G^{(n)}$, and the leading coefficient is unchanged. Since $e^{M/T} = 1 + O(z^{-1})$ and $Q$ is monic, one has $G_{p-q} = \mathrm{lc}(P) = \mathrm{lc}(P_\sharp) = \gamma_0$.

If $p < q$, the leading term of $G$ is $G_{p-q}\, z^{p-q}$, so

$$G^{(n)}(z) = G_{p-q}(-1)^n \frac{\Gamma(n + q - p)}{\Gamma(q - p)}\, z^{p-q-n} + O(z^{p-q-n-1}).$$

This yields part 2.

Assume now that $p \geq q$. For $0 \leq n \leq p - q$, the leading term of $G^{(n)}$ comes from the polynomial part:

$$G^{(n)}(z) = G_{p-q} \frac{(p - q)!}{(p - q - n)!}\, z^{p-q-n} + O(z^{p-q-n-1}),$$

which gives the first formula in part 3. For $n \geq p-q+1$, every nonnegative-power term of $G$ is annihilated by $n$ derivatives, so the leading term comes from $G_{-J} z^{-J}$:

$$G^{(n)}(z) = G_{-J}(-1)^n \frac{\Gamma(n + J)}{\Gamma(J)}\, z^{-J-n} + O(z^{-J-n-1}).$$

This completes part 3. The final asymptotics follow from Stirling's formula. □

# 4 Uniform convergence on open Voronoi cells

## 4.1 Translation generating function

We start with the following identity.

**Proposition 6.** *For $A_n = f^{(n)}/f$, $n = 0, 1, \dots$, and every $z \in \mathbb{C} \setminus (\Sigma \cup \mathcal{Z}(P_\sharp))$, one has*

$$\sum_{n\geq 0} A_n(z) \frac{\xi^n}{n!} = \frac{P(z + \xi)Q(z)}{P(z)Q(z + \xi)} \exp\big(E(z + \xi) - E(z)\big). \tag{21}$$

*The series on the left-hand side, called the* translation generating function, *has radius of convergence $\rho(z) = \mathrm{dist}(z, \Sigma)$.*

*Consequently,*

$$\limsup_{n\to\infty} \left|\frac{A_n(z)}{n!}\right|^{1/n} = \rho(z)^{-1}. \tag{22}$$

*Proof.* Since $\mathcal{Z}(P_T) \subseteq \Sigma$, the hypothesis implies $z \notin \mathcal{Z}(P)$, so $f(z) \neq 0$. For $|\xi| < \rho(z)$ one has $z+\xi \notin \Sigma$, hence Taylor's formula gives

$$f(z + \xi) = \sum_{n\geq 0} f^{(n)}(z)\frac{\xi^n}{n!} = f(z)\sum_{n\geq 0} A_n(z)\frac{\xi^n}{n!},$$

and therefore

$$\frac{f(z + \xi)}{f(z)} = \frac{P(z + \xi)Q(z)}{P(z)Q(z + \xi)} \exp\big(E(z + \xi) - E(z)\big).$$

This proves (21) on $|\xi| < \rho(z)$. As a function of $\xi$, the right-hand side is holomorphic for $z + \xi \notin \Sigma$. If $z + \xi = b_k \in \mathcal{Z}(Q) \setminus \mathcal{Z}(T)$, then $Q(z + \xi)^{-1}$ has a pole and $P(z + \xi) \neq 0$ by $\gcd(P, Q) = 1$. If $z + \xi = c_j \in \mathcal{Z}(T)$, then $E(z + \xi)$ has a pole, so $\exp\big(E(z + \xi) - E(z)\big)$ has an essential singularity that cannot be removed by the rational factor $P(z + \xi)/Q(z + \xi)$. Hence the singularities are exactly the points with $z+\xi \in \Sigma$, so the radius of convergence is $\rho(z)$; (22) then follows from the Cauchy–Hadamard formula. □

## 4.2 More notation

Set $C_n(z) := \frac{B_n(z)}{W(z)^n}$ for $n = 0, 1, \ldots$. By (17) and Proposition 6, for $z \in \mathbb{C} \setminus (\Sigma \cup \mathcal{Z}(P_\sharp))$ and $|\xi| < \rho(z)$,

$$\mathcal{C}(z,\xi) := \sum_{n\geq 0} C_n(z)\frac{\xi^n}{n!} = \frac{Q(z)}{P_T(z)}\,\frac{P(z+\xi)}{Q(z+\xi)}\,\exp\bigl(E(z+\xi) - E(z)\bigr). \tag{23}$$

Both sides are holomorphic on the domain $\{(z,\xi) \in (\mathbb{C}\setminus\Sigma)\times\mathbb{C} :\ |\xi| < \rho(z)\}$, so the identity extends to all $z \in \mathbb{C}\setminus\Sigma$. The right-hand side is holomorphic on $\mathbb{C}\setminus(\Sigma - z)$, so it furnishes the analytic continuation of $\mathcal{C}(z,\cdot)$ away from the translated singular set.

For a fixed site $a_i \in \Sigma$, set

$$m_i := \operatorname{ord}_{a_i} T, \qquad p_i := \operatorname{ord}_{a_i} P_T, \qquad r_i := \operatorname{ord}_{a_i} Q, \qquad \beta_i := p_i - r_i. \tag{24}$$

Here $m_i, p_i, r_i, \beta_i$ refer to the fixed site $a_i \in \Sigma$; they are the same local orders as before, now indexed by the unified enumeration $a_1, \ldots, a_N$ of $\Sigma$ rather than the separate enumerations $c_j$ and $b_k$. (Observe that $m_i = 0$ exactly on cells attached to poles of $P/Q$, and $m_i \geq 1$ exactly on cells attached to poles of $S/T$.) Also write

$$P_T(w) = (w-a_i)^{p_i}\widetilde{P}_i(w), \qquad Q(w) = (w-a_i)^{r_i}\widetilde{Q}_i(w), \qquad \widetilde{P}_i(a_i)\widetilde{Q}_i(a_i) \neq 0. \tag{25}$$

If $m_i \geq 1$, write

$$E(w) = \sum_{s=1}^{m_i} \lambda_{i,s}(w-a_i)^{-s} + E_i^{\text{reg}}(w), \qquad \lambda_{i,m_i} \neq 0, \tag{26}$$

whereas if $m_i = 0$ we set $E_i^{\text{reg}} := E$ and interpret sums over $1 \leq s \leq m_i$ as empty.

## 4.3 Main technical tools

For a domain $D \subset \mathbb{C}$, we write $\mathcal{O}(D)$ for the algebra of holomorphic functions on $D$, and for a function $G(\zeta) = \sum_{n\geq 0} g_n\zeta^n$ holomorphic near 0 we write $[\zeta^n]G(\zeta) := g_n$. We first state a parameter-uniform form of Wright's coefficient asymptotics for exponential singularities [27, 28]; the multi-saddle version is the one needed later, and the one-saddle formula away from the Stokes set is an immediate consequence.

Throughout this subsection, fix $m \geq 1$, $\beta \in \mathbb{Z}$, a simply connected domain $D \subset \mathbb{C}$, functions $\lambda_1, \ldots, \lambda_m \in \mathcal{O}(D)$ with $\lambda_m$ nonvanishing on $D$, and a function $R$ holomorphic on a neighborhood of $D \times \{1\}$ with $R(z,1) \neq 0$ for all $z \in D$. For $z \in D$, set

$$F_z(\zeta) := (1-\zeta)^\beta \exp\Bigl(\sum_{s=1}^{m} \lambda_s(z)(1-\zeta)^{-s}\Bigr) R(z,\zeta),$$

and assume that $F_z$ extends holomorphically to a neighborhood of $\{|\zeta| \leq 1\} \setminus \{1\}$, with $\zeta = 1$ as its unique singularity on $|\zeta| = 1$. Assume moreover that for every compact $L \Subset D$ there exists $\varrho_L > 1$ such that $(z,\zeta) \mapsto F_z(\zeta)$ is holomorphic on a neighborhood of $L \times \bigl(\{|\zeta| \leq \varrho_L\} \setminus \{1\}\bigr)$. Choose a holomorphic branch $\eta$ on $D$ with

$$\eta(z)^{m+1} = m\lambda_m(z),$$

choose a holomorphic square root $\eta^{1/2}$ on $D$, let $\omega_\nu := e^{2\pi i\nu/(m+1)}$, $0 \leq \nu \leq m$, and fix square roots $\omega_\nu^{1/2}$ once and for all. For $z \in D$ and $n \geq 1$, define

$$\Phi_{z,n}(t) := \sum_{s=1}^{m} \lambda_s(z)t^{-s} - (n+1)\log(1-t),$$

where $\log(1-t)$ denotes the principal branch near $t = 0$.

**Lemma 7** (parameter-uniform multi-saddle asymptotics)**.** *Let $K \Subset D$ be compact. Then there exists an open set $U$ with $K \Subset U \Subset D$ such that, for each $0 \leq \nu \leq m$ and all sufficiently large $n$, there is a holomorphic function $t_{n,\nu} : U \to \mathbb{C}$ satisfying*

$$\Phi'_{z,n}(t_{n,\nu}(z)) = 0, \qquad t_{n,\nu}(z) = \omega_\nu\eta(z)\,n^{-1/(m+1)} + O_U\Bigl(n^{-2/(m+1)}\Bigr).$$

*Set*

$$\theta := -\frac{2\beta+m+2}{2(m+1)}, \qquad A_\nu(z) := \frac{(\omega_\nu\eta(z))^\beta\,(\omega_\nu^{1/2}\eta^{1/2}(z))}{\sqrt{2\pi(m+1)}}\,R(z,1), \qquad \Xi_\nu(z;n) := \Phi_{z,n}(t_{n,\nu}(z)). \tag{27}$$

*Then each $A_\nu$ is holomorphic and nonvanishing on $U$, and each $\Xi_\nu(z;n)$ admits, by iterating the saddle equation, an asymptotic expansion to arbitrary finite order in descending powers of $n^{1/(m+1)}$, uniformly on $K$, with*

$$\Xi_\nu(z;n) = \frac{m+1}{m}\,\omega_\nu\eta(z)\,n^{m/(m+1)} + O_K\Big(n^{(m-1)/(m+1)}\Big) \tag{28}$$

*uniformly for $z \in K$. Moreover, uniformly for $z \in K$,*

$$[\zeta^n]F_z(\zeta) = n^\theta \sum_{\nu=0}^{m} A_\nu(z)\exp\!\big(\Xi_\nu(z;n)\big)\Big(1 + O_K\Big(n^{-1/(m+1)}\Big)\Big) + \mathcal{R}_n(z), \tag{29}$$

*where the $O_K$-term is relative to each fixed $\nu$-summand, and*

$$\mathcal{R}_n(z) = O_K\Big(n^\theta \exp\!\Big(\max_{0\le\nu\le m} \Re\Xi_\nu(z;n) - c_K n^{m/(m+1)}\Big)\Big) \tag{30}$$

*for some $c_K > 0$.*

*Proof.* Fix $K \Subset D$ and choose an open set $U$ with $K \Subset U \Subset D$. For $0 \le \nu \le m$, set $\tau_\nu(z) := \omega_\nu\eta(z)$. The scaled saddle equation is

$$\begin{aligned} 0 &= n^{-1}\Phi'_{z,n}\big(n^{-1/(m+1)}\tau\big) \\ &= 1 - \tau_\nu(z)^{m+1}\tau^{-m-1} + O_U\Big(n^{-1/(m+1)}\Big), \end{aligned}$$

uniformly for $\tau$ on compact subsets of $\mathbb{C}^\times$. Since

$$\partial_\tau\big(1 - \tau_\nu(z)^{m+1}\tau^{-m-1}\big)\Big|_{\tau=\tau_\nu(z)} = \frac{m+1}{\tau_\nu(z)} \neq 0,$$

the holomorphic implicit function theorem gives, for all sufficiently large $n$, a unique holomorphic function $\tau_{n,\nu} : U \to \mathbb{C}$ such that

$$\tau_{n,\nu}(z) = \tau_\nu(z) + O_U\Big(n^{-1/(m+1)}\Big), \qquad \Phi'_{z,n}\big(n^{-1/(m+1)}\tau_{n,\nu}(z)\big) = 0.$$

Setting $t_{n,\nu}(z) := n^{-1/(m+1)}\tau_{n,\nu}(z)$ gives the stated expansion. Iterating the implicit equation for $\tau_{n,\nu}$ yields the full asymptotic expansion of $\Xi_\nu(z;n) = \Phi_{z,n}(t_{n,\nu}(z))$, and its leading term is

$$\lambda_m(z)\tau_\nu(z)^{-m}n^{m/(m+1)} + \tau_\nu(z)n^{m/(m+1)} = \frac{m+1}{m}\,\tau_\nu(z)\,n^{m/(m+1)},$$

which is (28).

Fix a connected component $U_0$ of $U$. By the compact-uniform annulus hypothesis applied to $\overline{U}_0$, there exists $\varrho > 1$ such that $(z,\zeta) \mapsto F_z(\zeta)$ is holomorphic on a neighborhood of $\overline{U}_0 \times \big(\{|\zeta| \le \varrho\} \setminus \{1\}\big)$. Since $R$ is holomorphic on a neighborhood of $D \times \{1\}$, there exists $\delta_0 > 0$ such that the local representation of $F_z$ is holomorphic for $z \in U_0$ and $0 < |\zeta - 1| < 2\delta_0$. Choose $\Lambda > 0$ so large that $|\tau_\nu(z)| \le \Lambda/4$ on $U_0$ for every $\nu$, and set $\varepsilon_n := \Lambda n^{-1/(m+1)}$. For all sufficiently large $n$, one has $\varepsilon_n < \delta_0$, and Cauchy's theorem in the annulus with the deleted disc $|\zeta - 1| < \varepsilon_n$ gives

$$[\zeta^n]F_z(\zeta) = \frac{1}{2\pi i}\int_{|\zeta|=\varrho} F_z(\zeta)\zeta^{-n-1}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{\Gamma_{\varepsilon_n}} t^\beta R(z,1-t)e^{\Phi_{z,n}(t)}\,\mathrm{d}t, \tag{31}$$

where $\Gamma_{\varepsilon_n}$ is the negatively oriented circle $|t| = \varepsilon_n$ under the change of variables $\zeta = 1-t$. The outer-circle term is $O_{U_0}(\varrho^{-n})$, hence exponentially smaller than the saddle terms.

In the scaled variable $\tau = n^{1/(m+1)}t$, write

$$\widehat{\Phi}_{z,n}(\tau) := n^{-m/(m+1)}\Phi_{z,n}\big(n^{-1/(m+1)}\tau\big), \qquad \psi_z(\tau) := \lambda_m(z)\tau^{-m} + \tau.$$

Then $\widehat{\Phi}_{z,n} \to \psi_z$ with two $\tau$-derivatives on compact subsets of $U_0 \times \mathbb{C}^\times$. Since $|\eta|$ and $|(\omega_\nu - \omega_\mu)\eta|$ are bounded below on $U_0$, the limiting saddles $\tau_\nu(z)$, and hence also the exact saddles $\tau_{n,\nu}(z)$, stay in a fixed compact annulus and are pairwise separated by a uniform positive distance.

Wright's contour for the limiting phase $\psi_z$ in the punctured disc $0 < |\tau| < \Lambda$ is the union of $m+1$ disjoint local descent arcs, one through each saddle $\tau_\nu(z)$, and a background contour. Its topology is constant on $U_0$ because no saddle collides with $\tau = 0$ or with another saddle. Remove fixed symmetric neighborhoods of all saddle points. The union of the remaining pieces over $z \in K$ is compact, and on it one has

$$\Re\psi_z(\tau) < \max_{0\le\nu\le m} \Re\psi_z(\tau_\nu(z)).$$

Compactness therefore gives $3c_K > 0$ such that

$$\Re\psi_z(\tau) \le \max_{0\le\nu\le m} \Re\psi_z(\tau_\nu(z)) - 3c_K$$

on the limiting background contour for every $z \in K$.

Because $\widehat{\Phi}_{z,n} \to \psi_z$ with one $\tau$-derivative on compact subsets and $\tau_{n,\nu}(z) = \tau_\nu(z) + O_K\big(n^{-1/(m+1)}\big)$, the corresponding contour for $\widehat{\Phi}_{z,n}$ is a $C^1$-small perturbation of the limiting one for all sufficiently large $n$. In particular, its topology is the same, and its background part, denoted $\widehat{\Gamma}^{\mathrm{bg}}_{n,z}$, stays in a compact subset of $K \times \{0 < |\tau| \le \Lambda\}$. After perhaps decreasing $c_K$, the preceding gap for $\psi_z$ persists for $\widehat{\Phi}_{z,n}$:

$$\Re\widehat{\Phi}_{z,n}(\tau) \le \max_{0\le\nu\le m} \Re\widehat{\Phi}_{z,n}(\tau_{n,\nu}(z)) - c_K \qquad (z \in K,\ \tau \in \widehat{\Gamma}^{\mathrm{bg}}_{n,z}) \tag{32}$$

for all sufficiently large $n$.

For the local analysis, the holomorphic Morse lemma applies uniformly near each $\tau_{n,\nu}(z)$ because $\widehat{\Phi}''_{z,n}(\tau_{n,\nu}(z))$ stays bounded away from 0 and the higher derivatives of $\widehat{\Phi}_{z,n}$ are uniformly bounded on fixed neighborhoods of the saddles. Hence each local descent arc contributes

$$\frac{t_{n,\nu}(z)^\beta R(z,1)}{\sqrt{2\pi\,\Phi''_{z,n}(t_{n,\nu}(z))}} \exp\big(\Phi_{z,n}(t_{n,\nu}(z))\big) \Big(1 + O_K\Big(n^{-1/(m+1)}\Big)\Big), \tag{33}$$

where the square root is determined by the chosen Morse coordinate and therefore agrees with the fixed branch encoded by $\eta^{1/2}$ and $\omega_\nu^{1/2}$. Summing these $m+1$ local contributions yields the main term in (29). The bound (32) gives (30) after rescaling back to $t$.

Finally,

$$t_{n,\nu}(z) = \tau_\nu(z)\, n^{-1/(m+1)} \Big(1 + O_U\Big(n^{-1/(m+1)}\Big)\Big)$$

and

$$\Phi''_{z,n}(t_{n,\nu}(z)) = \frac{m+1}{\tau_\nu(z)}\, n^{(m+2)/(m+1)} \Big(1 + O_U\Big(n^{-1/(m+1)}\Big)\Big),$$

uniformly on $U$. Substituting these expansions into (33) gives (27) and (29). □

Set

$$N^{\mathrm{St}} := \bigcup_{0\le\nu<\mu\le m} \big\{z \in D : \Re\big((\omega_\nu - \omega_\mu)\eta(z)\big) = 0\big\}.$$

We call $N^{\mathrm{St}}$ the *Stokes set.*

**Lemma 8** (parameter-uniform one-saddle asymptotics)**.** *Let $K \Subset D \setminus N^{\mathrm{St}}$ be compact. Then there exists an open set $U$ with $K \Subset U \Subset D \setminus N^{\mathrm{St}}$ and a locally constant map $\nu_* : U \to \{0, \dots, m\}$ such that*

$$\Re\big(\omega_{\nu_*(z)}\eta(z)\big) = \max_{0\le\nu\le m} \Re\big(\omega_\nu \eta(z)\big) \qquad (z \in U).$$

*For all sufficiently large $n$, there exists a unique holomorphic function $t_n : U \to \mathbb{C}$ such that*

$$\Phi'_{z,n}(t_n(z)) = 0, \qquad t_n(z) = \omega_{\nu_*(z)}\eta(z)\, n^{-1/(m+1)} + O_U\Big(n^{-2/(m+1)}\Big).$$

*Set*

$$\theta = -\frac{2\beta+m+2}{2(m+1)}, \qquad A(z) := \frac{(\omega_{\nu_*(z)}\eta(z))^\beta\, (\omega^{1/2}_{\nu_*(z)}\eta^{1/2}(z))}{\sqrt{2\pi(m+1)}} R(z,1), \qquad \Xi(z;n) := \Phi_{z,n}(t_n(z)). \tag{34}$$

*Then $A$ is holomorphic and nonvanishing on $U$, and $\Xi(z;n)$ admits, by iterating the saddle-point equation, an asymptotic expansion to arbitrary finite order in descending powers of $n^{1/(m+1)}$, uniformly on $K$; in particular,*

$$\Xi(z;n) = \frac{m+1}{m}\,\omega_{\nu_*(z)}\eta(z)\,n^{m/(m+1)} + O_K\Big(n^{(m-1)/(m+1)}\Big). \tag{35}$$

*Moreover, uniformly for $z \in K$,*

$$[\zeta^n]F_z(\zeta) = A(z)\,n^{\theta}\exp\!\big(\Xi(z;n)\big)\Big(1 + O_K\Big(n^{-1/(m+1)}\Big)\Big). \tag{36}$$

*Proof.* For $z \in D \setminus N^{\mathrm{St}}$, the numbers $\Re(\omega_\nu\eta(z))$, $0 \le \nu \le m$, are pairwise distinct, so there is a unique dominant index $\nu_*(z)$. Since $K \Subset D \setminus N^{\mathrm{St}}$, after shrinking to an open set $U$ with $K \Subset U \Subset D \setminus N^{\mathrm{St}}$ we may regard $\nu_* : U \to \{0,\dots,m\}$ as locally constant, and compactness gives $\delta > 0$ such that

$$\Re\big((\omega_{\nu_*(z)} - \omega_\nu)\eta(z)\big) \ge 4\delta \qquad (z \in U,\ \nu \ne \nu_*(z)). \tag{37}$$

Apply Lemma 7 on $K$ and shrink $U$ if necessary so that the functions $t_{n,\nu}$ from that lemma are defined on $U$. Set

$$t_n(z) := t_{n,\nu_*(z)}(z), \qquad A(z) := A_{\nu_*(z)}(z), \qquad \Xi(z;n) := \Xi_{\nu_*(z)}(z;n).$$

The function $A$ is holomorphic on $U$ because $\nu_*$ is locally constant. The asymptotic expansion of $\Xi$ and the formula (35) are the corresponding assertions from Lemma 7. Uniqueness of $t_n$ follows because the points $\omega_\nu\eta(z)n^{-1/(m+1)}$ are uniformly separated on $U$, so for large $n$ exactly one critical point lies in a small neighborhood of the dominant one.

By (28) and (37), after decreasing $\delta$ and increasing the lower bound on $n$ if necessary,

$$\Re\Xi(z;n) \ge \Re\Xi_\nu(z;n) + 2\delta n^{m/(m+1)} \qquad (z \in K,\ \nu \ne \nu_*(z)).$$

Hence every non-dominant saddle term in (29) and the remainder (30) are exponentially smaller than the dominant contribution, uniformly on $K$. Absorbing them into the relative error gives (36). □

### 4.4 Main result on open Voronoi cells

**Theorem 9** (uniform coefficient extraction on Voronoi cells)**.** *Fix $i$ and let $D \Subset \mathcal{V}_i^\circ \setminus \Sigma$ be open. Set*

$$d_i(z) := a_i - z, \qquad w := z + d_i(z)\zeta, \qquad G_{i,z}(\zeta) := \mathcal{C}\big(z, d_i(z)\zeta\big) = \sum_{n\ge 0} \frac{C_n(z)}{n!}\,d_i(z)^n\zeta^n.$$

*Then $\zeta = 1$ is the unique singularity of $G_{i,z}$ on $|\zeta| = 1$. Consequently the coefficient asymptotics split into exactly two cases: the algebraic case $m_i = 0$, where Darboux extraction applies, and the essential case $m_i \ge 1$, where one obtains a Wright expansion with $m_i + 1$ saddle contributions. In both cases,*

$$G_{i,z}(\zeta) = (1-\zeta)^{\beta_i}\exp\Big(\sum_{s=1}^{m_i} \widetilde{\lambda}_{i,s}(z)(1-\zeta)^{-s}\Big)R_i(z,\zeta), \tag{38}$$

*where*

$$\widetilde{\lambda}_{i,s}(z) := \lambda_{i,s}(z - a_i)^{-s} \qquad (1 \le s \le m_i),$$

*and*

$$R_i(z,\zeta) := (-d_i(z))^{\beta_i}\,\frac{Q(z)}{P_T(z)}\,\frac{\widetilde{P}_i(w)\,P_\sharp(w)}{\widetilde{Q}_i(w)}\exp\!\big(E_i^{\mathrm{reg}}(w) - E(z)\big). \tag{39}$$

*The function $R_i$ is holomorphic on a neighborhood of $D \times \{1\}$ and satisfies*

$$R_i(z,1) = \frac{\widetilde{Q}_i(z)\,\widetilde{P}_i(a_i)\,P_\sharp(a_i)}{\widetilde{P}_i(z)\,\widetilde{Q}_i(a_i)}\,e^{E_i^{\mathrm{reg}}(a_i) - E(z)} \ne 0. \tag{40}$$

*Let $K \Subset D$ be compact.*

1. *If $m_i = 0$, then $\beta_i = -r_i \le -1$, and uniformly for $z \in K$,*

$$\frac{C_n(z)}{n!} = d_i(z)^{-n} n^{-\beta_i - 1} \frac{R_i(z,1)}{\Gamma(-\beta_i)} \big(1 + O_K(n^{-1})\big), \tag{41}$$

$$\frac{B_n(z)}{n!} = W(z)^n d_i(z)^{-n} n^{-\beta_i - 1} \frac{R_i(z,1)}{\Gamma(-\beta_i)} \big(1 + O_K(n^{-1})\big). \tag{42}$$

2. *Suppose $m_i \ge 1$ and $D$ is simply connected. Choose a holomorphic branch $\eta_D$ on $D$ with*

$$\eta_D(z)^{m_i+1} = m_i \lambda_{i,m_i} (z - a_i)^{-m_i},$$

*choose a holomorphic square root $\eta_D^{1/2}$ on $D$, let $\omega_\nu := e^{2\pi i \nu/(m_i+1)}$, $0 \le \nu \le m_i$, and fix square roots $\omega_\nu^{1/2}$ once and for all. Set*

$$\Phi_{i,z,n}(t) := \sum_{s=1}^{m_i} \widetilde{\lambda}_{i,s}(z) t^{-s} - (n+1)\log(1-t), \qquad \theta_i := -\frac{2\beta_i + m_i + 2}{2(m_i+1)}.$$

*Then there exists an open set $U$ with $K \Subset U \Subset D$ such that, for each $0 \le \nu \le m_i$ and all sufficiently large $n$, there is a holomorphic function $t_{i,n,\nu} : U \to \mathbb{C}$ satisfying*

$$\Phi'_{i,z,n}(t_{i,n,\nu}(z)) = 0, \qquad t_{i,n,\nu}(z) = \omega_\nu \eta_D(z)\, n^{-1/(m_i+1)} + O_U\Big(n^{-2/(m_i+1)}\Big).$$

*Define*

$$\mathcal{A}_{i,\nu}(z) := \frac{(\omega_\nu \eta_D(z))^{\beta_i} \, (\omega_\nu^{1/2} \eta_D^{1/2}(z))}{\sqrt{2\pi(m_i+1)}} R_i(z,1), \tag{43}$$

*and*

$$\Xi_{i,\nu}(z;n) := \Phi_{i,z,n}(t_{i,n,\nu}(z)). \tag{44}$$

*Then each $\mathcal{A}_{i,\nu}$ is holomorphic and nonvanishing on $U$, each $\Xi_{i,\nu}(z;n)$ admits an asymptotic expansion to arbitrary finite order in descending powers of $n^{1/(m_i+1)}$, uniformly on $K$, and*

$$\Xi_{i,\nu}(z;n) = \frac{m_i+1}{m_i} \omega_\nu \eta_D(z)\, n^{m_i/(m_i+1)} + O_K\Big(n^{(m_i-1)/(m_i+1)}\Big) \tag{45}$$

*uniformly for $z \in K$. Moreover, uniformly for $z \in K$,*

$$\frac{C_n(z)}{n!} = d_i(z)^{-n} n^{\theta_i} \sum_{\nu=0}^{m_i} \mathcal{A}_{i,\nu}(z) \exp\!\big(\Xi_{i,\nu}(z;n)\big) \Big(1 + O_K\Big(n^{-1/(m_i+1)}\Big)\Big) + \mathcal{R}_{i,n}(z), \tag{46}$$

$$\frac{B_n(z)}{n!} = W(z)^n d_i(z)^{-n} n^{\theta_i} \sum_{\nu=0}^{m_i} \mathcal{A}_{i,\nu}(z) \exp\!\big(\Xi_{i,\nu}(z;n)\big) \Big(1 + O_K\Big(n^{-1/(m_i+1)}\Big)\Big) + W(z)^n \mathcal{R}_{i,n}(z), \tag{47}$$

*where the $O_K$-term is relative to each fixed $\nu$-summand, and*

$$\mathcal{R}_{i,n}(z) = O_K\Big(d_i(z)^{-n} n^{\theta_i} \exp\!\Big(\max_{0 \le \nu \le m_i} \Re \Xi_{i,\nu}(z;n) - c_K n^{m_i/(m_i+1)}\Big)\Big) \tag{48}$$

*for some $c_K > 0$.*

*Proof.* For each $z \in D$, the singularities of $G_{i,z}$ occur precisely at

$$\zeta = \frac{a_j - z}{a_i - z}, \qquad 1 \le j \le N,$$

because (23) is holomorphic exactly on $\mathbb{C} \setminus (\Sigma - z)$. Since $z \in \mathcal{V}_i^\circ$, one has

$$\left| \frac{a_j - z}{a_i - z} \right| > 1 \qquad (j \ne i),$$

so $\zeta = 1$ is the unique singularity on $|\zeta| = 1$.

Using $P = P_T P_\sharp$, the local factorizations (25), and

$$w - a_i = -d_i(z)(1-\zeta),$$

we obtain from (23)

$$\begin{aligned} G_{i,z}(\zeta) &= (-d_i(z))^{\beta_i}(1-\zeta)^{\beta_i}\,\frac{Q(z)}{P_T(z)}\,\frac{\widetilde{P}_i(w)\,P_\sharp(w)}{\widetilde{Q}_i(w)}\exp\bigl(E(w)-E(z)\bigr) \\ &= (1-\zeta)^{\beta_i}\exp\Bigl(\sum_{s=1}^{m_i}\widetilde{\lambda}_{i,s}(z)(1-\zeta)^{-s}\Bigr)R_i(z,\zeta), \end{aligned}$$

which is (38)–(39). Evaluating at $\zeta = 1$ and using

$$(-d_i(z))^{\beta_i}\,\frac{Q(z)}{P_T(z)} = \frac{\widetilde{Q}_i(z)}{\widetilde{P}_i(z)}$$

gives (40).

Let

$$\varrho_K := \inf_{z\in K}\min_{j\neq i}\left|\frac{a_j - z}{a_i - z}\right| \in (1,\infty],$$

where the empty minimum is interpreted as $+\infty$. Fix $\varrho$ with $1 < \varrho < \varrho_K$ (or any $\varrho > 1$ if $\varrho_K = \infty$). Then, for every $z \in K$, all singularities of $G_{i,z}$ other than $\zeta = 1$ lie outside $|\zeta| \le \varrho$.

Since (23) is jointly holomorphic in $(z,\zeta)$ away from the moving singular set, it follows that $(z,\zeta) \mapsto G_{i,z}(\zeta)$ is holomorphic on a neighborhood of $K \times \bigl(\{|\zeta| \le \varrho\} \setminus \{1\}\bigr)$. The same argument applies to every compact $L \Subset D$, so the annulus hypothesis required in Lemmas 7 and 8 holds on $D$.

Assume first that $m_i = 0$. Then $r_i = -\beta_i \in \mathbb{N}$ and

$$G_{i,z}(\zeta) = (1-\zeta)^{-r_i}R_i(z,\zeta).$$

Expand $R_i(z,\zeta)$ at $\zeta = 1$ to order $r_i - 1$:

$$R_i(z,\zeta) = \sum_{u=0}^{r_i-1} c_{i,u}(z)(1-\zeta)^u + (1-\zeta)^{r_i}\widehat{H}_i(z,\zeta),$$

where the coefficients $c_{i,u}$ are holomorphic on a neighborhood of $K$, $c_{i,0}(z) = R_i(z,1)$, and $\widehat{H}_i$ is holomorphic near $K \times \{1\}$. Therefore

$$G_{i,z}(\zeta) = \sum_{u=0}^{r_i-1} c_{i,u}(z)(1-\zeta)^{-r_i+u} + \widehat{H}_i(z,\zeta),$$

and the right-hand side has no singularity at $\zeta = 1$ except in the displayed principal part. Hence $\widehat{H}_i$ extends holomorphically to a neighborhood of $K \times \{|\zeta| \le 1+\delta\}$ for every fixed $0 < \delta < \varrho_K - 1$. For integers $s \ge 1$,

$$[\zeta^n](1-\zeta)^{-s} = \binom{n+s-1}{s-1} = \frac{n^{s-1}}{\Gamma(s)}\bigl(1 + O(n^{-1})\bigr),$$

while Cauchy's estimate gives

$$[\zeta^n]\widehat{H}_i(z,\zeta) = O_K\bigl((1+\delta/2)^{-n}\bigr).$$

Thus the $u = 0$ term dominates uniformly on $K$, which yields (41); (42) follows by multiplying by $W(z)^n$.

Assume now that $m_i \ge 1$ and $D$ is simply connected. Then $\widetilde{\lambda}_{i,m_i}(z) = \lambda_{i,m_i}(z-a_i)^{-m_i}$ is holomorphic and nonvanishing on $D$, so the required branches $\eta_D$ and $\eta_D^{1/2}$ exist. Apply Lemma 7 to $F_z = G_{i,z}$ with $m = m_i$, $\beta = \beta_i$, $\lambda_s = \widetilde{\lambda}_{i,s}$, and $R = R_i$. This gives (43)–(48). □

*Remark* 10. On a compact subset of an essential cell where one saddle is uniquely dominant, the remaining saddle terms in Theorem 9(2) are exponentially smaller by (45); the theorem therefore collapses there to one-saddle asymptotics and, in particular, to eventual zero-freeness. We use only this elementary consequence in the proof of Corollary 11.

For the next corollary, with $\gamma_n = \mathrm{lc}(B_n)$ as in Proposition 5, set

$$s_n := \begin{cases} 0, & \text{if } h = 0, \\ \log n!, & \text{if } h > 0, \end{cases}$$

and, whenever $\deg B_n > 0$,

$$\widetilde{L}_n(z) := \frac{1}{\deg B_n}\Big(\log|B_n(z)| - \log|\gamma_n| - s_n\Big), \qquad \Psi_i(z) := \frac{1}{\kappa}\big(\log|W(z)| - \log|z-a_i| - \sigma\big).$$

**Corollary 11** (local $L^1$-rate on essential Voronoi interiors)**.** *Assume the notation of Theorem 9(2), and let $K \Subset D$ be compact. Then*

$$\big\|\widetilde{L}_n - \Psi_i\big\|_{L^1(K)} = O_K\Big(n^{-1/(m_i+1)}\Big). \tag{49}$$

*In particular, the same estimate holds on any compact subset of $\mathcal{V}_i^\circ \setminus \Sigma$ after covering that compact set by finitely many simply connected open subsets of the cell.*

*Proof.* Set

$$\alpha := \frac{m_i}{m_i+1}, \qquad F_n(z) := \frac{B_n(z)}{n!}\left(\frac{a_i - z}{W(z)}\right)^n n^{-\theta_i}.$$

Write

$$u_\nu(z) := \Re\big(\omega_\nu \eta_D(z)\big), \qquad 0 \le \nu \le m_i,$$

and let

$$\Gamma_D^{\mathrm{St}} := \bigcup_{0\le\nu<\mu\le m_i}\left\{z \in D:\ u_\nu(z) = u_\mu(z) = \max_{0\le\lambda\le m_i} u_\lambda(z)\right\}.$$

For each $\nu < \mu$, the function $z \mapsto \Re((\omega_\nu - \omega_\mu)\eta_D(z))$ is harmonic and nonconstant on $D$, because

$$\eta_D'(z) = -\frac{m_i}{m_i+1}\frac{\eta_D(z)}{z-a_i} \neq 0 \qquad (z \in D).$$

Hence every pairwise-equality set $\{u_\nu = u_\mu\}$ has empty interior. For each fixed $z$, the numbers $u_0(z), \dots, u_{m_i}(z)$ are the real parts of the vertices of a rotated regular $(m_i+1)$-gon scaled by $|\eta_D(z)|$, so a maximum can be attained by at most two adjacent vertices. Therefore $\Gamma_D^{\mathrm{St}}$ is closed and has empty interior. Since $D \setminus \Gamma_D^{\mathrm{St}}$ is open and dense, compactness gives discs $D_\ell := D(z_\ell, r_\ell)$ such that

$$K \subset \bigcup_{\ell=1}^{L} D_\ell, \qquad \overline{D(z_\ell, 4r_\ell)} \Subset D, \qquad z_\ell \in D \setminus \Gamma_D^{\mathrm{St}}.$$

Since $\bigcup_{n\ge0}\mathcal{Z}(B_n)$ is countable and $F_n$ and $B_n$ have the same zeros on $D$, we may in addition choose the radii so that neither $\partial D(z_\ell, 2r_\ell)$ nor $\partial D(z_\ell, 4r_\ell)$ contains a zero of any $F_n$.

Fix $\ell$. By Theorem 9(2) applied on $\overline{D(z_\ell, 4r_\ell)}$, there is $C_\ell > 0$ such that

$$\sup_{z\in D(z_\ell,4r_\ell)} \log|F_n(z)| \le C_\ell n^\alpha$$

for all sufficiently large $n$. Since $z_\ell \notin \Gamma_D^{\mathrm{St}}$, there is a unique dominant saddle index $\nu_*(\ell)$ at $z_\ell$ and a gap $\delta_\ell > 0$ such that

$$u_{\nu_*(\ell)}(z_\ell) \ge u_\nu(z_\ell) + 4\delta_\ell \qquad (\nu \neq \nu_*(\ell)).$$

By (45), after increasing the threshold on $n$ if necessary,

$$\Re\Xi_{i,\nu_*(\ell)}(z_\ell; n) \ge \Re\Xi_{i,\nu}(z_\ell; n) + 2\delta_\ell n^\alpha \qquad (\nu \neq \nu_*(\ell)).$$

Applying Theorem 9(2) at $z_\ell$ therefore gives

$$F_n(z_\ell) = \mathcal{A}_{i,\nu_*(\ell)}(z_\ell)\exp\big(\Xi_{i,\nu_*(\ell)}(z_\ell; n)\big)\left(1 + O_\ell\Big(n^{-1/(m_i+1)}\Big) + O_\ell\Big(e^{-\delta_\ell n^\alpha}\Big)\right).$$

In particular $F_n(z_\ell) \neq 0$ and

$$\big|\log|F_n(z_\ell)|\big| \le C_\ell n^\alpha$$

for all sufficiently large $n$.

Let $N_{\ell,n}$ be the number of zeros of $F_n$ in $D(z_\ell, 2r_\ell)$, counted with multiplicity. Jensen's formula on $D(z_\ell, 4r_\ell)$ yields

$$N_{\ell,n}\log 2 \le \frac{1}{2\pi}\int_0^{2\pi} \log|F_n(z_\ell + 4r_\ell e^{it})|\,dt - \log|F_n(z_\ell)| = O(n^\alpha),$$

so $N_{\ell,n} = O(n^\alpha)$. On $D(z_\ell, 2r_\ell)$, Poisson–Jensen writes

$$\log|F_n| = h_{\ell,n} + G_{\ell,n},$$

where $h_{\ell,n}$ is harmonic and $G_{\ell,n} \le 0$ is the Green potential of the zeros of $F_n$ in $D(z_\ell, 2r_\ell)$. The boundary values of $h_{\ell,n}$ are bounded above by $C_\ell n^\alpha$, and $h_{\ell,n}(z_\ell) = O(n^\alpha)$ because $G_{\ell,n}(z_\ell) = O(n^\alpha)$ by the same Jensen bound. Harnack's inequality applied to $C_\ell n^\alpha - h_{\ell,n}$ therefore gives

$$\|h_{\ell,n}\|_{L^\infty(D_\ell)} = O(n^\alpha).$$

Each Green kernel of $D(z_\ell, 2r_\ell)$ has $L^1(D_\ell)$-norm $O_\ell(1)$, uniformly in its pole, so $\|G_{\ell,n}\|_{L^1(D_\ell)} = O(n^\alpha)$. Hence

$$\|\log|F_n|\|_{L^1(D_\ell)} = O(n^\alpha).$$

Summing over $\ell$ gives

$$\|\log|F_n|\|_{L^1(K)} = O_K(n^\alpha).$$

Finally,

$$\widetilde{L}_n(z) - \Psi_i(z) = \frac{\log|F_n(z)|}{\deg B_n} + O_K\left(\frac{\log n}{n}\right)$$

uniformly on $K$. Indeed, Proposition 5 gives $\deg B_n = \kappa n + O(1)$ and

$$\frac{\theta_i \log n + \log n! - \log|\gamma_n| - s_n}{\deg B_n} + \frac{\sigma}{\kappa} = O\left(\frac{\log n}{n}\right),$$

while $\log|W| - \log|a_i - z|$ is bounded on $K$. Since $\alpha - 1 = -1/(m_i+1)$, dividing the preceding $L^1$-bound by $\deg B_n$ proves (49). □

# 5 Reduced local models at essential singularities

In this section we study the reduced local model at an essential singularity, obtained by retaining only the algebraic factor and the highest-order singular term in the exponent. The resulting microscopic laws refine the atoms of Theorem 21 on the natural blow-up scale.

Let $c_j$ be a zero of $T$ of multiplicity $m_j$, set $\alpha_j := p_j - \nu_j$, and write

$$E(z) = \sum_{s=1}^{m_j} \lambda_{j,s}(z-c_j)^{-s} + E_j^{\mathrm{reg}}(z), \qquad \lambda_{j,m_j} \neq 0,$$

where $E_j^{\mathrm{reg}}$ is holomorphic near $c_j$. Then

$$f(z) = (z-c_j)^{\alpha_j}\exp\left(\sum_{s=1}^{m_j}\lambda_{j,s}(z-c_j)^{-s}\right)\phi(z),$$

with $\phi$ holomorphic and nonvanishing near $c_j$. Discarding the lower-order singular terms and the holomorphic unit leads to the reduced model

$$g_{\alpha,m}(z) := (z-a)^\alpha \exp\left(\frac{\lambda}{(z-a)^m}\right), \qquad \lambda \neq 0,$$

which we study for all $m \ge 1$ and $\alpha \in \mathbb{Z}$. (The choice of $a \in \mathbb{C}$ is irrelevant.)

Recall that a polynomial sequence $\{P_n\}_{n\ge0}$ is a *Sheffer sequence* if its exponential generating function has the form $A(t)e^{xB(t)}$ with $A(0) \neq 0$, $B(0) = 0$, and $B'(0) \neq 0$. If $\mathbf{u} = (u_0, \ldots, u_{m-1})$ is an $m$-tuple of linear functionals on $\mathbb{C}[x]$, we say that $\{P_n\}_{n\ge0}$ is *$m$-orthogonal* with respect to $\mathbf{u}$ if

$$\langle u_j, x^\nu P_n\rangle = 0 \quad \text{for } n \ge m\nu + j + 1,$$

and

$$\langle u_j, x^\nu P_{m\nu+j}\rangle \neq 0 \quad \text{for every } \nu \geq 0 \text{ and } 0 \leq j \leq m-1.$$

We identify the associated polynomial family with the Varma–Taşdelen Laguerre-type Sheffer sequence; see [25]. For $\alpha < 0$ this sequence is $m$-orthogonal with respect to an explicit $m$-tuple of Laguerre-type linear functionals on $\mathbb{C}[x]$. For $\alpha \geq 0$ and every $n \geq \alpha + 1$, it has a fixed zero multiplicity at the origin. The case $m = 1$ reduces to generalized Laguerre polynomials, and after the reciprocal change of variables the microscopic zero distribution is the Marchenko–Pastur law. The full local factor, including the lower-order singular terms and the holomorphic unit, reappears in Section 7 as an open problem.

**Proposition 12** (reduced local model and Sheffer sequence)**.** *Let $m \geq 1$, $\alpha \in \mathbb{Z}$, $a \in \mathbb{C}$, and $\lambda \in \mathbb{C}^\times$, and define*

$$g_{\alpha,m}(z) := (z-a)^\alpha \exp\!\left(\frac{\lambda}{(z-a)^m}\right), \qquad x := -\frac{\lambda}{(z-a)^m}.$$

*Then the following hold.*

1. *For $z \neq a$, the derivatives satisfy*

$$g^{(n)}_{\alpha,m}(z) = (-1)^n (z-a)^{\alpha-n} \Pi_n^{(\alpha,m)}(x) \exp\!\left(\frac{\lambda}{(z-a)^m}\right), \qquad n \geq 0,$$

*where $\{\Pi_n^{(\alpha,m)}\}_{n\geq 0}$ is the unique polynomial Sheffer sequence satisfying*

$$\sum_{n\geq 0} \Pi_n^{(\alpha,m)}(x) \frac{t^n}{n!} = (1-t)^\alpha \exp\bigl(x(1-(1-t)^{-m})\bigr).$$

2. *The sequence is characterized equivalently by $\Pi_0^{(\alpha,m)} = 1$ and*

$$\Pi_{n+1}^{(\alpha,m)}(x) = mx\bigl(\Pi_n^{(\alpha,m)}\bigr)'(x) + (n-\alpha-mx)\Pi_n^{(\alpha,m)}(x).$$

*Explicitly,*

$$\Pi_n^{(\alpha,m)}(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (mj-\alpha)_n,$$

*where $(a)_0 := 1$ and $(a)_n := a(a+1)\cdots(a+n-1)$ for $n \geq 1$. With $\beta := -1-\alpha/m$, one has*

$$\Pi_n^{(\alpha,m)}(x) = P_n^{(\beta)}(x;m),$$

*where $P_n^{(\beta)}(x;m)$ is the Varma–Taşdelen Laguerre-type family defined by*

$$\sum_{n\geq 0} P_n^{(\beta)}(x;m)\frac{t^n}{n!} = (1-t)^{-(\beta+1)m} \exp\bigl(-x((1-t)^{-m}-1)\bigr),$$

*see [25, Eq. (3)].*

*For $m = 1$ the same generating relation reduces to the classical Laguerre generating function, i.e.,*

$$\Pi_n^{(\alpha,1)}(x) = n! L_n^{(-\alpha-1)}(x).$$

*In particular, $\deg \Pi_n^{(\alpha,m)} = n$ and $\mathrm{lc}\bigl(\Pi_n^{(\alpha,m)}\bigr) = (-m)^n$ for every $n$.*

*Proof.* Write $w := z-a$ and $x = -\lambda w^{-m}$. Since $x' = -mx/w$, differentiating

$$(-1)^n w^{\alpha-n} \Pi_n(x) e^{\lambda w^{-m}}$$

gives

$$\Pi_{n+1}^{(\alpha,m)}(x) = mx\bigl(\Pi_n^{(\alpha,m)}\bigr)'(x) + (n-\alpha-mx)\Pi_n^{(\alpha,m)}(x), \qquad \Pi_0^{(\alpha,m)} = 1.$$

Taylor's formula for $g_{\alpha,m}(z+\xi)$ with $t = -\xi/w$ gives

$$\sum_{n\geq 0} \Pi_n^{(\alpha,m)}(x)\frac{t^n}{n!} = (1-t)^\alpha \exp\bigl(x(1-(1-t)^{-m})\bigr).$$

Expanding that generating function gives

$$\Pi_n^{(\alpha,m)}(x) = n![t^n]\big((1-t)^\alpha \exp\!\big(x(1-(1-t)^{-m})\big)\big) = \sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (mj-\alpha)_n.$$

Comparing with the defining generating function of $P_n^{(\beta)}(x;m)$ and using $\beta = -1-\alpha/m$ gives $\Pi_n^{(\alpha,m)}(x) = P_n^{(\beta)}(x;m)$. For $m=1$ this reduces to the classical Laguerre generating function, hence $\Pi_n^{(\alpha,1)}(x) = n!L_n^{(-\alpha-1)}(x)$. The highest-degree term in the recurrence is $-mx\,\Pi_n^{(\alpha,m)}(x)$, so $\deg \Pi_n^{(\alpha,m)} = n$ and $\mathrm{lc}\big(\Pi_n^{(\alpha,m)}\big) = (-m)^n$. □

**Proposition 13** (Laguerre-type $m$-orthogonality for $\alpha < 0$)**.** *Assume $m \geq 1$ and $\alpha \in \mathbb{Z}$ with $\alpha < 0$, and let $\{\Pi_n^{(\alpha,m)}\}_{n\geq 0}$ be as in Proposition 12. For $0 \leq r \leq m-1$ set*

$$\lambda_r := \frac{r-\alpha}{m} > 0.$$

*For $0 \leq j \leq m-1$, define a linear functional on $\mathbb{C}[x]$ by*

$$\langle u_j, \varphi\rangle := \frac{1}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{1}{\Gamma(\lambda_r)}\int_0^\infty \varphi(x)x^{\lambda_r-1}e^{-x}\,\mathrm{d}x.$$

*Then*

$$\langle u_j, x^\nu \Pi_n^{(\alpha,m)}\rangle = 0 \quad \text{for } n \geq m\nu + j + 1,$$

*and*

$$\langle u_j, x^\nu \Pi_{m\nu+j}^{(\alpha,m)}\rangle \neq 0 \quad \text{for every } \nu \geq 0.$$

*Hence $\{\Pi_n^{(\alpha,m)}\}_{n\geq 0}$ is $m$-orthogonal with respect to $\mathbf{u} = (u_0,\ldots,u_{m-1})$. Equivalently, after the parameter change $\beta = -1-\alpha/m$, this is the Varma–Taşdelen Laguerre-type family from [25].*

*Proof.* Write $\Pi_n := \Pi_n^{(\alpha,m)}$. Fix $\nu \geq 0$. For $|t|$ sufficiently small one has $\Re((1-t)^{-m}) > 0$, so the integral below converges absolutely and termwise integration is justified. The generating function in Proposition 12 then gives

$$\begin{aligned}\sum_{n\geq 0}\langle u_j, x^\nu \Pi_n\rangle \frac{t^n}{n!} &= \frac{1}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{1}{\Gamma(\lambda_r)}\int_0^\infty y^\nu\left(\sum_{n\geq 0}\Pi_n(y)\frac{t^n}{n!}\right)y^{\lambda_r-1}e^{-y}\,\mathrm{d}y \\ &= \frac{1}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{1}{\Gamma(\lambda_r)}\int_0^\infty y^{\nu+\lambda_r-1}(1-t)^\alpha e^{-y(1-t)^{-m}}\,\mathrm{d}y \\ &= \frac{(1-t)^{m\nu}}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}(\lambda_r)_\nu(1-t)^r.\end{aligned}$$

The right-hand side is a polynomial in $t$ of degree $m\nu + j$. Its leading coefficient is

$$\frac{(-1)^{m\nu}}{j!}(\lambda_j)_\nu \neq 0,$$

because $\lambda_j > 0$. Therefore the coefficient of $t^n$ vanishes for $n > m\nu + j$ and is nonzero for $n = m\nu + j$, which is exactly the stated $m$-orthogonality. The final identification with the Varma–Taşdelen family is the parameter change already recorded in Proposition 12. □

**Proposition 14** (microscopic limit for the reduced local sequence)**.** *Let $m \geq 1$ and $\alpha \in \mathbb{Z}$, and let $\Pi_n^{(\alpha,m)}$ be as in Proposition 12. For $n \geq 1$, set*

$$R_n(\zeta) := \Pi_n^{(\alpha,m)}(n\zeta),$$

*and let $\nu_n$ be the normalized zero-counting measure of $R_n$. Then*

$$\nu_n \xrightarrow{w} \mu_m,$$

*where* $c_m := \left(\frac{m+1}{m}\right)^{m+1}$ *and* $\mu_m$ *is the probability measure supported on*

$$[0, c_m]$$

*whose Cauchy transform is*

$$C_m(\zeta) = 1 - v(\zeta)^{-m},$$

*with* $v$ *the branch, holomorphic on* $\mathbb{C} \setminus [0, c_m]$*, of*

$$v^{m+1} = m\zeta(v-1)$$

*satisfying* $v(\zeta) = 1 + \frac{1}{m\zeta} + O(\zeta^{-2})$ *as* $\zeta \to \infty$*. Equivalently, if*

$$\zeta(\phi) := \frac{1}{m} \frac{\sin^{m+1}((m+1)\phi)}{\sin\phi \, \sin^m(m\phi)}, \qquad 0 < \phi < \frac{\pi}{m+1},$$

*then*

$$\frac{d\mu_m}{d\zeta}(\zeta(\phi)) = \frac{\sin^{m+1}(m\phi)}{\pi \, \sin^m((m+1)\phi)}.$$

*Proof.* Cauchy's coefficient formula gives

$$\frac{\Pi_n^{(\alpha,m)}(n\zeta)}{n!} = \frac{1}{2\pi i} \int_\Gamma \frac{(1-t)^\alpha}{t^{n+1}} \exp\bigl(n\zeta\bigl(1-(1-t)^{-m}\bigr)\bigr) \, \mathrm{d}t, \tag{50}$$

where $\Gamma$ is a small positively oriented circle around 0. For $n \geq 1$, set

$$q_n(\zeta) := \Pi_n^{(\alpha,m)}(n\zeta), \qquad V_n(\zeta) := \frac{1}{n} \log|q_n(\zeta)| - \log(mn).$$

Since $\deg q_n = n$ and $\mathrm{lc}(q_n) = (-mn)^n$, one has

$$\nu_n = \frac{1}{2\pi} \Delta V_n.$$

Fix a compact set $K \subset \mathbb{C}$. Choosing in (50) a circle $|t| = r < 1$ gives

$$V_n(\zeta) \leq C_{K,r} \qquad (\zeta \in K),$$

so $\{V_n\}$ is locally uniformly bounded above.

Now fix $D \Subset \mathbb{C} \setminus [0, c_m]$. For the saddle-point analysis, keep the factor $t^{-n-1}$ in (50) on the global contour and introduce a branch of $\log t$ only in a simply connected neighborhood of the saddle used below. Set

$$\Phi_\zeta(t) := \zeta\bigl(1-(1-t)^{-m}\bigr) - \log t.$$

The critical points satisfy

$$m\zeta t + (1-t)^{m+1} = 0.$$

With $v = 1 - t$ this becomes

$$v^{m+1} = m\zeta(v-1).$$

Write

$$\zeta(v) := \frac{v^{m+1}}{m(v-1)}.$$

The branch $v(\zeta)$ is uniquely determined near $\infty$ by the condition $v(\zeta) \to 1$ as $\zeta \to \infty$. Since

$$\zeta'(v) = \frac{v^m(mv-(m+1))}{m(v-1)^2},$$

the only finite critical values are 0 and $c_m = \left(\frac{m+1}{m}\right)^{m+1}$. Because $\mathbb{C} \setminus [0, c_m]$ is simply connected and contains no finite critical values of $\zeta(v)$, analytic continuation of the inverse branch from $\infty$ gives a holomorphic branch on $\mathbb{C} \setminus [0, c_m]$. Let $t(\zeta) := 1 - v(\zeta)$. On every compact set $D \Subset \mathbb{C} \setminus [0, c_m]$, the point $t(\zeta)$ stays a positive distance from $\{0, 1\}$ and is a simple saddle, uniformly in $\zeta \in D$. The standard coefficient saddle-point deformation in $\mathbb{C} \setminus \{0, 1\}$ therefore moves the original small circle in (50) to a contour passing through $t(\zeta)$ with the same winding number about 0; the local contribution of this saddle gives the main term, and the remaining contour pieces are exponentially smaller uniformly on $D$. Consequently

$$\log|q_n(\zeta)| = \log n! + n\Re\Phi_\zeta(t(\zeta)) + O_D(\log n)$$

uniformly for $\zeta \in D$. Therefore

$$V_n(\zeta) \to V(\zeta) := \Re\Phi_\zeta(1 - v(\zeta)) - 1 - \log m$$

locally uniformly on $\mathbb{C} \setminus [0, c_m]$.

Let $\{V_{n_j}\}$ be an arbitrary subsequence. Since $V_n \to V$ locally uniformly on $\mathbb{C} \setminus [0, c_m]$, the sequence $\{V_{n_j}\}$ cannot converge to $-\infty$ uniformly on compact subsets of $\mathbb{C}$. The family $\{V_n\}$ is locally uniformly bounded above on $\mathbb{C}$, so the compactness theorem for subharmonic functions yields a further subsequence, still denoted by $\{V_{n_j}\}$, and a subharmonic function $\widetilde{V}$ such that $V_{n_j} \to \widetilde{V}$ in $L^1_{\mathrm{loc}}(\mathbb{C})$. On $\mathbb{C} \setminus [0, c_m]$ this subsequence also converges locally uniformly to $V$, hence $\widetilde{V} = V$ almost everywhere there, and therefore everywhere there by upper semicontinuous regularization. Thus $V$ has a unique subharmonic extension to $\mathbb{C}$, again denoted by $V$, and every subsequence of $\{V_n\}$ has a further subsequence converging to $V$ in $L^1_{\mathrm{loc}}(\mathbb{C})$. Consequently

$$V_n \longrightarrow V \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{C}),$$

and therefore

$$\nu_n \longrightarrow \frac{1}{2\pi}\Delta V$$

vaguely on $\mathbb{C}$. Since $V$ is harmonic on $\mathbb{C} \setminus [0, c_m]$, the limit measure is supported on $[0, c_m]$. On $\mathbb{C} \setminus [0, c_m]$,

$$2\partial V(\zeta) = \partial_\zeta \Phi_\zeta(1 - v(\zeta)) = 1 - v(\zeta)^{-m},$$

since $\partial_t \Phi_\zeta(1 - v(\zeta)) = 0$. Thus the limiting Cauchy transform is $C_m(\zeta) = 1 - v(\zeta)^{-m}$. As $\zeta \to \infty$, one has $v(\zeta) = 1 + \frac{1}{m\zeta} + O(\zeta^{-2})$, hence $C_m(\zeta) = \zeta^{-1} + O(\zeta^{-2})$; therefore $(2\pi)^{-1}\Delta V$ has total mass 1, and this measure is $\mu_m$. Since each $\nu_n$ is a probability measure, the preceding vague convergence is weak convergence.

On the cut, the upper boundary values of the distinguished branch are parameterized by

$$v_+(\phi) = \frac{\sin((m+1)\phi)}{\sin(m\phi)} e^{-i\phi}, \qquad 0 < \phi < \frac{\pi}{m+1},$$

for which

$$\zeta(\phi) = \frac{v_+(\phi)^{m+1}}{m(v_+(\phi) - 1)} = \frac{1}{m} \frac{\sin^{m+1}((m+1)\phi)}{\sin\phi \, \sin^m(m\phi)}.$$

Stieltjes inversion then gives

$$\frac{d\mu_m}{d\zeta}\big(\zeta(\phi)\big) = -\frac{1}{\pi}\Im C_{m,+}(\zeta(\phi)) = \frac{\sin^{m+1}(m\phi)}{\pi \, \sin^m((m+1)\phi)}.$$

As $\zeta \to 0$ on $\mathbb{C} \setminus [0, c_m]$, the relation $v^{m+1} = m\zeta(v-1)$ forces $v(\zeta) \to 0$, hence

$$v(\zeta)^{m+1} = -m\zeta\big(1 + o(1)\big) \qquad (\zeta \to 0).$$

Therefore

$$C_m(\zeta) = 1 - v(\zeta)^{-m} = O\big(|\zeta|^{-m/(m+1)}\big) \qquad (\zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0, c_m]).$$

A point mass at 0 would contribute a term $\mu_m(\{0\})/\zeta$ to the Cauchy transform, so $\mu_m(\{0\}) = 0$. ☐

*Remark* 15 (Marchenko–Pastur law at $m = 1$). At $m = 1$, the algebraic relation $v^2 = \zeta(v-1)$ has critical value $c_1 = 4$, and the measure $\mu_1$ supported on $[0, 4]$ has density

$$\frac{d\mu_1}{d\zeta} = \frac{1}{2\pi\zeta}\sqrt{\zeta(4-\zeta)},$$

which is the standard Marchenko–Pastur law. Since $\Pi_n^{(\alpha,1)}(x) = n!\, L_n^{(-\alpha-1)}(x)$, the $m = 1$ case of Proposition 14 recovers the classical Laguerre zero distribution.

**Corollary 16** (microscopic zero law for the higher-order local model)**.** *Assume $m \ge 2$ and keep the notation of Propositions 12 and 14. Let $N_n$ be the number of nonzero zeros of $\Pi_n^{(\alpha,m)}$. If $\alpha < 0$, then $N_n = n$ for every $n \ge 0$. If $\alpha \ge 0$ and $\alpha = qm + s$ with $q \in \mathbb{Z}_{\ge 0}$ and $0 \le s < m$, then*

$$\operatorname{ord}_0 \Pi_n^{(\alpha,m)} = q + 1, \qquad N_n = n - q - 1, \qquad n \ge \alpha + 1.$$

*In particular $N_n/n \to 1$, and $N_n > 0$ for all sufficiently large $n$. Fix $\eta$ with $\eta^m = -\lambda$, and for $0 \le \nu \le m-1$ define*

$$\phi_\nu : [0, c_m] \to \widehat{\mathbb{C}}, \qquad \phi_\nu(0) = \infty, \qquad \phi_\nu(x) = \varepsilon_\nu \eta\, x^{-1/m} \ (x > 0), \qquad \varepsilon_\nu := e^{2\pi i \nu/m},$$

*where $x^{1/m}$ denotes the positive real root for $x > 0$. Set*

$$\widehat{\mu}_{m,\lambda} := \frac{1}{m} \sum_{\nu=0}^{m-1} (\phi_\nu)_\# \mu_m.$$

*This measure is independent of the chosen root $\eta$. For all sufficiently large $n$, define*

$$\widehat{\nu}_n := \frac{1}{mN_n} \sum_{\substack{g_{\alpha,m}^{(n)}(z)=0 \\ z \ne a}} \delta_{n^{1/m}(z-a)},$$

*where the finite zeros are counted with multiplicity. Then*

$$\widehat{\nu}_n \xrightarrow{w} \widehat{\mu}_{m,\lambda}$$

*weakly on $\widehat{\mathbb{C}}$, hence on $\mathbb{C}$.*

*Proof.* If $\alpha < 0$, then the constant term in the explicit formula of Proposition 12 is $(-\alpha)_n \ne 0$. Hence $N_n = n$ for all $n$.

Assume $\alpha \ge 0$ and write $\alpha = mq + s$ with $q \ge 0$ and $0 \le s < m$. Fix $n \ge \alpha + 1$. In the explicit formula

$$\Pi_n^{(\alpha,m)}(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (mj - \alpha)_n,$$

the factor $(mj - \alpha)_n$ vanishes for every $j = 0, \dots, q$, because $\alpha - mj \in \{0, \dots, \alpha\} \subset \{0, \dots, n-1\}$. Therefore the coefficients of $x^k$ vanish for $0 \le k \le q$. For $k = q+1$, all terms with $j \le q$ vanish and the remaining term is

$$\frac{(-1)^{q+1}}{(q+1)!} (m - s)_n \ne 0.$$

Hence $\operatorname{ord}_0 \Pi_n^{(\alpha,m)} = q + 1$ and $N_n = n - q - 1$ for every $n \ge \alpha + 1$. In particular $N_n/n \to 1$. Changing $\eta$ to $\varepsilon_j \eta$ merely cyclically permutes the maps $\phi_\nu$, so $\widehat{\mu}_{m,\lambda}$ is independent of the chosen root.

For every continuous function $\varphi : \widehat{\mathbb{C}} \to \mathbb{C}$, define a continuous function $F_\varphi : \widehat{\mathbb{C}} \to \mathbb{C}$ by

$$F_\varphi(0) := \varphi(\infty), \qquad F_\varphi(\infty) := \varphi(0), \qquad F_\varphi(x) := \frac{1}{m} \sum_{w^m = -\lambda/x} \varphi(w) \quad (x \in \mathbb{C}^\times).$$

The sum over the $m$ roots of $w^m = -\lambda/x$ is symmetric, hence independent of their ordering; continuity at 0 and $\infty$ is immediate because those roots tend respectively to $\infty$ and 0. By the definition of $\widehat{\mu}_{m,\lambda}$,

$$\int F_\varphi \, \mathrm{d}\mu_m = \int \varphi \, \mathrm{d}\widehat{\mu}_{m,\lambda}.$$

Moreover,

$$\int F_\varphi \, \mathrm{d}\nu_n = \frac{N_n}{n} \int \varphi \, \mathrm{d}\widehat{\nu}_n + \frac{n - N_n}{n} \varphi(\infty).$$

Since $\nu_n \xrightarrow{w} \mu_m$, $N_n/n \to 1$, and $(n - N_n)/n \to 0$, it follows that $\widehat{\nu}_n \xrightarrow{w} \widehat{\mu}_{m,\lambda}$ on $\widehat{\mathbb{C}}$. Because $\mu_m(\{0\}) = 0$, the limit assigns zero mass to $\{\infty\}$, and the same convergence holds on $\mathbb{C}$. □

For $m = 1$, Proposition 12 gives

$$\Pi_n^{(\alpha,1)}(x) = n! L_n^{(-\alpha-1)}(x),$$

so the induced microscopic zero law in the original $z$-variable is as follows.

**Corollary 17** (Marchenko–Pastur scaling)**.** *Let $a \in \mathbb{C}$, $\alpha \in \mathbb{Z}$, and $\lambda \in \mathbb{C}^\times$, and set*

$$g_\alpha(z) := (z-a)^\alpha \exp\left(\frac{\lambda}{z-a}\right).$$

*Let $N_n$ be the number of finite zeros of $g_\alpha^{(n)}$, counted with multiplicity. Then $N_n/n \to 1$. For every sufficiently large $n$, let $x_{1,n}, \ldots, x_{N_n,n}$ be the nonzero zeros of $\Pi_n^{(\alpha,1)}(x) = n! L_n^{(-\alpha-1)}(x)$, counted with multiplicity. Then*

$$z_{k,n} = a - \frac{\lambda}{x_{k,n}}, \qquad k = 1, \ldots, N_n,$$

*are the finite zeros of $g_\alpha^{(n)}$, and*

$$\frac{1}{N_n} \sum_{k=1}^{N_n} \delta_{x_{k,n}/N_n} \xrightarrow{w} \mu_{\mathrm{MP}},$$

*where $\mu_{\mathrm{MP}}$ is the Marchenko–Pastur law on $[0,4]$. Consequently,*

$$\frac{1}{N_n} \sum_{k=1}^{N_n} \delta_{N_n(z_{k,n}-a)} \xrightarrow{w} (-\lambda/x)_{\#}\mu_{\mathrm{MP}}$$

*weakly on $\widehat{\mathbb{C}}$, hence on $\mathbb{C}$, and the same limit holds with $n$ in place of $N_n$.*

*Proof.* By Proposition 12 with $m = 1$,

$$g_\alpha^{(n)}(z) = (-1)^n (z-a)^{\alpha-n} \Pi_n^{(\alpha,1)}\left(-\frac{\lambda}{z-a}\right) \exp\left(\frac{\lambda}{z-a}\right),$$

so the finite zeros of $g_\alpha^{(n)}$ are obtained from the nonzero zeros of $\Pi_n^{(\alpha,1)}$ by $z = a - \lambda/x$. If $\alpha < 0$, then the constant term in the explicit formula for $\Pi_n^{(\alpha,1)}$ is $(-\alpha)_n \neq 0$, so $N_n = n$. If $\alpha \geq 0$ and $\varkappa := \alpha + 1$, then

$$\Pi_n^{(\alpha,1)}(x) = n! L_n^{(-\varkappa)}(x) = (-x)^\varkappa (n-\varkappa)!\, L_{n-\varkappa}^{(\varkappa)}(x) \qquad (n \geq \varkappa),$$

hence $\operatorname{ord}_0 \Pi_n^{(\alpha,1)} = \varkappa$ and $N_n = n - \varkappa$ for $n \geq \varkappa$. Thus $N_n/n \to 1$.

Proposition 14 with $m = 1$ and Remark 15 give the Marchenko–Pastur limit for the rescaled zero-counting measures of $\Pi_n^{(\alpha,1)}(n\zeta)$. When $\alpha \geq 0$, deleting the fixed zero at 0 and renormalizing by $N_n$ changes the integral against any bounded continuous test function by $O(n^{-1})$, so the same limit holds for the nonzero zeros. Since $N_n/n \to 1$, replacing $n$ by $N_n$ in either the $x$- or the $z$-scaling does not change the weak limit. The final statement then follows by applying the continuous mapping theorem on $\widehat{\mathbb{C}}$ to the extension of $x \mapsto -\lambda/x$ with $0 \mapsto \infty$, using $\mu_{\mathrm{MP}}(\{0\}) = 0$. □

For a nonvanishing meromorphic function $g$ of a local coordinate $x$, we call

$$d - d\log g$$

the *scalar connection trivialized by $g$*. Two such local connections are *holomorphically gauge equivalent at $x = \infty$* if their connection forms differ by $(\log h)'(x)\,dx$ for some holomorphic nonvanishing function $h$ near $x = \infty$.

**Proposition 18** (rigidity of the Laguerre reduction)**.** *Fix $c_j \in \mathcal{Z}(T)$, set $\alpha_j := p_j - \nu_j$, and let $E(z) = \sum_{s=1}^{m_j} \lambda_{j,s}(z-c_j)^{-s} + E_j^{\mathrm{reg}}(z)$ with $\lambda_{j,m_j} \neq 0$. In the coordinate $x = (z-c_j)^{-1}$, the scalar connection trivialized by $f$ becomes, after holomorphic gauge equivalence at $x = \infty$,*

$$d - \left(Q_j'(x) - \frac{\alpha_j}{x}\right) dx, \qquad Q_j(x) := \sum_{s=1}^{m_j} \lambda_{j,s} x^s.$$

*Since a holomorphic gauge at $x = \infty$ can only modify the connection coefficient by $O(x^{-2})$ terms, the polynomial degree $m_j$ is an invariant. In particular, after a linear rescaling of $x$, the local connection is holomorphically equivalent to*

$$d - \left(\lambda - \frac{\alpha_j}{x}\right) dx, \qquad \lambda \neq 0,$$

*if and only if $m_j = 1$.*

*Proof.* By Proposition 4 with $n = 0$, $f(z) = (z - c_j)^{\alpha_j} \phi_{j,0}(z) e^{E(z)}$ near $c_j$ with $\phi_{j,0}(c_j) \neq 0$. Setting $x = (z - c_j)^{-1}$ gives $f(z) = x^{-\alpha_j} e^{Q_j(x)} h_j(x)$ with $h_j(x) := \phi_{j,0}(c_j + x^{-1}) e^{E_j^{\mathrm{reg}}(c_j + x^{-1})}$ holomorphic and nonvanishing near $x = \infty$. The scalar connection trivialized by $f$ is $d - d \log f$; gauge by $h_j$ yields the stated form. Any further holomorphic gauge $g$ near $x = \infty$ with $g(\infty) \neq 0$ satisfies $(\log g)'(x) = O(x^{-2})$, so the polynomial part $Q_j'(x)$ is invariant, and hence so is $\deg Q_j = m_j$. □

*Remark* 19 (Riemann–Hilbert characterization of the local models). When $m_j = 1$, the local model at a simple pole of $S/T$ reduces to generalized Laguerre polynomials (Proposition 12), which admit the standard Fokas–Its–Kitaev $2 \times 2$ Riemann–Hilbert characterization [8, 5]. Proposition 18 shows that this reduction is rigid: for $m_j \geq 2$, the irregular type has degree $m_j$ and cannot be gauge-reduced to the Laguerre case. The $m_j$-orthogonality in Proposition 13 therefore points toward an $(m_j + 1) \times (m_j + 1)$ Riemann–Hilbert problem of multiple-orthogonal-polynomial type [24].

## 6 Global Voronoi law

In this section we describe the global zero asymptotics for the sequence of derivatives of a hyperexponential function $f$. The edge part of the limit measure comes from the competition of nearest singularities on open Voronoi cells, while the atomic part at $\mathcal{Z}(T)$ is the fixed-scale shadow of the microscopic clusters discussed in Section 5; Corollary 22 makes this comparison explicit. Recall from (6) and (13) that $h$ is the degree of the polynomial part $H$ of $E = S/T$ when $H$ is nonconstant, and $h = 0$ otherwise, while $\kappa = d + h - 1$. To isolate only the zero distribution, write $\gamma_n := \mathrm{lc}(B_n)$ and set

$$s_n := \begin{cases} 0, & \text{if } h = 0, \\ \log n!, & \text{if } h > 0. \end{cases} \tag{51}$$

For every $n$ with $\deg B_n > 0$, define

$$\widetilde{L}_n(z) := \frac{1}{\deg B_n} \Big( \log |B_n(z)| - \log |\gamma_n| - s_n \Big). \tag{52}$$

Recall that $(2\pi)^{-1} \Delta \widetilde{L}_n$ is the normalized zero-counting measure of $B_n$.

Define

$$\Psi(z) := \frac{1}{\kappa} \big( \log |W(z)| - \log \rho(z) - \sigma \big), \qquad z \in \mathbb{C} \setminus \Sigma. \tag{53}$$

Equivalently,

$$\Psi(z) = \max_{1 \leq i \leq N} \Psi_i(z), \qquad \Psi_i(z) := \frac{1}{\kappa} \big( \log |W(z)| - \log |z - a_i| - \sigma \big).$$

Since $\mathrm{ord}_{a_i} W \geq 1$, each $\Psi_i$ extends subharmonically across $a_i$, hence so does $\Psi$.

**Lemma 20** (local upper bounds)**.** *The family $\{\widetilde{L}_n : \deg B_n > 0\}$ is locally uniformly bounded from above on $\mathbb{C}$.*

*Proof.* Fix a compact set $K \subset \mathbb{C}$ and put $F := \Sigma \cup \mathcal{Z}(P_\sharp)$. Choose closed discs $D_1, \dots, D_L$ covering $K$ with $\partial D_\nu \cap F = \varnothing$ for every $\nu$, and set $\Gamma := \bigcup_\nu \partial D_\nu$. Since $\Gamma \cap \Sigma = \varnothing$, fix $r := \frac{1}{2} \operatorname{dist}(\Gamma, \Sigma) > 0$. Since $\Gamma \cap F = \varnothing$, for $z \in \Gamma$ and $|\xi| = r$ Proposition 6 applies and the function

$$\frac{P(z + \xi) Q(z)}{P(z) Q(z + \xi)} \exp\big(E(z + \xi) - E(z)\big)$$

is continuous on the compact set $\Gamma \times \{|\xi| = r\}$. Hence there is $\mathcal{M} > 0$ such that

$$\left| \frac{P(z + \xi) Q(z)}{P(z) Q(z + \xi)} \exp\big(E(z + \xi) - E(z)\big) \right| \leq \mathcal{M} \qquad (z \in \Gamma, \ |\xi| = r).$$

By Cauchy's estimate applied to (21),

$$\left|\frac{A_n(z)}{n!}\right| \le \mathcal{M}\, r^{-n} \qquad (z \in \Gamma,\ n \ge 0).$$

Using $B_n = P_\sharp W^n A_n$, one gets

$$\widetilde{L}_n(z) \le \frac{n}{\deg B_n}\big(\log \sup_\Gamma |W| - \log r\big) + \frac{\sup_\Gamma \log |P_\sharp| + \log \mathcal{M}}{\deg B_n} + \frac{\log n! - \log|\gamma_n| - s_n}{\deg B_n}.$$

By Proposition 5, the right-hand side is bounded above uniformly in $n$. Hence there is a constant $C_\Gamma$ with $\widetilde{L}_n(z) \le C_\Gamma$ for all $z \in \Gamma$ and all $n$ for which $\widetilde{L}_n$ is defined. Since each $\widetilde{L}_n$ is subharmonic, the maximum principle on every $D_\nu$ gives $\sup_K \widetilde{L}_n \le C_\Gamma$. □

**Theorem 21** (fixed-scale Voronoi law). *1. **(Convergence of potentials.)** For all sufficiently large $n$ one has $\deg B_n > 0$, and the resulting sequence $\{\widetilde{L}_n\}$ satisfies*

$$\widetilde{L}_n \longrightarrow \Psi \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{C}).$$

*2. **(Limit measure on $\mathbb{C}$.)** For those $n$, the normalized zero-counting measures $\mu_n := (2\pi)^{-1}\Delta \widetilde{L}_n$ converge vaguely on $\mathbb{C}$ to*

$$\mu_{\mathrm{fix}} := \frac{1}{2\pi}\Delta\Psi = \frac{1}{\kappa}\sum_{j=1}^{\check{t}} m_j\, \delta_{c_j} + \frac{1}{2\pi\kappa}\sum_{1\le i<j\le N} \frac{|a_i - a_j|}{|\cdot - a_i|\,|\cdot - a_j|}\, \mathrm{d}\ell\big|_{E_{ij}}, \tag{54}$$

*where the second term denotes the measure on $E_{ij}$ with density $\zeta \mapsto |a_i - a_j|/(|\zeta - a_i|\,|\zeta - a_j|)$ with respect to arclength.*

*3. **(Mass.)** The total finite-plane mass is*

$$\mu_{\mathrm{fix}}(\mathbb{C}) = \frac{d-1}{\kappa} = 1 - \frac{h}{\kappa}. \tag{55}$$

*In particular, if $h = 0$ then $\mu_n$ converges weakly to $\mu_{\mathrm{fix}}$ on $\mathbb{C}$.*

*4. **(Convergence on $\widehat{\mathbb{C}}$.)** Identify each $\mu_n$ from part (2) with the probability measure on $\widehat{\mathbb{C}}$ that assigns mass 0 to $\{\infty\}$. Then*

$$\mu_n \longrightarrow \mu_{\mathrm{fix}} + \frac{h}{\kappa}\,\delta_\infty$$

*weakly on $\widehat{\mathbb{C}}$, where $\delta_\infty$ denotes the unit point mass at $\infty$.*

*5. **(Cauchy transform on Voronoi interiors.)** Write $W(z) = \prod_{j=1}^N (z - a_j)^{\varpi_j}$, where $\varpi_j := \mathrm{ord}_{a_j} W$. On each Voronoi interior $\mathcal{V}_i^\circ \setminus \Sigma$,*

$$C_{\mu_{\mathrm{fix}}}(z) = 2\partial\Psi(z) = \frac{1}{\kappa}\left(\frac{W'(z)}{W(z)} - \frac{1}{z - a_i}\right) = \frac{1}{\kappa}\sum_{j=1}^N \frac{\varpi_j - \delta_{ij}}{z - a_j}, \tag{56}$$

*where $C_{\mu_{\mathrm{fix}}}$ is the Cauchy transform defined in § 2.3, $\delta_{ij}$ is the Kronecker delta, and $2\partial = \partial_x - i\partial_y$.*

*Proof of Theorem 21, parts* (1)–(4). Only two local inputs enter the argument. On pole cells Theorem 9(1) gives uniform asymptotics, and on essential-singularity cells Corollary 11 gives the required local $L^1$ control.

Choose $n_0$ so that $\deg B_n > 0$ for all $n \ge n_0$, and set

$$\Omega := \bigcup_{i=1}^N (\mathcal{V}_i^\circ \setminus \Sigma).$$

Let $K \Subset \Omega$. Then $K$ is covered by finitely many compact sets $K_1, \ldots, K_M$, each contained in a single Voronoi interior. If $K_\alpha \Subset \mathcal{V}_i^\circ \setminus \Sigma$ and $a_i \in \mathcal{Z}(Q) \setminus \mathcal{Z}(T)$, Theorem 9(1) gives

$$\widetilde{L}_n = \Psi_i + O_{K_\alpha}\left(\frac{\log n}{n}\right)$$

uniformly on $K_\alpha$. If $K_\alpha \Subset \mathcal{V}_i^\circ \setminus \Sigma$ and $m_i \geq 1$, Corollary 11 gives

$$\|\widetilde{L}_n - \Psi_i\|_{L^1(K_\alpha)} = O_{K_\alpha}\left(n^{-1/(m_i+1)}\right).$$

Summing over $\alpha$ yields

$$\widetilde{L}_n \to \Psi \qquad \text{in } L^1_{\mathrm{loc}}(\Omega).$$

More precisely, on compact subsets of pole cells the convergence is uniform with rate $O(\log n/n)$, while on compact subsets of an essential-singularity cell attached to $a_i$ it holds in $L^1$ with rate $O\big(n^{-1/(m_i+1)}\big)$.

Since $\mathbb{C} \setminus \Omega$ is the union of the Voronoi diagram and the finite set $\Sigma$, it has planar Lebesgue measure zero. Let $\{\widetilde{L}_{n_j}\}$ be any subsequence with $n_j \geq n_0$. By Lemma 20 and the compactness theorem for subharmonic functions [20], either $\widetilde{L}_{n_j} \to -\infty$ uniformly on compact subsets or a further subsequence converges in $L^1_{\mathrm{loc}}(\mathbb{C})$ to a subharmonic function $u$. The first alternative is impossible because $\Omega \neq \varnothing$ and the whole sequence converges in $L^1_{\mathrm{loc}}(\Omega)$ to the finite function $\Psi$. The same subsequence converges to $u$ in $L^1_{\mathrm{loc}}(\Omega)$, while the whole sequence converges there to $\Psi$; hence $u = \Psi$ almost everywhere on $\Omega$, and therefore almost everywhere on $\mathbb{C}$. Both $u$ and $\Psi$ are subharmonic, hence the upper-semicontinuous regularizations of the same $L^1_{\mathrm{loc}}(\mathbb{C})$-class. Therefore $u = \Psi$ everywhere. Thus every subsequence has a further subsequence converging to $\Psi$ in $L^1_{\mathrm{loc}}(\mathbb{C})$, and the whole sequence converges.

Since $\widetilde{L}_n \to \Psi$ in $L^1_{\mathrm{loc}}(\mathbb{C})$, one has $\mu_n = (2\pi)^{-1}\Delta\widetilde{L}_n \to (2\pi)^{-1}\Delta\Psi = \mu_{\mathrm{fix}}$ distributionally; because these are positive Radon measures, this is vague convergence on $\mathbb{C}$. Since

$$W(z) = z^d + O(z^{d-1}), \qquad \rho(z) = |z| + O(1) \qquad (|z| \to \infty),$$

one has

$$\Psi(z) = \frac{d-1}{\kappa}\log|z| + O(1).$$

By the standard mass formula for logarithmic potentials,

$$\mu_{\mathrm{fix}}(\mathbb{C}) = \frac{d-1}{\kappa},$$

which is (55). If $h = 0$, then $\mu_{\mathrm{fix}}(\mathbb{C}) = 1$. Since the measures $\mu_n$ are probabilities for $n \geq n_0$, their vague convergence to a probability measure is weak convergence on $\mathbb{C}$.

For the spherical statement, let $\nu$ be a subsequential weak limit of $\mu_n$ on the compact space $\widehat{\mathbb{C}}$, say $\mu_{n_j} \to \nu$ weakly on $\widehat{\mathbb{C}}$. For every continuous compactly supported function $\varphi : \mathbb{C} \to \mathbb{R}$, extend $\varphi$ to $\widehat{\mathbb{C}}$ by setting $\varphi(\infty) = 0$. Then weak convergence of $\mu_{n_j}$ on $\widehat{\mathbb{C}}$ and vague convergence of $\mu_n$ on $\mathbb{C}$ give

$$\int_{\widehat{\mathbb{C}}} \varphi \, \mathrm{d}\nu = \lim_{j\to\infty} \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_{n_j} = \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_{\mathrm{fix}}.$$

Hence $\nu|_{\mathbb{C}} = \mu_{\mathrm{fix}}$. Since $\nu(\widehat{\mathbb{C}}) = 1$ and $\mu_{\mathrm{fix}}(\mathbb{C}) = (d-1)/\kappa$, one gets $\nu(\{\infty\}) = 1 - \mu_{\mathrm{fix}}(\mathbb{C}) = h/\kappa$. Therefore $\nu = \mu_{\mathrm{fix}} + (h/\kappa)\delta_\infty$. Since every subsequential limit is the same, the whole sequence converges. □

*Proof of Theorem 21, explicit form of $\mu_{\mathrm{fix}}$, and the Cauchy-transform formula.* Write $W(z) = \prod_{j=1}^N (z - a_j)^{\varpi_j}$. On $\mathcal{V}_i^\circ \setminus \Sigma$ one has

$$\Psi(z) = \frac{1}{\kappa}\left((\varpi_i - 1)\log|z - a_i| + \sum_{j\neq i} \varpi_j \log|z - a_j| - \sigma\right).$$

Near $a_i$, the branch $\Psi_i$ dominates because $\log|z - a_i| \to -\infty$ and the coefficient of $\log|z - a_i|$ in $\Psi_i$ is $\varpi_i - 1 < \varpi_i$, whereas in $\Psi_j$ it is $\varpi_i$ for $j \neq i$. Hence $\Psi = \Psi_i$ in a punctured neighborhood of $a_i$, and the atomic coefficient there is $(\varpi_i - 1)/\kappa$, namely $m_j/\kappa$ when $a_i = c_j \in \mathcal{Z}(T)$ and 0 when $a_i \in \mathcal{Z}(Q) \setminus \mathcal{Z}(T)$.

Away from $\Sigma$ and the Voronoi edges the function is harmonic. On the relative interior of each nonempty edge $E_{ij}$, $\Psi = \max(\Psi_i, \Psi_j)$ and

$$\Psi_i - \Psi_j = \frac{1}{\kappa}\big(\log|z-a_j| - \log|z-a_i|\big).$$

Let $n_{ij}$ be a unit normal to $E_{ij}$, chosen so that $\Psi_i > \Psi_j$ on one side of the edge. Near a point of the relative interior of $E_{ij}$ that meets no other Voronoi edge, the functions $\Psi_i$ and $\Psi_j$ are harmonic, all other branches are strictly smaller, and $\Psi = \max(\Psi_i, \Psi_j)$. The jump formula for the normal derivative therefore gives

$$\Delta\Psi = \big(\partial_{n_{ij}}\Psi_i - \partial_{n_{ij}}\Psi_j\big)\, \mathrm{d}\ell\big|_{E_{ij}} = \big|\partial_{n_{ij}}(\Psi_i - \Psi_j)\big|\, \mathrm{d}\ell\big|_{E_{ij}}.$$

On $E_{ij}$ one has $|z-a_i| = |z-a_j| =: r$, and

$$|\nabla(\Psi_i - \Psi_j)(z)| = \frac{1}{\kappa}\left|\frac{z-a_j}{|z-a_j|^2} - \frac{z-a_i}{|z-a_i|^2}\right| = \frac{|a_i-a_j|}{\kappa r^2} = \frac{|a_i-a_j|}{\kappa|(z-a_i)(z-a_j)|}.$$

Since $\Psi_i - \Psi_j$ is constant on $E_{ij}$, this gradient is normal to $E_{ij}$. Hence the density of $\Delta\Psi$ along $E_{ij}$ is

$$\frac{|a_i-a_j|}{\kappa|(z-a_i)(z-a_j)|}\, \mathrm{d}\ell\big|_{E_{ij}}.$$

Dividing by $2\pi$ gives the edge term in (54). Voronoi vertices contribute no mass, since $\Psi$ is locally bounded there and an atomic term in $\Delta\Psi$ would force a logarithmic singularity. The integral defining $C_{\mu_{\mathrm{fix}}}(z)$ converges absolutely for $z \notin \operatorname{supp}\mu_{\mathrm{fix}}$: along each unbounded Voronoi edge the density is $O(|\zeta|^{-2})$, while $|z-\zeta|^{-1} = O(|\zeta|^{-1})$. Finally, $\Psi(z) = \mu_{\mathrm{fix}}(\mathbb{C})\log|z| + O(1)$ as $|z| \to \infty$, so the Riesz decomposition of $\Psi$ has no nonconstant harmonic part. Hence $C_{\mu_{\mathrm{fix}}}(z) = 2\partial\Psi(z)$ on $\mathbb{C} \setminus \operatorname{supp}\mu_{\mathrm{fix}}$. On $\mathcal{V}_i^\circ \setminus \Sigma$ one has $\Psi = \Psi_i$, which gives (56). ☐

**Corollary 22** (atomic mass and the reduced local model)**.** *Fix $c_j \in \mathcal{Z}(T)$ and choose $r > 0$ so small that $c_j$ is the unique nearest site throughout $\overline{D(c_j,r)} \setminus \{c_j\}$. Then $\mu_{\mathrm{fix}}(\partial D(c_j,r)) = 0$ and*

$$\sum_{\substack{\zeta:\, B_n(\zeta)=0\\ |\zeta-c_j|<r}} \mathrm{mult}_\zeta(B_n) = m_j n + o(n),$$

*where $\mathrm{mult}_\zeta(B_n)$ denotes the multiplicity of $\zeta$ as a zero of $B_n$. Equivalently, the same asymptotic count, with multiplicity, holds for the zeros of $f^{(n)}$ in $D(c_j,r) \setminus \{c_j\}$. Thus the atom $(m_j/\kappa)\delta_{c_j}$ in Theorem 21 represents a cluster of $m_j n + o(n)$ zeros collapsing to $c_j$. For the reduced local model*

$$g_{\alpha_j,m_j}(z) := (z-c_j)^{\alpha_j}\exp\left(\frac{\lambda_{j,m_j}}{(z-c_j)^{m_j}}\right), \qquad \alpha_j := p_j - \nu_j,$$

*Corollary 17 when $m_j = 1$ and Corollary 16 when $m_j \ge 2$ resolve this same cluster on the scale $n^{-1/m_j}$: $g^{(n)}_{\alpha_j,m_j}$ has $m_j n + O(1)$ finite zeros, and their rescaled counting measures converge to $(-\lambda_{j,1}/x)_{\#}\mu_{\mathrm{MP}}$ when $m_j = 1$ and to $\widehat{\mu}_{m_j,\lambda_{j,m_j}}$ when $m_j \ge 2$.*

*Proof.* Choose $r$ so small that $c_j$ is the unique nearest site throughout $\overline{D(c_j,r)} \setminus \{c_j\}$. Then $\overline{D(c_j,r)}$ meets no Voronoi edge and contains no site other than $c_j$, so

$$\mu_{\mathrm{fix}}\big(D(c_j,r)\big) = \frac{m_j}{\kappa}$$

and $\mu_{\mathrm{fix}}(\partial D(c_j,r)) = 0$. By Theorem 21,

$$\mu_n\big(D(c_j,r)\big) \longrightarrow \frac{m_j}{\kappa}.$$

Since $\mu_n$ is the normalized zero-counting measure of $B_n$ and Proposition 5 gives $\deg B_n = \kappa n + O(1)$, it follows that

$$\sum_{\substack{\zeta:\, B_n(\zeta)=0\\ |\zeta-c_j|<r}} \mathrm{mult}_\zeta(B_n) = \deg B_n\, \mu_n\big(D(c_j,r)\big) = m_j n + o(n).$$

By Proposition 4, one has $B_n(c_j) \neq 0$, and away from $\Sigma$ the zeros of $f^{(n)}$ are exactly the zeros of $B_n$, which gives the equivalent formulation for $f^{(n)}$.

For the reduced model, Corollary 17 gives $N_n = n + O(1)$ when $m_j = 1$, while Corollary 16 gives $N_n = n + O(1)$ when $m_j \ge 2$. In either case the number of finite zeros of $g^{(n)}_{\alpha_j,m_j}$ is $m_j N_n = m_j n + O(1)$, and the stated microscopic convergence follows from those corollaries. ☐

## 7 Open problems

The fixed-scale problem is settled in this paper, but several finer questions remain open.

**Escaping mass at infinity.** When the polynomial part $H$ of $S/T$ is nonconstant, Theorem 21 identifies only the mass $h/\kappa$ that escapes to $\infty$. A sharper result should describe the distribution of these escaping zeros after the natural rescaling determined by $H$ and should separate this outer scale from the finite-plane Voronoi geometry.

**Transition asymptotics.** Theorem 9 gives the full multi-saddle expansion, and Corollary 11 shows that this suffices for the fixed-scale law. What is still missing is a transition theory on the natural microscopic scale across dominant Stokes arcs and near Voronoi vertices, where several nearest singularities compete. Even in the simple-pole case one expects special transition kernels; for higher-order poles, higher-order analogues should appear.

**Microscopic laws beyond the reduced local model.** Section 5 studies the reduced model obtained by retaining only the algebraic factor and the highest-order singular term at an essential singularity. The next step is to determine the microscopic zero laws for the full local factor

$$(z-a)^{\alpha} \exp\left(\sum_{s=1}^{m} \lambda_s (z-a)^{-s}\right) \phi(z), \qquad \phi(a) \neq 0,$$

to seek the corresponding higher-rank Riemann–Hilbert formulation, and to decide which parts of the Voronoi law persist for more general meromorphic functions with finitely many poles and finitely many essential singularities.

## A Characterization of hyperexponential functions

**Proposition 23.** *Let $S \subset \mathbb{P}^1(\mathbb{C})$ be finite and set*

$$\Omega = \mathbb{P}^1(\mathbb{C}) \setminus S.$$

*Let $f$ be a nonzero meromorphic function on $\Omega$. Then the following are equivalent:*

1. *there exist $c \in \mathbb{C}^{\times}$ and rational functions $R, H \in \mathbb{C}(z)$ such that*

$$f(z) = c\, R(z) e^{H(z)} \quad \text{on } \Omega;$$

2. *the logarithmic derivative $f'/f$ belongs to $\mathbb{C}(z)$.*

*Proof.* The implication (1) ⇒ (2) is immediate, since

$$\frac{f'}{f} = \frac{R'}{R} + H' \in \mathbb{C}(z).$$

Now assume that

$$r(z) := \frac{f'(z)}{f(z)} \in \mathbb{C}(z).$$

Let $a \in \mathbb{C}$ be a finite pole of $r$. Choose $\varepsilon > 0$ so small that the positively oriented circle

$$\gamma_a(t) = a + \varepsilon e^{it}, \qquad 0 \le t \le 2\pi,$$

lies in $\Omega$ and contains no zero or pole of $f$ on the path. Since $f$ is single-valued and nonvanishing on $\gamma_a$, the map $f \circ \gamma_a$ is a loop in $\mathbb{C}^{\times}$, and therefore

$$\operatorname{Res}_{z=a}(r) = \frac{1}{2\pi i} \int_{\gamma_a} \frac{f'(z)}{f(z)}\, \mathrm{d}z$$

is the winding number of $f \circ \gamma_a$ about 0. Hence

$$n_a := \operatorname{Res}_{z=a}(r) \in \mathbb{Z}.$$

Let $A \subset \mathbb{C}$ be the finite set of finite poles of $r$, and define

$$R(z) := \prod_{a \in A} (z-a)^{n_a} \in \mathbb{C}(z)^{\times}.$$

Then

$$\frac{R'}{R} = \sum_{a \in A} \frac{n_a}{z-a},$$

so the rational function

$$s(z) := r(z) - \frac{R'(z)}{R(z)}$$

has no simple poles at any finite point. Therefore the partial fraction decomposition of $s$ contains only terms of order at least 2 at finite poles, together with a polynomial part. Termwise integration then gives a rational function $H \in \mathbb{C}(z)$ such that

$$H'(z) = s(z) = r(z) - \frac{R'(z)}{R(z)}.$$

Now put

$$g(z) := \frac{f(z)}{R(z)e^{H(z)}}.$$

This is a nonzero meromorphic function on $\Omega$, and

$$\frac{g'(z)}{g(z)} = \frac{f'(z)}{f(z)} - \frac{R'(z)}{R(z)} - H'(z) = 0.$$

Hence $g' \equiv 0$, so $g$ is constant on the connected set $\Omega$. Thus there exists $c \in \mathbb{C}^{\times}$ such that

$$f(z) = c\, R(z)e^{H(z)}.$$

This proves (2) $\Rightarrow$ (1). $\square$

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