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**Analysis and Mathematical Physics**

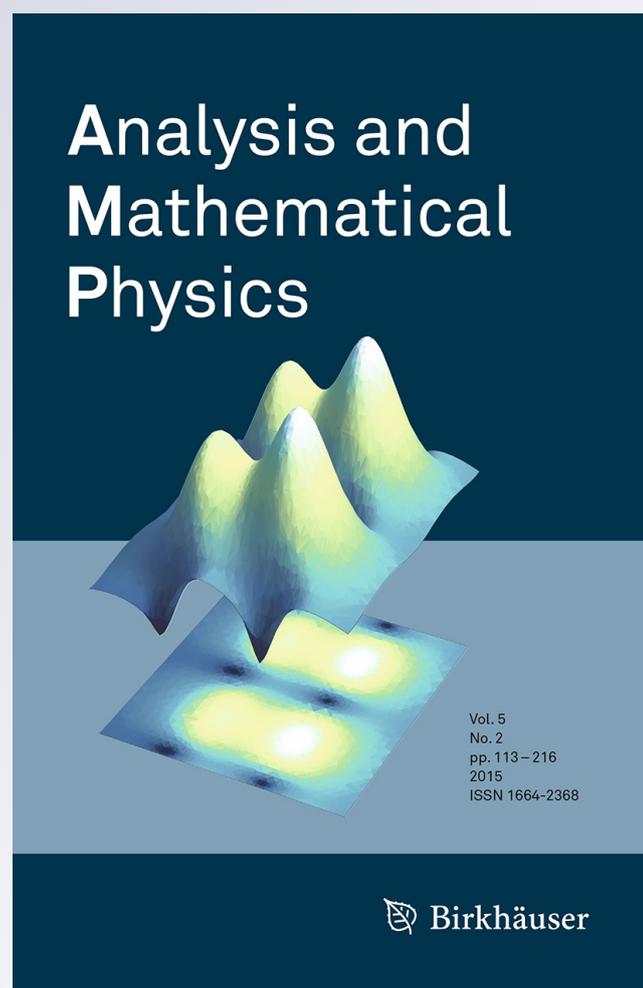
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# On Evgrafov–Fedoryuk’s theory and quadratic differentials

Boris Shapiro

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**Abstract** The purpose of this note is to recall the theory of the (homogenized) spectral problem for Schrödinger equation with a polynomial potential and its relation with quadratic differentials. We derive from results of this theory that the accumulation rays of the eigenvalues of the latter problem are in 1 – 1-correspondence with the short geodesics of the singular planar metrics induced by the corresponding quadratic differential. We prove that for a polynomial potential of degree  $d$ , the number of such accumulation rays can be any positive integer between  $(d - 1)$  and  $\binom{d}{2}$ .

**Keywords** Spectral asymptotics · Quadratic differentials · Singular planar metric · Geodesics

**Mathematics Subject Classification** Primary 34M40 · 34E20; Secondary 34E10

## 1 Introduction

This paper is motivated by a recent progress in the study of the spectral discriminant, [1,5,6]. We start by explaining some important results scattered through [7,9] §6 in ch. III of [8] and [20].

Consider a differential equation of the form

$$-y'' + P(z)y = 0 \quad (1.1)$$

where  $P(z) = a_0z^d + a_1z^{d-1} + \dots + a_d$ ,  $a_0 \neq 0$  is an arbitrary complex polynomial potential of degree  $d$ . Setting  $\phi_0 = \arg a_0$ , we define (open) *Stokes sectors*  $S_j$ ,  $j = 0, \dots, d + 1$  of (1.1) as given by the condition:

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$$S_j = \left\{ z : \left| \arg z - \frac{\phi_0}{d} - \frac{2\pi j}{d+2} \right| < \frac{\pi}{d+2} \right\}.$$

Observe that  $S_j, j = 0, \dots, d+1$  are cyclicly ordered on  $\mathbb{C}$ , i.e.  $S_{d+1}$  is neighboring to  $S_{d+2}$  and  $S_0$ . Notice that for an arbitrary  $P(z)$ , the definition of its Stokes sectors depends only on  $\phi_0$  since the multiplication of  $P(z)$  by a positive number preserves the set of the Stokes sectors.

It is well-known, see [13, 19], that for each open sector  $S_j$  there exists a unique up to a scalar ray non-trivial solution  $Y_j$  of (1.1) which is exponentially decreasing along any ray within this sector. Such  $Y_j$  is called *subdominant* in  $S_j$ . It is convenient to think of  $Y_j$ 's as points on  $\mathbb{CP}^1$  where the latter space is the projectivization of the linear space of solutions of (1.1). For a generic  $P(z)$ , all its  $Y_j, j = 0, \dots, d+1$  are distinct. Moreover, all conditions defining the set of all possible configurations of subdominant solutions considered as degree  $d+2$  divisors on  $\mathbb{CP}^1$  were described by Nevanlinna [16]. These restrictions are as follows:

- (i) this divisor has at least three geometrically distinct points;
- (ii) the multiplicity of each point of this divisor does not exceed  $\lfloor \frac{d+2}{2} \rfloor$ .

The above restrictions have a strong and so far unexplained resemblance with the notion of stability in the geometric invariant theory, see e.g. [15].

An important object of study is the following complex analytic hypersurface (and its restrictions to different linear subspaces).

**Definition 1** The (**extended**) **spectral discriminant**  $Spc_d \subset \mathcal{C}_d$  is the set of all potentials  $P(z)$  of degree  $d$  such that (1.1) possesses a solution subdominant in (at least) two different Stokes sectors. Here  $\mathcal{C}_d \simeq \mathbb{C}^* \times \mathbb{C}^d$  stands for the set of all coefficients  $(a_0, a_1, \dots, a_d), a_0 \neq 0$ .

For any fixed  $\tilde{a}_0$ , the restriction of  $Spc_d$  to the subspace  $(\tilde{a}_0, \mathbb{C}^d)$  where  $\mathbb{C}^d$  is spanned by coordinates  $(a_1, \dots, a_d)$  is given by an entire function of the latter coordinates. A description of the behavior of the spectrum when  $\tilde{a}_0 \rightarrow 0$  can be found in [6]. The group of affine coordinate transformations acts on the space of potentials preserving  $Spc_d$ . The space  $\mathcal{C}_d$  also carries the obvious free action of  $\mathbb{C}^*$  by multiplication of an arbitrary polynomial of degree  $d$  by a non-vanishing complex number. This action however does not preserve  $Spc_d$ .

The main goal of this paper is to present some results about the restriction of  $Spc_d$  onto the orbits of the latter action. In other words, we consider the following (homogenized) spectral problem.

**Problem 1** For a given polynomial  $P(z)$  of degree  $d$ , find the set of all non-vanishing  $\lambda \in \mathbb{C}^*$  for which the equation

$$-y'' + \lambda^2 P(z)y = 0, \tag{1.2}$$

has a solution subdominant in at least two distinct Stokes sectors.

This problem was consider in the writings of M. Fedoryuk (some of them joint) with M. Evgrafov as well as by Y. Sibuya, F. W. J. Olver and even in the PhD thesis of

G. D. Birkhoff from 1914 [3]. The spectrum of this problem is always discrete. The main results about the latter problem are:

- (\*) the description of the condition determining the accumulation rays of the spectrum, see Theorem 6.2, [9];
- (\*\*) the description of the asymptotic density of the eigenvalues along such an accumulation ray, see Theorem 6.3, [9].

To present these results in details, we need to recall the notion of the *Stokes graph* and of the *Stokes complexes* of equation (1.1), see [7,22] (Below we follow the terminology of M. Fedoryuk which apparently is not quite standard).

*Notation.* Each root of  $P(z)$  is called a *turning point* of (1.1). A *Stokes line* of (1.1) is a containing at most two turning points maximal under inclusion smooth segment of the real analytic curve solving the equation:

$$Re \xi_{z_0}(z) = 0 \quad \text{where} \quad \xi_{z_0}(z) = \int_{z_0}^z \sqrt{P(u)} du, \tag{1.3}$$

with respect to the variable  $z$ ,  $z_0$  being a turning point of (1.1) (A Stokes line can be either bounded or unbounded in  $\mathbb{C}$ ).

The *Stokes graph*  $ST_P$  of Eq. (1.1) is the union of all its Stokes lines. Connected components of  $ST_P$  will be referred to as *Stokes complexes* and connected components of  $\mathbb{C} \setminus ST_P$  will be called *admissible domains*.

One can distinguish between two natural types of admissible domains. Namely, we say that an admissible domain is of the *half-plane type* if the function  $\xi(z)$  maps it either to  $Re \xi(z) > a$  or to  $Re \xi(z) < a$  for some real  $a$ . Analogously, we say that an admissible domain is of the *strip type* if the function  $\xi(z)$  maps it to  $a < Re \xi(z) < b$  for some real  $a < b$ . For polynomial potentials  $P(z)$ , all admissible domains belong to one of these two types (which is no longer true for entire or rational potentials).

A Stokes complex is called *simple* if it contains exactly one turning point and *non-simple* otherwise. Note that the existence of a non-simple Stokes complex in the Stokes graph of  $P(z)$  is equivalent to the existence of a Stokes line connecting two turning points. Such Stokes lines will be called *short*.

In Fig. 1, the Stokes graph of  $z^2 + 1 + I$  (left picture) consists of two simple Stokes complexes. The narrow region in the middle is the strip type domain. The remaining four connected components are of the half-plane type. The Stokes graph of  $z^2 - 1$  (central picture) is a non-simple Stokes complex. It contains a short Stokes line

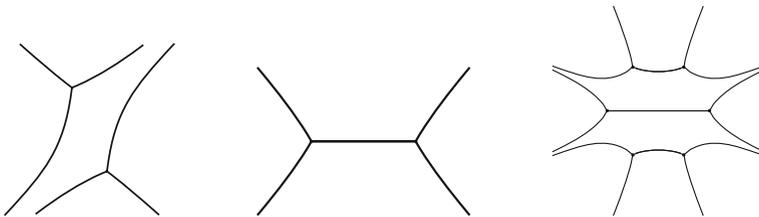


Fig. 1 Stokes graphs for potentials  $z^2 + 1 + I$ ,  $z^2 - 1$ , and  $z^6 - 1$  resp

between its turning points  $\pm 1$ . Finally, the Stokes graph of  $z^6 - 1$  consists of three nonsimple Stokes complexes since each of them contains a short Stokes line. This potential is not Fedoryuk-generic, see the next definition.

Given a polynomial  $P(z)$ , consider the family  $P_t(z) = e^{2t\sqrt{-1}}P(z)$ ,  $t \in [0, \pi)$ . Let us briefly discuss how the Stokes graph changes in this family. Theorem 6.1 of [9] claims that, for any  $P(z)$  with at least two distinct roots, there exist only finitely many values of  $t$  for which  $P_t(z)$  has a non-simple Stokes complex. We call a polynomial  $P(z)$  *Fedoryuk-generic* if it satisfies the following three conditions:

- (a) all its roots are simple;
- (b) all non-simple complexes arising in the family  $P_t(z)$  contain exactly two turning points;
- (c) for any value of  $t$ , the Stokes graph of  $P_t(z)$  has either none or exactly one non-simple Stokes complex.

A slight generalization and reformulation of Theorem 6.2 of [9] adjusted to our context reads as follows. Given a Fedoryuk-generic potential  $P(z)$ , denote by  $t_1, \dots, t_k$  the values of the parameter  $t \in [0, \pi)$  for which the Stokes graph of  $P_t(z)$  has (a unique) non-simple Stokes complex.

**Proposition 1** *In the above notation, for any  $\epsilon > 0$ , all eigenvalues of problem (1.2) except finitely many lie in the  $\epsilon$ -neighborhood of the union of  $k$  rays with the slopes  $\tan t_1, \dots, \tan t_k$ . Moreover, near each such ray (called the accumulation ray of (1.2)) lie countably many such eigenvalues.*

Generalizing Theorem 6.3 of [9], we can describe the asymptotic density of the eigenvalues. In the notation of Proposition 1, let  $t_j$  be one the values of the parameter  $t \in [0, \pi)$  for which the Stokes graph of  $P_t(z)$  has (a unique) non-simple Stokes complex. Fixing a sufficiently small  $\epsilon > 0$ , let  $\lambda_1^{(j)}, \lambda_2^{(j)}, \dots, \lambda_n^{(j)}, \dots$  be the sequence of the eigenvalues of (1.2) non-strictly ordered by their absolute values and lying  $\epsilon$ -close to the  $j$ th ray. By definition, the  $j$ th ray has the slope  $\tan t_j$ .

**Proposition 2** *When  $|\lambda| \rightarrow \infty$ , then*

$$\lambda_n^{(j)} \int_C \sqrt{P(\xi)} d\xi \sim 2\pi n + \pi + \sum_{i=1}^{\infty} \frac{1}{\left(\lambda_n^{(j)}\right)^i} \int_C \alpha_i(\xi) d\xi. \tag{1.4}$$

Here  $C$  is a simple closed curve containing the corresponding short Stokes line and no other turning points in its interior. The sequence  $\{\alpha_i(z)\}_{i=0}^{\infty}$  of functions is defined by:

$$\alpha_0(z) = -\frac{p'(z)}{4p(z)}, \quad \alpha_i(z) = -\frac{\left(\sum_{m=0}^{i-1} \alpha_m(z)\alpha_{i-m-1}(z) + \alpha'_{i-1}(z)\right)}{2\sqrt{p(z)}}, \quad j = 1, 2, \dots \tag{1.5}$$

Let us now reformulate the above statements in terms of quadratic differentials. Basic references for quadratic differentials are [21] and [14].

**Definition 2** A meromorphic quadratic differential  $\Psi$  on a (compact) Riemann surface  $\Gamma$  is a meromorphic section of the square  $(T_{\mathbb{C}}^*\Gamma)^{\otimes 2}$  of the holomorphic cotangent bundle  $T_{\mathbb{C}}^*\Gamma$ . The zeros and poles of  $\Psi$  are called its *critical points*. The set of all critical points of  $\Psi$  on  $\Gamma$  is denoted by  $Crit_{\Psi}$ .

An equivalent down-to-earth definition of a quadratic differential  $\Psi$  on  $\Gamma$  is as follows, see Def. 4.1. in [21]. If the conformal structure on the Riemann surface  $\Gamma$  is given by the family  $\{(U_{\nu}, h_{\nu})\}$ , then a meromorphic quadratic differential  $\Psi$  on  $\Gamma$  is a system of meromorphic function elements  $\phi_{\nu}$  in the local parameter  $z_{\nu} = h_{\nu}(p)$  for which the transformation law

$$\phi_{\nu}(z_{\nu})dz_{\nu}^2 = \phi_{\mu}(z_{\mu})dz_{\mu}^2, \quad dz_{\mu} = \frac{dz_{\mu}}{dz_{\nu}}dz_{\nu}$$

holds whenever  $z_{\mu}$  and  $z_{\nu}$  are parameter values corresponding to the same point  $p \in \Gamma$ .

**Definition 3** Given a meromorphic quadratic differential  $\Psi$  on  $\Gamma$  we define two distinct foliations on  $\Gamma \setminus Crit_{\Psi}$  as follows. At each non-critical point there are exactly two directions at which  $\Psi$  attains positive and negative values respectively. Integral curves of these direction fields are called **horizontal** and **vertical trajectories** of  $\Psi$  respectively.

Obviously, these direction fields are orthogonal at each non-critical point. In what follows by a *trajectory* we mean a horizontal trajectory.

**Definition 4** (comp. Definition 20.1 in [21]) A trajectory of  $\Psi$  is called **critical** if it starts or ends at a critical point.

**Definition 5** The **canonical length element** on  $\Gamma$  associated with a quadratic differential  $\Psi$  given locally as  $\Psi = f(z)(dz)^2$  is defined by

$$|dw| = |f(z)|^{\frac{1}{2}}|dz|.$$

(Local) minimizers of the latter length element are called **geodesics** of the quadratic differential  $\Psi$ . Geodesics whose both endpoints are critical points are called **short**.

**Definition 6** The **distinguished** or **canonical** parameter associated with a quadratic differential  $\Psi = f(z)(dz)^2$  is defined as

$$W = \int \sqrt{f(z)}dz$$

for some branch of the square root.

Notice that geodesics are (locally) straight lines in the canonical parameter and short Stokes lines in the family  $P_t(z)$  connecting the turning points are exactly the short geodesics of the quadratic differential  $P(z)dz^2$ . Thus, the following result holds.

**Corollary 1** For a Fedoryuk-generic polynomial  $P(z)$ , the accumulation rays of problem (1.2) are in 1 – 1-correspondence with short geodesics of  $P(z)dz^2$ .

The main result of this note is as follows.

**Theorem 1** For any polynomial  $P(z)$  of degree  $d$ , the number of short geodesics of the quadratic differential  $P(z)dz^2$  can be an arbitrary integer between  $d - 1$  and  $\binom{d}{2}$ .

*Remark* When a preliminary version of the present paper was posted on arXiv, the author was informed that Theorem 1 was formulated earlier in the physics literature, see §5.1 of [17], and proven there on the physics level of rigorousness. But, in my opinion, the connection of this result to the spectral theory of the Schrödinger equation is very important and it motivates the present publication.

## 2 Proofs

**Definition 7** Let  $\Psi$  be a quadratic differential on  $\mathbb{C}P^1$ . A  $\Psi$ -polygon is a simple closed curve consisting of a finite number of (possibly critical) geodesics of finite length.

Given a  $\Psi$ -polygon  $\Gamma$  with an interior domain  $D$ , assume that  $z_j$  are the critical points on  $\Psi$  having orders  $n_j$  and that  $\xi_i$  are the critical points inside  $D$  having orders  $n_i$ . Finally, denote by  $\theta_j$ ,  $0 \leq \theta_j \leq 2\pi$  the interior angles at the vertices of  $\Gamma$ . The following result can be found in e.g. Theorem 14.1 of [21].

**Theorem 2** (Teichmüller’s lemma) *In the above notation,*

$$\sum_j \left( 1 - \frac{(n_j + 2)\theta_j}{2\pi} \right) = 2 + \sum_i n_i. \tag{2.1}$$

Consider now a quadratic differential  $\Psi = P(z)dz^2$  on  $\mathbb{C}P^1$  where  $P(z)$  is a polynomial of degree  $d$ . We want to count the total number of unbroken short geodesics connecting pairs of roots of  $P(z)$  (By ‘an unbroken’ short geodesic we mean a short geodesic not passing through other roots of  $P(z)$  except its two endpoints).

**Lemma 1** *For any given pair of roots of  $P(z)$ , there exists at most one unbroken short geodesic connecting them.*

*Proof* Follows straightforwardly from Teichmüller’s lemma. Indeed, if there were two such short geodesics then they will form a  $\Psi$ -polygon  $\Gamma$  splitting  $\mathbb{C}P^1$  into two connected domains. Let  $D$  be the “bounded domain”, i.e.  $D$  is the component of  $\mathbb{C}P^1 \setminus \Gamma$  not containing  $\infty \in \mathbb{C}P^1$ . By assumption there are exactly two critical points on  $\Gamma$  and so the left-hand side of (2.1) is smaller than two, whereas the right-hand side is clearly at least two, a contradiction.  $\square$

This lemma immediately implies the inequalities in Theorem 1. Namely,

**Corollary 2** *A arbitrary quadratic differential  $\Psi = P(z)dz^2$ ,  $\deg P(z) = d$  has at least  $d - 1$  and at most  $\binom{d}{2}$  unbroken short geodesics.*

*Proof* The upper bound  $\binom{d}{2}$  is provided by the latter lemma. Moreover, it is clear that the set formed by all short unbroken geodesics is compact and connected. Indeed, any two roots of  $P(z)$  are connected by at least one geodesic (broken or unbroken). Assuming that all  $d$  roots of  $P(z)$  are distinct, one needs at least  $d - 1$  unbroken short geodesics to guarantee that this set is connected. The latter situation can be realized for instance by taking  $P(z)$  with all real and distinct roots.  $\square$

Let us now show that there exist polynomials  $P(z)$  having an arbitrary number of unbroken short geodesics between  $d - 1$  and  $\binom{d}{2}$ . We will use the known interpretation of a quadratic differential  $P(z)dz^2$  in terms of pairs of weighted chord diagrams on  $d + 2$  vertices, see § 4 of [2]. Here  $P(z)$  is a monic polynomial of degree  $d$  with the vanishing sum of its roots.

Indeed, given  $\Psi = P(z)dz^2$  as above, take its Stokes graph  $ST_P$  and its anti-Stokes graph  $AST_P$ . (The anti-Stokes graph of  $P(z)dz^2$  is, by definition, the Stokes graph of  $-P(z)dz^2$ .) Unbounded Stokes lines of  $ST_P$  tend at  $\infty \in \mathbb{CP}^1$  to  $d + 2$  standard directions called the Stokes rays. Analogously unbounded anti-Stokes lines of  $AST_P$  tend at  $\infty \in \mathbb{CP}^1$  to  $d + 2$  standard directions called the anti-Stokes rays. The set of all Stokes and anti-Stokes rays can be naturally considered as the sets of vertices of two regular  $(d + 2)$ -gons which are rotated w.r.t. each other by the angle  $\frac{\pi}{d+2}$ .

Recall that admissible domains, i.e. connected components of  $\mathbb{CP}^1 \setminus ST_P$  (resp.  $\mathbb{CP}^1 \setminus AST_P$ ), are of two types: the half-plane type and the strip type. Function  $\xi(z)$  defined by (1.3) maps the half-plane type components into half-planes and the strip type components to strips. For  $\mathbb{CP}^1 \setminus ST_P$  these image half-planes and image strips are bounded by the vertical lines and for  $\mathbb{CP}^1 \setminus AST_P$  they are bounded by the horizontal lines.

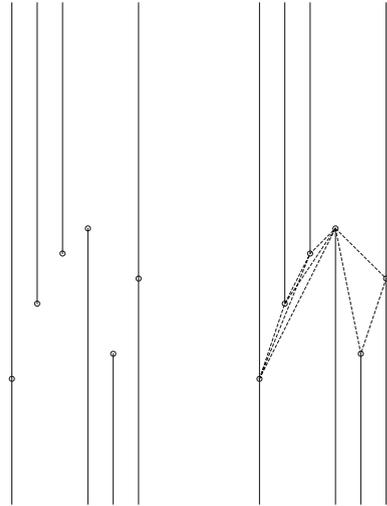
Each strip type domain is topologically an infinite strip bounded by two curves. Moreover, the two pairs of ends of its boundary approach two distinct and non-neighboring Stokes rays. Thus, one can interpret a strip type domain as a path connecting two vertices of the Stokes  $(d + 2)$ -gon and represent it by a corresponding chord (i.e. side or diagonal connecting these two vertices) in the Stokes  $(d + 2)$ -gon. Analogously, strip type domains for the anti-Stokes graph connect pairs of vertices of the anti-Stokes  $(d + 2)$ -gon.

Moreover, we can assign to each strip type domain a positive weight coinciding by definition with the width of its image under the map  $\xi(z)$ . These weights are the absolute values of the real and imaginary parts of the integrals  $\int \sqrt{P(t)}dt$  taken over certain paths connecting pairs of roots of  $P(z)$ . They are closely related to the periods of  $y^2 = P(z)$ .

For a generic polynomial  $P(z)$ , the number of its strip-type domains for  $ST_P$  and  $AST_P$  equals  $d - 1$  which means that one gets weighted triangulations of both the Stokes and anti-Stokes polygons. In other words, one obtains a pair of weighted chord diagrams of the Stokes and anti-Stokes  $(d + 2)$ -gons. The following statement can be found in §4 of [2].

**Proposition 3** *The above procedure gives a 1 – 1-correspondence between the set of all quadratic differentials of the form  $P(z)dz^2$  with  $P(z) = z^d + a_1z^{d-2} + \dots + a_{d-1}$  and the set of ordered pairs of weighted chord diagrams.*

**Fig. 2** A chopped strip and the same strip with its short geodesics



To find quadratic differentials  $P(z)dz^2$  with a given number  $k$  of short geodesics satisfying the inequality  $d - 1 \leq k \leq \binom{d}{2}$ , it suffices to use only a special class of quadratic differentials.

**Definition 8** A quadratic differential  $P(z)dz^2$ ,  $\deg P(z) = d$  is called **very flat** if (a) all roots of  $P(z)$  are simple; (b) the number of its horizontal strip-type domains equals  $d - 1$ ; (c) each root of  $P(z)$  lies in the closure of at most two strip-type domains.

Notice that condition c) above is equivalent to the property that each domain obtained by removing a chord from either of the two chord diagram of the differential has at most two neighboring domains.

To each very flat quadratic differential we can naturally associate the following object whose versions earlier appeared in e.g. [19], p. 269.

**Definition 9** A **chopped vertical strip** is a set of complex numbers  $z_1, z_2, \dots, z_d$  (called nodes) with distinct real and imaginary parts ordered by their real parts together with vertical rays going either up or down from each  $z_j$ ,  $j = 2, \dots, d - 1$ , see Fig. 2. These rays are called **cuts** of the strip.

The map associating to a very flat quadratic differential its chopped vertical strip is actually the same map  $\xi(z)$  introduced earlier whose domain is restricted to  $\mathbb{C}P^1 \setminus \bigcup_{j=1}^{d+2} O_j$  each  $O_j$  being the closed half-plane type domain. (It is usually convenient to assign the base point of integration in the definition of  $\xi(z)$  to the inverse image of  $z_1$  which implies  $z_1 = 0$ ).

**Lemma 2** For any chopped strip there exists a (non-unique) very flat quadratic differential  $P(z)dz^2$  mapped by its  $\xi(z)$  to this strip. Its short geodesics are in 1 - 1 correspondence with the straight segments connecting pairs of vertices of the strip and not intersecting its cuts.

*Proof* Just throw away all half-plane domains and use  $\xi(z)$ . □

The following statement finishes the proof of Theorem 1.

**Proposition 4** *There exist chopped strips with  $d - 2$  cuts and an arbitrary number  $k$  of unbroken short geodesics satisfying the inequality  $d - 1 \leq k \leq \binom{d}{2}$ .*

*Proof* We will use induction on  $d$ . Assume that for  $d - 1$  one can realize any number  $k'$  of short geodesics satisfying  $d - 2 \leq k' \leq \binom{d-1}{2}$ . Then it is easy to construct chopped strips with  $d$  nodes having exactly  $k' + 1$  short geodesics. Indeed, given a chopped strip with  $d - 1$  nodes and  $k'$  short geodesics, one can easily place the last node to the right of all other nodes in such a way that all straight segments from the previous nodes to the new one will intersect the cuts except the one from the  $(d - 1)$ st node which always exists. Thus, we get a chopped strip with  $d$  nodes and  $k' + 1$  short geodesics.

Therefore, by induction assumption, for  $d$  nodes we only need to realize  $k$  short geodesics where  $\binom{d-1}{2} + 2 \leq k \leq \binom{d}{2}$ . Let us first realize the maximal number  $k = \binom{d}{2}$ . To do this assume that the  $d$  nodes form the set of vertices of a convex  $d$ -gon. Direct the cut from each node (except the leftmost and the rightmost which lie on the outer boundary of the chopped strip) so that they do not intersect the latter  $d$ -gon. Then, obviously each side and each diagonal of the  $d$ -gon will be a short geodesic giving totally  $\binom{d}{2}$  such.

To realize the intermediate number of short geodesics between  $\binom{d-1}{2} + 2$  and  $\binom{d}{2} - 1$ , consider in the latter convex  $d$ -gon the second node from the left. Without loss of generality, one can assume that it lies above the leftmost node and, therefore, its cut is directed upwards. The leftmost node in the convex polygon is connected with all remaining nodes by short geodesics which are the sides and the diagonals of the  $d$ -gon starting at this node. Moving the second node down together with its cut and keeping the rest of the nodes intact, we can obtain a non-convex  $d$ -gon in which an arbitrary number of diagonals from the leftmost node are forbidden which is exactly what we need. □

### 3 Final remarks

1. The next question is motivated by Theorem 2 of [18] which gives necessary and sufficient conditions for the usual spectral problem for the Schrödinger equation with a polynomial potential to have infinitely many real eigenvalues (or, similarly, eigenvalues lying on a given accumulation ray). This result claims that (under some simple appropriate assumptions) in order to have infinitely many real eigenvalues, it is necessary and sufficient that there exists a point  $z_0 \in \mathbb{C}$  such that potential  $P(z - z_0)$  is PT-symmetric, see details in [18]. All tools used in the latter paper for the standard spectral problem have their natural counterparts for problem (1.2).

**Problem 2** Give necessary and sufficient conditions guaranteeing that spectral problem (1.2) has infinitely many eigenvalues which belong to some accumulation ray.

2. Obviously, not every polynomial  $P(z)$  gives rise to a very flat differential  $P(z)dz^2$ , but in many cases one can obtain a very flat differential by multiplying  $P(z)$  by a constant.

**Problem 3** Is it true that for any polynomial  $P(z)$  there exists  $t \in [0, 2\pi)$  such that the differential  $e^{t\sqrt{-1}}P(z)dz^2$  is very flat?

3. The number of unbroken short geodesics is an interesting invariant of a quadratic differential  $P(z)dz^2$ .

**Problem 4** Study the decomposition of the space of all monic polynomials of a given degree according to the number of unbroken short geodesics of their quadratic differentials?

4. It seems important to generalize Evgrafov–Fedoryuk’s theory from the case of polynomial potentials to the case of rational potentials. In particular, the following analog of the main problem considered in the present paper looks very tantalizing.

**Problem 5** Consider the set of all rational differentials of the form  $R(z)dz^2$  where  $R(z) = \frac{P(z)}{Q(z)}$  with  $\deg Q(z) = n > 0$ ,  $\deg P(z) = n + d$ ,  $d > 0$  and such that all its turning points are either simple zeros or poles (The pole at infinity is not supposed to be simple). Find the sharp lower and upper bounds for the number of short trajectories for such differentials.

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