DISCRIMINANTS OF CONVEX CURVES ARE HOMEOMORPHIC

B. Shapiro

Department of Mathematics, University of Stockholm,
S-10691, Sweden, shapiro@matematik.su.se

Abstract. For a given real generic curve $\gamma : S^1 \to \mathbb{P}^n$ let D_γ denote the ruled hypersurface in $\mathbb{P}^n$ consisting of all osculating subspaces to γ of codimension 2. In this short note we show that for any two convex real projective curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ the pairs $(\mathbb{P}^n, D_{\gamma_1})$ and $(\mathbb{P}^n, D_{\gamma_2})$ are homeomorphic.

§0. Preliminaries and results

Definition. A smooth curve $\gamma : S^1 \to \mathbb{P}^n$ is called *nondegenerate or locally convex* if the local multiplicity of its intersection with any hyperplane does not exceed n, i.e. in local terms $\gamma'(t), ..., \gamma^{(n)}(t)$ are linearly independent at every t or its osculating complete flag is well-defined at every point. A curve $\gamma : S^1 \to \mathbb{P}^n$ is called *convex* if the total multiplicity of its intersection with any hyperplane does not exceed n.

The set $\mathfrak{Con}_n$ of all convex curves in $\mathbb{P}^n$ forms 1 connected component of the space $\mathfrak{ND}_n$ of all nondegenerate curves if n is even and 2 connected components (since the osculating frame orients $\mathbb{P}^{2k+1}$) if $n = 2k+1$, see [MSh]. Different results about convex curves show that they have the most simple properties among all curves. In this paper we prove one more result of the same nature.

Definition. A curve $\gamma : S^1 \to \mathbb{P}^n$ is called *generic* if at every point $\gamma(t)$, $t \in S^1$ one has a well-defined osculating subspace of codimension 2, i.e. in local terms $\gamma'(t), ..., \gamma^{(n-1)}(t)$ are linearly independent at every t.

Note that any smooth curve $\gamma : S^1 \to \mathbb{P}^n$ can be made generic by a small smooth deformation of the map. The space $\mathfrak{ND}_n$ of all nondegenerate curves is enclosed in the space $\mathfrak{GEN}_n$ of all generic curves and consists of several connected components. (The number of connected components in $\mathfrak{ND}_n$ equals 10 for odd $n \geqslant 3$ and equals 3 for even $n \geqslant 2$, see [MSh].)

Definition. Given a generic $\gamma : S^1 \to \mathbb{P}^n$ we call by its *standard discriminant* $D_\gamma \subset \mathbb{P}^n$ the hypersurface consisting of all codimension 2 osculating subspaces to γ.

In many cases (algebraic, analytic etc) the assumption of genericity in the definition of discriminant can be omitted.

The following proposition answers the question posed by V.Arnold in [Ar2], p.37.

1991 *Mathematics Subject Classification.* Primary 14H50.
Key words and phrases. convex curves, discriminants.

Main proposition. *a) For any 2 convex curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ the pairs $(\mathbb{P}^n, D_{\gamma_1})$ and $(\mathbb{P}^n, D_{\gamma_2})$ are homeomorphic.*

b) For any convex curve γ the complement $\mathbb{P}^n \setminus D_\gamma$ consists of $[\frac{n}{2}]+1$ components. All components are contractible to S^1 for n even and all but one are contractible to S^1 for n odd. The remaining component is a cell.

Now we want to place this result into a more general context of associated discriminants in the spaces of (in)complete flags.

NOTATION. Let F_{n+1} denote the space of all complete flags in $\mathbb{P}^n$ (or, equivalently, in $\mathbb{R}^{n+1}$). Given a nondegenerate curve $\gamma : S^1 \to \mathbb{P}^n$ one can consider its *associated curve* $\tilde{\gamma} : S^1 \to F_{n+1}$ where $\tilde{\gamma}(t)$ is the complete osculating flag to γ at $\gamma(t)$. Note that any associated curve $\tilde{\gamma} : S^1 \to F_{n+1}$ is tangent to the special distribution of n-dimensional cones in F_{n+1} and any integral curve of this distribution is the associated curve of some nondegenerate projective curve, see e.g. [Sh1].

Given a complete flag $f \in F_{n+1}$ and some space $\mathfrak{G} = SL_{n+1}/P$ of (in)complete flags where P is some parabolic subgroup one gets the Schubert cell decomposition $\mathfrak{Sch}_f$ of $\mathfrak{G}$ as follows. Each cell of $\mathfrak{Sch}_f$ consists of all flags in $\mathfrak{G}$ subspaces of which have a given set of dimensions of intersections with the subspaces of f. Let $\mathfrak{D}_f$ denote the union of all cells in $\mathfrak{Sch}_f$ which have codimension at least 2. (Obviously, codim $\mathfrak{D}_f = 2$.)

EXAMPLES. 1) If $\mathfrak{G}$ equals $\mathbb{P}^n$ then $\mathfrak{D}_f$ is the subspace of f of codimension 2.

2) If $\mathfrak{G} = F_3$ then $\mathfrak{D}_f$ consists of 2 copies of $\mathbb{P}^1$ intersecting at f. The first $\mathbb{P}^1$ is the set of flags on $\mathbb{P}^2$ with the same point as that of f and the second $\mathbb{P}^1$ consists of all flags with the same line as that of f.

DEFINITION. For a given curve $c : S^1 \to F_{n+1}$ and a space $\mathfrak{G} = SL_{n+1}/P$ of (in)complete flags we denote by its $\mathfrak{G}$-*discriminant* $\mathfrak{GD}_c$ the union $\bigcup_{t\in S^1} \mathfrak{D}_{c(t)} \subset \mathfrak{G}$. (If c is not a constant map then $\mathfrak{GD}_c$ is a hypersurface in $\mathfrak{G}$.)

Note that the standard discriminant D_γ of a nondegenerate curve $\gamma : S^1 \to \mathbb{P}^n$ can be considered as the $\mathfrak{G}$-discriminant for $\mathfrak{G} = \mathbb{P}^n$.

DEFINITION. Two nondegenerate curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ are called $\mathfrak{G}$-*equivalent* if the pairs $(\mathfrak{G}, \mathfrak{GD}_{\tilde{\gamma}_1})$ and $(\mathfrak{G}, \mathfrak{GD}_{\tilde{\gamma}_2})$ are homeomorphic. (Recall that $\tilde{\gamma}$ denotes the associated curve of γ.)

REMARK. The notion of $\mathfrak{G}$-equivalence of nondegenerate curves is intrinsically related with the qualitative theory of linear ODE since each nondegenerate curve in $\mathbb{P}^n$ can be represented as the projectivization of the fundamental solution of some linear ODE of order $n+1$, see [Sh2]. The problem of enumeration of $\mathfrak{G}$-equivalent generic curves is apparently a very interesting and difficult question even for $n = 2$.

The following conjecture is formulated in [Sh2].

Conjecture. *Any 2 convex curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ are $\mathfrak{G}$-equivalent for any $\mathfrak{G} = SL_{n+1}/P$.*

Note that it suffices to prove this conjecture in the case of the space F_{n+1} of complete flags, i.e. to the case $P = B$ where B is the Borel subgroup of uppertriangular matrices.

REMARK. After the first version of the paper was written the author discussed the topic with Vl. Zakalyukin who later proved the following stronger result (conjected by the author).

Proposition. *For any pair of convex curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ the pairs $(\mathbb{P}^n, D_{\gamma_1})$ and $(\mathbb{P}^n, D_{\gamma_2})$ are diffeomorphic.*

The main idea of his proof is to show the equivalence of the standard generating functions for the Legendre submanifolds which are lifts of the standard discriminants in the space of tangent elements $\mathbb{P}T^*\mathbb{P}^n$. Unfortunately this method does not give the topological part b) of the above main proposition.

Notice that in a sufficiently small neighborhood of any point p of a locally convex curve $\gamma \subset \mathbb{P}^n$ its discriminant D_γ is diffeomorphic to the standard discriminant $Disc_n$, i.e. the set of all monic degree n polynomials in one variable with real coefficients which have at least one real multiple root, see [Ish]. (In singularity theory $Disc_n$ is also called the swallowtail.) Thus in spite of the fact that D_γ is highly singular it has no local moduli. Still the existence of a global diffeomorphism of different D_γs is quite a nontrivial fact.

The main motivation of this paper was an attempt to formalize the idea that any 2 convex curves as qualitatively equivalent in any natural sense. It is difficult to overestimate the role of my visit to the Max-Planck Institute during the summer 1996 where the main bulk of this project was carried out. Stimulating discussions with M. Shapiro and Vl. Zakalyukin are highly appreciated.

§1. Proofs.

Some generalities on convex curves.

Definition. For any $t \in S^1$ and $1 \le k \le n-1$ let L_t^k denote the osculating subspace to γ at $\gamma(t)$ of dimension k.

1.1. Theorem (criterion of convexity). *A curve $\gamma : S^1 \to \mathbb{P}^n$ is convex if and only if for any r-tuple of positive integers $k_1, ..., k_r$ such that $\sum k_i = n$ and any r-tuple of pairwise different moments $t_1, ..., t_r$ the intersection $L_{t_1}^{n-k_1} \cap ... \cap L_{t_r}^{n-k_r}$ is a point.*

Proof. In order to save the space we refer the interested reader to [Co] and further references. □

Definition. Given a nondegenerate curve $\gamma : S^1 \to \mathbb{P}^n$ we call by its dual $\gamma : S^1 \to (\mathbb{P}^n)^*$ the curve consisting of all osculating hyperplanes to γ.

Remark. If γ is convex then γ^* is also convex, see [Ar1], [Ar2].

Notation. If a point p lies on some osculating hyperplane H_τ to γ we say that *the order of tangency* $\sharp_p(\gamma(\tau))$ of p at $\gamma(\tau)$ equals to i if p belongs to the osculating subspaces at $\gamma(\tau)$ of the codimension at most i. (For example, for every point p on a line l tangent to a circle c at $c(1)$ on $\mathbb{P}^2$ except for the tangency point $c(1)$ one has $\sharp_p(c(1)) = 1$. On the other side, $\sharp_{c(1)}(c(1)) = 2$.)

Given a nondegenerate $\gamma : S^1 \to \mathbb{P}^n$ and a point $p \in \mathbb{P}^n$ we call by *the number of roots* $\sharp_p(\gamma)$ of p the sum of the orders of tangency $\sharp_p(\gamma(t_i))$ taken over all osculating hyperplanes H_{t_i} through p.

(The term 'number of roots' comes from the example when γ is the rational normal curve in $\mathbb{P}^n$. In this case all points in $\mathbb{P}^n$ can be interpreted as homogeneous polynomials in 2 variables of degree n with real coefficients (considered up to a constant factor) and γ is the family of polynomials of the form $(ax_1 + bx_2)^n$, $a^2 + b^2 \neq 0$. In this situation $\sharp_p(\gamma)$ coincides with the total number of real roots of such a polynomial on $\mathbb{P}^1$ counted with multiplicies.)

Observation. *The number of roots $\sharp_p(\gamma)$ coincides with the total multiplicity (i.e. sum of local multiplicities) of the intersection of $\tilde{H}_p$ with γ^*. Here $\tilde{H}_p$ denotes the hyperplane in $(\mathbb{P}^n)^*$ corresponding to the point $p \in \mathbb{P}^n$.*

1.2. Corollary. *A curve γ is convex if and only if for any $p \in \mathbb{P}^n$ one has $\sharp_p(\gamma) \le n$.*

Projection.

Given a convex curve $\gamma : S^1 \to \mathbb{P}^n$ and its osculating hyperplane H_τ at the point $\gamma(\tau)$ let us denote by $\gamma^\tau : S^1 \to H_\tau$ the curve obtained by projection of γ onto H_τ along the pencil of tangent lines to γ, i.e. for any $t \in S^1$ one has $\gamma^\tau(t) = H_\tau \cap l_t$ where l_t is the tangent line to γ at $\gamma(t)$.

1.3. Lemma. *For any $\tau \in S^1$ the curve γ^τ is a convex curve in H_τ. Osculating hyperplanes to γ^τ and its discriminant D_{γ^τ} are obtained by intersection of the osculating hyperplanes and D_γ with H_τ.*

Proof. The argument splits into 2 principal parts. At first we show that γ^τ is nondegenerate, i.e. $(\gamma^\tau)'(t), ..., (\gamma^\tau)^{(n-1)}(t)$ are linearly independent at any $t \in S^1$. Then we prove that γ^τ is convex, i.e. its total multiplicity of intersection with any hyperplane in H_τ does not exceed $n-1$. Observe that γ has the only intersection point with H_τ, namely $\gamma(\tau)$. Assume first that $t \neq \tau$. In this case one can choose a system of affine coordinates $x_1, ..., x_n$ in $\mathbb{P}^n$ such that H_τ coincides with the hyperplane $\{x_n = 0\}$; $\gamma(t)$ is the point with coordinates $(0, ..., 0, 1)$ and the tangent line l_t to γ at $\gamma(t)$ is the x_n-axis. In these coordinates the curve γ^τ has the form

$$\gamma^\tau(t) = \gamma(t) - \frac{\gamma_n(t)}{\gamma_n'(t)}\gamma'(t)$$

where γ_n is the last coordinate of γ. (Under our assumptions $\gamma_n'(t) \neq 0$.) Therefore

$$(\gamma^\tau)^{(i)}(t) = (-1)^i \frac{\gamma_n(t)}{\gamma_n'(t)}\gamma^{(i)}(t) + ...$$

where ... denotes the terms containing derivatives of γ of order lower than i. By the above assumptions $\frac{\gamma_n(t)}{\gamma_n'(t)} \neq 0$ and since $\gamma'(t), ..., \gamma^{(n)}(t)$ are linearly independent one gets that the derivatives $(\gamma^\tau)^{(i)}(t),\ i = 1, ..., n-1$ are linearly independent as well.

The alternative geometric argument is as follows. Since the point $\gamma(t)$ does not lie on H^τ one has that the osculating complete flag $f(t)$ to γ at $\gamma(t)$ is transversal to H_τ and the same holds for all t' close to t. Therefore the complete flags obtained by intersection of $f(t')$ with H_τ are well defined. But in their turn these flags coincide with the osculating flags to γ^τ which are therefore well-defined in some neighborhood of t.

It is left to show that γ^τ is nondegenerate at $t = \tau$. This follows from the local calculation given below. In this case we can choose a system of coordinates such that in the neighborhood of τ (we assume $\tau = 0$) the curve γ has the form

$$x_1 = t + ...,\ x_2 = t^2 + ...,\ ...,\ x_n = t^n + ...$$

The osculating hyperplane H^0 at $\tau = 0$ is given by $\{x_n = 0\}$. The projected curve $\gamma^0(t)$ is given by

$$\gamma^0(t) = \gamma(t) - \frac{\gamma_n(t)}{\gamma_n'(t)}\gamma' = (t + ..., t^2 + ..., ..., t^n + ...) - \frac{t^n + ...}{nt^{n-1} + ...}(1 + ..., 2t + ..., ..., nt^{(n-1)} + ...)$$
$$= \frac{1}{n}((n-1)t + ..., (n-2)t^2 + ..., ..., t^{(n-1)} + ..., 0)$$

which shows that γ^0 is nondegenerate at $t = 0$.

Now we show that γ^τ is convex. By Corollary 1.2. one has to prove that for any $p \in H_\tau$ the number of roots $\sharp_p(\gamma^\tau)$ is less or equal $n-1$. This follows from the equality

$$\sharp_p(\gamma^\tau) + 1 = \sharp_p(\gamma)$$

which together with convexity of γ gives the required result. Indeed, assume that some $p \in H_\tau$ lies in the intersection $L^{k_1}_{t_1}(\gamma) \cap L^{k_2}_{t_2}(\gamma) \cap ... \cap L^{k_r}_{t_r}(\gamma)$ where $t_1 = \tau$. Since each subspace $L^{k_i}_{t_i}$ for $i \neq 1$ is transversal to H_τ (see criterion of convexity) one has that p lies in the intersection $L^{k_1}_{t_1}(\gamma^\tau) \cap L^{k_2-1}_{t_2}(\gamma^\tau) \cap ... \cap L^{k_r-1}_{t_r}(\gamma^\tau)$. Therefore, by definition of the number of roots, one gets the above equality.

□

For any k-tuple of moments $(t_1, ..., t_k)$, $t_i \in S^1$ let $H_{t_1} \cap ... \cap H_{t_k}$ denote the intersection of the osculating hyperplanes H_{t_i}, $i = 1, ..., k$. In what follows we use the following convention. If some of the moments $t_{j_1}, t_{j_2}, ..., t_{j_r}$ coincide we define the intersection $H_{t_{j_1}} \cap H_{t_{j_2}} \cap ... \cap H_{t_{j_r}}$ as the osculating subspace to γ at $\gamma(t_{j_1})$ of codimension r. Under this convention one has that $H_{t_1} \cap ... \cap H_{t_k}$ always has codimension k, see 1.1.

1.4. Corollary. *The projection $\gamma^{t_1,...,t_k}$ of γ onto any intersection of osculating hyperplanes $H_{t_1} \cap ... \cap H_{t_k}$ by a pencil of k-dimensional osculating subspaces to γ is a convex curve. For any point $p \in H_{t_1} \cap ... \cap H_{t_k}$ one has*

$$\sharp_p(\gamma^{t_1,...,t_k}) + k = \sharp_p(\gamma).$$

Proof. Apply the above lemma several times.

□

Elliptic hull of γ and root filtration of $\mathbb{P}^n$.

Definition. For a convex $\gamma : S^1 \to \mathbb{P}^n$ we denote by *its elliptic hull Ell_γ* the set of all $p \in \mathbb{P}^n$ with

$$\begin{cases} \sharp_p(\gamma) = 0, & \text{if } n \text{ is even} \\ \sharp_p(\gamma) = 1, & \text{if } n \text{ is odd.} \end{cases}$$

1.5. Lemma. *a) If n is even then Ell_γ is a nonempty convex set in some affine chart of $\mathbb{P}^n$, comp.* [ShS]*;*

b) If n is odd then Ell_γ is a disjoint union of $\bigcup_{\tau \in S^1} Ell_{\gamma^\tau}$ and, therefore, is fibered over γ with a contractible fiber.

Proof. a) Note that if $\gamma : S^1 \to \mathbb{P}^{2k}$ is convex then γ lies in some affine chart in $\mathbb{P}^{2k}$. Indeed, take some osculating hyperplane H_τ. The curve γ is tangent to H_τ only at $\gamma(\tau)$ with the multiplicity $2k$. Locally γ lies on one side w.r.t. H_τ. Therefore, one can make a small shift of H_τ in order to get rid of the intersection points with γ near $\gamma(\tau)$. But no new intersection can appear for a sufficiently small shift since the only intersection point of γ and H_τ is $\gamma(\tau)$.

Assume now that $\mathbb{R}^{2k} \subset \mathbb{P}^{2k}$ is the affine chart containing γ. We claim that Ell_γ coincides with the intersection $\bigcap_{\tau \in S^1} Half_\tau$. Here $Half_\tau$ is the open halfspace in $\mathbb{R}^{2k}$ containing γ and bounded by the osculating hyperplane H_τ. First of all, $\bigcap_{\tau \in S^1} Half_\tau$ is nonempty since it is an open convex set containing the interior of the convex hull of γ in $\mathbb{R}^{2k}$. Then $\bigcap_{\tau \in S^1} Half_\tau$ is contained in Ell_γ. Indeed, every hyperplane through a point $p \in \bigcap_{\tau \in S^1} Half_\tau$ is transversal to any osculating hyperplane H_τ since $p \notin H_\tau$. On the other side, $Ell_\gamma \subseteq \bigcap_{\tau \in S^1} Half_\tau$. Indeed, for every $p \notin Half_\tau$, $\tau \in S^1$ there exists a hyperplane L_p through p not intersecting

γ at all. Take the affine chart $\mathbb{P}^n \setminus L_p$ containing γ and some pencil $\mathcal{L}_p$ of 'parallel' hyperplanes through p. Since γ is a closed curve in $\mathbb{P}^n \setminus L_p$ one gets that some hyperplane in $\mathcal{L}_p$ does not intersect γ. Therefore, there exists a hyperplane in $\mathcal{L}_p$ tangent to γ at some $\gamma(t_p)$. But this exactly means that the osculating hyperplane H_{t_p} contains p.

b) Take a 1-parameter family of osculating hyperplanes. According to the proof of lemma 1.3. for any $\tau \in S^1$ the curve γ^τ is convex in H_τ and one has $\sharp_p(\gamma^\tau) + 1 = \sharp_p(\gamma)$. Therefore the elliptic domain Ell_{γ^τ} of every curve γ^τ belongs to Ell_γ, i.e. $\bigcup_{\tau \in S^1} Ell_{\gamma^\tau} \subset Ell_\gamma$. (Note that the union $\bigcup_{\tau \in S^1} Ell_{\gamma^\tau}$ is disjoint.) Conversely, by definition, for odd n every point p in Ell_γ has exactly one tangent hyperplane to γ and thus p belongs exactly to one osculating H_τ. By the equality $\sharp_p(\gamma^\tau) + 1 = \sharp_p(\gamma)$ the point p lies in the elliptic hull of γ^τ. Moreover, by the first part of this proof, Ell_{γ^τ} is a convex domain in H_τ and, therefore, is contractible which gives the necessary result.

□

Definition. By *the root filtration* of $\mathbb{P}^n$ w.r.t. a convex curve $\gamma : S^1 \to \mathbb{P}^n$

$$\mathbb{P}_0(\gamma) \subset ... \subset \mathbb{P}_{[\frac{n}{2}]}(\gamma) = \mathbb{P}^n$$

we denote the filtration where each $\mathbb{P}_i(\gamma)$ consists of all $p \in \mathbb{P}^n$ for which the number of roots $\sharp_p(\gamma)$ is greater or equal $n - 2i$.

Let $\mathcal{T}^j = (S^1)^j$ denote the j-dimensional torus and let $\mathcal{T}^j/\mathfrak{S}_j$ be its quotient mod the natural action of the symmetric group $\mathfrak{S}_j$ by permutation of copies of S^1.

1.6. Lemma. *a) For any n and $0 \le i \le [\frac{n}{2}]$ the set $\mathbb{P}_i(\gamma) \setminus \mathbb{P}_{i-1}(\gamma)$ is naturally fibered over $\mathcal{T}^{n-2i}/\mathfrak{S}_{n-2i}$ with a contractible fiber. (For $n = 2k$ and $i = k$ the set $\mathbb{P}_k(\gamma) \setminus \mathbb{P}_{k-1}(\gamma)$ is contractible, see 1.5.a);*

b) This fibration is trivial.

Proof. a) Every point $p \in \mathbb{P}_i(\gamma) \setminus \mathbb{P}_{i-1}(\gamma)$ can be described as follows. There exists and unique $(n-2i)$-tuple of osculating hyperplanes $H_{t_1}, ..., H_{t_{n-2i}}$ to γ (with probably coinciding moments $t_1, ..., t_{n-2i}$ in which case we use the same convention as above) such that p belongs to the intersection $H_{t_1} \cap H_{t_2} \cap ... \cap H_{t_{n-2i}}$ and, moreover, lies in the elliptic hull of the curve $\gamma^{t_1,...,t_{n-2i}}$. (Here $\gamma^{t_1,...,t_{n-2i}}$ is the projection of γ onto $H_{t_1} \cap H_{t_2} \cap ... \cap H_{t_{n-2i}}$ by the pencil of osculating subspaces of dimension $n-2i$.) Indeed, we have that $\sharp_p(\gamma^{t_1,...,t_{n-2i}}) + 2i = \sharp_p(\gamma)$, see 1.2. Therefore p must lie in the elliptic hull of $\gamma^{t_1,...,t_{n-2i}}$. On the other side, any intersection $H_{t_1} \cap H_{t_2} \cap ... \cap H_{t_{n-2i}}$ has codimension $n - 2i$, see 1.1. and any curve $\gamma^{t_1,...,t_{n-2i}}$ is convex. Therefore applying 1.5. we get the necessary result.

b) The fibration of elliptic components $Ell_{\gamma^{t_1,...,t_{n-2i}}}$ of the curves $\gamma^{t_1,...,t_{n-2i}}$ over the set of moments $(t_1, ..., t_{n-2i}) \in \mathcal{T}^{n-2i}/\mathfrak{S}_{n-2i}$ depends continuously on $\gamma \in \mathfrak{Con}_n$. Since $\mathfrak{Con}_n$ consists of 1 connected component (up to orientation for odd n) it suffices to show that the fibration sending $Ell_{\gamma^{t_1,...,t_{n-2i}}}$ to $(t_1, ..., t_{n-2i})$ is trivial for some $\gamma \in \mathfrak{Con}_n$.

The simplest example showing triviality is the case when γ is the rational normal curve. Indeed, in this case the space under consideration is the fibration of the space $\Pi_n(i)$ all homogeneous forms of degree n in 2 variables (up to a scalar multiple) which have exactly $n - 2i$ real zeros counted with multiplicities over the space $\mathcal{T}^{n-2i}/\mathfrak{S}_{n-2i}$ of their real zeros. But $\Pi_n(i)$ has the obvious structure of the product of the space of degree $n - 2i$ polynomials with all real zeros (considered up to a scalar multiple) and the space of degree $2i$ polynomials with no real zeros (considered up to a scalar multiple). This shows that the fibration $\Pi_n(i) \to \mathcal{T}^{n-2i}/\mathfrak{S}_{n-2i}$ is trivial.

□

Proof of the main proposition.

a) We will construct the homeomorphism of pairs $(\mathbb{P}^n, D_{\gamma_1})$ and $(\mathbb{P}^n, D_{\gamma_2})$ in $[\frac{n}{2}]+1$ steps. On the ith step, $i = 0, ..., [\frac{n}{2}]$ we obtain the partial homeomorphism h_i of the terms $\mathbb{P}_i(\gamma_1)$ and $\mathbb{P}_i(\gamma_2)$ of the above filtration.

The initial step. We construct the homeomorphism $h_0 : \mathbb{P}_0(\gamma_1) \to \mathbb{P}_0(\gamma_2)$. Indeed, each of $\mathbb{P}_0(\gamma_1)$ and $\mathbb{P}_0(\gamma_2)$ is homeomorphic to $\mathcal{T}^n/\mathfrak{S}_n$ as follows. Every element in $\mathcal{T}^n/\mathfrak{S}_n$ is a pair $(t_1, ..., t_r) \in (\mathcal{T}^r \setminus Diag)/\mathfrak{S}_r, r \leq n$ and $(k_1, ..., k_r)$, $\sum k_i = n$. We map such a pair $(t_1, ..., t_r), (k_1, ..., k_r)$ onto the intersection point $L_{t_1}^{n-k_1}(\gamma_j) \cap ... \cap L_{t_r}^{n-k_r}(\gamma_j), j = 1, 2$. This identification provides the homeomorphism $h_0 : \mathbb{P}_0(\gamma_1) \to \mathbb{P}_0(\gamma_2)$ by 1.1.

The typical step. Each point in $\mathbb{P}_i(\gamma_j) \setminus \mathbb{P}_{i-1}(\gamma_j)$ lies in the eliptic hull of the unique curve $\gamma^{t_1,...,t_{n-2i}} \subset H_{t_1} \cap ... \cap H_{t_{n-2i}}$, i.e. the set of (nonnecessarily pairwise different) moments $(t_1, ..., t_{n-2i}) \in \mathcal{T}^{n-2i}/\mathfrak{S}_{n-2i}$ is uniquely defined. For each individual intersection $H_{t_1} \cap ... \cap H_{t_{n-2i}}$ the homeomorphism h_{i-1} is already defined on the complement to the elliptic hulls of the curves $\gamma_1^{t_1,...,t_{n-2i}}$ and $\gamma_2^{t_1,...,t_{n-2i}}$. Since the elliptic hulls are convex domains and the fibrations of the elliptic hulls over $\mathcal{T}^{n-2i}/\mathfrak{S}_{n-2i}$ are trivial we can extend h_{i-1} fiberwise to h_i by identifying the points of the elliptic hull of $\gamma_1^{t_1,...,t_{n-2i}}$ with points of the elliptic hull of $\gamma_2^{t_1,...,t_{n-2i}}$ for all tuples $(t_1, ..., t_{n-2i}) \in \mathcal{T}^{n-2i}/\mathfrak{S}_{n-2i}$

b) The corresponding component $Comp_i$ of $\mathbb{P}^n \setminus D_\gamma$ contained in $\mathbb{P}_i \setminus \mathbb{P}_{i-1}$ is fibered over $(\mathcal{T}^{n-2i} \setminus Diag)/\mathfrak{S}_{n-2i}$ with the contractible fiber. Since $(\mathcal{T}^{n-2i} \setminus Diag)/\mathfrak{S}_{n-2i}$ is contractible to S^1 for any $n-2i > 0$ one gets that $Comp_i$ is contractible to S^1 for all n and $i \leq [\frac{n}{2}]$ except for Ell_γ for even n which is contractible to a point.

□

REMARK. For convex algebraic curves the above homeomorphism (constructed rather explicitly) can be made (piecewise) real algebraic. This leads to an interesting problem of studying properties of the complexification of this homeomorphism in the case when the initial convex curves are not projectively equivalent.

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