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RESEARCH

Introducing isodynamic points for binary forms and their ratios

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Abstract
The isodynamic points of a plane triangle are known to be the only pair of its centers invariant under the action of the Möbius group $\mathcal{M}$ on the set of triangles, Kimberling (Encyclopedia of Triangle Centers, http://faculty.evansville.edu/ck6/encyclopedia). Generalizing this classical result, we introduce below the *isodynamic* map associating to a univariate polynomial of degree $d \geq 3$ with at most double roots a polynomial of degree (at most) $2d - 4$ such that this map commutes with the action of the Möbius group $\mathcal{M}$ on the zero loci of the initial polynomial and its image. The roots of the image polynomial will be called the *isodynamic points* of the preimage polynomial. Our construction naturally extends from univariate polynomials to binary forms and further to their ratios.

Keywords Isodynamic points · Projective invariance · Polar derivative · Triangle centers

1 Introduction

One of the classical problems about triangles in the Euclidean plane is to find all points such that the distances to the vertices are inversely proportional to the lengths of the opposite sides. There are typically two such points called the *first and second isodynamic points* of the triangle under consideration. (Every equilateral triangle however has just one isodynamic point; the second one can be thought as lying at infinity.)

To Morris Marden, for his contributions to geometry of polynomials.

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An elementary construction of the isodynamic points using Apollonian circles of a triangle can be found in Fig. 1 below.[1]

Besides the invariance of the isodynamic points under the action of the Möbius group on the vertices of triangles, see the footnote, a large number of their intriguing properties can be found in two recent publications [12,13] and references therein. An interesting old source of information about the isodynamic points is the dissertation [11].

As we will see below, the classical geometric recipe associating to a plane triangle its isodynamic point(s) can be substituted by a pleasant explicit formula (3.1) associating to the unique monic cubic polynomial whose roots are the vertices of the triangle an appropriate polynomial of degree at most two vanishing at its isodynamic point(s). This formula seems to be missing in the existing literature.

Inspired by the latter observations, we consider the following question.

Problem 1 Find a (natural) map from an open and dense subset of complex-valued monic polynomials of a given degree $d \geq 3$ to some space of univariate polynomials which com-

[1] As we mentioned in the abstract, the isodynamic points can be also described as unique triangle centers which are invariant under Möbius transformations. The formal definition of triangle centers as well as more information about classical isodynamic points can be found in the Encyclopedia of Triangles Centers, [7] and in Sect. 4.

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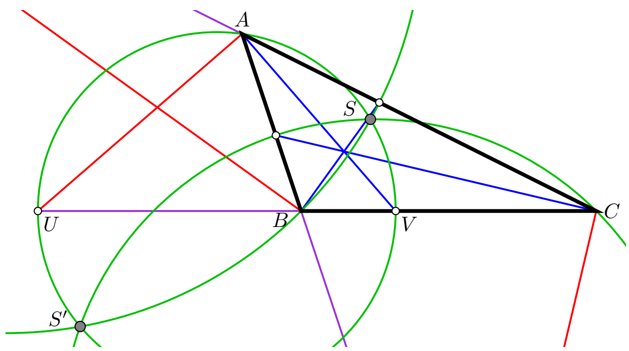

Fig. 1 Let U and V be the points on sideline BC met by the exterior and interior bisectors of angle A. The circle having diameter UV is called the *A-Apollonian circle*. The B- and C- Apollonian circles are constructed similarly. Each circle passes through one vertex and both isodynamic points S and S', see [7]

mutes with the action of the Möbius group $\mathcal{M} \simeq PGL_2(\mathbb{C})$ on the respective zero loci of the preimages and of the images.

Remark 1 Observe that the Möbius group $\mathcal{M}$ does not quite act on the space of monic polynomials or on their zero loci since a Möbius transformation typically has a pole in $\mathbb{C}$, i.e., it sends one point in $\mathbb{C}$ to ∞. The same difficulty already occurs in case of isodynamic points of triangles. Therefore, to ensure a well-defined action of the Möbius group $\mathcal{M}$ and for our Problem 1 to be correctly stated, we will later instead of the space of monic polynomials of a given degree d consider the space of homogeneous binary forms of the same degree d.

In what follows, we present one non-trivial solution to Problem 1 which for $d = 3$ sends cubic monic polynomials to polynomials of degree at most two and which, on the level of their zero loci, associates to a triple of points in the Euclidean plane the above classical isodynamic point(s) of their convex hull, see Proposition 10. We will call this map *isodynamic*.

More exactly, denote by Pol_d the affine space of monic complex-valued polynomials of degree d and by $Pol_d^\star \subset Pol_d$ its subset consisting of monic polynomials with roots of multiplicity at most 2. Further denote by $\mathcal{P}_n$ the linear space of all complex-valued univariate polynomials of degree at most n. (Observe that $\mathcal{P}_n$ is isomorphic to the linear space $\mathcal{B}_n$ of binary forms of degree n.) Denote by $\mathcal{B}_n^\star \subset \mathcal{B}_n$ the subset of homogeneous binary forms with roots of multiplicity at most 2 (considered as points in $\mathbb{C}P^1$).

Given a monic polynomial $P(z)$ of a given degree d, consider its *polar derivative*[2]

$$\mathcal{D}(z,u) = d\,P(z) + (u-z)P'(z). \tag{1.1}$$

[2] Polar derivative has been already considered by E. Laguerre and G. Pólya jointly with G. Szegö, who used different terminology. A nice survey of its properties is Ch. 3 of the famous treatise [10]. One the most recent application of polar derivative to geometry of polynomials can be found in [14].

Note that $\mathcal{D}(z,u)$ is a bivariate polynomial in (z,u) which is linear non-homogeneous in the variable u and has degree at most $d-1$ in z. The *(univariate) isodynamic map*

$$ID_d : Pol_d^\star \to \mathcal{P}_{2d-4} \tag{1.2}$$

sends a monic polynomial $P(z)$ of degree d with roots of multiplicity at most 2 to the polynomial $ID_d(P)$ in the variable u given by

$$ID_d(P) := \mathrm{Discr}(\mathcal{D}(z,u), z). \tag{1.3}$$

In other words, ID_d sends $P(z)$ to the discriminant of its polar derivative $\mathcal{D}(z,u)$ with respect to the variable z. The roots of $ID_d(P)$ are, by definition, the *isodynamic points* of $P(z)$.

Remark 2 Recall that for a bivariate polynomial $U(z,u)$, its discriminant $\mathrm{Discr}(U(z,u),z)$ with respect to the variable z is defined as follows. Assume that $U(z,u) = a_0(u)z^n + \cdots + a_n(u)$ has degree n in the variable z. Then,

$$\mathrm{Discr}(U(z,u),z) := \frac{(-1)^{\binom{n}{2}}}{a_0(u)}\mathrm{Resultant}(U(z,u), U_z'(z,u), z)$$

where Resultant stands for the standard resultant of two polynomials, see e.g. [5]. A similar definition is valid for polynomials in any number of variables.

Further let $\mathcal{I}_P$ be the zero locus of $ID_d(P)$ considered as a divisor of degree at most $2d-4$ in the complex plane of the variable u. We will call $\mathcal{I}_P$ the *(affine) isodynamic divisor* and its points the *(affine) isodynamic points* of $P(z)$. (In Fig. 1 the isodynamic points are shown in the same plane as the zeros of a cubic polynomial $P(z)$.)

We can also formulate a slightly different recipe of obtaining the isodynamic divisor $\mathcal{I}_P$. Namely, given a polynomial $P(z) = z^d + a_1z^{d-1} + a_2z^{d-2} + \cdots + a_d$ of degree d as above, consider the rational function

$$\begin{aligned} R_P(z) &:= z - \frac{d\,P(z)}{P'(z)} \\ &= -\frac{a_1z^{d-1} + 2a_2z^{d-2} + \cdots + (d-1)a_{d-1}z + da_d}{dz^{d-1} + (d-1)a_1z^{d-2} + \cdots + 2a_{d-2}z + a_{d-1}} \end{aligned} \tag{1.4}$$

which we call the *associated rational function* of P and take the divisor of the critical values of $R_P(z)$. One can show that the latter divisor coincides with $\mathcal{I}_P$, i.e. the critical values of $R_P(z)$ are the isodynamic points of $P(z)$, see Lemma 3 below. Thus the map (1.2) can be thought as a version of the *Lyashko-Looijenga map* sending a monic polynomial of degree d with simple roots to the univariate polynomial whose zero locus is the set of all critical values of its associated rational function, comp. [1].

Remark 3 There might exist natural extensions of the map ID_d to the space of all polynomials of a fixed degree or another appropriate compactification of $Pol_d^\star$, but we do not pursue this direction of study here.

Remark 4 One intriguing detail of our construction is that the associated rational function $R_P(z)$ is exactly the one used in the relaxed Newton method for finding roots of $P(z)$ for the relaxation parameter equal to d, see e.g. [6]. However, in our set-up we are interested in the critical values $R_P(z)$ and not in its intersection points with the diagonal line one typically approaches using iterations of $R_P(z)$. At the moment we do not see any obvious connection of our problem to complex dynamics of the relaxed Newton method although such connection is quite plausible.

Let us now modify the above construction to accommodate the case of binary forms and to ensure a well-defined action of the Möbius group $\mathcal{M}$ on the preimage and the image spaces. Given $P(z)$ as above, define its homogenization $P(x, y) := y^d P(\frac{x}{y})$, where $z = \frac{x}{y}$. Using this notation, in the homogeneous coordinates $(x : y)$ on $\mathbb{C}P^1$ one has the following alternative expression for the homogenization of the associated rational function

$$R_P\,(x, y) = -\frac{P_y'(x, y)}{P_x'(x, y)}, \tag{1.5}$$

where $R_P\,(x, y)$ is obtained by the homogenization of the enumerator and denominator of the above $R_P(z)$, see Lemma 3(ii). Noticing that $R_P\,(x, y)$ has the same degrees of its numerator and denominator and therefore defines a rational function on $\mathbb{C}P^1$, we can introduce the *(bivariate) isodynamic map*

$$ID_d : Proj(\mathcal{B}_d^\star) \to Proj(\mathcal{B}_{2d-4}) \tag{1.6}$$

sending a binary form $P(x, y) \in \mathcal{B}_d^\star$ (considered up to a constant factor) to the binary form ID_d^P (considered up to a constant factor) whose projectivized zero locus is the set of critical values of the rational function $-\frac{P_y'(x,y)}{P_x'(x,y)}$. The latter set of critical values is *(projective) isodynamic divisor* and its points are *(projective) isodynamic points* of $P(x, y)$. As above $\mathcal{B}_d^\star$ is the space of binary forms of degree d with roots of multiplicity at most 2 on $\mathbb{C}P^1$.

Our first result is as follows.

Theorem 1 *The natural action of the Möbius group $\mathcal{M}$ on $Proj(\mathcal{B}_d^\star)$ and $Proj(\mathcal{B}_{2d-4})$ commutes with the bivariate isodynamic map ID_d.*

Note that binary forms of degree d can be thought of as holomorphic sections of the sheaf $\mathcal{O}(d)$ on $\mathbb{C}P^1$; this circumstance sparkles the idea that the above construction of isodynamic map might extend to meromorphic sections of $\mathcal{O}(d)$ as well, i.e. to the ratios of binary forms $\frac{P(x,y)}{Q(x,y)}$, where $\deg P - \deg Q = d$. And indeed, such an extension exists.

Namely, consider the space $B_{d+\partial,\partial}^\star$ of pairs of binary forms $(P(x, y), Q(x, y))$ where P and Q are coprime, both P and Q have roots of multiplicity at most 2, $\deg P = d + \partial$, $\deg Q = \partial$, $d > 0$, and $\partial \geq 0$. We can interpret the pair $(P(x, y), Q(x, y))$ as the bivariate rational function $W(x, y) = \frac{P(x,y)}{Q(x,y)}$. (Note that $W(x, y)$ is not a rational function on $\mathbb{C}P^1$ in the usual sense, but is naturally a meromorphic section of $\mathcal{O}(d)$.) We can associate to $W(x, y)$ its divisor $D(W)$ of degree d whose positive part, i.e. divisor of zeros $D^0(W)$ has degree $d + \partial$ and whose negative part, i.e. divisor of poles $D_\infty(W)$ has degree $-\partial$.

Next, let us define the associated rational function of W as

$$R_W(x, y) = -\frac{W_y'(x, y)}{W_x'(x, y)}. \tag{1.7}$$

We will show that $R_W(x, y)$ has the same degree of its denominator $W_x'(x, y)$ as of its numerator $W_y'(x, y)$ implying that it is a well-defined rational function on $\mathbb{C}P^1$ or, equivalently, a meromorphic section of $\mathcal{O}(0)$. Namely, for generic binary forms $P(x, y)$ and $Q(x, y)$ as above the numerator and denominator both have degrees equal to $2d + 4\partial - 4$. This circumstance explains why although formula (1.7) looks the same for all meromorphic sections of $\mathcal{O}(d)$ it makes sense not only to fix d, but also the degree ∂ of the pole divisor.

Let us finally associate to the divisor $D(W)$ the positive divisor $D(R_W)$ of all critical values of the associated rational function R_W.

Formula (1.7) can also be rewritten in a way similar to (1.4). Indeed, restriction of the above section $W(x, y)$ of $\mathcal{O}(d)$ to the local chart $y = 1$ on $\mathbb{C}P^1$ can be identified with the univariate rational function

$$w(z) = \frac{p(z)}{q(z)}$$

where $p(t) = P(t, 1)$ and $q(t) = Q(t, 1)$. Typically, $\deg p(t) = \deg P(x, y) = d + \partial$ and $\deg q(t) = \deg Q(x, y) = \partial$. Moreover we can define the associated (univariate) rational function

$$R_w(z) := z - \frac{d\, w(z)}{w'(z)} = z - \frac{d\, p(z)q(z)}{p'(z)q(z) - p(z)q'(z)}.$$

One can easily see that $R_w(z) = R_W(x, y)|_{y=1}$ and its set of critical values can be defined similarly to (1.1).

Namely, for $w(z) = \frac{p(z)}{q(z)}$ with coprime $p(z)$ and $q(z)$ having roots of multiplicity at most 2 such that the degree $\deg p = d + \partial$ and $\deg q = \partial$ where $d \geq 1$, define its polar derivative as

$$\mathcal{D}(z, u) = d\, w(z) + (u - z)w'(z). \tag{1.8}$$

The numerator of $\mathcal{D}(z, u)$ is given by the expression

$$N\mathcal{D}(z, u) = d\, p(z)q(z) + (u - z)(p'(z)q(z) - p(z)q'(z)). \tag{1.9}$$

We define the *(univariate) rational isodynamic map*

$$ID_{d,\partial} : Rat^{\star}_{d+\partial,\partial} \to \mathcal{P}_{2d+4\partial-4} \tag{1.10}$$

as sending a rational function $w(z)$ to the polynomial given by

$$ID_{d,\partial}(u) := \mathrm{Discr}(N\mathcal{D}(z, u), z). \tag{1.11}$$

Here $Rat^{\star}_{d+\partial,\partial}$ is the space of rational functions $w = \frac{p}{q}$ with coprime pairs of polynomials (p, q) having only roots of multiplicity at most 2 , $\deg p = d + \partial$ and $\deg q = \partial$ for $\partial \geq 1$. If $\partial = 0$ then $Rat^{\star}_{d,0} = Pol^{\star}_{d}$. The fact that $ID_{d,\partial}$ is well-defined will be proven in Proposition 5 below.

Now define the *rational isodynamic divisor* $\mathcal{I}_w$ of the rational function w as the zero divisor of $ID_{d,\partial}(w)$. Similarly to the above, we can show that $\mathcal{I}_w$ is the divisor of the criticial values of $R_w(z)$.

Theorem 2 *In the above notation, the action of the Möbius group $\mathcal{M} \simeq PGL_2(\mathbb{C})$ on $Rat^{\star}_{d+\partial,\partial}$ and $\mathcal{P}_{2d+4\partial-4}$ commutes with the operation of taking the divisor of isodynamic points, i.e. with the map $ID_{d,\partial}$.*

The structure of the paper is as follows. In Sect. 2 we settle the above mentioned results and discuss several properties of isodynamic maps and their discriminants. In Sect. 3 we present several explicit formulas for the isodynamic maps for polynomials and rational functions in low degree cases. In Sect. 4 we recall the construction of the classical isodynamic points of triangles and show that they fit as a special case of our general framework. Finally, in Sect. 5 we indicate some further possible directions of study.

2 Proofs of basic results and some properties of isodynamic maps

2.1 Proofs of Theorems 1 and 2

We start with the following statement. As above set $w(z) = \frac{p(z)}{q(z)}$ with coprime p and q having roots of multiplicity at most 2 and $\deg p = d + \partial$, $\deg q = \partial$, $d \geq 1$. Further, let $R_w(z) = z - \frac{d\, w(z)}{w'(z)}$ and $W(x, y) = \frac{P(x,y)}{Q(x,y)}$, where $P(x, y) = y^{d+\partial} p\left(\frac{x}{y}\right)$ and $Q(x, y) = y^{\partial} q\left(\frac{x}{y}\right)$. Finally, set $\mathcal{D}(z, u) = d\, w(z) + (u - z)w'(z)$.

Lemma 3 *In the above notation,*
(i) *u^* is a value of the variable u such that the polar derivative $\mathcal{D}(z, u^*)$ has a multiple root in z if and only if u^* is a critical value of $R_w(z)$;*
(ii) *The associated rational function $R_W(x, y)$ obtained by homogenizing the numerator and denominator of $R_w(z)$ coincides with $-\frac{W'_y(x,y)}{W'_x(x,y)}$.*

Proof To settle (i), note that if u^* is a value of the variable u for which the polar derivative $\mathcal{D}(z, u)$ has a multiple root in the variable z then denoting this root by z^* we get

$$\begin{aligned} 0 &= d\, w(z^*) + (u^* - z^*)w'(z^*) \Leftrightarrow u^* \\ &= z^* - \frac{d\, w(z^*)}{w'(z^*)} = R_P(z^*). \end{aligned}$$

Note that the latter expression is well-defined if $w'(z^*) \neq 0$. Since z^* is a multiple root of $R_w(z)$, u^* has to be a critical value of $R_w(z)$ at z^*.

To settle (ii), let us rewrite the ratio $-\frac{W'_y}{W'_x}$. By definition, $W(x, y) = y^d \frac{p\left(\frac{x}{y}\right)}{q\left(\frac{x}{y}\right)}$. Thus

$$\begin{aligned} W'_x &= y^d \frac{\frac{1}{y}\left(p'\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right) q'\left(\frac{x}{y}\right)\right)}{q^2\left(\frac{x}{y}\right)} \\ &= y^{d-1} \frac{\left(p'\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right) q'\left(\frac{x}{y}\right)\right)}{q^2\left(\frac{x}{y}\right)}. \end{aligned}$$

Similarly,

$$\begin{aligned} W'_y &= d \cdot y^{d-1} \frac{p\left(\frac{x}{y}\right)}{q\left(\frac{x}{y}\right)} \\ &\quad + \frac{y^d\left(-\frac{x}{y^2}\right)\left(p'\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right) q'\left(\frac{x}{y}\right)\right)}{q^2\left(\frac{x}{y}\right)}. \end{aligned}$$

The above implies

$$\begin{aligned} -\frac{W'_y}{W'_x} &= -\frac{d\, p\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right) - \frac{x}{y}\left(p'\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right) q'\left(\frac{x}{y}\right)\right)}{p'\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right) q'\left(\frac{x}{y}\right)} \\ &= \frac{x}{y} - \frac{d\, p\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right)}{p'\left(\frac{x}{y}\right) q\left(\frac{x}{y}\right) - p\left(\frac{x}{y}\right) q'\left(\frac{x}{y}\right)} \end{aligned}$$

$$= z - \frac{d\,p(z)q(z)}{p'(z)q(z) - p(z)q'(z)},$$

where $z = \frac{x}{y}$. □

Since Theorem 1 is a special case of Theorem 2 we present below only the proof of the latter result.

Proof Using the above notation, set $R_W(x, y) = -\frac{W'_y}{W'_x}$ and make a change of coordinates $u = \alpha x + \beta y;\ v = \gamma x + \delta y$. Using the chain rule we obtain

$$\begin{cases} W'_x = \alpha W'_u + \gamma W'_v \\ W'_y = \beta W'_u + \delta W'_v \end{cases} \Leftrightarrow \begin{cases} W'_u = \frac{1}{\mathfrak{D}}\left(\delta W'_x - \gamma W'_y\right) \\ W'_v = \frac{1}{\mathfrak{D}}\left(-\beta W'_x + \alpha \frac{\partial}{\partial y}\right), \end{cases} \tag{2.1}$$

where $\mathfrak{D} = \alpha\delta - \beta\gamma$. Thus introducing $R_W(u, v) = -\frac{W'_v}{W'_u}$, we obtain

$$\begin{aligned} R_W(u, v) &= -\frac{-\beta W'_x + \alpha W'_y}{\delta W'_x - \gamma W'_y} = \frac{\alpha\left(\frac{-W'_y}{W'_x}\right) + \beta}{\gamma\left(\frac{-W'_y}{W'_x}\right) + \delta} \\ &= \frac{\alpha R_W(x, y) + \beta}{\gamma R_W(x, y) + \delta}. \end{aligned}$$

Thus, the action of the Möbius group $\mathcal{M}$ in the homogeneous coordinates $(x : y)$ results in the same action of $\mathcal{M}$ on the associated rational functions which implies that the locus of critical values experiences the same action. The result follows. □

2.2 Discriminant of the isodynamic map

Below we discuss possible situations when the image of the isodynamic map $ID_{d,\partial}$ is degenerate. Namely, we describe

(i) For which pairs $(p(z), q(z))$ the map $ID_{d,\partial}$ is well-defined (respectively, for which polynomials $P(z)$ the map ID_d is well-defined);

(ii) For which pairs $(p(z), q(z))$ the corresponding divisor $\mathcal{I}_{p,q}$ has degree less than $2d + 4\partial - 4$ in $\mathbb{C}$ (respectively, for which polynomials $P(z)$ of degree d the divisor $\mathcal{I}_P$ has degree less than $2d - 4$);

(iii) For which pairs $(p(z), q(z))$, the corresponding divisor $\mathcal{I}_{p,q}$ has multiple roots (respectively, for which polynomials $P(z)$ of degree d, the corresponding divisor $\mathcal{I}_P$ has multiple roots).

The latter problem can be reformulated as the question of when the associated rational function $R_P(z)$ does not have $2d - 4$ distinct critical values. (In [2] dealing with a similar situation this set of parameters is called the *Hurwitz discriminant*.)

In order to answer the latter questions we need an additional statement.

Lemma 4 *A polynomial pencil $f(z, u) = A(z) + uB(z)$ has a multiple root w.r.t the variable z for each value of the variable u if and only if $A(z)$ and $B(z)$ have a common root which has multiplicity at least 2 for both $A(z)$ and $B(z)$.*

Proof Denote by $z^*(u)$ some multiple root of the pencil $f(z, u)$ for a given value of the variable u. It has to satisfy the following system of algebraic equations:

$$\begin{cases} f(z^*(u), u) = 0 \\ \frac{\partial f}{\partial z}(z^*(u), u) = 0 \end{cases} \leftrightarrow \begin{cases} A(z^*(u)) + uB(z^*(u)) = 0 \\ A'(z^*(u)) + uB'(z^*(u)) = 0. \end{cases}$$

(By the assumptions of the lemma, the set of multiple roots $z^*(u)$ is a complex algebraic curve in the space $\mathbb{C}^2$ with coordinates (u, z) which projects surjectively on the u-axis.) Differentiating the first equation in the latter system w.r.t the variable u we get

$$B(z^*(u)) + \left(A'(z^*(u)) + uB'(z^*(u))\right)\frac{\mathrm{d}z^*(u)}{\mathrm{d}u} = 0.$$

Using the second equation from the latter system we conclude that $B(z^*(u)) = 0$ which implies that $z^*(u) = z^*$ is a constant and this constant is a root of $B(z)$. Thus z^* is also a root of $A(z)$. Finally, by our assumptions z^* is a multiple root of both $A(z)$ and $B(z)$. □

The following claim now answers the above question (i).

Proposition 5 *The map $ID_{d+\partial,\partial}$ applied to a rational function $w(z) = \frac{p(z)}{q(z)}$, $\deg p = d + \partial$, $\deg q = \partial$ given by (1.11) is well-defined if and only if*

either

(*) $\partial > 0$, *and the polynomials p, q are coprime and have roots of multiplicity at most 2;*

or

(**) $\partial = 0$ (i.e. $q \equiv 1$), *and all roots of the polynomial P have multiplicity at most 2.*

Proof To settle (*), let us first show that in case $\partial > 0$, if $p(z)$ and $q(z)$ have a common root z^* then $N\mathcal{D}(z, u)$ and $N\mathcal{D}'_z(z, u)$ have a common root w.r.t z for any choice of u. Thus for such $(p(z), q(z))$, the map $ID_{d+\partial,\partial}$ is not defined. Indeed, we have

$$\begin{cases} N\mathcal{D}(z^*, u) = d \cdot p(z^*)q(z^*) + (u - z^*)(p'(z^*)q(z^*) \\ \qquad\qquad -p(z^*)q'(z^*)) \equiv 0 \\ N\mathcal{D}'_z(z^*, u) = (d-1)p'(z^*)q(z^*) + (d+1)p(z^*)q(z^*) \\ \qquad\qquad +(u - z^*)(p''(z^*)q(z^*) - p(z^*)q''(z^*)) \equiv 0, \end{cases}$$

i.e., both $N\mathcal{D}(z^*, u)$ and $N\mathcal{D}'_z(z^*, u)$ vanish identically in the variable u. Exactly the same argument holds if either p or q has a root z^* of multiplicity exceeding 2.

Let us now show the converse implication, i.e., that $ID_{d+\partial,\partial}$ is well-defined if $p(z)$ and $q(z)$ are coprime and have no roots of multiplicity bigger than 2. Assume that $ID_{d+\partial,\partial}$ is not defined for the pair (p, q) which is equivalent to the fact that $N\mathcal{D}(z, u)$ has a multiple root w.r.t z for all values of u. Observe that $N\mathcal{D}(z, u)$ is a polynomial pencil of the form $F(z)+u\Phi(z)$ where $F(z) = d\, pq-z(p'q-pq')$ and $\Phi(z) = (p'q - pq')$. Thus, by Lemma 4, the univariate polynomials $F(z)$ and $\Phi(z)$ must have a common root of multiplicity at least 2 at some point z^*. But if $F(z)$ and $\Phi(z)$ have a multiple common root at z^* then pq and $p'q - pq'$ have a multiple common root at z^* as well. But the latter claim is impossible since p and q are coprime and therefore pq has no multiple roots.

To settle (**) let us first show that if $P(z)$ has a root z^* of multiplicity at least 3, then $\mathcal{D}(z, u) = d\, P(z)+(u-z)P'(z)$ has a multiple root w.r.t. z for any choice of u. Indeed,

$$\begin{cases}\mathcal{D}(z^*, u) = d\, P(z^*) + (u - z^*)P'(z^*) \equiv 0,\\ \mathcal{D}'_z(z^*, u) = (d - 1)\, P'(z^*) + (u - z^*)P''(z^*) \equiv 0.\end{cases}$$

since P has a least a triple root at z^*. Now assume the converse, i.e., that $\mathcal{D}(z, u)$ has a multiple root w.r.t z for any value of u. Since $\mathcal{D}(z, u)$ is a polynomial pencil of the form $dP(z) - zP'(z) + uP'(z)$, Lemma 4 implies that $P(z)$ and $P'(z)$ must at least have a common double root z^*, i.e., $P(z)$ has a least a triple root. □

Next define the polynomial families $p(z) = z^{d+\partial} + a_1 z^{d+\partial-1} + \cdots + a_{d+\partial}$ and $q(z) = z^{\partial} + b_1 z^{\partial-1} + \cdots + b_{\partial}$. Using these families, consider the expression (1.9) and the map (1.11). The following Lemma settles question (ii).

Lemma 6 (i) *For $\partial > 0$, in the above notation and up to a constant factor, the expression for the map $ID_{d,\partial}$ has a factor* Resultant(p, q)*, where* Resultant(p, q) *stands for the resultant of p and q. The leading coefficient of $ID_{d,\partial}$, i.e. the coefficient of $u^{2d+4\partial-4}$ equals* Discr$(p'q - pq')$*, where* Discr$(p'q - pq')$ *stands for the discriminant of $\Phi(z) = p'q - pq'$. (The latter discriminant factors into* Resultant(p, q) *and an additional factor, see Remark 6.)*

(ii) *For $\partial = 0$ the leading coefficient of ID_d equals the discriminant* Discr(P') *of $P'(z)$. In other words, the polynomial $ID_d(P)$ has degree less than $2d - 4$ if and only if the derivative of the initial polynomial P has multiple zeros.*

Proof To settle (i), observe that by Proposition 5, if $p(z)$ and $q(z)$ have a common root z^* then the bivariate polynomial $N\mathcal{D}(z, u)$ given by (1.10) has a multiple root at z^* in the variable z for all values of the second variable u which implies that the map $ID_{d,\partial}$ vanishes identically. To show that the leading coefficient of $ID_{d,\partial}$ equals Discr$(p'q - pq')$ let us calculate $ID_{d,\partial}$ using the standard determinantal formula for the resultant of the Sylvester matrix, see e.g. [5], ch. 12. In our situation each non-vanishing entry of the Sylvester matrix will be a linear non-homogeneous polynomial in u with the leading coefficient and the constant term being polynomials in z. If we drop all the constant terms and just keep the terms containing u in the Sylvester matrix, then the determinant of this matrix will be equal to $u^{2d+4\partial-4}$ times the discriminant of the coefficient of u in the original expression $N\mathcal{D}(z, U)$. Since this coefficient equals $p'q - q'p$, the claim follows.

The argument for (ii) is exactly the same as for (i). □

Remark 5 Observe that Lemma 6 explains why in the classical situation there exists only one isodynamic point for a triple of non-collinear points in the plane if and only if they form an equilateral triangle; see Sect. 3 below.

To solve question (iii), let us first discuss a more general set-up. Assume that we consider a family $\Phi_\lambda(z) = \frac{U_\lambda(z)}{V_\lambda(z)}$ of univariate rational functions depending on some multi-dimensional parameter λ taking values in some connected complex algebraic variety Λ. We assume that

(a) Univariate polynomials $U_\lambda(z)$ and $V_\lambda(z)$ are coprime for generic values of $\lambda \in \Lambda$, but they can have a common factor for special values of λ belonging to some complex algebraic subvariety of Λ;
(b) For generic λ, $\Phi_\lambda(z)$ has distinct and simple critical values.

By our assumptions, the number of distinct critical values of $\Phi_\lambda(z)$ is constant for almost all $\lambda \in \Lambda$. Denoting this number by κ, let us define the Lyashko-Looijenga map $\mathcal{L} : \Lambda \to \mathbb{C}^\kappa$ sending every point $\lambda \in \Lambda$ to the divisor of the critical values of $\Phi_\lambda(z)$. We define the *Hurwitz discriminant* $\mathcal{H}D \subset \Lambda$ as the set of all $\lambda \in \Lambda$ for which the divisor $\mathcal{L}(\lambda)$ does not consist of κ simple points. According to [2], set-theoretically, the Hurwitz discriminant $\mathcal{H}D$ typically contains three irreducible components $\mathcal{H}D^0 \cup \mathcal{H}D^W \cup \mathcal{H}D^M$, where

(i) $$\mathcal{H}D^0 = \left\{\lambda \in \Lambda | \exists z^* \text{ such that } U_\lambda(z^*) = V_\lambda(z^*) = 0\right\};$$

(ii) $\mathcal{H}D^W$ is the closure of $\mathcal{H}D^W_o$ where

$$\mathcal{H}D^W_o = \left\{\lambda \in \Lambda \setminus \mathcal{H}D^0 | \exists z^* \text{ such that } W_\Lambda(U; V) \text{ has (at least) a double root at } z^*\right\},$$

and $W_\Lambda(U;V) = U'_\lambda(z)V_\lambda(z) - U_\lambda(z)V'_\lambda(z)$ is the Wronskian of $U_\lambda(z)$ and $V_\lambda(z)$;

(iii) $\mathcal{H}D^M$ is the closure of $\mathcal{H}D^M_o$ where

$$\begin{aligned}\mathcal{H}D^M_o &= \{\lambda \in \Lambda | \exists z_1 \neq z_2 \text{ such that } \Phi'_\lambda(z_1) = \Phi'_\lambda(z_2) \\ &= 0 \text{ and } \Phi_\lambda(z_1) = \Phi_\lambda(z_2)\}.\end{aligned}$$

Remark 6 The union $\mathcal{H}D^0 \cup \mathcal{H}D^W$ considered as a subset of Λ coincides with the zero locus of the discriminant of the Wronskian $W_\Lambda(U;V)$ with respect to z. The third irreducible component $\mathcal{H}D^M$ is usually referred to as the *Maxwell stratum*, see e.g. [8].

In our specific situation the family of rational functions $R_P(z)$ under consideration is given by (1.4) where $P(z)$ runs over the space Pol_d. Set

$$\begin{cases}\mathcal{U}_P(z) = a_1 z^{d-1} + 2a_2 z^{d-2} + \cdots + (d-1)a_{d-1}z \\ \qquad +da_d = zP'(z) - d\,P(z); \\ \mathcal{V}_P(z) = dz^{d-1} + (d-1)a_1 z^{d-2} + \cdots + 2a_{d-2}z \\ \qquad +a_{d-1} = P'(z)\end{cases} \tag{2.2}$$

i.e., $\mathcal{U}_P(z)$ is the numerator and $\mathcal{V}_P(z)$ is the denominator of the associated rational function $R_P(z)$.

Set $\mathcal{D}_d :=$ Resultant $\left(ID_d, \frac{\partial ID'_d}{\partial u}, u\right)$. Numerical experiments with Mathematica strongly support the following guess.

Conjecture 1 *(a) In the above notation, for any $d \geq 3$, $\mathcal{D}_d = (\mathcal{D}^0_d)^{j^0_d}(\mathcal{D}^W_d)^{j^W_d}(\mathcal{D}^M_d)^{j^M_d}$, where $\mathcal{D}^0_d, \mathcal{D}^W_d, \mathcal{D}^M_d$ are irreducible polynomials whose zero loci satisfy the above conditions* (i), (ii), (iii) *respectively and j^0_d, j^W_d, j^M_d are some non-negative multiplicities.*

(b) The non-negative multiplicities j^0_d, j^W_d, j^M_d are all positive for $d \geq 5$, see Sect. 3.

Lemma 7 *In the above notation, $\mathcal{U}_P(z)$ and $\mathcal{V}_P(z)$ are linearly dependent, i.e., $R_P(z)$ is a constant if and only if $P(z) = (z+t)^d$ for some constant t.*

Proof Direct calculation of proportionality. □

Lemma 8 *In the above notation, $\mathcal{U}_P(z)$ and $\mathcal{V}_P(z)$ have a common zero if and only $P(z)$ has a multiple root, i.e., the discriminant $\mathcal{D}^0_d$ coincides with the discriminant of $P(z)$.*

Proof Indeed, if $\mathcal{U}_P(z^*) = \mathcal{V}_P(z^*) = 0$ then $P(z^*) = P'(z^*) = 0$ which means that $P(z)$ has a multiple root at z^*. □

Remark 7 Equation (2.2) implies that for a given degree d, the Wronskian $W(\mathcal{U}_P, V_P)$ is given by $(d-1)(P')^2 - dPP''$. The latter expression has previously occurred in [9] in connection with the stronger form of Laguerre inequality, see also [4].

Table 1 Known centroids of the zeros of $\mathrm{Discr}(D^\alpha_P(z,u),z)$ for cubic P, for $\alpha = -50, -49, \ldots, 50$

α	m
-12	$X(316)$
-6	$X(31173)$
-3	$X(625)$
-1	$X(10150)$
0	$X(2)$
2	$X(5215)$
3	$X(187)$
4	$X(26613)$
6	$X(187)$
24	$X(14712)$

Note that the definition (1.8) of polar derivative of rational functions hints the following definition of a more general polar derivative

$$D^\alpha_f(z,u) = \alpha f(z) + (u-z)f'(z) \tag{2.3}$$

where $f(z)$ belongs to some suitable class of functions and α is a real parameter. This expression appears naturally e.g. in the calculation of the asymptotic root-counting measure of the sequence of polynomials $\{(P^n)^{(\lfloor \alpha n \rfloor)}\}$, see [3]. In our situation, for $\alpha = 1$ and polynomials $P(z)$ of degree d, we can define the degree $d-2$ polynomial $\mathcal{W}_P(u) = \mathrm{Discr}(D^1_P(z,u),z)/P(u)$ with zeros $u_1, \ldots, u_{d-2}$, which has the following connection to the isodynamic points of P.

Lemma 9 *For any triple of non-collinear points z_1, z_2, z_3, i.e. for $d = 3$,*

$$u_1 = \left(\frac{2}{3}\right)^2 \frac{(z_1+z_2+z_3)^3 - 3^3 z_1 z_2 z_3}{\mathrm{Discr}(P'(z),z)}$$

is the centroid of the isodynamic points $\mathcal{I}_1, \mathcal{I}_2$ and the mass center $(z_1+z_2+z_3)/3$. Furthermore, u_1 is listed as $X(26613)$ in the Encyclopedia of Triangle Centers [7], and is the Dao-6-point-circle-inverse of $(z_1+z_2+z_3)/3$.

Proof Trivial calculations. □

Additionally, if $m = m(\alpha)$ denotes the centroid of the zeros of $\mathrm{Discr}(D^\alpha_P(z,u),z)$, then for cubic polynomials P, $m(\alpha)$ coincides with several known triangle centers in [7] for various values of $\alpha \in \mathbb{Z}$, see Table 1. (The connection between $m(\alpha)$ and these points can be proved by converting their corresponding trilinear or absolute barycentric coordinates in [7] to complex numbers using a reference triangle with vertices z_1, z_2, and z_3. Mathematica files containing these calculations are available upon request.)

Remark 8 Due to the considerable number of known triangle centers in Table 1, and the fact that $m(\alpha)$ traverses a real line in $\mathbb{C}$ as α runs over the real line for any polynomial P of degree $d \geq 2$ with simple zeros, the authors think that this line is worth further study.

3 Examples of isodynamic maps

To illustrate our results and in order to give some intuition about the objects of our study, we present below explicit formulas for the isodynamic maps and their discriminants for polynomials and rational functions of low degrees.

3.1 Isodynamic maps for polynomials

3.1.1 Cubics

In the classical case $d = 3$, direct calculations give the following explicit formula for the isodynamic map

$$ID_3 : z^3 + az^2 + bz + c \mapsto (a^2 - 3b)u^2 + (ab - 9c)u + b^2 - 3ac. \tag{3.1}$$

Equation (3.1) implies that if the image $ID_3(P)$ of a cubic polynomial P is linear, i.e., $a^2 = 3b$ then $P(z) = (z + a/3)^3 + c - a^3/27$ which means that its zero locus is necessarily an equilateral triangle.

In the above notation, the discriminant $\mathcal{D}_3$ of the family $ID_3(P)$ w.r.t. the variable u equals

$$\mathcal{D}_3 = 27c^2 + (4a^3 - 18ab)c + 4b^3 - a^2b^2$$

which is exactly the discriminant $\mathcal{D}_3^0$ of the original polynomial family $P(z) = z^3 + az^2 + bz + c$ w.r.t. the variable z. In other words, two isodynamic points of a triple of points in the plane coincide if and only if at least two of the points in the triple are equal. Thus for $d = 3$, $j_3^0 = 1$ and the discriminants $\mathcal{D}_3^W$ and $\mathcal{D}_3^M$ are empty, see Conjecture 1.

3.1.2 Quartics

To simplify our formulas, note that the affine shift $z \to z + t$ acts trivially on the isodynamic map ID_4. Using this fact, let us restrict our considerations to the standard reduced polynomial family

$$P_4(z) = z^4 + az^2 + bz + c$$

which is the classical versal deformation of the singularity z^4. The isodynamic map restricted to the latter family is explicitly given by

$$\begin{aligned} ID_4 : z^4 + az^2 + bz + c \mapsto &(32a^3 + 108b^2)u^4 \\ &+(72a^2b + 864bc)u^3 \\ &+(-4a^4 + 108ab^2 - 288a^2c + 1728c^2)u^2 \\ &+(-4a^3b + 108b^3 - 432abc)u - 9a^2b^2 + 32a^3c. \end{aligned} \tag{3.2}$$

Up to a constant factor, the leading coefficient $(32a^3 + 108b^2)$ in the right-hand side of (3.2) is the discriminant of the derivative of $P_4(z)$ with respect of z.

The discriminant of the right-hand side of (3.2) w.r.t. u is given by

$$\begin{aligned} \mathcal{D}_4 = &(2a^3 + 27b^2 - 72ac)^6(-4a^3b^2 - 27b^4 + 16a^4c + 144ab^2c \\ &-128a^2c^2 + 256c^3). \end{aligned} \tag{3.3}$$

The second factor $-4a^3b^2 - 27b^4 + 16a^4c + 144ab^2c - 128a^2c^2 + 256c^3$ of the right-hand side of (3.3) is $\mathcal{D}_4^0$ which is the standard discriminant of the above family $P_4(z)$. The first factor $2a^3 + 27b^2 - 72ac$ is the Wronski discriminant $\mathcal{D}_4^W$. (Both discriminants are quasi-homogeneous with weights $w(a) = 2$, $w(b) = 3$, $w(c) = 4$ of the variables. The total weight of $\mathcal{D}_4$ equals 12; the total weights of $\mathcal{D}_4^0$ and of $\mathcal{D}_4^W$ equal 6.) The Maxwell discriminant $\mathcal{D}_4^M$ is empty and $j_4^0 = 1$, $j_4^W = 6$. Figure 2 shows these discriminants separately and together.

3.1.3 Quintics

Here we present the formulas for ID_5 and its discriminant $\mathcal{D}_5$ since $d = 5$ is the minimal value for which $\mathcal{D}_d$ contains all three irreducible components including the Maxwell stratum $\mathcal{D}_d^M$. (All formulas were calculated in Mathematica.)

For $P(z) = z^5 + az^3 + bz^2 + cz + e$, one has

$$\begin{aligned} ID_5 = &(-405a^4c + 135a^3b^2 + 1800a^2c^2 - 2700ab^2c \\ &+675b^4 - 2000c^3)u^6 - 16(-2025a^4e - 180a^3bc \\ &+135a^2b^3 + 18000a^2ce - 13500ab^2e \\ &-6000abc^2 + 2700b^3c - 30000c^2e)u^5 \\ &+(27a^5c - 9a^4b^2 - 3600a^3be + 1800a^3c^2 \\ &-600a^2b^2c + 45000a^2e^2 + 135ab^4 \\ &-6000abce - 8400ac^3 - 13500b^3e + 7200b^2c^2 \\ &-150000ce^2)u^4 + (135a^5e - 9a^4bc \\ &-2a^3b^3 + 3600a^3ce \\ &-3000a^2b^2e + 1560a^2bc^2 - 720ab^3c \\ &+120000abe^2 - 30000ac^2e + 135b^5 \\ &-36000b^2ce + 10800bc^3 \\ &-250000e^3)u^3 + (135a^4be - 117a^4c^2 + 51a^3b^2c \\ &-13500a^3e^2 - 9a^2b^4 + 13200a^2bce - 2040a^2c^3 \\ &-3600ab^3e + 180ab^2c^2 + 60000ace^2 \\ &+135b^4c + 45000b^2e^2 - 54000bc^2e + 10800c^4)u^2 \\ &+(-270a^4ce + 135a^3b^2e + 12a^3bc^2 - 9a^2b^3c \end{aligned}$$

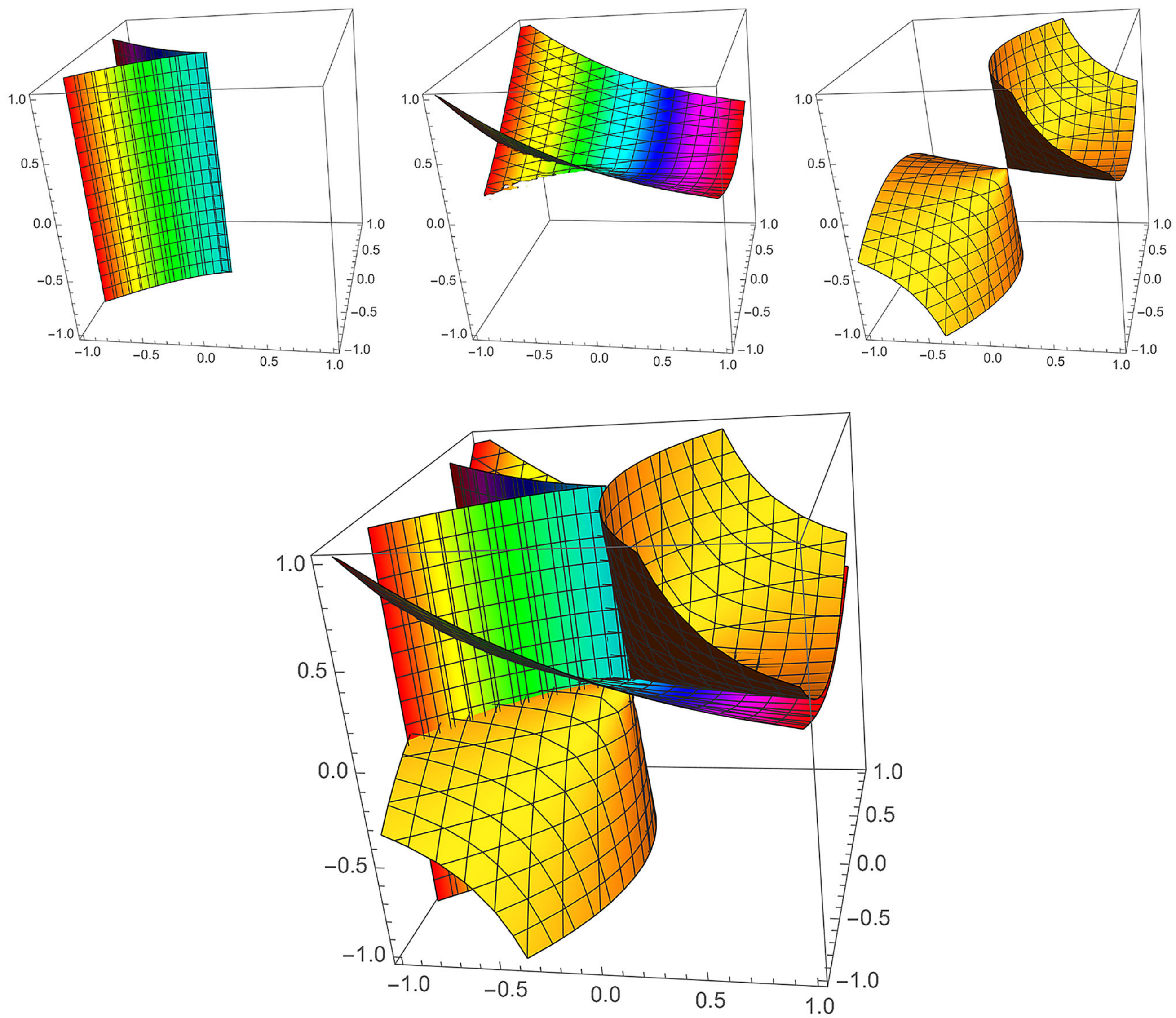

Fig. 2 Discriminants of the reduced isodynamic family ID_4 and their union (bottom). The top-left subfigure is the discriminant of P', the top-central subfigure is $\mathcal{D}_4^0$, i.e. the discriminant of P, and the top-right subfigure is $\mathcal{D}_4^W$. (The vertical axis is c, the horizontal one is a and the one pointing northeast is b.)

$$
\begin{aligned}
&-13500a^2be^2 + 600a^2c^2e + 9000ab^2ce \\
&-2160abc^3 - 2025b^4e + 540b^3c^2)u + (675a^4e^2 \\
&-540a^3bce + 128a^3c^3 + 135a^2b^3e - 36a^2b^2c^2).
\end{aligned}
$$

Standard discriminant:

$$
\begin{aligned}
\mathcal{D}_5^0 &= 108a^5e^2 - 72a^4bce + 16a^4c^3 + 16a^3b^3e \\
&-4a^3b^2c^2 - 900a^3ce^2 + 825a^2b^2e^2 + 560 \\
&a^2bc^2e - 128a^2c^4 - 630ab^3ce \\
&+144ab^2c^3 - 3750abe^3 + 2000ac^2e^2 + 108b^5 \\
&e - 27b^4c^2 + 2250b^2ce^2 - 1600bc^3e \\
&+256c^5 + 3125e^4.
\end{aligned}
$$

Wronskian discriminant:

$$
\begin{aligned}
\mathcal{D}_5^W &= 675a^{10}e^2 - 450a^9bce - 8a^9c^3 + 100a^8b^3e \\
&+83a^8b^2c^2 - 36000a^8ce^2 - 36a^7b^4c + 19500a^7b^2e^2 \\
&+22400a^7bc^2e + 640a^7c^4 + 4a^6b^6 \\
&-17100a^6b^3ce - 4400a^6b^2c^3 - 150000a^6be^3 \\
&+620000a^6c^2e^2 + 2650a^5b^5e + 3920a^5b^4c^2 \\
&-540000a^5b^2ce^2 - 328000a^5bc^3e - 19200a^5c^5 \\
&-10000000a^5e^4 - 1110a^4b^6c + 157500a^4b^4e^2 \\
&+394000a^4b^3c^2e + 77600a^4b^2c^4 + 18000000a^4bce^3 \\
&-3200000a^4c^3e^2 + 100a^3b^8 - 162000a^3b^5ce \\
&-88000a^3b^4c^3 - 5500000a^3b^3e^3 - 5000000a^3b^2c^2e^2
\end{aligned}
$$

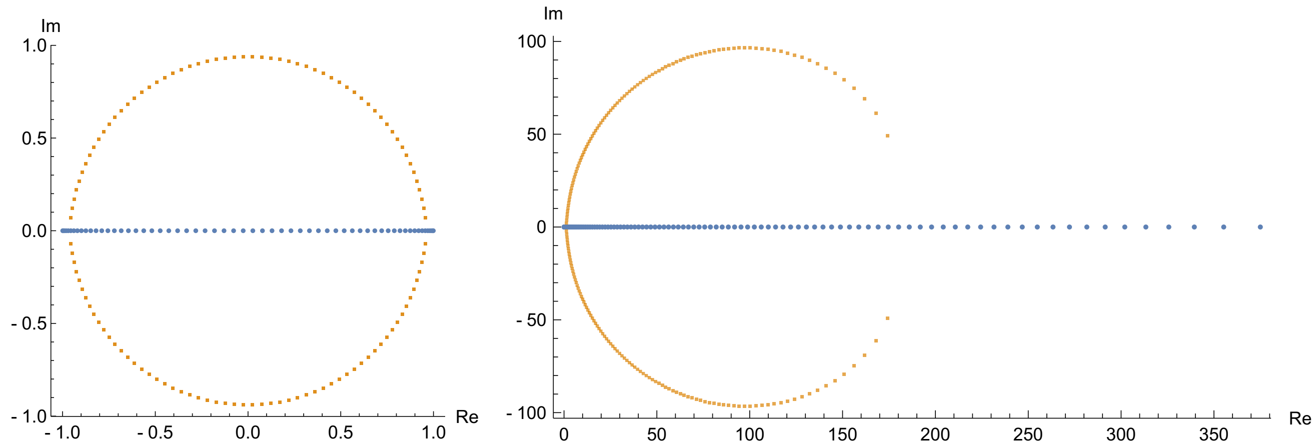

Fig. 3 The zeros of the 60th Legendre polynomial (left, blue) and the 100th Laguerre polynomial (right, blue), shown along with the isodynamic points of these polynomials (brown)

$$\begin{aligned}&+640000a^3bc^4e+256000a^3c^6+19500a^2b^7e\\&+42800a^2b^6c^2+2700000a^2b^4ce^2\\&+920000a^2b^3c^3e-480000a^2b^2c^5-25000000a^2b^2e^4\\&+20000000a^2bc^2e^3-4000000a^2c^4e^2\\&-9000ab^8c-150000ab^6e^2-700000ab^5c^2e\\&+224000ab^4c^4+30000000ab^3ce^3-32000000ab^2c^3e^2\\&+11200000abc^5e-1280000ac^7+675b^{10}\\&+90000b^7ce-28000b^6c^3-10000000b^5e^3\\&+11000000b^4c^2e^2-4000000b^3c^4e+480000b^2c^6.\end{aligned}$$

Maxwell discriminant:

$$\begin{aligned}\mathcal{D}_5^M &= 9c^2a^6-3375e^2a^5-6b^2ca^5+b^4a^4-380c^3a^4\\&+2325bcea^4+40b^2c^2a^3+30000ce^2a^3\\&-525b^3ea^3+4400c^4a^2-26250b^2e^2a^2+15b^4ca^2\\&-17000bc^2ea^2-4800b^2c^3a+125000be^3a\\&-50000c^2e^2a+19500b^3cea-8000c^5+900b^4c^2\\&-75000b^2ce^2-3375b^5e+50000bc^3e.\end{aligned}$$

Finally, $\mathcal{D}_5=\mathcal{D}_5^0(\mathcal{D}_5^W)^3(\mathcal{D}_5^M)^2$. In other words, $j_5^0=1,\ j_5^W=3,\ j_5^M=2$.

3.2 Isodynamic maps for rational functions

3.2.1 Case 1–1

Let us consider the isodynamic map for the simplest non-trivial family of rational functions $w(z)=\frac{p(z)}{q(z)}$ where $p(z)=z^2+az+b,\ q(z)=z+c$. (In this case $d=\partial=1$.) Then

$$ID_{1,1}:\frac{z^2+az+b}{z+c}\mapsto 4(b-ac+c^2)(u^2+au+b)$$

and its discriminant equals

$$\mathcal{D}_{1,1}=16(a^2-4b)(b-ac+c^2)^2.$$

3.2.2 Case 2–1

Consider $w(z)=\frac{p(z)}{q(z)}$ where $p(z)=z^3+az^2+bz+c,\ q(z)=z+e$. (In this case $d=2$ and $\partial=1$.) Then

$$\begin{aligned}&ID_{2,1}:\frac{z^3+az^2+bz+c}{z+e}\mapsto\\&4(e^3-c+be-ae^2)((-a^3+27c+9a^2e-27be)u^4\\&+(-6a^2b+54ac+8a^3e-18abe-54ce)u^3\\&+(18a^2c-12ab^2+54bc\\&+12a^2be-18b^2e-54ace)u^2\\&+(18abc-8b^3+54c^2+6ab^2e-54bce)u\\&-9b^2c+27ac^2+b^3e-27c^2e).\end{aligned}$$

The discriminant of $ID_{2,1}$ is given by

$$\begin{aligned}\mathcal{D}_{2,1}=&-1289945088(-a^2b^2+4b^3+4a^3c-18abc\\&+27c^2)^4(c-be+ae^2-e^3)^8.\end{aligned}$$

3.2.3 Case 1–2

Consider $w(z)=\frac{p(z)}{q(z)}$ where $p(z)=z^3+az^2+bz+c,\ q(z)=z^2+ez+f$. (In this case $d=1$ and $\partial=2$.) Then

$$\begin{aligned}&ID_{1,2}:\frac{z^3+az^2+bz+c}{z^2+ez+f}\mapsto\\&-16(c+bu+au^2+u^3)(c^2-bce+ace^2-ce^3\\&+b^2f-2acf-abef+3cef+be^2f+a^2f^2-2bf^2\\&-aef^2+f^3)((-b^3+27c^2+3ab^2e-27bce\end{aligned}$$

$$
\begin{aligned}
&-3a^2be^2+27ace^2+a^3e^3-27ce^3\\
&+18b^2f-54acf-9abef+81cef-9a^2e^2f\\
&+27be^2f+27a^2f^2-81bf^2)u^3\\
&+(-9b^2c+27ac^2+3b^3e-9abce\\
&-6ab^2e^2+18a^2ce^2\\
&+3a^2be^3-27ace^3+24ab^2f\\
&-54a^2cf-54bcf-21a^2bef\\
&+18b^2ef+135acef-3a^3e^2f\\
&+9abe^2f+27a^3f^2-72abf^2\\
&-81cf^2-9a^2ef^2+27bef^2)u^2+(-9b^2ce+27ac^2e\\
&-3b^3e^2+9abce^2+3ab^2e^3-27bce^3+27b^3f\\
&-72abcf-81c^2f-21ab^2ef+18a^2cef+135bcef\\
&-6a^2be^2f+18b^2e^2f+24a^2bf^2-54b^2f^2\\
&-54acf^2+3a^3ef^2-9abef^2-9a^2f^3+27bf^3)\\
&u-9b^2ce^2+27ac^2e^2\\
&+b^3e^3-27c^2e^3+27b^2cf-81ac^2f\\
&-9abcef+81c^2ef-3ab^2e^2f\\
&+27bce^2f+18a^2cf^2\\
&-54bcf^2+3a^2bef^2-27acef^2-a^3f^3+27cf^3).
\end{aligned}
$$

The discriminant of $ID_{1,2}$ equals

$$
\begin{aligned}
\mathcal{D}_{1,2}&=21641687369515008(-a^2b^2+4b^3+4a^3c\\
&-18abc+27c^2)(e^2-4f)\times(b^4-6ab^2c+9a^2c^2\\
&-ab^3e+3a^2bce+9b^2ce-27ac^2e+b^3e^2+a^3ce^2\\
&-9abce^2+27c^2e^2+3a^2b^2f-10b^3f-10a^3cf+\\
&36abcf-27c^2f-a^3bef+3ab^2ef+9a^2cef\\
&-27bcef+a^4f^2-6a^2bf^2+9b^2f^2)^2\\
&(c^2-bce+ace^2-ce^3+b^2f-2acf-abef\\
&+3cef+be^2f+a^2f^2-2bf^2-aef^2+f^3)^1 2\\
&(-2b^3c+9abc^2-27c^3+b^4e-3ab^2ce\\
&-9a^2c^2e+27bc^2e-ab^3e^2+6a^2bce^2\\
&-9b^2ce^2+b^3e^3-a^3ce^3\\
&-ab^3f+6a^2bcf-9b^2cf+6b^3ef\\
&-6a^3cef+a^3be^2f-6ab^2e^2f+9a^2ce^2f+a^3bf^2\\
&-6ab^2f^2+9a^2cf^2-a^4ef^2+3a^2bef^2\\
&+9b^2ef^2-27acef^2+2a^3f^3-9abf^3+27cf^3)^6.
\end{aligned}
$$

4 Appendix I. Classical isodynamic points for plane triangles in our context

As we mentioned in the introduction classical isodynamic points for plane triangles have been studied for more than a century and for their properties are described in e.g. [12, 13,15]. The main result of this appendix is the following statement which relates the construction of the present paper to one of the definitions of the classical isodynamic points.

Proposition 10 *The zeros of $ID_3(P)$ given by (3.1) are the first and second isodynamic points of the triangle $T\subset\mathbb{C}$ whose vertices are the (noncollinear) zeros of $P(z)=z^3+az^2+bz+c$.*

Proof Let z_1, z_2 and z_3 denote the zeros of $P(z)$. We will show that the zeros u_1, u_2 of $ID_3(P)$ satisfy the equations

$$|u-z_1||z_3-z_2|=|u-z_2||z_3-z_1|=|u-z_3||z_2-z_1| \tag{4.1}$$

which is one of the definitions for the isodynamic points of T. Due to symmetry of the vertices of T, it is sufficient to show that the first of these equations is satisfied. Furthermore, we can restrict ourselves to the case $z_1=0$, $z_2=1$ and $z_3=\rho\in\{\zeta\in\mathbb{C}\mid \mathrm{Im}(\zeta)>0\}$ since these points are the vertices of any triangle $T\subset\mathbb{C}$ under the action of affine transformations.

Under these conditions we have that

$$u_1=\frac{\rho(\rho+1)-\sqrt{-3(\rho(\rho-1))^2}}{2+2\rho(\rho-1)}$$

and

$$u_2=\frac{\rho(\rho+1)+\sqrt{-3(\rho(\rho-1))^2}}{2+2\rho(\rho-1)}.$$

Thus, for u_1 we need to show that

$$|u_1-z_1||z_3-z_2|=|u_1-z_2||z_3-z_1|, \tag{4.2}$$

or, equivalently (for all $\rho\neq\frac{1\pm\sqrt{3}i}{2}$; the pole $\rho=\frac{1+\sqrt{3}i}{2}$ can be seen to satisfy equation (4.2)),

$$
\begin{aligned}
&\left|\frac{(u_1-z_1)(z_3-z_2)}{(u_1-z_2)(z_3-z_1)}\right|\\
&=1 \iff \left|\frac{\left(\frac{\rho(\rho+1)-\sqrt{-3(\rho(\rho-1))^2}}{2+2\rho(\rho-1)}-0\right)(\rho-1)}{\left(\frac{\rho(\rho+1)-\sqrt{-3(\rho(\rho-1))^2}}{2+2\rho(\rho-1)}-1\right)(\rho-0)}\right|=1.
\end{aligned}
\tag{4.3}
$$

Equation (4.3) simplifies to

$$\left|\frac{\sqrt{3}\rho(\rho-1)}{\sqrt{-(\rho(\rho-1))^2}}-1\right|=2 \tag{4.4}$$

which is equivalent to $|\pm\sqrt{-3}-1|=2$. Since $\pm\sqrt{-3}=\pm i\sqrt{3}$ the later fact is trivial. Calculations for u_2 are completely similar.

□

5 Final remarks

1. The following analog of the famous theorem of E. Lague rre, see e.g. § 13 of Ch. 3 in [10], about the location of the roots of the polar derivative in our current setting is supported by our numerical experiments with polynomials of degrees ≥ 3 with randomly distributed roots in various rectangles.

Conjecture 2 *For a univariate polynomial P of degree $d \geq 3$ with at most double roots, no circle or line in $\mathbb{C}$ separates the zero locus of $P(z)$ from $\mathcal{I}_P$.*

2. Our numerical experiments with the isodynamic points for the Legendre and the Laguerre polynomials resulted in the following intriguing pictures which need to be explained, see Fig. 3 below.

3. Is our construction of the isodynamic map unique in some appropriate sense?

4. Is it possible to find a natural algebraic—geometric interpretation of the divisor of isodynamic points? Taking into account that the linear spaces $\mathcal{B}_n$ of degree n binary forms are the standard irreducible representation of $sl_2(\mathbb{C})$ is there a representation—theoretic meaning of our constructions.

5. Prove Conjecture 1 and its analog for rational functions.

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References

1. Arnold, V.I.: Topological classification of trigonometric polynomials and combinatorics of graphs with an equal number of vertices and edges. Funct. Anal. Appl. **30**, 1–14 (1996)
2. Borcea, J., Shapiro, B.: Classifying real polynomial pencils. Int. Math. Res. Not. **69**, 3689–3708 (2004)
3. Bøgvad, R., Hägg, C.H., Shapiro, B.: Rodrigues' descendants of a polynomial and Boutroux curves. Constr. Approx. (to appear). arXiv:2107.05710. Accessed 5 Sept 2022
4. Foster, W.H., Krasikov, I.: Inequalities for real rooted polynomials and entire functions. Adv. Appl. Math. **29**, 102–114 (2002)
5. Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants, Resultants, and Multidimensional Determinants, Mathematics: Mathematics: Theory & Applications, p. 523. Birkhäuser Boston, Inc., Boston, MA (1994)
6. Gilbert, W.J.: Generalizations of Newton's method. Fractals **9**(3), 251–262 (2001)
7. Kimberling, C.: *Encyclopedia of Triangle Centers*. http://faculty.evansville.edu/ck6/encyclopedia. Accessed 10 Sept 2022
8. Lando, S., Zvonkin, A.: Graphs on Surfaces and Their Applications. With an appendix by Don B. Zagier. Encyclopaedia of Mathematical Sciences, 141. Low-Dimensional Topology, II, p. 455. Springer, Berlin (2004)
9. Love, J.B.: Problem E 1532. Am. Math. Mon. **69**, 668 (1962)
10. Marden, M.: Geometry of Polynomials. Mathematical Surveys, No. 3, Second edn, p. 243. American Mathematical Society, Providence, RI (1966)
11. Moll, M.J.: The isodynamic points of a triangle. Dissertation, Catholic University of America (1942)
12. Pamfilos, P.: Isodynamic Points of the Triangle. http://users.math.uoc.gr/~pamfilos/eGallery/problems/Isodynamic.pdf. Accessed 10 Sept 2022
13. Rabinowitz, S.: Catalog of properties of the first isodynamic point of a triangle. Int. J. Comput. Discov. Math. (IJCDM) **6**, 108–136 (2021)
14. Sendov, B.I., Sendov, H.R.: Duality between loci of complex polynomials and the zeros of polar derivatives. Math. Proc. Camb. Philos Soc. **167**, 65–87 (2019)
15. https://en.wikipedia.org/wiki/Isodynamic_point. Accessed 10 Jan 2023