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Could René Descartes Have Known This?

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CONTENTS

Below we discuss the partition of the space of real univariate polynomials according to the number of positive and negative roots and signs of the coefficients. We present several series of non-realizable combinations of signs together with the numbers of positive and negative roots. We provide a detailed information about possible non-realizable combinations up to degree 8 as well as a general conjecture about such combinations.

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Cogito ergo sum (I think, therefore I am): The father of modern philosophy and mathematician René Descartes (in Latin Renatus Cartesius), who spent most of his life in the Dutch Republic, died on February 11, 1650, in Stockholm, Sweden. He had been invited by Queen Christina of Sweden to tutor her. The cause of death was said to be pneumonia. One theory claims that accustomed to working in bed until noon, he may have suffered damage to his health from Christina's study regime, which began early in the morning at 5 a.m. Another theory says that he might have been poisoned with arsenic for the following reason. At this time Queen Christina intended to convert to Catholicism and later she actually did that and abdicated her throne as Swedish law requires a Protestant ruler. The only Catholic with whom she had prolonged contact had been Descartes which might have caused the intense hatred by the Swedish Protestant clergy. On the other hand, another lead says that he might have been poisoned by a local Catholic priest who was afraid that Descartes' radical religious ideas might interfere with Christina's intention to convert. In any case already in 1663, the Pope placed his works on the Index Librorum Prohibitorum (Index of Prohibited Books). As a Catholic in a Protestant nation, he was interred in a graveyard used mainly for unbaptized infants in Adolf Fredriks Kyrka in Stockholm. In 1667, his remains were taken to France and first buried in the Abbey of Sainte Geneviève. Although the National Convention in 1792 had planned to transfer his remains to the Panthéon, in 1819, they were moved to the Abbey of Saint-Germain-des-Prés in Paris. Two centuries later, they are still resting in a chapel of the latter abbey while his memorial, erected in the 18th century, remains in the Swedish church.
As if all that was not enough already, it seems that Descartes' body was shipped without his head due to the small size of the coffin sent for him from France, and his cranium was delivered to "Musée de l'Homme" in Paris only in early 19th century by the famous Swedish chemist J. Berzelius who bought it on an auction in Stockholm! One wonders if there is any reasonable evidence behind any of these myths.

1. INTRODUCTION

The famous Descartes' rule of signs claims that the number of positive roots of a real univariate polynomial does not exceed the number of sign changes in its sequence of coefficients. In what follows, we only consider polynomials with all non-vanishing coefficients. An arbitrary ordered sequence $\overline{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_d)$ of $\pm$-signs is called a *sign pattern*. Given a sign pattern $\overline{\sigma}$, we call its *Descartes' pair* $(p_{\overline{\sigma}}, n_{\overline{\sigma}})$ the pair of non-negative integers counting sign changes and sign preservations of $\overline{\sigma}$. The Descartes' pair of $\overline{\sigma}$ gives the upper bound on the number of positive and negative roots of any polynomial of degree d whose signs of coefficients are given by $\overline{\sigma}$. (Observe that, for any $\overline{\sigma}$, $p_{\overline{\sigma}} + n_{\overline{\sigma}} = d$.) To any polynomial $q(x)$ with the sign pattern $\overline{\sigma}$, we associate the pair (pos_q, neg_q) giving the numbers of its positive and negative roots counted with multiplicities. Obviously the pair (pos_q, neg_q) satisfies the standard restrictions

$$pos_q \le p_{\overline{\sigma}},\ pos_q \equiv p_{\overline{\sigma}} (\mathrm{mod}\, 2),\ neg_q \le n_{\overline{\sigma}},\quad neg_q \equiv n_{\overline{\sigma}} (\mathrm{mod}\, 2). \tag{1–1}$$

We call pairs (pos, neg) satisfying (1–1) *admissible* for $\overline{\sigma}$. Conversely, for a given pair (pos, neg), we call a sign pattern $\overline{\sigma}$ such that (1–1) is satisfied *admitting* the latter pair. It turns out that not for every pattern $\overline{\sigma}$, all its admissible pairs (pos, neg) are realizable by polynomials with the sign pattern $\overline{\sigma}$. Below we address this very basic question.

Problem 1. *For a given sign pattern $\overline{\sigma}$, which admissible pairs (pos, neg) are realizable by polynomials whose signs of coefficients are given by $\overline{\sigma}$?*

To the best of our knowledge, this natural question was for the first time raised in [Anderson et al. 98] published in 1998. Soon after that [Grabiner 99] found the first example of non-realizable combination for polynomials of degree 4. Namely, he has shown that the sign pattern $(+,-,-,-,+)$ does not allow to realize the pair $(0,2)$ and the sign pattern $(+,+,-,+,+)$ does not allow to realize $(2,0)$. Observe that their Descartes' pairs equal $(2,2)$.

His argument is very simple. (Due to symmetry induced by $x \mapsto -x$ it suffices to consider only the first case.) Observe that a fourth-degree polynomial with only two negative roots for which the sum of roots is positive could be factored as $a(x^2+bx+c)(x^2-sx+t)$ with $a,b,c,s,t>0$, $s^2<4t$, and $b^2 \geq 4c$. The product of these factors equals $a(x^4+(b-s)x^3+(t+c-bs)x^2+(bt-cs)x+ct)$. To get the correct sign pattern, we need $b<s$ and $bt<cs$, which gives $b^2t<s^2c$ and thus $b^2/c<s^2/t$. But we have $b^2/c \geq 4 > s^2/t$.

A recent paper [Albouy and Fu 14] deals with the same question and contains a complete description of non-realizable patterns and pairs (pos, neg) for polynomials up to degree 6, see Theorem 8 below.

To formulate our results in this direction, we need to introduce some notation. Consider the (affine) space Pol_d of all real monic univariate polynomials of degree d and define the *standard real discriminant* $\mathcal{D}_d \subset Pol_d$ as the subset of all polynomials having a real multiple root. (Detailed information about a natural stratification of $\mathcal{D}_d$ can be found in e.g., [Khesin and Shapiro 92].) It is a well-known and simple fact that $Pol_d \setminus \mathcal{D}_d$ consists of $\left[\frac{d}{2}\right]+1$ components distinguished by the number of real simple roots. Moreover, each such component is contractible in Pol_d.

When working with monic polynomials, we will mainly use their shortened sign patterns $\widehat{\sigma}$ representing the signs of all coefficients except the leading term which equals 1. For the actual sign pattern $\overline{\sigma}$, we then write $\overline{\sigma}=(1,\widehat{\sigma})$ to emphasize that we consider monic polynomials.

For any pair (d,k) of non-negative integers with $d-k \geq 0$; $d-k \equiv 0 \mod 2$, denote by $Pol_{d,k}$ the set of all monic real polynomials of degree d with k real simple roots. Denote by $Pol_d(\widehat{\sigma}) \subset Pol_d$ the set (orthant) of all monic polynomials $p=x^d+a_1x^{d-1}+\ldots+a_d$ whose coefficients $(a_1,\ldots,a_d)$ have the (shortened) sign pattern $\widehat{\sigma}=(\sigma_1,\ldots,\sigma_d)$, respectively. Finally, set $Pol_{d,k}(\widehat{\sigma}) = Pol_{d,k} \cap Pol_d(\widehat{\sigma})$. To the best of our knowledge, the topology of $Pol_{d,k}(\widehat{\sigma})$ for arbitrary (d,k) and $\widehat{\sigma}$ has not been studied previously.

We have the natural $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on the space of monic polynomials and on the set of all sign patterns, respectively. The first generator acts by reverting the signs of all monomials of odd degree (which for polynomials means $P(x) \to (-1)^d P(-x)$); the second generator acts by reading the pattern backward (which for polynomials means $P(x) \to x^d P(1/x)$). If one wants to preserve the set of monic polynomials, one has to divide $x^d P(1/x)$ by its leading term. We will refer to the latter action as the *standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action*. (Up to some trivialities) the properties we will study below are invariant under this action.

We start with the following simple result (which should be known).

Theorem 2. (i) *If d is even, then $Pol_{d,0}(\widehat{\sigma})$ is nonempty if and only if $\sigma_d = +$ (i.e., the constant term is positive).*

(ii) *For any pair of positive integers (d,k) with $d-k \geq 0$ and $d-k \equiv 0 \mod 2$ and any sign pattern $\widehat{\sigma} = (\sigma_1,\ldots,\sigma_d)$, the set $Pol_{d,k}(\widehat{\sigma})$ is nonempty.*

Observe that, in general, the set $Pol_{d,k}(\widehat{\sigma})$ does not have to be connected. The total number k of real zeros can be distributed between m positive and n negative in different ways satisfying the inequalities $m+n=k, m \leq p_{\widehat{\sigma}},\ n \leq n_{\widehat{\sigma}}$, and $m \equiv p_{\widehat{\sigma}} \mod 2, n \equiv n_{\widehat{\sigma}} \mod 2$, see examples below.

On the other hand, some specific sets $Pol_{d,k}(\widehat{\sigma})$ must be connected. In particular, the following holds.

Proposition 3. (i) *For any d and $\widehat{\sigma}$, the sets $Pol_{d,d}(\widehat{\sigma})$ and $Pol_{d,0}(\widehat{\sigma})$ are contractible. (The latter set is empty for odd d.)*

(ii) *For the (shortened) sign pattern $\widehat{+} = (+,+,\ldots,+)$ consisting of all pluses, the set $Pol_{d,k}(\widehat{+})$ is contractible, for any $k \leq d$, $k \equiv d \mod 2$. (The same holds for the shortened alternating sign pattern $(-,+,-,\ldots)$.)*

(iii) *For any sign pattern $\overline{\sigma} = (1,\widehat{\sigma})$ with just one sign change, all sets $Pol_{d,k}(\widehat{\sigma})$ are nonempty. For $k=d$ (which is the case of real-rooted polynomials having one positive and $d-1$ negative roots), this set is contractible.*

Concerning non-realizable combinations of $\overline{\sigma}$ and (pos, neg), we are able to prove the following two results.

Proposition 4. *For d even, consider patterns satisfying the following three conditions: the sign of the constant term (i.e., the last entry) is $+$; the signs of all odd monomials are $+$; among the remaining signs of even monomials there are $l \geq 1$ minuses (at arbitrary positions). Then, for any such sign pattern, the pairs $(2,0),(4,0),\ldots,(2l,0)$, and only they, are*

non-realizable. (Using the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action one obtains more such examples.)

Problem 5. *Does there exist a version of Proposition 4 for odd d?*

Proposition 6. *Consider a sign pattern $\overline{\sigma}$ with 2 sign changes, consisting of m consecutive pluses (including the leading 1) followed by n consecutive minuses and then by p consecutive pluses, where $m + n + p = d + 1$. Then*

(i) *for the pair $(0, d-2)$, this sign pattern is not realizable if*

$$\kappa := \frac{d-m-1}{m} \cdot \frac{d-p-1}{p} \geq 4; \qquad (1\text{–}2)$$

(ii) *the sign pattern $\overline{\sigma}$ is realizable with any pair of the form $(2, v)$.*

Remark 7. Inequality (1–2) provides only sufficient conditions for non-realizability of the pattern $\overline{\sigma}$ with the pair $(0, d-2)$. One can ask how sharp this condition is. But at the moment we do not have examples of non-realizable pairs $(0, d-2)$ with inequality (1–2) violated.

Let us now reproduce the main result of [Albouy and Fu 14] which includes Grabiner's example.

Theorem 8. (i) *Up to degree $d \leq 3$, for any sign pattern $\overline{\sigma}$, all admissible pairs (pos, neg) are realizable.*

(ii) *For $d = 4$ (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) the only non-realizable combination is $(1, -, -, -, +)$ with the pair $(0, 2)$;*

(iii) *For $d = 5$ (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) the only non-realizable combination is $(1, -, -, -, -, +)$ with the pair $(0, 3)$;*

(iv) *For $d = 6$ (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) the only non-realizable combinations are $(1, -, -, -, -, -, +)$ with $(0, 2)$ and $(0, 4)$; $(1, +, +, +, -, +, +)$ with $(2, 0)$; $(1, +, -, -, -, -, +)$ with $(0, 4)$.*

Trying to extend Theorem 8, we obtained a computer-aided classification of all non-realizable sign patterns and pairs for $d = 7$ and almost all for $d = 8$, see below.

Theorem 9. *For $d = 7$, among the 1472 possible combinations of a sign pattern and a pair (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action), there exist exactly 6 which are non-realizable. They are:*

$(1, +, -, -, -, -, -, +)$ *with (0,5);*
$(1, +, -, -, -, -, +, +)$ *with (0,5);*
$(1, +, -, +, -, -, -, -)$ *with (3,0);*
$(1, +, +, -, -, -, -, +)$ *with (0,5);*
and $(1, -, -, -, -, -, -, +)$ *with (0,3) and (0,5).*

Theorem 10. *For $d = 8$, among the 3648 possible combinations of a sign pattern and a pair (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action), there exist 13 which are known to be non-realizable. They are:*

$(1, +, -, -, -, -, -, +, +)$ *with (0,6);*
$(1, -, -, -, -, -, -, +, +)$ *with (0,6);*
$(1, +, +, +, -, -, -, -, +)$ *with (0,6);*
$(1, +, +, -, -, -, -, -, +)$ *with (0,6);*
$(1, +, +, +, -, +, +, +, +)$ *with (2,0);*
$(1, +, +, +, +, +, -, +, +)$ *with (2,0);*
$(1, +, +, +, -, +, -, +, +)$ *with (2,0) and (4,0) ;*
$(1, -, -, -, +, -, -, -, +)$ *with (0,2) and (0,4);* $(1, -, -, -, -, -, -, -, +)$ *with (0,2), (0,4), and (0,6).*

Remark 11. For $d = 8$, there exist exactly 7 (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) combinations of a sign pattern and a pair for which it is still unknown whether they are realizable or not. They are:

$(1, +, -, +, -, -, -, +, +)$ *with (4,0);*
$(1, +, -, +, -, +, -, -, +)$ *with (4,0);*
$(1, +, +, -, -, -, -, +, +)$ *with (0,6);*
$(1, +, +, -, -, +, -, +, +)$ *with (4,0);*
$(1, +, +, +, -, +, -, -, +)$ *with (4,0);*
$(1, +, -, +, -, -, -, -, +)$ *with (4,0) and (0,4).*

Based on the above results, we formulate the following claim.

Conjecture 12. *For an arbitrary sign pattern $\overline{\sigma}$, the only type of pairs (pos, neg) which can be non-realizable has either pos or neg vanishing. In other words, for any sign pattern $\overline{\sigma}$, each pair (pos, neg) satisfying (1–1) with positive pos and neg is realizable.*

Rephrasing the above conjecture, we say that the only phenomenon implying non-realizability is that "real roots on one half-axis force real roots on the other half-axis." At the moment, this conjecture is verified by computer-aided methods up to $d = 10$.

Theorem 13. *For any sign pattern $\overline{\sigma}$ and integer m, the closure of the union of sets*

$$Pol_{d,\geq m}(\overline{\sigma}) := \bigcup_{j=m}^{d} \overline{Pol_{d,j}(\overline{\sigma})} \qquad (1\text{–}3)$$

is simply connected.

2. PROOFS

The next two lemmas are very useful in our arguments proving the realizability of a given pair (pos, neg) with a given sign pattern $\overline{\sigma}$.

Lemma 14. [*First concatenation lemma*] *Suppose that the monic polynomials P_1 and P_2 of degrees d_1 and d_2 with sign patterns $\overline{\sigma}_1 = (1, \widehat{\sigma}_1)$ and $\overline{\sigma}_2 = (1, \widehat{\sigma}_2)$, respectively, realize the pairs (pos_1, neg_1) and (pos_2, neg_2). (Here $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are the shortened sign patterns of P_1 and P_2 respectively.) Then*

- *if the last position of $\widehat{\sigma}_1$ is $+$, then for any $\epsilon > 0$ small enough, the polynomial $\epsilon^{d_2} P_1(x)P_2(x/\epsilon)$ realizes the sign pattern $(1, \widehat{\sigma}_1, \widehat{\sigma}_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$.*
- *if the last position of $\widehat{\sigma}_1$ is $-$, then for any $\epsilon > 0$ small enough, the polynomial $\epsilon^{d_2} P_1(x)P_2(x/\epsilon)$ realizes the sign pattern $(1, \widehat{\sigma}_1, -\widehat{\sigma}_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$. (Here $-\widehat{\sigma}$ is the sign pattern obtained from $\widehat{\sigma}$ by changing each $+$ by $-$ and vice versa.)*

Proof. Set $P_2(x) = x^{d_2} + b_1 x^{d_2-1} + b_2 x^{d_2-2} + \cdots + b_{d_2}$. Then $\epsilon^{d_2} P_2(x/\epsilon) = x^{d_2} + \epsilon b_1 x^{d_2-1} + \epsilon^2 b_2 x^{d_2-2} + \cdots + \epsilon^{d_2} b_{d_2}$ and for $\epsilon > 0$ small enough, the first $d_1 + 1$ coefficients of $\epsilon^{d_2} P_1(x)P_2(x/\epsilon)$ are close to the respective coefficients of $P_1(x)$. Thus, they have the same signs. Then if the last entry of $\widehat{\sigma}_1$ is $+$, the remaining coefficients of $\epsilon^{d_2} P_1(x)P_2(x/\epsilon)$ (up to higher order terms in ϵ) are equal to the respective coefficients of $\epsilon^{d_2} P_1(0)P_2(x/\epsilon)$. If this entry is $-$, the remaining coefficients of $\epsilon^{d_2} P_1(x)P_2(x/\epsilon)$ (up to higher order terms in ϵ) are equal to the opposite of the respective coefficients of $\epsilon^{d_2} P_1(0)P_2(x/\epsilon)$. □

Example 15. Denote by τ the last entry of $\widehat{\sigma}_1$. We consider the cases $P_2(x) = x - 1,\ x + 1,\ x^2 + 2x + 2,\ x^2 - 2x + 2$ with $(pos_2, neg_2) = (1, 0), (0, 1), (0, 0), (0, 0)$. When $\tau = +$, then one has respectively $\widehat{\sigma}_2 = (-), (+), (+, +), (-, +)$ and the sign pattern of $\epsilon^{d_2} P_1(x)P_2(x/\epsilon)$ equals $(1, \widehat{\sigma}_1, -)$, $(1, \widehat{\sigma}_1, +)$, $(1, \widehat{\sigma}_1, +, +)$, $(1, \widehat{\sigma}_1, -, +)$. When $\tau = -$, then one has respectively $\widehat{\sigma}_2 = (+), (-), (-, -), (+, -)$ and the sign pattern of $\epsilon^{d_2} P_1(x)P_2(x/\epsilon)$ equals $(1, \widehat{\sigma}_1, +)$, $(1, \widehat{\sigma}_1, -)$, $(1, \widehat{\sigma}_1, -, -)$, $(1, \widehat{\sigma}_1, +, -)$.

Example 16. The sign pattern $(1, -, -, -, +, +, +)$ is realizable with the pair $(0, 4)$. Indeed by Lemma 14 with $P_2(x) = x + 1$, this follows from the realizability of the pattern $(1, -, -, -, +, +)$ for $d = 5$ and the pair $(0, 3)$ in which case one can set $P(x) = x(x^2 - 1)^2 + \epsilon - \epsilon^2(x^2 + x^4)$, where $\epsilon > 0$ is small.

Lemma 17. [*Second concatenation lemma*] *Take (not necessarily monic) polynomials $P_1(x) = \sum_{k=0}^{d_1} a_k x^k$ and $P_2(x) = \sum_{k=0}^{d_2} b_k x^k$ of degrees d_1 and d_2, respectively, with all nonvanishing coefficients. Assume that they have sign patterns $\overline{\sigma}_1 = (\widehat{\sigma}_1, +)$ and $\overline{\sigma}_2 = (+, \widehat{\sigma}_2)$, respectively, and realize the pairs (pos_1, neg_1) and (pos_2, neg_2). (Here $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are arbitrary sequences of $\pm$ of lengths d_1 and d_2.) Then, for $\epsilon > 0$ small enough, the polynomial*

$$P(x) = \left(\frac{1}{a_{d_1}} \sum_{k=0}^{d_1-1} a_k x^k\right) + x^{d_1} + \frac{x^{d_1}}{b_0}\left(\sum_{k=1}^{d_2} b_k (\epsilon x)^k\right)$$

realizes the sign pattern $(\widehat{\sigma}_1, +, \widehat{\sigma}_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$.

Proof. Since $a_{d_1}, b_0 > 0$ by assumption, the polynomial P has the sign pattern $(\widehat{\sigma}_1, +, \widehat{\sigma}_2)$ for all $\epsilon > 0$. Notice that, pointwise (and uniformly on compact subsets),

$$P(x) \to \frac{P_1(x)}{a_{d_1}}, \quad \epsilon \to 0, \quad \text{and} \quad \epsilon^{d_1} P(x/\epsilon) \to \frac{P_2(x)}{d_0}, \quad \epsilon \to 0.$$

Therefore, it is clear that for ϵ sufficiently small, P has at least $pos_1 + pos_2$ positive roots, and at least $neg_1 + neg_2$ negative roots.

It remains to show that for ϵ small enough, the number of non-real roots of $P(x)$ is equal to the sum of the numbers of non-real roots of P_1 and P_2. By continuity of roots, for each neighborhood N_p of a non-real root p of P_1 of multiplicity m_p, there is a $t = t(p) > 0$ such that $P(x)$ has m_p roots in N_p if $\epsilon < t(p)$. Similarly, for each neighborhood N_q of a non-real root q of P_2 of multiplicity m_q, there is a $t = t(q) > 0$ such that $P(x/\epsilon)$ has m_q roots in N_q. This implies that $P(x)$ has m_q roots in the dilated set ϵN_q, for $\epsilon < t(q)$. For each non-real root p of P_1, choose its neighborhood N_p such that all N_p's are pairwise disjoint and do not intersect the real axis. Choose the neighborhoods N_q of the non-real roots q of P_2 similarly. If P_1 and P_2 has a common non-real root, then we cannot choose the neighborhoods N_p's and N_q's as above so that N_p is disjoint from N_q for every pair p and q. However, for ϵ sufficiently small, the dilated sets ϵN_q are disjoint from N_p for any p and q. Indeed, since the open sets N_p do not meet $\mathbb{R}$, there is a neighborhood N_0 of the origin disjoint from each N_p; for ϵ small enough we have that $\epsilon N_q \subset N_0$ implying the latter claim.

The fact that $N_q \cap \mathbb{R} = \emptyset$ implies that $\epsilon N_q \cap \mathbb{R} = \emptyset$ as well. Therefore, we can conclude that, for ϵ small enough, all roots of $P(x)$ contained in any of the sets ϵN_q or N_p are non-real, which finishes the proof. □

Proof of Theorem 2. Part (i) is straightforward. Indeed, the necessity of the positivity of the constant term is obvious for monic polynomials of even degree with no real roots. Moreover fix any even degree monic polynomial with coefficients of the necessary signs and increase its constant term until the whole graph of the polynomial will lie strictly above the x-axis. The resulting polynomial has the required signs of its coefficients and no real roots.

For the need of the rest of the proof, observe that in the same way one constructs polynomials for d odd which realize an arbitrary sign pattern with exactly one real root (positive or negative depending on the sign of the constant term). As above, one starts with an arbitrary odd-degree polynomial with a given pattern and then one either increases or decreases the constant term until the polynomial has a single simple real root.

Now we prove part (ii) by induction on d and k. For $d = 1, 2, 3$, the fact can be easily checked. For $k = 0$ with d even and $k = 1$ with d odd, the proof is given above.

Suppose first that $k = d$, i.e., the polynomial has to be real-rooted. In this case, one applies Lemma 14 and Example 15 $d - 1$ times with $P_2 = x \pm 1$.

If $k < d$, then consider the last three signs of the sign pattern. If they are $(+, +, +)$, $(+, -, +)$, $(-, -, -)$, or $(-, +, -)$, then one can apply Lemma 14 and Example 15 with $P_2 = x^2 \pm 2x + 2$. This preserves k and reduces d by 2.

Suppose that they are $(+, -, -)$ or $(+, +, -)$. If d is odd, then one applies Lemma 14 and Example 15 with $P_2 = x \pm 1$ and reduces the proof to the case with $d - 1$ and $k - 1$ in the place of d and k. As d is odd and $k > 1$, one actually has $k > 2$. If d is even, then one applies Lemma 14 and Example 15 twice, with $P_2 = x + 1$ and with $P_2 = x - 1$ or vice versa. One obtains the case of $d - 2$, $k - 2$. If $k - 2 = 0$, then the proof of the theorem follows from part (i). If $k > 2$, then the reduction can continue.

Suppose that the last three signs are $(-, +, +)$ or $(-, -, +)$. If d is odd and $k = 1$, the proof follows from part (i). If d is odd and $k > 2$, then one can apply Lemma 14 and Example 15 with $P_2 = x \pm 1$ and reduce the proof to the case $d - 1, k > 0$.

If d is even and $k = 0$, then the proof follows from part (i). If d is even and $k > 0$, then one applies Lemma 14 and Example 15 with $P_2 = x \pm 1$ and one reduces the proof to the case $d - 1$. □

To prove Proposition 3, we need the following lemma having an independent interest.

Lemma 18. *For any shortened sign pattern $\overline{\sigma}$, the intersection $Pol_{d,d}(\overline{\sigma})$ is path-connected.*

Proof. Recall that a real polynomial $p(x)$ is called sign-independently real-rooted if every polynomial obtained from $p(x)$ by an arbitrary sign change of its coefficients is real-rooted. It is shown in [Passare et al. 11] that the logarithmic image of the set of all sign-independently real-rooted polynomials is convex. Hence, the set of all sign-independently real-rooted polynomials itself is logarithmically convex, and, in particular, it is path-connected. The following criterion of sign-independently real-rootedness is straightforward.

A real polynomial p is sign-independently real-rooted if and only if, for every monomial $a_k x^k$ of $p(x)$, there exists a point x_k such that

$$|a_k x_k^k| > \sum_{j \neq k} |a_j x_k^j|. \tag{2–4}$$

Using induction on the degree d, we will now prove that, for any polynomial $p \in Pol_{d,d}(\overline{\sigma})$, there is a path $t \mapsto p_t$ such that (i) $p_0 = p$; (ii) p_1 is sign-independently real-rooted; (iii) $p_t \in Pol_{d,d}(\overline{\sigma})$ for all $t = [0, 1]$. Since the set of all sign-independently real-rooted polynomials is path-connected, this claim settles Lemma 18. The case $d = 1$ is trivial, since any linear polynomial is sign-independently real-rooted.

Let p be a real-rooted polynomial of degree d. Then, $q = p'$ is a real-rooted polynomial of degree $d - 1$. Hence, by the induction hypothesis, there is a path $t \mapsto q_t$ as above. Furthermore, since p is real-rooted, so is its polar derivative $p'_\alpha(x) := p(x) + \frac{x}{\alpha} p'(x)$ for all $\alpha \in \mathbb{R}^+$.

For each $t \in [0, 1]$, let $\alpha_t > 0$ be such that $Q_{t,\alpha}(x) := p(x) + \frac{x}{\alpha} q_t(x)$ is real-rooted for any $0 < \alpha < \alpha_t$. By continuity of roots, $Q_{\hat{t},\alpha_t}$ is real-rooted for $\hat{t}$ in a small neighborhood of t. Since $[0, 1]$ is compact, we can find a finite set $\alpha_{t_1}, \ldots, \alpha_{t_N}$ such that $Q_{t,\alpha}(x)$ is real-rooted for all $t \in [0, 1]$ if $\alpha < \min(\alpha_{t_1}, \ldots, \alpha_{t_N})$.

Since $xq_1(x)$ is sign-independently real-rooted, for all k and all monomials $b_k x^k$ of $xq_1(x)$, there exists a point x_k such that (2–4) holds. Since the signs of $p(x)$ are equal to the signs of $xq_1(x)$, then, for each $k = 1, \ldots, d$, there exists an $\alpha_k > 0$ such that (2–4) holds for $Q_{1,\alpha}(x)$ at x_k for every given k. However, since (2–4) always holds for the constant term with x_0 sufficiently small, we conclude that $Q_{1,\alpha}$ is sign-independently real-rooted when $\alpha < \min_{k=1,\ldots,d-1} \alpha_k$.

Now fix a positive number $\alpha^* < \min(\alpha_{t_1}, \ldots, \alpha_{t_N}, \alpha_1, \ldots \alpha_{d-1})$ and consider the path composed of the two paths

$$\alpha \mapsto p'_\alpha, \quad \alpha \in [\infty, \alpha^*] \quad \text{and} \quad t \mapsto Q_{t,\alpha^*}, \quad t \in [0, 1].$$

By construction, this path is contained in $Pol_{d,d}(\overline{\sigma})$. Its starting point is $p(x)$ and its endpoint Q_{1,α^*} is sign-independently real-rooted. This concludes the induction step. □

Proof of Proposition 3. To settle the fact that $Pol_{d,0}(\widehat{\sigma})$ is contractible (see (i)), we notice that the set $Pol_{d,0}$ of all positive monic polynomials is a convex cone. (Here d is even.) Therefore, its intersection with any orthant is convex and contractible (if nonempty).

To settle the fact that $Pol_{d,d}(\widehat{\sigma})$ is contractible (see (i)), take a real-rooted polynomial Q realizing a given pattern. Consider the family $Q + \lambda x^a$, $a = 0, 1, \ldots, n-1$. Polynomials in this family are real-rooted and with the given sign pattern until either there is a confluence of roots of the polynomial or its ath derivative vanishes at the origin. In both cases, further increase or decrease of the parameter λ never brings us back to the set of real-rooted polynomials.

Thus the set $Pol_{d,d}(\overline{\sigma})$ has what we call *Property A*:

The intersection of every connected component intersected with any line parallel to any coordinate axis in the space of coefficients is either empty, or a point, or, finally, an interval whose endpoints are continuous functions of the other coefficients.

(Indeed, they are values of the polynomial or of its derivatives at roots of the polynomial or its derivatives; therefore these roots are algebraic functions depending continuously on the coefficients.)

Maxima and minima of such functions are also continuous. Therefore, the projection of each connected component of $Pol_{d,d}(\overline{\sigma})$ on each coordinate hyperplane in the space of the coefficients also enjoys Property A. (It suffices to fix the values of all coefficients but one and study the endpoints of the segments as functions of that coefficient.)

Now replace $Pol_{d,d}(\overline{\sigma})$ by a smaller set obtained as follows. Choose some coefficient and, for fixed values of all the other coefficients, substitute every nonempty intersection of $Pol_{d,d}(\overline{\sigma})$ with lines parallel to the axis corresponding to the chosen coefficient by the half-sum of the endpoints, i.e., substitute the intersection segment by its middle point. This operation produces the graph of a continuous function depending on the other coefficients. The projection of this graph to the coordinate hyperplane of the other coefficients is a domain having Property A, but belonging to a space of dimension $n-1$. Continuing this process, one contracts each connected component of the set $Pol_{d,d}(\overline{\sigma})$ to a point. Using Lemma 18, we conclude that $Pol_{d,d}(\overline{\sigma})$ is path-connected and therefore contractible.

To prove (ii), it is enough to settle the case $\widehat{\sigma} = \widehat{+} = (+, +, +, \ldots, +)$. Let us show that any compact subset in $Pol_{d,k}(\widehat{+})$ can be contracted to a point inside $Pol_{d,k}(\widehat{+})$. Observe that for any polynomial $p(x)$ with positive coefficients, the family of polynomials $p(x+t)$, where t is an arbitrary positive number, consists of polynomials with all positive coefficients and the same number of real roots all being negative. Given a compact set $K \subset Pol_{d,k}(\widehat{+})$, consider its image K_t obtained by applying to K the above shift to the left on the distance t, for t sufficiently large. Then, all real roots of all polynomials in the compact set K_t will be very large negative numbers and all complex conjugate pairs will have very large negative real parts. Therefore, one can choose any polynomial $\tilde{p}$ in K_t and contract the whole K_t to $\tilde{p}$ along the straight segments, i.e., $\tau\tilde{p} + (1-\tau)p$ for any $p \in K_t$. Obviously such a contraction takes place inside $Pol_{d,k}(\widehat{+})$.

Let us prove (iii). It is clear that there exists just one component (which is contractible) of real-rooted polynomials with all roots of the same sign. Suppose that they are all negative. To pass from degree d to degree $d+1$ polynomials, and from the pair $(0, d)$ to the pair $(1, d)$, one adds a positive root. One considers the polynomial

$$\left(x^d + a_1 x^{d-1} + \cdots + a_d\right)(x - b),\ a_j > 0,\ b \geq 0.$$

Its coefficients are of the form $c_j = a_j - b a_{j-1}$, $a_0 = 1$ (i.e., they are linear functions of the parameter $b > 0$). Hence, each of the coefficients except the first and the last one vanishes for some $b > 0$ and then remains negative. As one must have for any $b > 0$ exactly one sign change and never two consecutive zeros, it is always the last positive coefficient c_j that vanishes. The value of b for which a given coefficient c_j vanishes depends continuously on a_i which implies the contractibility and uniqueness of the component with the pair $(1, d)$ with the different sign patterns.

We have just settled the real-rooted case with one sign change. Now we treat the non-real-rooted case. Fix a sign pattern with one sign change and with the pair $(1, d)$. One can realize it by a polynomial having all distinct critical values. Hence when one decreases the constant term (it is negative, so the pattern does not change), the positive root goes to the right and the negative roots remain within a fixed interval $[-u, -v]$, $u > 0$, $v > 0$. When the constant term decreases, the polynomial loses consecutively $[d/2]$ pairs of real negative roots and the realizable pairs become $(1, d-2), (1, d-4), \ldots, (1, d-2[d/2])$, respectively. □

Let us now prove the realizability for a certain general class of pairs (pos, neg). For a given sign pattern $\overline{\sigma}$, consider all possible sign patterns $\widetilde{\overline{\sigma}}$ obtained from $\overline{\sigma}$ by removing an arbitrary subset of its entries except for the leading 1 and the last entry (constant term). On the level of polynomials, this corresponds to requiring that the corresponding coefficients vanish. For any such $\widetilde{\overline{\sigma}}$, let $(\widetilde{pos, neg})$ be its Descartes' pair,

i.e., the number of its sign changes and the number of sign changes of the flip of $\widetilde{\overline{\sigma}}$ (which corresponds to $P(-x)$).

Lemma 19. *Given an arbitrary sign pattern $\overline{\sigma}$, all pairs $(\widetilde{pos}, \widetilde{neg})$ as above are realizable.*

Proof. Recall that a sign-independently real-rooted polynomial is a real univariate polynomial such that it has only real roots and the same holds for an arbitrary sign change of its coefficients, see [Passare et al. 11]. As we already mentioned, a polynomial $p(x) = \sum_{k=0}^{d} a_k x^k$ is sign-independently real-rooted if and only if, for each $k = 0, \ldots, d$, there exists $x_k \in \mathbb{R}_+$ such that

$$|a_k x_k^k| \geq \sum_{l \neq k} |a_l x_k^l|.$$

Let $P(x)$ be a sign-independently real-rooted polynomial with the given sign pattern $\overline{\sigma}$. Removal of components of $\overline{\sigma}$ corresponds to the deletion of monomials from $P(x)$. For each $\widetilde{\overline{\sigma}}$, let $\widetilde{P}(x)$ denote the polynomial obtained by deleting those monomials from $P(x)$ which correspond to components of $\overline{\sigma}$ deleted when constructing $\widetilde{\overline{\sigma}}$. Clearly, the above inequality holds for $\widetilde{P}(x)$ as well since we are removing monomials from its right-hand side. Therefore, the sign of $\tilde{p}(x_k)$ equals that of $a_k x_k^k$. Since $x_0 < x_1 < \cdots < x_d$, this implies that $\widetilde{P}(x)$ has at least $\widetilde{pos}$ sign changes in $\mathbb{R}_+$. Similarly, we find that $\widetilde{P}(x)$ has at least $\widetilde{neg}$ sign changes in $\mathbb{R}_-$. However, by Descartes' rule of signs, this is the maximal number of positive and negative roots, respectively. Hence, this is the exact number of positive and negative roots of $\widetilde{P}(x)$. Therefore, perturbations of the coefficients do not change the number of real roots. □

Proposition 20. *Given an arbitrary sign pattern $\overline{\sigma}$, any its admissible pair (pos, neg) satisfying the condition*

$$\min(pos, neq) > \left\lfloor \frac{d-4}{3} \right\rfloor$$

is realizable.

Proof. Notice first that, if $d \leq 3$, then $\left\lfloor \frac{d-4}{3} \right\rfloor < 0$. Thus, we need to prove that any admissible pair is realizable in this case. Indeed, using Lemma 19, this is straightforward to check.

For arbitrary d, let us decompose $\overline{\sigma}$ in the following manner. Let

$$\tau_k = (\sigma_{3k+1}, \ldots, \sigma_{3k+4}), \quad k = 0, \ldots, \left\lfloor \frac{d-4}{3} \right\rfloor,$$

(where we use slight abuse of notation—the last pattern does not have to be of length 4). Then, for each τ_k, the admissible pairs are among the pairs

$$(1, 0), (1, 2), (3, 0), (0, 1), (2, 1), \text{ and } (0, 3),$$

and for each τ_k, all admissible pairs are realizable because they correspond to the case $d \leq 3$.

For each τ_k, choose initially an admissible pair $u_k = (1, 0)$ or $u_k = (0, 1)$ depending on whether τ_k admits an odd number of positive roots and an even number of negative roots, or vice versa. By assumption,

$$\sum_k u_k \leq (pos, neg),$$

where the inequality is understood componentwise. If this is not an equality, then the difference is of the form $(2a, 2b)$, where $a + b \leq \lfloor \frac{d-4}{3} \rfloor$, since the original pair (pos, neg) is admissible. Define

$$\begin{aligned} v_k &= u_k + (2, 0), \quad k = 0, \ldots, a-1, \\ v_k &= u_k + (0, 2), \quad k = a, \ldots, a+b-1, \\ v_k &= u_k, \quad k = a+b, \ldots, \left\lfloor \frac{d-4}{3} \right\rfloor. \end{aligned}$$

Then, v_k is an admissible pair for τ_k, and in addition

$$\sum_k v_k = (pos, neg).$$

Applying Lemma 17 repeatedly to the patterns τ_k, we prove Proposition 20. □

For d odd, consider the sign patterns $\overline{\sigma} = (1, \widehat{\sigma})$ of the form: (a) the last entry is $+$; (b) all other entries at even positions are $-$; (c) there is at most one sign change in the group of signs at odd positions. Example, $(1, -, +, -, -, -, -, -, -, +)$.

Lemma 21. *If, under the above assumptions, the pair has no positive roots, then it has exactly one negative, i.e., of all possible pairs $(0, s)$ only $(0, 1)$ is realizable.*

Proof. Let us decompose a polynomial $P(x)$ with the sign pattern $\overline{\sigma}$ as above into $P_{od}(x)$ and $P_{ev}(x)$, where $P_{od}(x)$ (resp. $P_{ev}(x)$) contains all odd (resp. even) monomials of $P(x)$. Then obviously, P_{ev}, P_{od}, and P'_{od} have one positive root each; we denote them by x_{ev}, x_{od}, and x'_{od} respectively. We first claim that $x'_{od} < x_{od} < x_{ev}$. Indeed, assume that $x_{ev} \leq x_{od}$. Then both P_{od} and P_{ev} are non-positive on the interval $[x_{ev}, x_{od}]$. Therefore , $P(x)$ will also be non-positive on the same interval which contradicts the assumption that $P(x)$ is positive, for all positive x. Now we prove that $x'_{od} < x_{od}$. Present $P_{od} = P_{od}^+ - P_{od}^-$, where P_{od}^+ is the sum of all odd-degree monomials with positive coefficients and P_{od}^- is the negative of the sum of all odd-degree monomials with negative coefficients. Observe that the degree of the smallest monomial in P_{od}^+ is larger than $\delta = \deg P_{od}^-$ by assumption.

Now if $P(x) \geq 0$, i.e., $x \geq x_{od}$, then

$$\left(P_{od}^+\right)'(x) > \delta P_{od}^+(x) \geq \delta P_{odd}^-(x) > \left(P_{od}^-\right)'(x),$$

which implies that $P'_{od}(x) > 0$, and hence $x > x'_{od}$.

Finally, we show that $P(x)$ has at most one negative root. Consider the interval $[0, x_{od}]$. Since $x_{ev} > x_{od}$, then $P_{ev} > 0$

in $[0, x_{od}]$. Additionally, P_{od} is non-positive in this interval, implying that $P(-x) = P_{ev}(x) - P_{od}(x)$ is positive in the interval $[0, x_{od}]$. In the interval $[x'_{od}, +\infty)$, the polynomial P'_{od} is positive which together with the fact that P'_{ev} is negative implies that $P'(-x) = P'_{ev} - P'_{od}$ is negative. Thus being positive in $[0, x_{od}]$ and monotone decreasing to $-\infty$ in $[x'_{od}, +\infty)$, $P(-x)$ necessarily has exactly one positive root. □

Proof of Proposition 4. Suppose that a polynomial P realizes a given sign pattern $\overline{\sigma}$ with the pair $(2k, 0)$, where $0 < k \le l$. Then $P(0) > 0$ and there exists $a > 0$ such that $P(a) < 0$. Hence $P(-a) < 0$ since the monomials of even degree attain the same value at a and $-a$ while odd-degree monomials have smaller values at $-a$ than at a by our assumption on the signs. Thus, there exists at least one negative root which is a contradiction.

We prove that any pair of the form $(2s, 2t)$, $0 < s \le l$, $0 < t \le d/2 - l$ is realizable with the given sign pattern satisfying the assumption of Proposition 4. We make use of Lemma 14 and Example 15. Represent the considered sign pattern $\overline{\sigma}$ in the form $\overline{\sigma} = (1, \widehat{\sigma}_1, +)$, where the last entry of $\widehat{\sigma}_1$ is $+$. (For $d = 2$, $\widehat{\sigma}_1 = +$.) Then the realizability of $\overline{\sigma}$ with the pair $(2s, 2t)$ follows from that of $\overline{\sigma}' = (1, \widehat{\sigma}_1)$ with the pair $(2s, 2t - 1)$, see Example 15 with $P_2(x) = x + 1$.

The sign pattern $\overline{\sigma}'$ has an even number of entries (including the leading 1). Denote by $\overline{\sigma}'(r)$ the sign pattern obtained by truncation of the last $2r$ entries of $\overline{\sigma}'$. (In particular, $\overline{\sigma}'(0) = \overline{\sigma}'$.) Below we will provide an algorithm which shows how realizability of $\overline{\sigma}'(r)$ implies that of $\overline{\sigma}'(r - 1)$. We also indicate how the pairs change in this process. It is clear that $\overline{\sigma}'(d/2 - 1) = (1, +)$. The corresponding pair is $(0, 1)$ and it is realizable by the polynomial $x + 1$.

The sign pattern $\overline{\sigma}'$ contains l minuses, all of which occupy only odd positions. We mark the leftmost s of them. We distinguish between the following three cases according to the last two entries of $\overline{\sigma}'(r - 1)$. Notice that the last entry of $\overline{\sigma}'(r)$ is always $+$.

Case (a) $\overline{\sigma}'(r - 1) = (\overline{\sigma}'(r), -, +)$ and the minus sign in the last but one position is marked. In this case one applies Example 15 twice, each time with $P_2(x) = x - 1$. If the pair of $\overline{\sigma}'(r)$ equals (u, v), then the one of $\overline{\sigma}'(r - 1)$ equals $(u + 2, v)$.

Cbse (b) $\overline{\sigma}'(r - 1) = (\overline{\sigma}'(r), -, +)$ and the minus sign in the last but one position is not marked. In this case, one applies Example 15 with $P_2(x) = x^2 - 2x + 2$. The pairs of $\overline{\sigma}'(r - 1)$ and of $\overline{\sigma}'(r)$ are the same.

Ccse (c) $\overline{\sigma}'(r - 1) = (\overline{\sigma}'(r), +, +)$. If $v < 2t$, then one applies Example 15 twice, each time with $P_2(x) = x + 1$. The pair of $\overline{\sigma}'(r - 1)$ equals $(u, v + 2)$. If $v = 2t$, then one applies Example 15 with $P_2(x) = x^2 + 2x + 2$. The pairs of $\overline{\sigma}'(r - 1)$ and of $\overline{\sigma}'(r)$ are the same.

Observe that any sign pattern $\overline{\sigma}$ satisfying the assumptions of Proposition 4 can be obtained from the initial $(1, +)$ by applying consecutively $d/2 - 1$ times the appropriate of the Cases (a)–(c). Notice that we add exactly $2s$ positive and $2(t - 1)$ negative roots. Another negative root comes from $\overline{\sigma}'(d/2 - 1)$ and the last one is obtained when passing from $\overline{\sigma}'$ to $\overline{\sigma}$. Hence the pattern and the pair are realizable. □

Proof of Proposition 6. To prove (i) we show that the three-part sign pattern $\overline{\sigma}$ satisfying the assumptions of Proposition 6 is not realizable by a polynomial $P(x)$ having $d - 2$ negative and a double-positive root. By a linear change of x, the latter can be assumed to be equal to 1:

$$P(x) = \left(x^2 - 2x + 1\right) S(x), \quad \text{where}$$
$$S(x) = x^{d-2} + a_1 x^{d-3} + \cdots + a_{d-2}.$$

Here $a_j > 0$ and the factor $S(x)$ has $d - 2$ negative roots. The coefficients of $P(x)$ are equal to

$$1, a_1 - 2, a_2 - 2a_1 + 1, a_3 - 2a_2 + a_1, \ldots, a_{d-2} - 2a_{d-3} + a_{d-4}, -2a_{d-2} + a_{d-3}, a_{d-2}.$$

We want to show that it is impossible to have both inequalities:

$$a_m - 2a_{m-1} + a_{m-2} < 0 \quad (*) \quad \text{and}$$
$$a_{m+n-1} - 2a_{m+n-2} + a_{m+n-3} < 0 \quad (**)$$

satisfied.

Now consider a polynomial having $d - 2$ negative roots and a complex conjugate pair. If the polynomial has at least one negative coefficient, then its factor having complex roots must be of the form $x^2 - 2\beta x + \beta^2 + \gamma$, where $\beta > 0$ and $\gamma > 0$. A linear change of x brings the polynomial to the form

$$Q(x) = (x^2 - 2x + 1 + \delta)S(x), \ \delta > 0.$$

The coefficients of $Q(x)$ are obtained from that of $P(x)$ by adding the ones of the polynomial $\delta S(x)$. If inequality $(**)$ fails, then the coefficient of $x^{d-m-n+1}$ in $Q(x)$ is positive (it equals $a_{m+n-1} - 2a_{m+n-2} + a_{m+n-3} + \delta a_{m+n+1} > 0$). So the sign pattern of $Q(x)$ is different from $\overline{\sigma}$. If inequality $(**)$ holds, then inequality $(*)$ fails and the coefficient of x^{d-m} in $Q(x)$ is non-negative, so $Q(x)$ does not have the sign pattern $\overline{\sigma}$.

The polynomial $S(x)$ being real-rooted, its coefficients satisfy the Newton inequalities:

$$\frac{a_k^2}{\binom{d-2}{k}^2} \ge \frac{a_{k-1}a_{k+1}}{\binom{d-2}{k+1}\binom{d-2}{k-1}}, \ k = 1, \ldots, d-3 \quad (\text{we set} \quad a_0 = 1).$$

Here

$$\kappa = \frac{\binom{d-2}{m}\binom{d-2}{m+n-3}}{\binom{d-2}{m-1}\binom{d-2}{m+n-2}} = \frac{d-m-1}{m} \cdot \frac{d-p-1}{p},$$

i.e., $a_m a_{m+n-3} \geq \kappa a_{m-1} a_{m+n-2}$. Inequalities $(*)$ and $(**)$ imply, respectively,

$$a_m < 2a_{m-1} \quad \text{and} \quad a_{m+n-3} < 2a_{m+n-2}.$$

Thus, $a_m a_{m+n-3} \geq \kappa a_{m-1} a_{m+n-2} > \frac{\kappa}{4} a_m a_{m+n-3}$, which is a contradiction since $\kappa \geq 4$ by assumption.

To prove (ii), we use Lemma 14 and Example 15. We construct sign patterns $\overline{\sigma}(0) = \overline{\sigma}, \overline{\sigma}(1), \ldots,$ each being a truncation of the previous one (by one, two, or three according to the case as explained below), and corresponding pairs (u_j, v_j), $j = 0, 1, 2, \ldots,$ where $(u_0, v_0) = (2, v)$, such that realizability of $\overline{\sigma}(j-1)$ with (u_{j-1}, v_{j-1}) follows from the one of $\overline{\sigma}_j$ with (u_j, v_j). For convenience, we write instead of the pair (u_j, v_j) the triple (u_j, v_j, w_j), where w_j is the number of complex conjugate pairs of roots (hence $u_j + v_j + 2w_j = d_j$, where $d_j + 1$ is the number of entries of $\overline{\sigma}(j)$).

We consider first the case $v \geq 2$. The necessary modifications in the cases $v = 0$ and $v = 1$ are explained at the end of the proof.

If $\overline{\sigma}(j-1)$ has not more than three entries, then we do not need to construct the sign pattern $\overline{\sigma}(j)$. Two cases are to be distinguished:

Case A. If $\overline{\sigma}(j-1)$ has only two entries, then these are either $(1, +)$ or $(1, -)$; and $\overline{\sigma}(j-1)$ is realizable by the polynomials $x+1$ or $x-1$ with $(u_{j-1}, v_{j-1}, w_{j-1}) = (0, 1, 0)$ or $(1, 0, 0)$, respectively.

Case B. Case B. If $\overline{\sigma}(j-1)$ has only three entries, then they can be only $(1, +, +)$, $(1, +, -)$, or $(1, -, -)$. In the first case, $(\overline{\sigma}(j-1), (0, 2, 0))$ is realizable by $(x+1)(x+2)$ and $(\overline{\sigma}(j-1), (0, 0, 1))$ by $x^2 + 2x + 2$. In the second case, $(\overline{\sigma}(j-1), (1, 1, 0))$ is realizable by $(x+2)(x-1)$ and in the third case $(\overline{\sigma}(j-1), (1, 1, 0))$ by $(x+1)(x-2)$.
Suppose that $\overline{\sigma}(j-1)$ contains more than three entries. The following cases are possible:

Case C. The last three entries of $\overline{\sigma}(j-1)$ are $(+, +, +)$ or $(-, -, -)$. If $w_{j-1} > 0$, then we apply Lemma 14 with $P_2(x) = x^2 + 2x + 2$ and we set $(u_j, v_j, w_j) = (u_{j-1}, v_{j-1}, w_{j-1} - 1)$. If $w_{j-1} = 0$, then we apply Lemma 14 twice, both times with $P_2(x) = x + 1$. We set $(u_j, v_j, w_j) = (u_{j-1}, v_{j-1} - 2, 0)$.

Case D. The last three entries of $\overline{\sigma}(j-1)$ are $(-, +, +)$ or $(+, -, -)$. One applies Lemma 14 twice, the first time with $P_2(x) = x + 1$ and the second time with $P_2(x) = x - 1$. One sets $(u_j, v_j, w_j) = (u_{j-1} - 1, v_{j-1} - 1, w_j)$.

Case E. The last three entries of $\overline{\sigma}(j-1)$ are $(-, -, +)$ or $(+, +, -)$. One applies Lemma 14 with $P_2(x) = x - 1$ and sets $(u_j, v_j, w_j) = (u_{j-1} - 1, v_j, w_j)$.

In Cases C and D, $\overline{\sigma}(j)$ has two entries and in Case E it has one entry less than $\overline{\sigma}(j-1)$.

Further explanations. In the pair $(2, v)$ obtained as the result of this algorithm, the first component equals 2 because one encounters exactly once Case D with $(-, +, +)$ or Case E with $(-, -, +)$ (in both of them u_{j-1} decreases by 1); and exactly once Case D with $(+, -, -)$ or Case E with $(+, -, -)$ (when u_{j-1} also decreases by 1); or Case A with $(1, -)$ or Case B with $(1, +, -)$ or $(1, -, -)$.

When $v = 0$ or $v = 1$, one does not have the possibility to apply Lemma 14 with $P_2(x) = x + 1$ and Cases D and E have to be modified. One has to consider the last four entries of $\overline{\sigma}(j-1)$. If they are $(+, \pm, \mp, -)$ or $(-, \mp, \pm, +)$, then one applies Lemma 14 with $P_2(x) = x^3 \pm \epsilon_1 x^2 \mp \epsilon_2 x - 1$, where $\epsilon_i > 0$ are small. One sets $(u_j, v_j, w_j) = (u_{j-1} - 1, v_{j-1}, w_{j-1} - 1)$ and $\overline{\sigma}(j)$ has three entries less than $\overline{\sigma}(j-1)$. □

Proof of Theorem 9. The fact that the patterns given in the formulation of Theorem 9 are non-realizable follows from Proposition 6 and Lemma 21. It remains to show that all other admissible patterns and pairs are realizable.

Using Lemma 17 and a Mathematica script [Forsgård] written by the first author, this question is reduced to checking the cases:

$$\begin{aligned}
&(1, +, -, +, +, -, +, +) \quad \text{with} \quad (4,1);\\
&(1, +, +, -, -, -, +, +) \quad \text{with} \quad (0,5);\\
&(1, +, +, -, -, +, -, -) \quad \text{with} \quad (3,0);\\
&(1, +, +, +, -, -, -, +) \quad \text{with} \quad (0,5);\\
&(1, +, +, +, -, +, +, -) \quad \text{with} \quad (3,0);\\
&(1, +, -, +, -, +, +, -) \quad \text{with} \quad (3,0);\\
&(1, +, -, +, +, +, -, -) \quad \text{with} \quad (3,0); \quad \text{and}\\
&(1, +, -, +, +, +, +, -) \quad \text{with} \quad (3,0).
\end{aligned}$$

The first five cases can be settled by using either Lemma 14 or Lemma 17. For the realizability of the remaining three cases, we provide the following concrete examples:

$$\begin{aligned}
P_1(x) &= (x - 0.1690)(x - 1.4361)(x - 2.0095)\\
&\quad \times \left(x^2 + 0.0218x + 6.2846\right)\left(x^2 + 3.6029x + 3.2609\right),\\
P_2(x) &= (x - 2.6713)(x - 2.6087)(x - 0.6059)\\
&\quad \times \left(x^2 + 0.5495x + 0.3304\right)\left(x^2 + 5.3464x + 7.1668\right),
\end{aligned}$$

$$P_3(x) = (x - 0.6056)(x - 2.6105)(x - 2.6696) \times \left(x^2 + 0.5493x + 0.3305\right)\left(x^2 + 5.3465x + 7.1672\right).$$

□

Proof of Theorem 10. The fact that the patterns given in the formulation of Theorem 9 are non-realizable follows from Proposition 6 and Lemma 21 except for the case

$$(1, -, -, -, +, -, -, -, +) \quad \text{with } (0, 2) \text{ and } (0, 4).$$

Substituting $x \mapsto -x$, we obtain the sign pattern $(1, +, -, +, +, +, -, +, +)$. That is $P(x) < P(-x)$ for $x \in \mathbb{R}_+$. In particular, if P has a negative root, then it has at least two positive roots.

Using Lemma 17 and Lemma 19 and the above-mentioned Mathematica script, all patterns except those of Remark 11 can be shown to be realizable. □

Proof of Theorem 13. We will follow the steps of the proof of Lemma 18. For any polynomial p, the set K_p consisting of all exponents k such that there exists a $x_k \in \mathbb{R}_+$ for which (2–4) holds provides a lower bound on the number of real roots of p. This lower bound is called the number of *lopsided induced zeros* of p. Fixing an arbitrary set of exponents K, let us denote by S_K the set of all polynomials such that $K \subseteq K_p$. It is shown in [Forsgård 15] that S_K is logarithmically convex. For example, if $K = \{0, 1, \ldots, d\}$, then S_K is the set of all sign-independently real-rooted polynomials. Consider the family F_m consisting of all exponent sets K such that the number of lopsided-induced zeros of polynomials in S_K is at least m. The set $S_m = \cup_{K \in F_m} S_K$ is a union of logarithmically convex sets, whose intersection contains the set of all sign-independently real-rooted polynomials. In particular, S_m is path-connected.

As in the proof of Lemma 18, for any polynomial p which has at least m real roots, all polynomials in the path

$$\alpha \mapsto p(x) + \frac{x}{\alpha} p'(x), \quad \alpha \in [\infty, \alpha^*]$$

have at least m real roots. Exactly the same argument as in the proof of Lemma 18 gives path-connectedness of the set (1–3) of polynomials with at least m real roots.

Let us now prove simply connectedness of the set (1–3) by induction on the degree d. Consider a closed loop in $Pol_{d,\geq m}(\overline{\sigma})$, i.e., a path ℓ given by $\theta \mapsto p_\theta(x)$, $\theta \in [0, 1]$, such that $p_0(x) = p_1(x)$, and such that $p_\theta(x)$ has at least m real roots, for all θ.

Consider the induced loop ℓ' given by $\theta \mapsto p'_\theta(x)$, where we use the notation $p'_\theta(x) = \frac{d}{dx} p_\theta(x)$. It is contained in the set $Pol_{d-1,\geq m-1}(\widehat{\overline{\sigma}})$, where $\widehat{\overline{\sigma}}$ is obtained from $\overline{\sigma}$ by deleting its last entry. By the induction hypothesis, the loop ℓ' can be contracted to a point within the set of all polynomials of degree $d - 1$ with at least $m - 1$ real roots. In other words, we have a map $(\theta, \phi) \mapsto p'_{(\theta,\phi)}$, for $(\theta, \phi) \in [0, 1]^2$, satisfying the conditions: (i) $p'_\theta(x) = p'_{(\theta,0)}(x)$; (ii) $p'_{(\theta,1)}$ is independent of θ; (iii) $p'_{(\theta,\phi)}$ has at least $m - 1$ real roots, for all θ and ϕ. The last property implies that $xp'_{(\theta,\phi)}$ has at least m real roots for all θ and ϕ. Define $p_{(\theta,\phi)}$ by the conditions that $\frac{d}{dx} p_{(\theta,\phi)} = p'_{(\theta,\phi)}$ and that the constant term of $p_{(\theta,\phi)}$ is independent of ϕ.

Since the loop ℓ' is compact, we can find an $\alpha^* \in \mathbb{R}_+$ such that the polar derivative

$$p'_{(\theta,\phi,\alpha)}(x) := p_{(\theta,\phi)}(x) + \frac{x}{\alpha} p'_{(\theta,\phi)}(x)$$

has at least m roots for each $\alpha < \alpha^*$ and all $(\theta, \phi) \in [0, 1]^2$. Thus, similar to the proof of Lemma 18, the composition of the maps

$$\alpha \mapsto p'_{(\theta,0,\alpha)}, \quad \alpha \in [\infty, \alpha^*], \quad \text{and} \quad \phi \mapsto p'_{(\theta,\phi,\alpha^*)}, \quad \phi \in [0, 1]$$

provides a contraction of the loop ℓ in the set $Pol_{d,\geq m}(\overline{\sigma})$. □

3. FINAL REMARKS

Above, we mainly discussed the question which pairs (pos, neg) of the numbers of positive and negative roots satisfying the obvious compatibility conditions are realized by polynomials with a given sign pattern. Our main Conjecture 12 presents restrictions observed in consideration of all non-realizable pairs up to degree 10. However, the following important and closely related questions remained unaddressed above.

Problem 22. *Is the set of all polynomials realizing a given pair (pos, neg) and having a sign pattern $\overline{\sigma}$ path-connected (if nonempty)?*

Given a real polynomial p of degree d with all non-vanishing coefficients, consider the sequence of pairs

$$\{(pos_0(p), neg_0(p)), (pos_1(p), neg_1(p)), (pos_2(p), neg_2(p)), \ldots, (pos_{d-1}(p), neg_{d-1}(p))\},$$

where $(pos_j(p), neg_j(p))$ is the numbers of positive and negative roots of $p^{(j)}$, respectively. Observe that if one knows the above sequence of pairs then one knows the sign pattern of a polynomial p which is assumed to be monic. Additionally, it is easy to construct examples that the converse fails.

Problem 23. *Which sequences of pairs are realizable by real polynomials of degree d with all non-vanishing coefficients?*

Notice that similar problem for the sequence of pairs of real roots (without division into positive and negative) was considered in [Kostov 07].

Our final question is as follows.

Problem 24. *Is the set of all polynomials realizing a given sequence of pairs as above path-connected (if nonempty)?*

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