A FIFTEEN-QUADRIC DELETION ARRANGEMENT IN $\mathbb{RP}^3$

ANDREI GABRIELOV AND BORIS SHAPIRO

Abstract. We study the first real projective case in which deletion of one point from a matroid-generic point configuration produces a non-trivial discriminantal locus. For a matroid-generic configuration of six labelled points in $\mathbb{RP}^3$, the fibre of the deletion map for a seventh point is the complement of the twenty planes spanned by triples of the six points. After normalizing five points, we prove that the locus where this adjoint plane arrangement acquires an additional fourfold point is the union of fifteen explicit quadrics in the normalized copy of $\mathbb{RP}^3$. These quadrics are naturally indexed by the fifteen partitions of six labels into three pairs. We describe the pairwise intersections needed for the real chamber calculation and prove that the complement of the fifteen quadrics in the normalized configuration space has 720 connected components. The coincidence of the two equations associated with a pair partition is the classical theorem of Möbius on mutually inscribed tetrahedra, written here in explicit coordinates.

1. Introduction

Let X be a connected manifold and let $F(X, n)$ be the space of ordered configurations of n pairwise distinct points of X. The usual deletion map

$$F(X, n) \longrightarrow F(X, n-1), \qquad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}),$$

is a locally trivial fibration; this is the Fadell–Neuwirth theorem [3, 4]. The situation changes if one replaces the full configuration space by an open subspace cut out by incidence conditions. This note studies the corresponding deletion problem for labelled point configurations in real projective spaces.

Let

$$\Psi(k, n) = \{(Q_1, \ldots, Q_n) \in (\mathbb{RP}^k)^n : \text{no } k+1 \text{ of the } Q_i \text{ lie in a hyperplane}\}.$$

We call such configurations matroid-generic. If

$$\mathcal{V} = (Q_1, \ldots, Q_m) \in \Psi(k, m),$$

then the fibre over $\mathcal{V}$ of the deletion map

$$\Psi(k, m+1) \longrightarrow \Psi(k, m)$$

is the complement in $\mathbb{RP}^k$ of the adjoint arrangement

$$\mathcal{A}_{\mathcal{V}} = \big\{\langle Q_{i_1}, \ldots, Q_{i_k}\rangle : 1 \le i_1 < \cdots < i_k \le m\big\}.$$

By Randell's lattice-isotopy theorem [8], this deletion map is locally topologically trivial over every connected locus on which the intersection lattice of $\mathcal{A}_{\mathcal{V}}$ is constant. Thus the discriminantal locus for deletion is controlled by jumps of the intersection lattice of this adjoint arrangement.

2020 *Mathematics Subject Classification.* Primary 14N20, 52C35; Secondary 51A20, 05B35, 52B40.

Key words and phrases. point configurations, projective geometry, hyperplane arrangements, discriminantal arrangements, Möbius configuration, oriented matroids.

The first real projective case in which a genuinely new phenomenon occurs is $m = 6$, $k = 3$. One starts with six matroid-generic labelled points in $\mathbb{RP}^3$ and considers the twenty planes

$$P_{ijk} = \langle Q_i, Q_j, Q_k \rangle.$$

There are many forced incidences: for example, four planes contain each line $L_{ij} = \langle Q_i, Q_j \rangle$, and ten planes contain each labelled point Q_i. We prove that the first additional incidence away from the forced lines is governed by a concrete arrangement of fifteen quadric surfaces.

Theorem 1.1. *After fixing five of the six points in standard position, the normalized space of matroid-generic six-tuples is*

$$W = \{(x_1 : x_2 : x_3 : x_4) \in \mathbb{RP}^3 : x_i \neq 0,\ x_i \neq x_j \text{ for } i \neq j\}.$$

The locus in W where the adjoint arrangement of the six points acquires an additional point incident with at least four planes is the union of fifteen quadrics

$$\bigcup_{\lambda} S_\lambda \cap W,$$

where λ runs over the fifteen partitions of $\{1, \ldots, 6\}$ into three pairs. Each quadric meets exactly sixteen of the sixty chambers of W, each chamber meets exactly four quadrics, and in each chamber these four quadrics cut the chamber into twelve contractible connected components. Consequently

$$W \setminus \bigcup_{\lambda} S_\lambda$$

has 720 connected components.

The result is an explicit low-dimensional calculation in the spirit of discriminantal arrangements of Manin–Schechtman [6], adjoint arrangements, and single-element extensions of oriented matroids [1, 9]. The incidence theorem behind the coincidence of two a priori different equations is classical: it is equivalent to Möbius' theorem on mutually inscribed tetrahedra [7]; see also Coxeter's discussion of self-dual configurations [2]. The point of the present note is the explicit deletion-fibre interpretation and the chamber-level description of the resulting fifteen-quadric arrangement.

2. Deletion and adjoint arrangements

The deletion map

$$\mathfrak{d} : \Psi(k, n) \longrightarrow \Psi(k, n-1)$$

forgets the last point. For $\mathcal{V} = (Q_1, \ldots, Q_{n-1}) \in \Psi(k, n-1)$, the fibre is

$$\mathfrak{d}^{-1}(\mathcal{V}) = \mathbb{RP}^k \setminus \mathcal{A}_{\mathcal{V}},$$

where $\mathcal{A}_{\mathcal{V}}$ consists of the hyperplanes spanned by all k-subsets of $\mathcal{V}$.

Lemma 2.1. *On every connected stratum on which the intersection lattice of $\mathcal{A}_{\mathcal{V}}$ is constant, the deletion map is locally topologically trivial.*

Proof. Along such a stratum the family of projective hyperplane arrangements is lattice-isotopic. Randell's lattice-isotopy theorem [8] gives a locally trivial family of complements. Applying this theorem to the family of adjoint arrangements gives the claim. $\square$

We now specialize to six points in $\mathbb{RP}^3$.

3. Six points in real projective three-space and admissible quadruples

Let $Q_1, \dots, Q_6$ be a matroid-generic configuration in $\mathbb{RP}^3$. Write

$$L_{ij} = \langle Q_i, Q_j \rangle, \qquad P_{ijk} = \langle Q_i, Q_j, Q_k \rangle.$$

The twenty planes P_{ijk} form the adjoint arrangement $\mathcal{A}_{\mathcal{V}}$.

We first record that there are no hidden rank-two incidences away from the forced labelled lines.

Lemma 3.1. *If three distinct planes among the P_{ijk} contain a common line, then that line is one of the labelled lines L_{ij}.*

Proof. If two of the three planes have two labels in common, their intersection is the labelled line spanned by these two points; any line common to all three planes must therefore be this labelled line. Suppose, therefore, that no two of the triples have two labels in common.

If two triples have exactly one label in common, say a, then the intersection of the two corresponding planes is a line through Q_a. Under the standing assumption that no pair of triples has two labels in common, the third triple cannot also contain a: its two other labels would have to avoid all labels already paired with a, but only one such label is available. If the intersection line of the first two planes were contained in the third plane, then the third plane would contain Q_a. Since the third triple does not contain a, this would put four of the original six points in a plane, contradicting matroid-genericity.

It remains to consider the case where two of the triples are disjoint, say $\{a, b, c\}$ and $\{d, e, f\}$. A third triple must meet at least one of them, say it contains a but is not equal to $\{a, b, c\}$. If the common line of the first two planes were contained in the third plane, then this line would also be the intersection of the first and third planes, hence would contain Q_a. But then Q_a would lie in P_{def}, again contradicting matroid-genericity. □

Thus an additional rank-three incidence away from the labelled lines occurs when four planes P_{ijk} have a common point not lying on any L_{ij}. Such a quadruple has a rigid combinatorial form.

Definition 3.2. A set Y of four triples from $\{1, \dots, 6\}$ is called *admissible* if no two triples in Y have a common pair of labels.

For example,

$$Y = \{135, 146, 236, 245\}$$

is admissible. Its complementary quadruple is

$$Y^c = \{246, 235, 145, 136\}.$$

Both are associated with the same pair partition

$$(12)(34)(56).$$

Figure 1 shows the same combinatorics in the dual cube picture: the eight triples avoiding the three paired edges split into two parity classes, i.e. into two complementary admissible quadruples.

Lemma 3.3. *The admissible quadruples of triples on six labels occur in complementary pairs indexed by the fifteen partitions of $\{1, \dots, 6\}$ into three pairs. Equivalently, each pair partition gives exactly two admissible quadruples, and there are thirty admissible quadruples in all.*

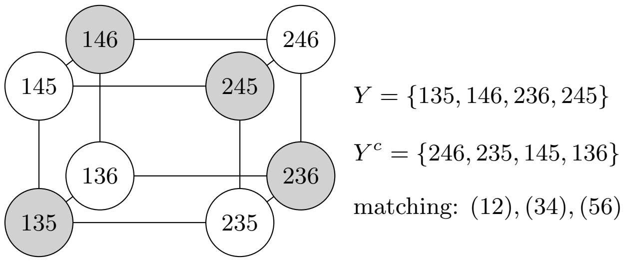

FIGURE 1. The two admissible quadruples associated with the pair partition (12)(34)(56). Vertices of the cube are the triples containing one element from each pair; the two parity classes give the two complementary quadruples.

Proof. An admissible quadruple uses $4 \cdot 3 = 12$ pairs of labels, counted over its four triples. Since no pair is repeated, these are twelve distinct edges of the complete graph K_6. If a label occurred in three triples, then the six edge-incidences from this label to the other labels would force a repeated edge, because there are only five other labels. Hence each label occurs at most twice. Since the total number of label occurrences is 12, every label occurs exactly twice.

It follows that the three unused edges form a perfect matching of K_6, namely a partition of the six labels into three pairs. Conversely, after removing a perfect matching from K_6, one obtains the one-skeleton of an octahedron. Its triangular faces split into the two complementary classes of four faces. These two classes are precisely the two admissible quadruples associated with the matching. $\square$

Proposition 3.4. *Four planes $P_{i_1j_1k_1}, \ldots, P_{i_4j_4k_4}$ can meet at a point outside the union of the lines L_{ij} only if the four triples form an admissible quadruple.*

Proof. If two triples share two labels, the corresponding planes meet along the line spanned by those two labelled points. Therefore any common point of all four planes lies on some L_{ij} unless no pair of triples shares two labels. This is exactly admissibility. $\square$

Conversely, for an admissible quadruple, linear dependence of the four plane equations is the condition that the four planes have a common point. In the coordinate calculation below one checks, on the normalized open set W, that this common point is outside the labelled lines.

4. THE FIFTEEN QUADRICS

Use projective automorphisms to put the first five points in the standard position

$$Q_1 = (1:0:0:0), \quad Q_2 = (0:1:0:0), \quad Q_3 = (0:0:1:0),$$
$$Q_4 = (0:0:0:1), \quad Q_5 = (1:1:1:1).$$

and write

$$Q_6 = (x_1 : x_2 : x_3 : x_4).$$

Matroid-genericity is equivalent to

$$x_i \neq 0 \quad (1 \leq i \leq 4), \qquad x_i \neq x_j \quad (i \neq j).$$

Let

$$W = \{(x_1 : x_2 : x_3 : x_4) : x_i \neq 0,\, x_i \neq x_j\} \subset \mathbb{RP}^3.$$

Thus W is the normalized space of matroid-generic six-tuples with the first five points fixed.

Theorem 4.1. *The locus in W where at least four planes P_{ijk} meet at a point outside the labelled lines L_{ij} is the union of fifteen quadrics*

$$\bigcup_{\lambda} S_\lambda \cap W,$$

where λ runs over the fifteen partitions of $\{1,\dots,6\}$ into three pairs. The two admissible quadruples associated with the same pair partition give the same quadric.

Proof. It is enough, by the action of the symmetric group on the first five labels, to compute the equation for the partition $(12)(34)(56)$. The two associated admissible quadruples are

$$Y = \{135, 146, 236, 245\}, \qquad Z = \{246, 235, 145, 136\}.$$

The four planes corresponding to Y meet in a point exactly when their four normal vectors are linearly dependent. In the above coordinates these normals may be chosen as

$$\begin{aligned} n_{135} &= (0,-1,0,1), & n_{146} &= (0,-x_3,x_2,0),\\ n_{236} &= (-x_4,0,0,x_1), & n_{245} &= (-1,0,1,0). \end{aligned}$$

Their determinant is, up to sign,

$$x_1x_2 - x_3x_4.$$

For the complementary admissible quadruple Z one can choose normals

$$\begin{aligned} n_{246} &= (-x_3,0,x_1,0), & n_{235} &= (-1,0,0,1),\\ n_{145} &= (0,-1,1,0), & n_{136} &= (0,-x_4,0,x_2), \end{aligned}$$

and their determinant is again, up to sign, $x_1x_2 - x_3x_4$. Hence both admissible quadruples define the same quadric

$$S_{(12)(34)(56)}: \quad x_1x_2 = x_3x_4.$$

On this quadric the four planes in Y meet at

$$R_Y = (x_3 : x_2 : x_3 : x_2).$$

A direct substitution in the fifteen labelled lines L_{ij} shows that $R_Y \in L_{ij}$ would force one of the excluded relations $x_i = 0$ or $x_i = x_j$. Thus, on W, the common point is genuinely away from the forced rank-two flats. Permuting the labels gives the remaining fourteen equations. Proposition 3.4 shows that every additional fourfold point away from the labelled lines arises in this way. $\square$

For reference, Table 1 lists the fifteen equations in the above normalization. The equations are only used on W, where the factors x_i and $x_i - x_j$ which occur in intermediate normal-vector calculations are non-zero.

Remark 4.2. The equality of the two equations associated with complementary admissible quadruples is a coordinate form of Möbius' theorem on mutually inscribed tetrahedra. In the example above, the common point of the four planes in Y and the common point of the four planes in Z give the two vertices completing the two mutually inscribed tetrahedra.

5. Pairwise intersections of the quadrics

We record the part of the intersection pattern needed for the chamber computation. Since the group S_5 acts transitively on the normalized chambers and on the quadrics, it is enough to compare $S_{(12)(34)(56)}$ with three representative types.

Proposition 5.1. *Let λ and μ be two distinct partitions of six labels into three pairs.*

TABLE 1. The fifteen quadrics in the normalized coordinates.

Pair partition λ	Equation of S_λ
(12)(34)(56)	$x_1x_2 - x_3x_4 = 0$
(12)(35)(46)	$x_1x_2 - x_1x_4 - x_2x_4 + x_3x_4 = 0$
(12)(36)(45)	$x_1x_2 - x_1x_3 - x_2x_3 + x_3x_4 = 0$
(13)(24)(56)	$x_1x_3 - x_2x_4 = 0$
(13)(25)(46)	$x_1x_3 - x_1x_4 + x_2x_4 - x_3x_4 = 0$
(13)(26)(45)	$x_1x_2 - x_1x_3 + x_2x_3 - x_2x_4 = 0$
(14)(23)(56)	$x_1x_4 - x_2x_3 = 0$
(14)(25)(36)	$x_1x_3 - x_1x_4 - x_2x_3 + x_3x_4 = 0$
(14)(26)(35)	$x_1x_2 - x_1x_4 - x_2x_3 + x_2x_4 = 0$
(15)(23)(46)	$x_1x_4 + x_2x_3 - x_2x_4 - x_3x_4 = 0$
(15)(24)(36)	$x_1x_3 - x_2x_3 + x_2x_4 - x_3x_4 = 0$
(15)(26)(34)	$x_1x_2 - x_2x_3 - x_2x_4 + x_3x_4 = 0$
(16)(23)(45)	$x_1x_2 + x_1x_3 - x_1x_4 - x_2x_3 = 0$
(16)(24)(35)	$x_1x_2 - x_1x_3 + x_1x_4 - x_2x_4 = 0$
(16)(25)(34)	$x_1x_2 - x_1x_3 - x_1x_4 + x_3x_4 = 0$

(i) *If λ and μ have a common pair not containing label* 6, *then $S_\lambda \cap S_\mu$ consists, in the projective closure, of two boundary lines and one conic whose intersection with W is non-empty.*

(ii) *If λ and μ have a common pair containing label* 6, *then $S_\lambda \cap S_\mu$ consists of three boundary lines and one line whose intersection with W is non-empty.*

(iii) *If λ and μ have no common pair, then $S_\lambda \cap S_\mu$ consists of one boundary line and an irreducible rational cubic curve meeting W. In this case there is a third partition ν such that*

$$S_\lambda \cap S_\mu = S_\lambda \cap S_\nu = S_\mu \cap S_\nu$$

as sets in the projective closure.

Here a boundary line means a component contained in the complement of W.

Proof. Take $\lambda = (12)(34)(56)$, so that S_λ has equation

$$q = x_1x_2 - x_3x_4 = 0.$$

First consider $\mu = (12)(35)(46)$, which has a common pair not containing 6. The corresponding quadric is

$$q_\mu = x_1x_2 - x_1x_4 + x_3x_4 - x_2x_4 = 0.$$

Modulo q, this becomes

$$x_4(2x_3 - x_1 - x_2) = 0.$$

Thus the intersection consists of the two boundary lines contained in $x_4 = 0$ and the conic in the plane $2x_3 = x_1 + x_2$ cut out by $q = 0$. This conic meets W, for example near $(3 : 1 : 2 : 3/2)$.

Next take $\mu = (13)(24)(56)$, which has a common pair containing 6. Then

$$q_\mu = x_1x_3 - x_2x_4 = 0.$$

Together with $q = 0$, this factors into the four lines

$$\begin{aligned} &x_1 = x_4 = 0, && x_2 = x_3 = 0, \\ &x_1 = x_4,\ x_2 = x_3, && x_1 = -x_4,\ x_2 = -x_3. \end{aligned}$$

The first three are boundary lines; the last one meets W.

Finally take $\mu = (14)(25)(36)$, which has no common pair with λ. Its quadric is

$$q_\mu = x_1x_3 - x_1x_4 - x_2x_3 + x_3x_4 = 0.$$

The intersection with $q = 0$ contains the boundary line $x_1 = x_3 = 0$. On the affine chart $x_4 = 1$, the remaining component is parametrized by

$$x_3 = x_1x_2, \qquad x_1 = x_2 - 1 + \frac{1}{x_2},$$

which gives an irreducible rational cubic in the projective closure. The third partition is $\nu = (16)(23)(45)$, with equation

$$q_\nu = x_1x_2 + x_1x_3 - x_1x_4 - x_2x_3 = 0.$$

Since $q_\nu - q_\mu = q$, the three pairwise intersections coincide as claimed. □

6. Chambers and the local decomposition

The space W is the complement in $\mathbb{RP}^3$ of the hyperplanes

$$x_i = 0, \qquad x_i = x_j, \qquad 1 \le i < j \le 4.$$

It has sixty contractible connected components. Indeed, in the affine chart $x_4 = 1$, these chambers correspond to the cyclic orderings of the five points $0, x_1, x_2, x_3, 1$ on the real projective line, hence to $5!/2 = 60$ chambers. The group S_5 permuting $Q_1, \ldots, Q_5$ acts transitively on them by projective linear transformations.

Theorem 6.1. *The fifteen quadrics have the following chamber-level behaviour.*

(i) *Each quadric meets exactly sixteen chambers, and in each such chamber the intersection is an open disc whose boundary lies in the boundary of the chamber.*
(ii) *Each chamber meets exactly four quadrics.*
(iii) *In each chamber, these four quadrics partition the chamber into twelve contractible connected components.*

Proof. No point of W lies on $x_4 = 0$, so we work in the affine chart $x_4 = 1$. For (i), by symmetry it is enough to consider

$$S_{(12)(34)(56)} : \quad x_3 = x_1x_2.$$

Projecting this graph to the (x_1, x_2)-plane, the chamber walls on the graph are

$$x_1 = 0, \quad x_2 = 0, \quad x_1 = 1, \quad x_2 = 1, \quad x_1 = x_2, \quad x_1x_2 = 1.$$

Figure 2 gives the affine picture. The four vertical and horizontal lines first divide the plane into nine rectangles. The diagonal $x_1 = x_2$ splits three of them, and the hyperbola $x_1x_2 = 1$ splits four of the resulting regions. Hence the graph has sixteen chamber pieces, all open discs. This proves (i).

Since there are fifteen quadrics, each meeting sixteen chambers, and since the S_5-action is transitive on the sixty chambers, every chamber meets

$$\frac{15 \cdot 16}{60} = 4$$

quadrics. This proves (ii).

For (iii), by transitivity it is enough to inspect the chamber

$$C = \{0 < x_2 < x_3 < 1 < x_1\}.$$

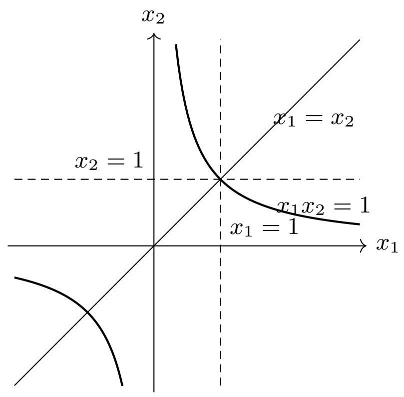

FIGURE 2. The chamber walls induced on the quadric $S_{(12)(34)(56)}$ after projecting the graph $x_3 = x_1x_2$ to the (x_1, x_2)-plane. The full picture has sixteen regions; the drawing shows a finite window of the affine plane.

The four quadrics meeting C are

$$\begin{aligned} H_1 : &\quad x_3 = x_1x_2, \\ H_2 : &\quad x_1 = x_3(x_1 - x_2 + 1), \\ H_3 : &\quad x_1(x_2 + x_3 - 1) = x_2x_3, \\ H_4 : &\quad x_1x_2 = x_3(x_1 + x_2 - 1). \end{aligned}$$

Put $u = x_2$, $v = x_3$, and $h = x_1$. The chamber is

$$0 < u < v < 1 < h.$$

The four surfaces are graphs over open discs in the base triangle $B = \{0 < u < v < 1\}$:

surface	$h =$	domain in B
H_1	v/u	B
H_2	$v(1-u)/(1-v)$	$v > 1/(2-u)$
H_3	$uv/(u+v-1)$	$u+v > 1$
H_4	$v(1-u)/(v-u)$	B.

Their intersections project to the following arcs in B:

$$\begin{aligned} H_1 \cap H_2 \cap H_3 : &\quad v = 1 - u + u^2, \\ H_1 \cap H_4 : &\quad v = 2u - u^2, \\ H_2 \cap H_4 : &\quad v = \frac{1+u}{2}, \\ H_3 \cap H_4 : &\quad u = \frac{1}{2}. \end{aligned}$$

The configuration of arcs is shown in Figure 3. Notice that the first projected arc is shared by the three intersections $H_1 \cap H_2$, $H_1 \cap H_3$, and $H_2 \cap H_3$, so the planar picture has fewer curves than a generic four-surface arrangement.

All four surfaces meet at the single point

$$(x_1, x_2, x_3) = \left(\frac{3}{2}, \frac{1}{2}, \frac{3}{4}\right).$$

Now add the four discs successively. The first disc $H_1 \cap C$ has one piece. The disc $H_4 \cap C$ is cut by H_1 along one proper arc, hence into two pieces. The disc

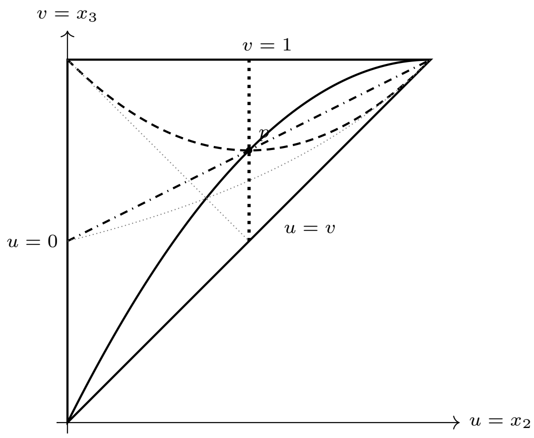

FIGURE 3. Projection to the base triangle $B = \{0 < u < v < 1\}$ of the intersections among the four quadrics meeting the chamber C. The grey dotted curves indicate the domain boundaries of H_2 and H_3; the black curves are the intersection arcs listed above.

$H_2 \cap C$ is cut by its intersections with H_1 and H_4 along two proper arcs crossing once, hence into four pieces. Finally, $H_3 \cap C$ is cut by $H_1 \cup H_2 \cup H_4$ along two proper arcs crossing once: the intersections with H_1 and H_2 coincide along the arc $v = 1 - u + u^2$, while the intersection with H_4 is the arc $u = 1/2$. Thus $H_3 \cap C$ is also cut into four pieces.

Each of these pieces is a properly embedded disc in the complement of the previously inserted pieces. By the three-dimensional Schoenflies theorem, each such disc increases the number of regions by one. Therefore the number of regions in C is

$$1 + 1 + 2 + 4 + 4 = 12.$$

The same argument shows that all twelve regions are contractible. $\square$

Corollary 6.2. *The complement of the fifteen quadrics in W has* $60 \cdot 12 = 720$ *connected components.*

Proof. The sixty chambers of W are disjoint open three-balls. By Theorem 6.1, each is cut into twelve connected components by the quadrics. Since the chamber walls are not part of W, components lying in different chambers cannot be connected inside W. $\square$

7. Contraction and the planar case

The projective contraction operation is slightly more delicate than deletion, because the quotient of $\mathbb{R}^{k+1}$ by the line representing the contracted point has no canonical identification with a fixed projective space. One should therefore either work modulo projective transformations or formulate contraction as a map to a bundle of quotient projective spaces. For this reason the main body of the paper has focused on deletion.

There is nevertheless a closely related contraction picture. If six points in $\mathbb{RP}^3$ are projected from a centre C to $\mathbb{RP}^2$, the image fails to be arrangement-generic precisely when, for some partition of the six labels into three pairs, the three projected joining lines are concurrent. Lifting this condition back to $\mathbb{RP}^3$ says that the centre C lies on the ruled quadric containing the three skew lines corresponding to the three pairs. Thus one again obtains fifteen quadrics indexed by pair partitions. A detailed comparison between this contraction version and the deletion quadrics above requires additional bookkeeping and is left for future work.

For six points in $\mathbb{RP}^2$, special configurations of the fifteen joining lines are classical and are related to Falk's examples in the theory of discriminantal arrangements [5]. These examples provide useful background but are not needed in the proof of Theorem 4.1.

8. Further directions

The Möbius incidence theorem itself is classical, and the general language of adjoints and extensions belongs to oriented matroid theory. The contribution of the present note is the explicit deletion-fibre interpretation and the organization of the exceptional locus as a fifteen-quadric arrangement with a chamber-level description. We include the following concrete continuations to delimit the scope of the present paper.

Problem 8.1. Describe the full intersection poset of the fifteen quadrics, including all boundary components, and compare it with the combinatorics of pair partitions of six labels.

Problem 8.2. Determine the adjacency graph of the 720 components of

$$W \setminus \bigcup_{\lambda} S_{\lambda}.$$

Problem 8.3. Formulate the contraction analogue intrinsically as a map to a quotient-projective-space bundle and compare its fifteen quadrics with the deletion quadrics described above.

References

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, *Oriented Matroids*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, 1999.

[2] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* **56** (1950), 413–455.

[3] E. Fadell, Homotopy groups of configuration spaces and the string problem of Dirac, *Duke Math. J.* **29** (1962), 231–242.

[4] E. Fadell and L. Neuwirth, Configuration spaces, *Math. Scand.* **10** (1962), 111–118.

[5] M. Falk, A note on discriminantal arrangements, *Proc. Amer. Math. Soc.* **122** (1994), 1221–1227.

[6] Y. I. Manin and V. V. Schechtman, Arrangements of hyperplanes, higher braid groups and higher Bruhat orders, in *Algebraic Number Theory*, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, 1989, 289–308.

[7] A. F. Möbius, Kann von zwei dreiseitigen Pyramiden eine jede in Bezug auf die andere um- und eingeschrieben zugleich heißen?, *J. Reine Angew. Math.* **3** (1828), 273–278.

[8] R. Randell, Lattice-isotopic arrangements are topologically isomorphic, *Proc. Amer. Math. Soc.* **107** (1989), no. 2, 555–559.

[9] B. Sturmfels and G. M. Ziegler, Extension spaces of oriented matroids, *Discrete Comput. Geom.* **10** (1993), 23–45.

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
Email address: `agabriel@math.purdue.edu`

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
Email address: `shapiro@math.su.se`