ON THE NUMBER OF CONNECTED COMPONENTS IN THE SPACE OF CLOSED NONDEGENERATE CURVES ON $\mathbf{S}^n$

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The main definition. A parametrized curve $\gamma : \mathbf{I} \to \mathbf{R}^n$ is called **nondegenerate** if for any $t \in \mathbf{I}$ the vectors $\gamma'(t), \dots, \gamma^{(n)}(t)$ are linearly independent. Analogously $\gamma : \mathbf{I} \to \mathbf{S}^n$ is called **nondegenerate** if for any $t \in I$ the covariant derivatives $\gamma'(t), \dots, \gamma^{(n)}(t)$ span the tangent hyperplane to $\mathbf{S}^n$ at the point $\gamma(t)$ (compare with the notion of n-freedom in [G]).

Fixing an orientation in $\mathbf{R}^n$ or $\mathbf{S}^n$ we call a nondegenerate curve γ right-oriented if the orientations of $\gamma', \dots, \gamma^{(n)}$ coincide with the given one and left-oriented otherwise.

Nondegenerate curves on $\mathbf{S}^n$ are closely related with linear ordinary differential equations of $(n+1)$-th order. Such an equation defines two nondegenerate curves on $\mathbf{S}^n \subset V^{(n+1)*}$, where $V^{(n+1)*}$ is the $(n+1)$-dimensional vector space dual to the space of its solutions as follows. For each moment $t \in \mathbf{I}$ we choose the linear hyperplane in V^{n+1} of all solutions vanishing at t i.e. obtain a unique curve in the projective space $\mathbf{P}^n$. Raising it to $\mathbf{S}^n$ we obtain a pair of curves; both of them are right-oriented if n is odd and have opposite orientations if n is even (nondegeneracy follows from nonvanishing of its Wronskian).

A nondegenerate curve $\gamma : [0,1] \to \mathbf{S}^n$ defines a monodromy operator $M \in \mathbf{GL}^+_{n+1}$ which maps $\gamma(0), \gamma'(0), \dots, \gamma^{(n)}(0)$ to $\gamma(1), \gamma'(1), \dots, \gamma^{(n)}(1)$.

In the paper [K-O] there is given the complete set of invariants for symplectic leaves of the second Gelfand-Dikii bracket; namely its leaves are enumerated by pairs consisting of 1) monodromy operator, and 2) the connected component of the space of right-oriented curves in the sphere with the given monodromy operator.

In this paper we study the number of connected components for closed nondegenerate right-oriented curves (corresponding to the identity monodromy operator). Nondegeneracy is also interesting in connection with the general theory of the h-principle (see [G]).

Let $\mathbf{NR}^n$ $(\mathbf{NS}^n)$ be the space of all nondegenerate closed right-oriented curves in $\mathbf{R}^n(\mathbf{S}^n$ respectively).

The question we consider is how to calculate $\pi_o(\mathbf{NS}^n)$ and $\pi_o(\mathbf{NR}^n)$. The first paper studying a similar question is [F]. Later J.Little [L1,L2] studied $\mathbf{NS}^2$ and $\mathbf{NR}^3$ and proved the following (W.Pohl' conjecture): $card(\pi_o(\mathbf{NS}^2)) = 3$ and $card(\pi_o(\mathbf{NR}^3)) = 2$. (The invariant which distinguishes closed nondegenerate curves is an element of π_1 of the image of the natural map $\nu : \mathbf{NR}^n \to \mathbf{SO}_n$, where

1991 *Mathematics Subject Classification.* 1980 Mathematics Subject Classification (1985 Revision) Primary 53A04.

$\nu(\gamma(t))$ equals the matrix obtained by orthogonalization of $\gamma'(t), \dots, \gamma^{(n)}(t)$; and $s\nu : \mathbf{NS}^n \to \mathbf{SO}_{n+1}$, where $s\nu(\gamma(t))$ equals the matrix obtained by orthogonalization of $\gamma'(t), \dots, \gamma^{(n)}(t)$ considered as vectors in $\mathbf{R}^{n+1}$ via the standard embedding $\mathbf{S}^n \subset \mathbf{R}^{n+1}$.

U.Hamenstadt [H] continued the study of this question and formulated the following erroneous result (mentioned in [G]).

Proposition. $\pi_o(\mathbf{NR}^k) = \pi_o(\mathbf{NS}^k)$ *and consists of 2 elements for any* $k > 2$.

The true statement which was recently proved by the second author is as follows:

Theorem.

(1) $\pi_o(\mathbf{NR}^k) = \pi_o(\mathbf{NS}^k)$ *and consists of 2 elements for any odd* $k > 2$, *i.e. in this case the only invariant equals* π_1 *of the above mentioned map.*
(2) $\pi_o(\mathbf{NR}^k) = \pi_o(\mathbf{NS}^k)$ *and consists of 3 elements for any even* $k > 3$.

In this paper we'll prove the following result:

Theorem 1. $card(\pi_o(\mathbf{NS}^{2k})) \geq 3$

We need several definitions.

Definition 1. By a **linear subspace** of $\mathbf{S}^n$ we will denote any intersection of $\mathbf{S}^n$ with any linear subspace in $\mathbf{R}^{n+1}$ (where $\mathbf{S}^n$ is considered to be standardly embedded). Any k-tuple $(v_1, \dots, v_k)$ of vectors tangent to $\mathbf{S}^n$ at some point p defines a linear subspace in $\mathbf{S}^n$ as its intersection with the subspace in $\mathbf{R}^{n+1}$ spanned by $p, v_1, \dots, v_k$.

Definition 2. Consider a closed nondegenerate curve γ on $\mathbf{S}^n$. The curve γ will be called **disconjugate** if the sum of local multiplicities of its intersection over all intersection points of the curve with any linear hyperplane (see above) is at most n and **conjugate** otherwise. (Local multiplicity is the degree of the restriction of the divisor on the curve; in the considered case it equals the maximal dimension of the osculating subspace to the curve contained in the hyperplane.)

Remark 2. One can easily see that the local multiplicity of such intersection for a nondegenerate curve is at most n. It equals the maximal dimension of the osculating subspace to the curve at this point and lying in the hyperplane.

Definition 3. Denote by $\mathbf{S}^{n*}$ the sphere dual to the considered $\mathbf{S}^n$, i.e. the set of all oriented hyperplanes on $\mathbf{S}^n$. If γ is a nondegenerate curve on $\mathbf{S}^n$ then we can define the curve $\gamma^* \in \mathbf{S}^{n*}$ dual to γ as the set of all right-oriented osculating hyperplanes tangent to γ.

Remark 3. Now we are able to give the dual formulation of disconjugacy. For the curve γ^* dual to the disconjugate curve γ and an arbitrary point $x \in (\mathbf{S}^{n*})$ the sum of tangency orders over all hyperplanes passing through x and tangent to γ is at most n.

Notation. $\mathbf{ND}^{2k}$ $(\mathbf{NC}^{2k})$ will denote the space of all closed disconjugate (resp. conjugate) curves on $\mathbf{S}^{2k}$.

One can see that $\mathbf{ND}^{2k}$ is nonempty. For example, the properties of trigonometrical polynomials imply that it contains the curve whose coordinates are $\sqrt{\frac{1}{k+1}}(1, \sin t, \cos t, \sin 2t, \cos 2t, \dots, \sin kt, \cos kt)$ (it lies on standard $\mathbf{S}^{2k}$).

Since $\mathbf{NC}^{2k}$ consist of at least 2 components differing by the element of π_1 of $\mathbf{GL}_{2k+1}$ (one can easily construct two curves with trigonometrical coordinates realizing different elements of π_1 of $\mathbf{GL}_n^+$). Theorem 1 follows from the next result.

Lemma 1. $\mathbf{ND}^{2k}$ *is disconnected with* $\mathbf{NC}^{2k}$.

The proof is divided into 2 parts:

(1) $\mathbf{ND}^{2k}$ is open;
(2) $\mathbf{NC}^{2k}$ is open.

Instead of (1) we will prove the following more general fact. For any $m \geq n$ the set $\mathbf{NS}^n_m$ of all nondegenerate closed curves whose sum of intersection multiplicities with any hyperplane does not exceed m is open. Let $t_1, \dots, t_p$ be pairwise different moments of intersection of γ with an arbitrary hyperplane $\mathbf{L}$ and $1 \leq k_1, \dots, k_p \leq n$ be the set of corresponding local multiplicities ($\sum k_i \leq m$). Since γ is nondegenerate, then by the definition of multiplicity for each t_i there exist δ_i and ξ_i such that if $|\gamma - \bar{\gamma}|_{\mathbf{C}^n} \leq \xi_i$ and the sum $\sharp$ of intersection multiplicities of $\bar{\gamma}$ with $\mathbf{L}$ if $\bar{\gamma}$ belongs to the δ_i-neighborhood of $\gamma(t_i)$ is at most k_i (the lower index $\mathbf{C}^n$ means that the distance is taken with respect to the metrics with n derivatives). Let U_i denote the δ_i-neighborhoods of the points $\gamma(t_i)$. Denote $\bar{\xi} = \min\{\xi_1, \dots, \xi_p\}$ and let ρ be the distance between $\gamma \setminus \{U_1, \dots, U_p\}$ and $\mathbf{L}$ in the ordinary metrics of $\mathbf{R}^n$. Finally, take $\xi = \min\{\bar{\xi}, \rho/2\}$; we see that if $|\gamma - \bar{\gamma}|_{\mathbf{C}^n} < \xi/2$ then for any hyperplane $\mathbf{L}$ $\sharp(\mathbf{L} \cap \bar{\gamma}) \leq m$.

The idea of the proof of (2) is as follows. A hyperplane $\mathbf{L}$ is called **conjugate** relative to a conjugate curve $\gamma \subset \mathbf{S}^n$ if the sum of local intersection multiplicities of $\mathbf{L}$ with γ exceeds n. If a curve is conjugate, i.e. there exists a conjugate hyperplane then (by the result of Sherman ([S]) there exists a hyperplane with $\geq 2k$ transversal intersections. Consequently, for arbitrary sufficiently small deformation $\bar{\gamma}$ of γ the same hyperplane also intersects transversally γ at least 2k times. Thus $\bar{\gamma}$ is also conjugate and $\mathbf{NC}^{2k}$ is open.

The geometrical idea of the proof of Sherman's result is as follows.

If $\gamma : \mathbf{I} = [a, b] \to \mathbf{S}^n$ is a conjugate nondegenerate curve then there exists a conjugate hyperplane $\mathbf{L}$ such that the intersection points of $\mathbf{L}$ and γ lie on $[a, b)$. Let as above $\gamma(t_1), \gamma(t_2), \dots, \gamma(t_p)$ be the set of pairwise different intersection points of γ and $\mathbf{L}$ and let $k_1, k_2, \dots, k_p$ be the set of multiplicities of intersections. For any intersection point $\gamma(t_i),\ t_i \in [a, b)$ with multiplicity of intersection $k_i > 1$ we can consider the $(k_i - 1)$-dimensional linear system of hyperplanes preserving intersection points $\gamma(t_1), \gamma(t_{i-1}), \gamma(t_{i+1}), \dots, \gamma(t_p)$ but with different multiplicities $\tilde{k}_1, \tilde{k_{i-1}}, \tilde{k_{i+1}} \dots, \tilde{k_p}$ where $\sum_j \tilde{k}_j = n - k_i$. Then in this family of hyperplanes we can find a hyperplane $\tilde{\mathbf{L}}$ having the same multiplicities of intersections $\tilde{k}_1, \tilde{k_{i-1}}, \tilde{k_{i+1}} \dots, \tilde{k_p}$ where $\sum_j \tilde{k}_j = n - k_i$ in the points $\gamma(t_1), \gamma(t_{i-1}), \gamma(t_{i+1}), \dots, \gamma(t_p)$ and k_i simple zeros in the neighborhood of $\gamma(t_i)$.

Conjecture. *The space* $\mathbf{NS}^n$

(1) *is homotopically equivalent to the space of all closed curves on* $\mathbf{SO}_n$ *passing through a given point for even n;*
(2) *consists of two parts one of which is contractible and consists of (nonstrictly) disconjugate curves and the other one is homotopically equivalent to the space of all closed curves on* $\mathbf{SO}_n$ *passing through the given point for odd n.*

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