To our Teacher
on the occasion of his 60-th birthday

RAMIFIED COVERINGS OF S^2 WITH ONE DEGENERATE BRANCHING POINT AND ENUMERATION OF EDGE-ORDERED GRAPHS

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Abstract. In this short note we extend the results of Lyashko, Looijenga, and Arnold on the number of nonequivalent rational functions on the sphere with 1 or 2 poles and simple finite branching points to several other cases. In particular, we calculate the number of meromorphic functions on the torus with the same properties.

0. Introduction

Let M be a Riemann surface of genus g, S^2 be a Riemann sphere with a fixed infinity, $f\colon M \to S^2$ be a meromorphic function of degree n with l poles of orders $n_1 \geqslant n_2 \geqslant \dots \geqslant n_l$, $\sum_{i=1}^{l} n_i = n$. Such a function is said to be *primitive* if all its finite critical values are pairwise distinct and in a neighborhood of each critical point with a finite critical value it is equivalent to a quadratic function. Two primitive functions f and f' are said to be equivalent if there exists a homeomorphism π of M such that $f' = f \circ \pi$. We are interested in finding the number μ_ν^g of pairwise nonequivalent primitive meromorphic functions, where ν is the partition of n defined by the orders of the poles.

Our problem can be considered as a particular case of a more general problem, which belongs to Hurwitz [H] and consists in counting all nonequivalent ramified coverings of a Riemann surface N by another Riemann surface M having a given set of ramification orders. Reference [M1] contains a solution to the Hurwitz problem; however, the expression for the number of coverings presented there is extremely difficult to use. It involves complicated multiple sums and products, and substantial efforts are needed to derive more suitable formulas even for simplest particular cases [M2].

On the other hand, several particular cases of the Hurwitz problem were studied recently in the conformal field theory [CT, D, GT]. Some of the results obtained there are apparently new, while the others are just rediscoveries of classic formulas. What is most surprising, is the simplicity of the answers obtained in the case $g = 0$. Reformulating the main theorem of [GJ] one gets the following nice multiplicative formula for the number of primitive rational functions on the sphere:
the number μ^0_ν of nonequivalent primitive rational functions of degree n on the sphere with l poles of orders $n_1, \dots, n_l$ is equal to

$$\mu^0_\nu = n^{l-3}(n+l-2)!\frac{n_1^{n_1}\cdots n_l^{n_l}}{n_1!\cdots n_l! p_1!\cdots p_q!},$$

where $p_1, \dots, p_q$ are the cardinalities of the groups of pairwise equal numbers among n_i's.

A possible reason for such a simple multiplicative form of the answer was explained by Arnold in [A2]. He started from the observation that in the simplest case of polynomials on S^2 the number in question (that is, $\mu^0_{(n)}$) coincides, up to factor n, with the multiplicity of the Lyashko–Looijenga map [A1, L], which assigns to a polynomial of degree $n+1$ an unordered collection of its critical values. This map turns out to be quasihomogeneous in appropriate coordinates in the source and the target and has an isolated singularity at the origin; thus, its multiplicity is obtained easily (in a multiplicative form!) from the Bezout theorem. Similar ideas, though in a far more complicated situation, were used in [A2] to get a nice multiplicative formula for the number $\mu^0_{(n-k,k)}$ of nonequivalent meromorphic functions from S^2 to S^2 with two poles of given orders.

Arnold then exploits the connection between meromorphic functions and edge–ordered graphs to derive certain purely combinatorial statements about such graphs. The simplest result of this kind is a famous Cayley theorem on the number of labeled trees, which corresponds to the case of polynomials on S^2. In this note we go in the opposite direction, and find μ^g_ν in several particular cases starting from a purely combinatorial setting. Some of the answers below are no longer multiplicative, and apparently there is no way to obtain them via a properly constructed quasihomogeneous map. However, those which are multiplicative deserve special attention, and though we failed to extend the Arnold's construction to these cases, we are almost sure that such a construction should exist.

The problem, along with several conjectures concerning the values of μ^g_ν for certain g and ν, was communicated to the authors by V. Arnold in summer, 1995. Later we had several stimulating discussions with him on various aspects of the problem. We cannot overestimate the role of T. Ekedahl, who explained to us the essence of the classic approach, and taught us several useful facts in the representation theory of the symmetric group. We are also grateful to S. Natanzon, who pointed out the references [M1, M2].

1. Edge–ordered graphs

Let $G = (V, E)$ be a multigraph without loops, $|V| = n$, $|E| = m$. To each edge $e \in E$ we assign a mapping $\pi_e\colon V \to V$ that transposes the ends of e. Assume now that the edges of G are ordered (say, labeled by the numbers $1,\dots,m$). We then define a mapping $\pi_G\colon V\to V$ as the product of the transpositions π_e in the increasing order of labels. To represent π_G as an element of the symmetric group S_n, one have to choose a numbering of the elements of V. Evidently, all the permutations obtained in such a way for different numberings of the same graph belong to the same conjugacy class, and thus have the same cycle type. This cycle type is said to be the *cycle type* of the edge–ordered multigraph G. In the same way we define the *cycle partition* of G. Finally, if $\nu=(n_1,\dots,n_l)\vdash n$ is the cycle partition of an edge–ordered multigraph G, then the number of its parts l is said to be the *cycle length* of G.

Connected edge–ordered multigraphs arise naturally in the study of meromorphic functions on Riemann surfaces. The following statement is a reformulation of the main result of §2 in [A2].

Theorem 1. *Let M be a Riemann surface of genus g, $\nu=(n_1,n_2,\dots,n_l)$ be a partition of n. Then the number of pairwise nonequivalent primitive meromorphic functions on M of degree n with l poles of orders $n_1,n_2,\dots,n_l$ is equal to the number of connected edge–ordered multigraphs with n vertices, $n+2g+l-2$ edges, and cycle partition ν.*

Theorem 1 provides a reduction of our initial problem to a purely combinatorial one: find the number of connected edge–ordered multigraphs with n vertices, m edges, and cycle partition ν. and a given cycle partition. In what follows we denote this number by $N_c(n,m,\nu)$.

General edge–ordered multigraphs (not necessarily connected) possess a natural interpretation in the representation theory of the symmetric group. To formulate the corresponding statement, we recall some standard notation. Let $\rho,\sigma\vdash n$ be partitions of n. By f^ρ we denote the multiplicity of the irreducible representation of S_n labeled by ρ, by C_σ the conjugacy class of S_n with cycle partition σ, and by χ^ρ_σ the value of the character of the irreducible representation labeled by ρ on the class C_σ.

Theorem 2. *The number $N(n,m,\nu)$ of edge–ordered multigraphs with n vertices, m edges, and cycle partition ν is given by the following expression:*

$$N(n,m,\nu)=\frac{|C_\nu|}{(n!)^2}\sum_{\rho\vdash n} f^\rho(h(\rho')-h(\rho))^m\chi^\rho_\nu, \tag{1}$$

where $h(\rho)=\sum(i-1)\rho_i$ and ρ' is the partition conjugate to ρ.

Proof. Let Z denote the center of the group algebra of the symmetric group. It is well known that Z, as an algebra, is generated by the conjugacy classes of S_n (or, more exactly, by the sums of all the elements of a conjugacy class). Therefore, for each $z\in Z$ one can define $\delta(z)$ as the coefficient of the unity in the decomposition of z in a weighted sum of conjugacy classes. The following proposition is an easy consequence of the above definitions.

Proposition 3. *The number $N(n,m,\nu)$ satisfies the following relation:*

$$N(n,m,\nu)=\frac{1}{n!}\delta(z_2^m z_\nu), \tag{2}$$

where z_2 is a transposition and z_ν belongs to the conjugacy class C_ν.

To evaluate the right hand side of (2) we use several results in the representation theory of the symmetric group. As follows from the main theorem of this theory,

$$\delta(z) = \frac{1}{n!}\sum_{\rho\vdash n}(f^\rho)^2\psi^\rho(z), \tag{3}$$

where ψ^ρ is the central character of the irreducible representation labeled by μ. Observe that central characters are multiplicative; thus from (2) and (3) one gets

$$N(n,m,\nu) = \frac{1}{(n!)^2}\sum_{\rho\vdash n}(f^\rho)^2(\psi^\rho(C_2))^m\psi^\rho_\nu, \tag{4}$$

where C_2 stands for the conjugacy class of transpositions. Recall now that central characters satisfy the relation $f^\rho\psi^\rho_\nu = |C_\nu|\chi^\rho_\nu$, and in particular,

$$\psi^\rho(C_2) = \frac{\binom{n}{2}\chi^\rho(C_2)}{f^\rho}.$$

By a Frobenius theorem ([Ma, p.64]), the right hand side of the last expression equals $h(\rho') - h(\rho)$. Substituting this into (4) and taking into account the above relation for central characters, one gets (1). □

The four numeric parameters of an edge–ordered multigraph, namely, the number of edges, number of vertices, number of connected components, and cycle length, are not independent. Relation between these parameters are described by the following statement.

Theorem 4. *Let G be an edge–ordered multigraph with n vertices, m edges, and c connected components, then its cycle length l can assume an arbitrary value satisfying the following conditions*:

$$c \leqslant l \leqslant \min\{n, m-n+2c\}, \tag{5}$$
$$l = m-n \mod 2. \tag{6}$$

Proof. First of all, let us introduce some notation. Let us denote by $\Pi_G(u)$ the *orbit* of a vertex u under iterations of π_G, that is, $\Pi_G(u) = \{v\colon \pi_G^k(u) = v,\ k\in\mathbb{Z}\}$.

We prove the theorem by induction on the number of edges in G. For an empty graph, one has $c=n$, $m=0$, and thus the only value of l satisfying (5) and (6) is n, which is indeed assumed, since π_G in this case is the identity.

Suppose now that the statement of the theorem holds for all multigraphs G with m edges, and we add a new edge (u,v) labeled by $m+1$ to obtain a multigraph G'. We than have the following three possibilities:

1) u and v lie in distinct connected components of G;
2) u and v lie in the same connected component of G, and $\Pi_G(u)\neq\Pi_G(v)$;
3) u and v lie in the same connected component of G, and $\Pi_G(u)=\Pi_G(v)$.

In the first case, one gets from the definitions $c' = c-1$ and $m' = m+1$. Besides, it is easy to see that $\Pi_{G'}(u) = \Pi_{G'}(v) = \Pi_G(u)\cup\Pi_G(v)$, and thus $l' = l-1$. We therefore see that conditions (5) and (6) for the numbers c', m', and l' are yielded by the same conditions for the numbers c, r, and l, which are satisfied by the inductive hypothesis. Moreover, any l' satisfying these conditions can be obtained in such a way from the corresponding l, and thus can be realized as the cycle length of an edge-labeled multigraph with $m+1$ edges.

In the second and the third cases, one has $c' = c$ and $m' = m+1$. Besides, in the second case again $\Pi_{G'}(u) = \Pi_{G'}(v) = \Pi_G(u) \cup \Pi_G(v)$, and thus $l' = l-1$, while in the third case, to the contrary, $\Pi_G(u) = \Pi_G(v) = \Pi_{G'}(u) \cup \Pi_{G'}(v)$, and thus $l' = l+1$. Assume first that $c < l < n$. In this case there exists a connected component of G that contains at least two orbits, and there exists an orbit that contains at least two vertices. Hence, both cases 2) and 3) can be realized, and we can thus obtain any l' satisfying conditions (5) and (6) with c' and m'. The remaining cases $l = c$ and $l = n$ do not add anything new. Indeed, if $c = l < n$, then each connected component is an orbit, and there exists an orbit with at least two vertices. Thus, only case 3) is possible, and we get $l' = c'+1 \leqslant \min\{n, m'-n+2c'\}$. If $c < l = n$, the each orbit contains only one vertex, and there exists a connected component containing at least two orbits. Thus, only case 2) is possible, and we get $c' \leqslant n-1 = l'$. Finally, if $c = l = n$, then G is an empty graph, and thus both cases 2) and 3) are impossible. □

2. Enumeration

A typical expression one encounters while trying to evaluate the right hand side of (1) is

$$\sigma(t,p,\alpha) = \sum_{m=0}^{t} \binom{t}{m} (-1)^m (\alpha - m)^p,$$

where $p, t \in \mathbb{N}$, $\alpha \in \mathbb{R}$. Let us introduce the generating function

$$\Sigma(t,\alpha;x) = \sum_{p=0}^{\infty} \sigma(t,p,\alpha) \frac{x^p}{p!}.$$

Then the following proposition holds.

Proposition 5.

$$\Sigma(t,\alpha;x) = e^{\alpha x}(1-e^{-x})^t. \tag{7}$$

Proof. Indeed, one has

$$\Sigma(t,\alpha;x) = \sum_{p=0}^{\infty}\sum_{m=0}^{t} \binom{t}{m} (-1)^m \frac{[(\alpha-m)x]^p}{p!} = \sum_{m=0}^{t} \binom{t}{m} (-1)^m e^{x(\alpha-m)}$$
$$= e^{\alpha x} \sum_{m=0}^{t} \binom{t}{m} (e^{-x})^m = e^{\alpha x}(1-e^{-x})^t.$$

□

It is easy to see that $\Sigma(t,\alpha;x)$ has a zero of order t at the origin. Thus, introducing coefficients $\Delta_q^t(\alpha)$ by

$$\sum_{q=0}^{\infty} \Delta_q^t(\alpha) x^q = e^{\alpha x} \left(\frac{1-e^{-x}}{x}\right)^t,$$

we can rewrite (7) as

$$\sigma(t,p,\alpha) = p!\Delta^t_{p-t}(\alpha). \tag{8}$$

Now we are ready to start computing $N_c(n,m,\nu)$ for several simple cases. The simplest situation occurs when $l=1$ and $\nu=(n)$, which corresponds to primitive meromorphic functions with one pole. In this case the permutation π_G is a cycle, and thus graph G is forced to be connected. Thus, $N_c(n,m,(n)) = N(n,m,(n))$. As an immediate consequence of Theorem 4 we get $N(n,n+2g,(n))=0$ for $g=0,1,\dots$; so we are now interested only in $N(n,n+2g-1,(n))$, $g=0,1,\dots$, which, by Theorem 1, are just $\mu^g_{(n)}$.

Theorem 6. *The number of pairwise nonequivalent primitive meromorphic functions of degree $n \geqslant 3$ with one pole on a Riemann surface of genus g is equal to*

$$\mu^g_{(n)} = n^{n+2g-2}\frac{(n+2g-1)!}{n!}\delta^n_{2g},$$

where $\delta^n_{2g} = \Delta^{n-1}_{2g}(\frac{n-1}{2})$ satisfy

$$\sum_{g=0}^{\infty}\delta^n_{2g}x^{2g} = \left(\frac{\sinh x/2}{x/2}\right)^{n-1}. \tag{9}$$

Proof. It follows from the Murnaghan–Nakayama rule (see e.g. [Ma, p.64]) that $\chi^\rho_{(n)}$ vanishes if ρ is not a hook. For the hook ρ_m with the leg of length m, $0\leqslant m \leqslant n-1$, one gets easily

$$\chi^{\rho_m}_{(n)} = (-1)^m, \qquad f^{\rho_m} = \binom{n-1}{m}, \qquad h(\rho'_m) = \binom{n-m}{2}, \qquad h(\rho_m) = \binom{m+1}{2}.$$

Besides, $|C_{(n)}| = (n-1)!$. Thus, Theorems 1 and 2 imply the following formula:

$$\begin{aligned}\mu^g_{(n)} &= N(n,n+2g-1,(n))\\ &= \frac{1}{n\cdot n!}\sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m\left(\binom{n-m}{2}-\binom{m+1}{2}\right)^{n+2g-1}\\ &= \frac{n^{n+2g-2}}{n!}\sigma\left(n-1, n+2g-1, \frac{n-1}{2}\right).\end{aligned}$$

An equivalent statement was actually proved in [J] by similar representation– theoretic methods as above (see also [G] for a direct combinatorial proof). However, both [J] and [G] do not mention the interpretation of the right–hand side of the above formula in terms of meromorphic functions.

The statement of the Theorem follows immediately from the above formula and Proposition 5. □

Theorem 6 yields very simple expressions for the number of nonequivalent primitive meromorphic functions with one pole for surfaces of small genus. In particular, for $g = 0,\ 1,\ 2$ we get the following

Corollary 7. *Let $n \geqslant 3$, then*

$$\mu^0_{(n)} = n^{n-3}, \qquad \text{(Lyashko–Looijenga)}$$
$$\mu^1_{(n)} = \frac{n^n(n^2-1)}{24},$$
$$\mu^2_{(n)} = \frac{n^{n+2}(n^2-1)(n+3)(n+2)(5n-7)}{5760}.$$

Let us now consider the case when the cycle length of π_G equals 2, which corresponds to primitive meromorphic functions having two poles. Let $\nu = (n-r, r)$, $r \leqslant n/2$. From Theorem 4 we immediately get $N_c(n, n+2g-1, \nu) = 0$, $g = 0, 1, \dots$. For the case of an odd cyclomatic number one gets the following result.

Proposition 8. *Let $\nu = (n-r, r)$, $0 < r < n-r$. Then*

$$N(n, n+2g, \nu) = N_c(n, n+2g, \nu) + \sum_{s=0}^{g+1} \binom{n+2g}{r+2s-1} N(r, r+2s-1, (r)) N(n-r, n-r+2g-2s+1, (n-r)).$$

In the case $r = n-r$ the sum in the right hand side of the above expression should be taken over s varying from 0 to $\lfloor \frac{g+1}{2} \rfloor$.

Proof. Indeed, by Theorem 4, G is either connected, or contains exactly two connected components. In the latter case one of the components has r vertices, and the other one has $n-r$ vertices, and the cycle length of each component is exactly one. Moreover, the difference between the numbers of edges and vertices in each component is even by (6), which gives the desired result. □

Taking into account Theorems 1 and 6, we get the following

Theorem 9. *The number of pairwise nonequivalent primitive meromorphic functions of degree $n \geqslant 3$ with two poles of orders $n-r$ and r on a Riemann surface of genus g is equal to*

$$\text{(10)} \quad \mu^g_{n-r,r} = N(n, n+2g, (n-r,r)) - \binom{n}{r} \frac{(n+2g)!}{n!} r^{r-2} (n-r)^{n-r+2g} \times \sum_{s=0}^{g+1} \left(\frac{r}{n-r}\right)^{2s} \delta^r_{2s} \delta^{n-r}_{2g+2-2s}, \qquad 0 < r < n-r,$$

where δ^l_{2k} are defined by (9). In the case $r = n-r$ the sum in the right hand side of the above expression should be taken over s varying from 0 to $\lfloor \frac{g+1}{2} \rfloor$.

The formula for $\mu^g_{n-r,r}$ given in Theorem 9 is still very complicated. However, when the genus of the surface M is small, it is possible to simplify it substantially.

For the case of the sphere ($g = 0$) we get the following result.

Corollary 10 (see also [A2])**.** *The number of pairwise nonequivalent primitive meromorphic functions of degree $n \geqslant 3$ with two poles of orders $n-r$ and r on the sphere is equal to*

$$\mu^0_{n-r,r} = \binom{n}{r} \frac{r^r (n-r)^{n-r}}{n}, \qquad 0 < r < n-r,$$
$$\mu^0_{r,r} = \binom{2r}{r} \frac{r^{2r-1}}{4}.$$

Proof. Let $0 \leqslant r \leqslant n-r$. Then from (10) we get

$$\mu^0_{n-r,r} = N(n,n,(n-r,r)) - \binom{n}{r} r^{r-2}(n-r)^{n-r}\left(\delta_2^{n-r} + \left(\frac{r}{n-r}\right)^2 \delta_2^r\right).$$

Since $\delta_2^k = \frac{n-1}{24}$ by (9), we have just to find $N(n,n,(n-r,r))$. To do that, we use (1) and apply the Murnaghan–Nakayama rule, as in the proof of Theorem 6. It turns out that in this case the character χ^ρ_ν vanishes on all diagrams ρ that cannot be expressed as a union of at most two hooks. As a result of lengthy calculations we get an expression for $N(n,n,(n-r,r))$ as a linear combinations of the sums $\sigma(t,p,\alpha)$ for $\alpha = (n-7)/2,\ (n-5)/2,\ (n-1)/2,\ (n+1)/2$. We than apply Proposition 5 to get the desired formula. □

Remark. Corollary 10 follows also from the result of Goulden and Jackson mentioned in the introduction.

Observe that an expression for the total number of connected edge–ordered graphs on n vertices and n edges, that is, for $\sum_{r=1}^{n-1} N_c(n,n,(n-r,r))$, was obtained earlier in [AFPR]. Comparing the two results we get the following identity.

Corollary 11.

$$\sum_{k=0}^{n-2} \frac{n^k}{k!} = \sum_{k=1}^{n-1} \frac{k^k}{k!}\frac{(n-k)^{n-k}}{(n-k)!}.$$

For the case of the torus, the calculations become more involved. Using a similar technique as above, we get the following partial results.

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Proposition 12. *The number of pairwise nonequivalent primitive meromorphic functions of degree n on the torus with two poles of orders $n-r$ and r is equal to*

$$\mu^1_{n-r,r} = \binom{n+2}{r+1}(n-r)^{n-r} r^r P_r(n),$$

where

$$P_1(n) = \frac{n^3}{12},$$
$$P_2(n) = \frac{(n-1)(n^2-3n+4)}{8},$$
$$P_3(n) = \frac{(n-2)(n^2-4n+9)}{6}.$$

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