Constructive Approximation
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CONSTRUCTIVE APPROXIMATION

CORRECTION

Correction: Non-Self-Adjoint Toeplitz Matrices Whose Principal Submatrices Have Real Spectrum

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Abstract
We announce an error in the proof of Theorem 8 of *Constr. Approx.* **48**(2) (2019) 191–226.

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In [3, Eq. (7)], the false equality

$$\langle u, T_n(\overline{a})u\rangle = \langle f_u, \overline{a} f_u\rangle_\gamma$$

has been used in the course of the proof of Theorem 8. If the terms are interpreted using the notation from [3], the right-hand side equals the integral

$$\frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \overline{a(\gamma(t))}\, f_u(\gamma(t))\, \overline{f_u(\gamma^*(t))}\, \frac{\dot{\gamma}(t)}{\gamma(t)} \mathrm{d}t,$$

while the left-hand side coincides with $\langle u, T_n^*(a)u\rangle$ and can be expressed as the integral

$$\frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \overline{a(\gamma^*(t))}\, f_u(\gamma(t))\, \overline{f_u(\gamma^*(t))}\, \frac{\dot{\gamma}(t)}{\gamma(t)} \mathrm{d}t.$$

The two integrals do not coincide in general. Unfortunately, this error is an essential problem for the idea of the proof of Theorem 8, that was based on a contour integral

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representation of the quadratic form of a Toeplitz matrix $T_n(a)$, where the integration path is chosen as the Jordan curve γ on which the symbol a is real-valued.

Recall that [3, Thm. 1], which is the main result of the paper, claims that the following 3 statements are equivalent:

(i) $\Lambda(b) \subset \mathbb{R}$,
(ii) $b^{-1}(\mathbb{R})$ contains a Jordan curve,
(iii) $\operatorname{spec}(T_n(b)) \subset \mathbb{R}$ for all $n \in \mathbb{N}$,

where b is a Laurent polynomial, $T_n(b)$ the $n \times n$ Toeplitz matrix given by the symbol b, and $\Lambda(b)$ is the set of limit points of eigenvalues of $T_n(b)$, as $n \to \infty$; see [3] for details. The logic of the proof of Theorem 1 was to establish implications (i)⇒(ii)⇒(iii)⇒(i). The currently unproven Theorem 8 was used to prove (ii)⇒(iii). As we are not able to find an alternative proof at the moment, the implication (ii)⇒(iii) remains unproven.

However, we strongly believe that the currently unproven implication, or at least its weaker form (ii)⇒(i), is true. The implication (ii)⇒(iii) is supported by numerous numerical experiments that we have performed. Professors Yi Zhang and Hao-Yan Chen, to whom we are sincerely grateful for noticing the latter error in the proof, expressed a similar opinion.

Recall, by [3, Def. 2], that the Laurent polynomial b is said to belong to the class $\mathscr{R}$, i.e. $b \in \mathscr{R}$, if and only if $\Lambda(b) \subset \mathbb{R}$. We list several places in [3], where arguments based on the claim of [3, Thm. 8] have been used:

(a) Remark 10.
(b) Sec. 4.1: Example 1. The symbol $b(z) = z^{-1} + az$, where $a \in \mathbb{C}\backslash\{0\}$, belongs to the class $\mathscr{R}$, if and only if $a > 0$.
(c) Sec. 4.2: Example 2. The symbol $b(z) = z^{-1} + \alpha z + \beta z^2$, where $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}\backslash\{0\}$, belongs to the class $\mathscr{R}$, if and only if $\beta \in \mathbb{R}\backslash\{0\}$ and $\alpha^3 \geq 27\beta^2$.
(d) Sec. 4.3: Example 3. The symbol $b(z) = z^{-r}(1 + az)^{r+s}$, where $r, s \in \mathbb{N}$ and $a \in \mathbb{R}\backslash\{0\}$, belongs to the class $\mathscr{R}$.
(e) The second paragraph of Sec. 4.4.

The claim of [3, Rem. 10] was drawn as a direct consequence of Theorem 8 and remains open, too. The other points (b)–(e) comprise concrete examples of symbols, which belong to $\mathscr{R}$ for specific restrictions of the parameters. As the main argument for these claims, we found in [3] an explicit parametrization of the Jordan curve γ, for which $b \circ \gamma$ is real-valued in each of the cases. We will prove at least partly the claims without using Theorem 8.

The equivalence in (b) can be verified directly since the eigenvalues of $T_n(b)$ can be computed fully explicitly

$$\lambda_k = 2(-1)^n \sqrt{a} \cos \frac{\pi k}{n+1}, \quad k = 1, 2, \dots, n,$$

with the standard branch of the square root. This example also exhibits (iii), if $a > 0$.

The point (c) is left conjectural as inessential for the paper in its generality. Its relevant particular case, $b \in \mathscr{R}$ for $\alpha = 3a^2$ and $\beta = a^3$, with $a \in \mathbb{R}\backslash\{0\}$, will be checked as a a special case of the point (d) in the end of this corrigendum. Lastly, a description of a possible construction of further examples of not necessarily banded Toeplitz matrices with real spectra from the point (e) is heavily based on the falsely proven Theorem 8 and also remains open at this point.

As a final correction, we briefly indicate the proof of the implication

$$b(z) = \frac{1}{z^r}(1+az)^{r+s} \text{ with } r, s \in \mathbb{N} \text{ and } a \in \mathbb{R}\backslash\{0\} \quad \Rightarrow \quad b \in \mathscr{R}.$$

without using the fact that b is real on a Jordan curve. This is the claim (d). In fact, one can prove the stronger claim that the eigenvalues of $T_n(b)$ are all real, for all $n \in \mathbb{N}$, i.e. (iii). The argument relies on the theory of osciallatory matrices [2] and has been used for the special case $r = s = 1$ in the proof of [1, Thm. 2.8]. Without loss of generality, we may assume $a = 1$ since two Toeplitz matrices $T_n(b)$ and $T_n(b_a)$, where $b_a(z) := b(az)$, are similar, if $a \neq 0$. Thus, suppose $b(z) = z^{-r}(1+z)^{r+s}$. Notice the bidiagonal matrix $T_n(1+z)$ is totally non-negative, see [2, Def. 4, p. 74]. Since $T_n(b)$ is a submatrix of $T_{n+r+s}^{r+s}(1+z)$, it is also totally non-negative; see [2, p. 74]. Further, without going into details, let us mention the determinant of $T_n(b)$ ban be explicitly computed in terms of binomial coefficients as follows:

$$\det T_n(b) = \prod_{j=0}^{r-1} \binom{n+s+j}{s} \Big/ \binom{s+j}{j} = \prod_{j=0}^{r-1} \frac{(n+s+j)!j!}{(n+j)!(s+j)!}.$$

The determinant is obviously non-vanishing and therefore $T_n(b)$ is non-singular. By [2, Thm. 10, p. 100], $T_n(b)$ is oscillatory and hence eigenvalues of $T_n(b)$ are all positive (and simple), see [2, Thm. 6, p. 87].

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