Confocal Families of Plane Algebraic Curves

Abstract

We study families of plane algebraic curves sharing the same set of foci. We reformulate confocality via a focal map on equiclassical families and analyze its fibers using deformation theory.

1 Introduction

The main goal of this paper is to study families of real plane algebraic curves admitting confocal deformations, that is, positive-dimensional families of curves sharing the same set of real foci.

In the nineteenth century, foci of ellipses were generalized to foci of higher-degree curves: the (real) *foci* of a real algebraic curve $\Gamma \subset \mathbb{C}P^2$ are the real points on isotropic tangent lines to Γ, see [24]. In traditional terminology associated with an algebraic curve $P(X,Y,Z) = 0$ in the complex projective plane $\mathbb{C}P^2 = [X,Y,Z]$, an isotropic line is one containing either of the circular points $c_\pm = [1, \pm i, 0]$ (the pair of points common to all circles). Although this notion is not especially intuitive at first sight, it appears in a number of research areas including approximation theory, numerical analysis, and fluid models. This is the main reason for discussing foci of real algebraic curves again.

Short historical account

The article [21] from 1936 contains the following information. The points now called foci first arose in connection with conic sections. According to the classical book [9] on the history of mathematics, Apollonius of Perga, who lived in the third century B.C., incidentally discovered the foci of the ellipse and hyperbola. Five hundred years later Pappus of Alexandria found the focus of the parabola. The theory of foci was first worked out systematically by J. Kepler while formulating his famous laws of planetary motion in the early seventeenth century; he also introduced the term *focus*. The general notion of a focus of an algebraic curve of degree higher than two does not seem to have been established until 1832 by J. Plücker, see [20, 5].

A general discussion of foci and their properties may be found in [1, 24, 2, 26, 10, 6]. Hilton and Basset describe graphically the foci of several cubics and quartics. Among the papers devoted specifically to cubics, Roberts studies foci and confocal systems of plane curves, with applications to confocal cubics, see [22, 23]. Jeffrey considers cubics of class three with prescribed focal configurations and cubics with a double and a simple focus, see [14]. Emch studies curves with a prescribed system of foci and computes the foci of a special cubic, see [7].

Summarizing, the theory of foci of plane algebraic curves is classical, going back to the works of Roberts [22, 23], Emch [7], Rice [21], and Ichida [13]. These works study both the geometry of foci of individual curves and inverse problems with prescribed foci.

More recently, the subject has been revisited from different perspectives, including geometric approaches to real foci (Langer–Singer) and connections with complex polynomials (Casas-Alvero), as well as reconstruction results in special families (Poncelet–Darboux and Kippenhahn-type curves).

However, the deformation-theoretic problem of describing families of curves sharing the same foci does not appear to have been systematically studied. In particular, we address the following question:

> Which plane algebraic curves admit nontrivial confocal deformations?

The main conceptual contribution of this paper is the introduction of a focal map on equiclassical families and the interpretation of confocal families as its fibers. This allows us to study confocality using deformation theory.

2 Classical preliminaries

It is well-known that every circle intersects the line at infinity in two imaginary points $p_\pm = [1, \pm i, 0]$. These points are called the *circular points at infinity.* In the case of a curve with real coefficients the two points $p_\pm$ are related in a similar manner to the curve; for example, if p_+ is a multiple point on a curve, then p_- is also a multiple point of the same multiplicity. Lines passing thorugh the circular points, other than the line at infinity, are called *circular lines* or *isotropic lines.*

If from a focus of a conic we draw two tangents to the curve, these pass respectively through the two circular points at infinity. This fact led Plücker, [20] to his generalization of the foci of curves of higher degree. His definition as stated by Cayley in [5], is as follows: "If from each of the circular points at infinity ... the tangents are drawn to the curve, the intersections of each tangent from the one point with eeach tangent from the other point are the foci of the curve".

This definition not only gives real loci but also imaginary foci corresponding to the imaginary intersections of the tangents from the circular points to the curve.[1]

If the circular points $p_\pm$ do not lie on the curve, the finite intersections of ordinary tangent (tangent having ordinary contact) from p_+ and p_- are called *ordinary foci.* If p_+ and p_- are on the curve, the finite intersections of the tangents to the curve at p_+ and p_- are called *singular foci* of the curve. The intersections of the remaining tangents from p_+ and p_- to the curve are ordinary foci.

Consider a curve of class c. Since c tangents can be drawn from a generic point to such a curve c tangents can be typically drawn to the curve from p_+ and p_-. These isotropic tangents will intersect in c^2 points. If such a tangent from p_+ is given by $x + iy = a + ib$ then $x - iy = a - ib$ is a tangent from p_-. These tangents interesct in a real point (a, b), so there are c real foci. There are no more than c real foci, for no tangent from $p_\pm$ can contain more than one real point. Hence a curve of class c has in general c real and $c^2 - c$ unreal foci.

The number c of real foci will be decreased if the curve is tangent to the line at infinity or passes through the circular points. (More information how the number of real loci decreases can be found in [21].)

3 Which degree-d curves can admit confocal families?

We start with consideration of generic plane curves of a given degree. The complete linear system of plane curves of degree d has dimension

$$N_d = \dim \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = \frac{d(d+3)}{2}.$$

For a smooth curve of degree d, the class is $c = d(d-1)$, so the focal polynomial $G(-1, -i, z)$ has degree c. Fixing the foci amounts to fixing

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[1] With this definition the ellipse, for instance, has four foci, two real and two imaginary.

this degree-c polynomial, i.e. a choice of $2c$ real parameters. Thus a naive dimension count gives

$$N_d - 2c = \frac{d(d+3)}{2} - 2d(d-1) = -\frac{3}{2}d^2 + \frac{7}{2}d.$$

This is negative for $d \geq 3$. So one should expect that a general smooth curve of degree $d \geq 3$ does not belong to a positive-dimensional confocal family; among smooth curves, only conics have room for such a family.

More generally, let $V_{d,g,c}$ denote an equiclassical family of irreducible plane curves of degree d, genus g, and class c. Its expected dimension is

$$d - g + c + 1.$$

Fixing the foci should again impose $2c$ real conditions, so the expected dimension of the confocal locus is

$$d - g + c + 1 - 2c = d - g - c + 1.$$

This suggests that a positive-dimensional confocal family can only occur when

$$c \leq d - g.$$

In particular, curves admitting confocal families should be very special singular curves of unusually small class.

Remark 3.1. Thus the basic existence problem with prescribed real foci is naturally a problem about class rather than degree: every c-tuple of distinct real points occurs as the set of real foci of some real curve of class c, while the rational case considered below is much more rigid.

The rational case is the most favorable one. If $g = 0$, the inequality becomes $c \leq d$. Thus rational curves are the natural first case to study systematically, although in principle one could also look for singular curves of higher genus with exceptionally small class.

A particularly transparent test occurs for rational nodal-cuspidal curves. Suppose C is a rational plane curve of degree d with δ nodes and κ cusps. Then

$$\delta + \kappa = \frac{(d-1)(d-2)}{2},$$

and the Plücker formula gives

$$c = d(d-1) - 2\delta - 3\kappa = 2(d-1) - \kappa.$$

Hence the expected dimension of the confocal locus becomes

$$d+1-c=d+1-(2d-2-\kappa)=3-d+\kappa.$$

Therefore a rational nodal-cuspidal curve can be expected to admit a positive-dimensional confocal family only if

$$\kappa \geq d-2.$$

For instance, nodal rational curves ($\kappa=0$) should give a positive-dimensional family only for $d=2$, a finite confocal locus for $d=3$, and no positive-dimensional confocal family for $d \geq 4$. Each cusp increases the expected confocal dimension by one.

Proposition 3.2. *Let W be an irreducible real equiclassical family of dual plane curves of degree c, genus g, and class d. On the affine chart where the coefficient of w^c is 1, define the focal map*

$$\Phi : W \to \mathbb{C}[z]_{\leq c-1} \cong \mathbb{R}^{2c}, \qquad \Phi(G) := G(-1,-i,z)-z^c.$$

Then the locus in W of curves with prescribed real foci is a fiber of Φ. Hence, at a smooth point $G \in W$, its local real dimension is

$$\dim_{\mathbb{R}} W - \operatorname{rank}(d\Phi_G).$$

In particular, the expected dimension $d-g-c+1$ is equivalent to the maximal-rank condition

$$\operatorname{rank}(d\Phi_G)=2c$$

at a general smooth point of W.

Proof. Two real curves are confocal exactly when their dual equations G have the same polynomial $G(-1,-i,z)$, because the roots of this polynomial are precisely the foci, counted with multiplicity. Therefore the locus of curves with prescribed foci is a fiber of Φ. The dimension statement is then the usual fiber-dimension formula at a smooth point. $\square$

Theorem 3.3. *Let W be an irreducible real equiclassical family of dual plane curves of degree c and class d. Assume that a general smooth point $G \in W$ parametrizes a rational nodal-cuspidal curve $D=\{G=0\}$. Then*

$$\operatorname{rank}(d\Phi_G)=2c.$$

Equivalently, the local confocal locus through G has the expected real dimension $d-c+1$.

Proof. Let $Z \subset \mathbb{P}^2_\vee$ be the equisingular scheme of D. Since D has only nodes and cusps, the equiclassical and equisingular deformations coincide, so at a smooth point of W one has

$$T_G W \cong H^0(\mathcal{I}_Z(c))/\langle G\rangle.$$

The differential of the focal map is induced by restriction to the isotropic line $L = \{v = iu\}$:

$$d\Phi_G : T_G W \longrightarrow H^0(\mathcal{O}_L(c))/\langle z^c\rangle.$$

Thus it is enough to prove that the restriction map

$$H^0(\mathcal{I}_Z(c)) \longrightarrow H^0(\mathcal{O}_L(c))$$

is surjective. By the exact sequence

$$0 \to \mathcal{I}_Z(c-1) \to \mathcal{I}_Z(c) \to \mathcal{O}_L(c) \to 0,$$

it suffices to show that

$$H^1(\mathcal{I}_Z(c-1)) = 0.$$

Let $\nu : \widetilde{D} \to D$ be the normalization. Since D is rational, $\widetilde{D} \cong \mathbb{P}^1$. Write δ for the number of nodes and κ for the number of cusps of D. If $p_1, \ldots, p_\delta$ are the nodes, with preimages $p_i', p_i'' \in \widetilde{D}$, and $q_1, \ldots, q_\kappa$ are the cusps, define the divisor

$$E = \sum_{i=1}^{\delta} (p_i' + p_i'') + 3\sum_{j=1}^{\kappa} q_j.$$

Locally, the equisingular ideal pulls back to $(x) \oplus (y)$ at a node and to (t^3) at a cusp. Therefore one has an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2_\vee}(-1) \to \mathcal{I}_Z(c-1) \to \nu_*\mathcal{O}_{\widetilde{D}}(\nu^*\mathcal{O}_D(c-1) - E) \to 0.$$

Now

$$\deg \nu^*\mathcal{O}_D(c-1) = c(c-1),$$

while $\deg E = 2\delta + 3\kappa$. Since the class of D is

$$d = c(c-1) - 2\delta - 3\kappa,$$

we get

$$\deg\big(\nu^*\mathcal{O}_D(c-1) - E\big) = d.$$

Hence

$$\mathcal{O}_{\widetilde{D}}(\nu^*\mathcal{O}_D(c-1)-E) \cong \mathcal{O}_{\mathbb{P}^1}(d),$$

so

$$H^1\big(\widetilde{D}, \mathcal{O}_{\widetilde{D}}(\nu^*\mathcal{O}_D(c-1)-E)\big) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = 0.$$

Also $H^1(\mathbb{P}^2_\vee, \mathcal{O}(-1)) = 0$, hence $H^1(\mathcal{I}_Z(c-1)) = 0$. Therefore the restriction map is surjective and $d\Phi_G$ has complex rank c. Since Φ is viewed as a map to the real vector space $\mathbb{C}[z]_{\leq c-1} \cong \mathbb{R}^{2c}$, this means that its real rank is $2c$. □

Remark 3.4. This theorem overlaps with the later discussion of rational curves, but it is worth recording here because it proves the maximal-rank statement directly in the nodal-cuspidal rational case.

The same argument shows that, in general, the maximal-rank problem is controlled by a single cohomology group. Indeed, if Z^{ec} denotes the equiclassical scheme of a dual curve D and $\nu : \widetilde{D} \to D$ is the normalization, then one expects an exact sequence of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2_\vee}(-1) \to \mathcal{I}_{Z^{ec}}(c-1) \to \nu_* A \to 0,$$

where A is the class line bundle on $\widetilde{D}$. Thus the maximal-rank statement should follow from the vanishing of $H^1(A)$.

Conjecture 3.5 (Refined conjecture)**.** *Let W be an irreducible real equiclassical family of dual curves of degree c, genus g, and class d, and let G be a general smooth point of W, corresponding to a dual curve D with normalization $\nu : \widetilde{D} \to D$. Let A be the class line bundle on $\widetilde{D}$ associated with D. Then A is nonspecial, equivalently*

$$H^1(A) = 0.$$

Equivalently, one has

$$H^1(\mathcal{I}_{Z^{ec}}(c-1)) = 0,$$

and hence the differential of the focal map has maximal rank $2c$ at G.

4 Known examples of confocal families

4.1 Rational curves

Let $n = 2(m-1)$ (general) points in $\mathbb{R}^2$ be given. Then there exists a rational curve of degree m and class n with these points as real foci if and only if $m \leq 3$. For $m = 2$ there is a one-dimensional family of such curves, and for $m = 3$ there is a finite number. This follows from a simple dimension count. The dimension of the space of rational curves of degree m is $\frac{1}{2}m(m+3) - \frac{1}{2}(m-1)(m-2) = 3m - 1$. To have the given points as real foci impose $2n = 4(m-1)$ conditions, and $3m - 1 - 4m + 4 = 3 - m$.

Proposition 4.1. *Let $P_j = (x_j, y_j) \in \mathbb{R}^2$, $j = 1, \ldots, n$, be distinct points, and put $z_j = x_j + iy_j$. Then there exists a real plane curve of class n whose real foci are precisely $P_1, \ldots, P_n$. Moreover, class n is minimal possible. More precisely, the dual curves of class n with these prescribed foci form an affine space of dimension $\binom{n}{2}$, namely*

$$G(u, v, w) = \prod_{j=1}^{n} (w + x_j u + y_j v) + (u^2 + v^2) Q_{n-2}(u, v, w),$$

where Q_{n-2} is an arbitrary real homogeneous polynomial of degree $n - 2$.

Proof. Set

$$G_0(u, v, w) := \prod_{j=1}^{n} (w + x_j u + y_j v).$$

Then

$$G_0(-1, -i, z) = \prod_{j=1}^{n} (z - x_j - iy_j) = \prod_{j=1}^{n} (z - z_j),$$

so $G_0 = 0$ has the prescribed foci. If

$$G(u, v, w) = G_0(u, v, w) + (u^2 + v^2) Q_{n-2}(u, v, w),$$

then G is still a real homogeneous polynomial of degree n, and since $u^2 + v^2 = 0$ on the isotropic lines $v = \pm iu$, we get

$$G(-1, -i, z) = G_0(-1, -i, z) = \prod_{j=1}^{n} (z - z_j).$$

Hence every such G defines a real curve of class n with real foci $P_1, \ldots, P_n$. For general Q_{n-2}, the curve $G = 0$ is irreducible and smooth in the dual plane, hence its bidual is an irreducible real curve of class n with these foci. The space of choices of Q_{n-2} has dimension

$$\dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-2)) = \binom{n}{2}.$$

Finally, if a curve has class m, then its focal polynomial $G(-1,-i,z)$ has degree m, so it has at most m foci counted with multiplicity. Therefore n distinct prescribed foci force $m \geq n$, and the above construction shows that $m = n$ is attainable. $\square$

Example 4.2. Let $P_i = (x_i : y_i : 1)$, $i = 1, 2$, be two points in $\mathbb{R}P^2$. Then there is a one-dimensional family of real conics with P_1 and P_2 as real foci.

Consider a real conic, given by $F(x, y, z) = 0$, and its dual conic, given in line coordinates u, v, w by

$$G(u, v, w) = w^2 + \alpha_1 wv + \alpha_2 wu + \alpha_3 v^2 + \alpha_4 vu + \alpha_5 u^2 = 0,$$

where $\alpha_i \in \mathbb{R}$. That the P_i are foci of the conic means that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are the roots of the equation

$$G(-1, -i, z) = 0,$$

i.e.,

$$G(-1,-i,z) = z^2 - (\alpha_1 i + \alpha_2)z - \alpha_3 + \alpha_4 i + \alpha_5 = (z - x_1 - iy_1)(z - x_2 - iy_2).$$

We conclude that this is satisfied if $\alpha_1 = y_1 + y_2$, $\alpha_2 = x_1 + x_2$, $\alpha_4 = x_1 y_2 + x_2 y_1$, and $\alpha_3 = \alpha_5 - x_1 x_2 + y_1 y_2$. Hence there is a one-dimensional family of confocal real conics.

By choosing $x_1 = 1$, $x_2 = -1$, $y_1 = y_2 = 0$, we obtain the example in [7]*p. 160, 5..

Example 4.3. Consider a real curve of class 3, given by the equation of its dual curve

$$\begin{aligned} G(u,v,w) = w^3 + \alpha_1 w^2 v + \alpha_2 w^2 u + \alpha_3 wv^2 + \alpha_4 wvu + \alpha_5 wu^2 \\ + \alpha_6 v^3 + \alpha_7 v^2 u + \alpha_8 vu^2 + \alpha_9 u^3 = 0. \end{aligned}$$

We can fix three real foci (x_i, y_i), $i = 1, 2, 3$, and write

$$\begin{aligned} G(-1,-i,z) = z^3 - (i\alpha_1 + \alpha_2)z^2 + (-\alpha_3 + i\alpha_4 + \alpha_5)z \\ + (i\alpha_6 + \alpha_7 - i\alpha_8 - \alpha_9) = (z - x_1 - iy_1)(z - x_2 - iy_2)(z - x_3 - iy_3). \end{aligned}$$

We see that $\alpha_1 = y_1 + y_2 + y_3$ and $\alpha_2 = x_1 + x_2 + x_3$, and that also $\alpha_3 - \alpha_5$, α_4, $\alpha_6 - \alpha_8$, and $\alpha_7 - \alpha_9$ are determined. Thus we get a three-dimensional family of confocal curves. If the curve $G(u, v, w) = 0$ is nonsingular, then the confocal curves are sextics with nine cusps.

Example 4.4. Emch [7] considers an example of a circular nonsingular cubic, and computes the equation of its dual curve, and finds its six foci. He also studies curves admitting the nth roots of unity as foci.

Example 4.5. Roberts [22, 23] treats *rational* curves, such as nodal and cuspidal cubics, or bicircular nodal quartics, where the line at infinity and the circular points are not in general position with respect to the curves. A rational bicircular quartic of class 4 of the type considered has cusps at the circular points and an additional "finite" node, such curves are among those considered in [23]. Clearly, one can find lots of confocal families, for special curves. For example, if we consider a cuspidal cubic, then its dual curve is also a cuspidal cubic. When three real foci (in general position) are given, we see that we obtain a one-dimensional family of confocal cuspidal cubics. Roberts [22] says that there are in fact 36 different one-dimensional families.

4.2 Siebeck curves

Siebeck [25] showed the following theorem: If $g \in \mathbb{C}[z]$ is a cubic polynomial such that its three roots z_1, z_2, z_3 are not aligned (considered as points in $\mathbb{R}^2$), then the roots of its derivative $\partial g/\partial z$ are the foci of the unique conic which is tangent to the sides of the triangle with vertices z_1, z_2, z_3 at their midpoints. Linfield [17] generalized this theorem to polynomials g of arbitrary degree, in particular he showed the following (see [4]):

Let $f \in \mathbb{C}[z]$ be a polynomial of degree m, with roots $z = x_j + iy_j$, $j = 1, \ldots, m$. Let $H(u, v, w) := \prod_{j=1}^{m}(y_j u + x_j v + w) = 0$ be the equation of the union of the lines in $(\mathbb{P}^2)^\vee$ dual to the points z_j. The *polar curve* of $H = 0$ with respect to the point $(0 : 0 : 1) \in (\mathbb{P}^2)^\vee$ (dual to the line at ∞ in $\mathbb{P}^2$) is given by $\partial H/\partial w = 0$. Then the curve $\partial H/\partial w = 0$ is the dual of a curve of class $m - 1$, whose foci are the roots of $\partial f/\partial z = 0$.

Casas-Alvero [3, 4] generalized these results further. A special case of his more general result [4]*Thm. 6.1, p. 236 is the following: Let $f \in \mathbb{C}[z]$ be a polynomial of degree m with distinct roots $z_1, \ldots, z_m$. Let $p_{j,k}$ denote the midpoint of the line segment between z_j and z_k (considered as points in $\mathbb{R}^2$). Then there is a unique curve S, the *Siebeck curve*, of class $m - 1$, tangent to each of the lines $z_j z_k$ at the point $p_{j,k}$. Moreover, the foci of S are the roots of $\partial f/\partial z = 0$.

4.3 Poncelet curves

In [12]*Def. 3.2, p. 15 the authors consider algebraic curves that are envelopes of polygons supported on the unit circle $\mathbb{T} \subset \mathbb{C} = \mathbb{R}^2$. They define a family of n-Poncelet polygons as a set of n-sided polygons $\mathcal{P}(z)$, $z \in \mathbb{T}$, inscribed in $\mathbb{T}$, such that z is one of the vertices of $\mathcal{P}(z)$, and satisfying $w \in \mathbb{T}$ is a vertex of $\mathcal{P}(z)$ implies $\mathcal{P}(w) = \mathcal{P}(z)$. A closed curve in the unit disc $\mathbb{D}$ is an *n-Poncelet curve* which envelopes a family of n-Poncelet polygons in a specified way. The definition implies that an n-Poncelet curve determines the family of n-Poncelet polygons uniquely.

The authors define *complete* Poncelet curves [12]*Def. 3.11, p. 21. A *minimal class* complete n-Poncelet curve determines a real algebraic curve Γ of class $n-1$, whose real foci are all inside $\mathbb{D}$. Hence there is a bijection between $n-1$ points in $\mathbb{D}$ and complete n-Poncelet curves, and Γ can be reconstructed from its real foci. They ask whether the assumption that the curve is of class $n-1$ is superfluous [12]*Rmk. 4.2, p. 29.

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