CONFOCAL FAMILIES OF PLANE ALGEBRAIC CURVES

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Abstract. We study real plane algebraic curves with prescribed focal divisors. Confocality is reformulated by means of a focal map on real equiclassical families of dual curves. The differential of this map is governed by a focal adjoint line bundle on the normalization of the dual curve; this bundle is obtained from the class adjoint bundle by subtracting one hyperplane section. As an application, we prove a maximal-rank statement for rational nodal-cuspidal dual curves and obtain the expected local dimension of their confocal loci. We also compare this deformation-theoretic picture with several classical constructions of curves with prescribed foci.

To the late Vladimir Arnold on the occasion of his 90th birthday

1. Introduction

The main goal of this paper is to study families of real plane algebraic curves admitting confocal deformations, that is, positive-dimensional families of curves with the same focal divisor. Equivalently, for curves with simple real foci, this means families sharing the same set of real foci.

In the nineteenth century, foci of ellipses were generalized to foci of higher-degree curves: the (real) *foci* of a real algebraic curve $C \subset \mathbb{P}^2$ are the real points on isotropic tangent lines to C, see [22, p. 120]. In traditional terminology associated with an algebraic curve $F(x, y, z) = 0$ in the complex projective plane $\mathbb{P}^2$, an isotropic line is one containing either of the circular points $p_\pm = (1 : \pm i : 0)$ (the pair of points common to all circles). Although this notion is not especially intuitive at first sight, it appears in a number of research areas including approximation theory, numerical analysis, and fluid models. This is the main reason for discussing foci of real algebraic curves again.

Short historical account. The points now called foci first arose in connection with conic sections. According to the classical book [8, p. 252] on the history of mathematics, Apollonius of Perga, who lived in the third century B.C., incidentally discovered the foci of the ellipse and hyperbola. Five hundred years later Pappus of Alexandria found the focus of the parabola.

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The theory of foci was first worked out systematically by J. Kepler while formulating his famous laws of planetary motion in the early seventeenth century; he also introduced the term *focus*. The general notion of a focus of an algebraic curve of degree higher than two does not seem to have been established until 1832 by J. Plücker, see [18, p. 85].

A general discussion of foci and their properties may be found in [1, 2, 6, 9, 12, 19, 22, 25]. Hilton and Jervis describe graphically the foci of several cubics and quartics, see [10]. These works study both the geometry of foci of individual curves and inverse problems with prescribed foci. Roberts studies foci and confocal systems of various plane curves of degree ≤ 4, see [20, 21]. Jeffrey considers cubics of class three with prescribed focal configurations and cubics with a double and a simple focus, see [13, 14]. Emch studies curves with a prescribed system of foci and computes the foci of a special cubic, see [7].

More recently, the subject has been revisited from different perspectives, including geometric approaches to real foci (Langer–Singer [16]) and connections with complex polynomials (Casas-Alvero [3, 4]), as well as reconstruction results in special families (Poncelet–Darboux [11] and Kippenhahn-type curves [15]).

However, the deformation-theoretic problem of describing families of curves with the same focal divisor does not appear to have been systematically studied. In particular, we address the following question:

> Which plane algebraic curves admit nontrivial confocal deformations?

The main conceptual contribution of this paper is the introduction of a focal map on equiclassical families and the interpretation of confocal families as its fibers. This allows us to study confocality using deformation theory. The key point is that, over the reals, the infinitesimal condition of fixing the focal divisor is the condition of vanishing on the union of the two conjugate isotropic lines

$$Y = \{u^2 + v^2 = 0\} = L_+ \cup L_-.$$

Consequently the relevant cohomological object is the focal adjoint bundle

$$\mathcal{B}_D = \mathcal{A}_D \otimes \nu^*\mathcal{O}_D(-1)$$

on the normalization of the dual curve. Under the usual equiclassical tangent-space hypotheses, the kernel of the differential of the focal map is identified with

$$H^0(\widetilde{D}, \mathcal{B}_D).$$

For rational nodal-cuspidal dual curves this gives the maximal-rank formula

$$\operatorname{rank}(d\Phi_G) = \min(2c, c + d + 1),$$

and the expected local dimension of the confocal locus is

$$\max(0, d - c + 1).$$

The general maximal-rank statement is formulated below as a Brill–Noether type conjecture for $\mathcal{B}_D$.

2. Classical preliminaries

Throughout the paper, a *tangent* to curve at a point p is a line whose intersection number with the curve at p is strictly greater than the multiplicity of p as a point on the curve. Hence, the tangents are the lines corresponding to the points on the dual curve. In particular, not every line passing through a singular point is considered to be a tangent.

It is well-known that every circle intersects the line at infinity in two imaginary points $p_\pm = (1 : \pm i : 0)$. These points are called the *circular points at infinity.* In the case of a *real* curve, the two circular points at infinity are related to the curve by an analogous projective construction, namely via isotropic lines and tangents through these points. Unlike the case of a circle, the curve need not pass through the circular points; the analogy is only at the level of the projective construction. Lines passing through the circular points, other than the line at infinity, are called *circular lines* or *isotropic lines.*

If from a focus of a conic we draw two tangents to the curve, these pass respectively through the two circular points at infinity. This fact led Plücker [18, p. 85] to his generalization of the foci of curves of higher degree. His definition, as stated by Cayley in [5, p. 515], is as follows: "If from each of the circular points at infinity . . . tangents are drawn to the curve, the intersections of each tangent from the one point with each tangent from the other point are the foci of the curve".

This definition not only gives real foci but also imaginary foci corresponding to the imaginary intersections of the tangents from the circular points to the curve.[1]

If the circular points $p_\pm$ do not lie on the curve, the finite intersections of ordinary tangents (tangents having ordinary contact) from p_+ and p_- are called *ordinary foci.* If p_+ and p_- are on the curve, the finite intersections of the tangents to the curve at p_+ and p_- are called *singular foci* of the curve. The intersections of the remaining tangents from p_+ and p_- to the curve are ordinary foci. If one of the two isotropic tangents is a tangent at a circular point lying on the curve and the other is not, then their intersection is counted among the ordinary foci. The term singular focus is reserved for the intersection points arising from tangents at the circular points themselves.

The *class* of a plane curve is the number of tangents that can be drawn to the curve from a general point. Equivalently, the class is the degree of the dual curve. Consider a real curve of class c. Then c tangents can typically be drawn to the curve from p_+ and p_-. These isotropic tangents will intersect in c^2 points. If such a tangent from p_+ is given by $x + iy = (a + ib)z$, then $x - iy = (a - ib)z$ is a tangent from p_-. These two tangents intersect in a

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[1] With this definition the ellipse, for instance, has four foci, two real and two imaginary.

real point $(a : b : 1)$, so there are c real foci. There are no more than c real foci, for no tangent from $p_\pm$ can contain more than one real point. Hence a curve of class c has in general c real and $c^2 - c$ unreal foci [9, p. 69].

The number c of real foci will be decreased if the curve is tangent to the line at infinity or passes through the circular points. Another possible drop occurs when a double tangent of the curve passes through one of the circular points. (More information how the number of real foci decreases can be found in [19].)

3. Which degree-d curves can admit confocal families?

We start with the consideration of general plane curves of given degree d. The complete linear system of plane curves of degree d has dimension

$$N_d = \dim \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = \frac{d(d+3)}{2}.$$

For a given smooth curve of degree d, the class is $c = d(d-1)$. If $G(u, v, w) = 0$ is the equation of the dual curve, the focal data are encoded by the two degree-c polynomials

$$G_+(w) := G(-1, -i, w) \quad \text{and} \quad G_-(w) := G(-1, i, w).$$

Indeed, if $x + iy = r_+ z$ (resp. $x - iy = r_- z$) is the equation of an isotropic line through p_+ (resp. p_-), then the line is tangent to the curve iff r_+ (resp. r_-) is a root of G_+ (resp. G_-). The intersection

$$\left(\frac{1}{2}(r_+ + r_-) : -\frac{i}{2}(r_+ - r_-) : 1\right)$$

of two such lines is a focus of the curve. To ask that a given point $(f_+ : f_- : 1)$ be a focus, amounts to fixing one root r_+ of G_+ and one root r_- of G_-:

$$r_+ = f_+ + if_- \text{ and } r_- = f_+ - if_-.$$

Hence, to ask for a fixed focus, amounts to imposing two conditions on the dual curve, hence on the original curve.

The focus is real iff the two roots r_+ and r_- are complex conjugate. If the curve is real, then also its dual is real, so that G can be chosen to have real coefficients. In this case, G_- is the complex conjugate of G_+, so if r_+ is a root of G_+, then $r_- := \overline{r_+}$ is a root of G_-, and hence the corresponding focus is a real point. More precisely, it follows from the above that if $r_+ = a + ib$ is a root of G_+, then $(a : b : 1)$ is a real focus of the curve.

It follows that a general real curve of degree d will have at least $c = d(d-1)$ real foci. It cannot have more, since two isotropic lines that are not conjugate cannot intersect in a real point, since each line only contains one real point.

A naive dimension count then gives that the set of curves with c fixed foci should have dimension

$$N_d - 2c = \frac{d(d+3)}{2} - 2d(d-1) = -\frac{1}{2}d(3d-7)$$

since fixing c foci is expected to impose $2c$ real conditions. Now $N_d - 2c$ is negative for $d \geq 3$. So we should expect that a general smooth curve of degree $d \geq 3$ does not belong to a positive-dimensional confocal family; among smooth curves, only conics have room for such a family.

More generally, let $V_{d,g,c}$ denote the equiclassical family of irreducible plane curves of degree d, genus g, and class c [23, p. 196]. Its expected dimension is

$$d - g + c + 1.$$

Fixing c foci should again impose $2c$ conditions, so the expected dimension of the confocal locus is

$$d - g + c + 1 - 2c = d - g - c + 1.$$

This suggests that, generically, positive-dimensional confocal families can occur only when the inequality

$$c \leq d - g$$

holds. Very exceptional positive-dimensional families are not excluded by this dimension count. In particular, curves admitting confocal families should be very special singular curves of unusually small class.

Shustin proved that if $2g \leq d + 1$, then $V_{d,g,c}$ is an irreducible variety of dimension $d - g + c + 1$, and a general point corresponds to a curve with only nodes and cusps as singularities [23, Cor. 1.2, p.197]. A classical example where irreducibility fails, was given by Zariski: $V_{6,4,12}$ consists of two 15-dimensional irreducible families of six-cuspidal sextics [26, p. 223].

Remark 3.1. Thus the basic existence problem with prescribed real foci is naturally a problem about class rather than degree: every c-tuple of distinct real points occurs as the set of real foci of some real curve of class c, while the rational case considered below is much more rigid.

The rational case is the most favorable one. If $g = 0$, the inequality becomes $c \leq d$. Thus rational curves are the natural first case to study systematically, although in principle one could also look for singular curves of higher genus with exceptionally small class.

A particularly transparent test occurs for rational nodal-cuspidal curves. Suppose C is a rational plane curve of degree d with δ nodes and κ cusps. Then

$$\delta + \kappa = \frac{(d-1)(d-2)}{2},$$

and the Plücker formula gives

$$c = d(d-1) - 2\delta - 3\kappa = 2(d-1) - \kappa.$$

Hence the expected dimension of the confocal locus becomes

$$d + 1 - c = d + 1 - (2d - 2 - \kappa) = 3 - d + \kappa.$$

Therefore a rational nodal-cuspidal curve can be expected to admit a positive-dimensional confocal family only if

$$\kappa \geq d-2.$$

For instance, nodal rational curves ($\kappa = 0$) should give a positive-dimensional family only for $d=2$, a finite confocal locus for $d=3$, and no positive-dimensional confocal family for $d \geq 4$. Each cusp increases the expected confocal dimension by one.

Let $\mathbb{P}(H^0((\mathbb{P}^2)^\vee, \mathcal{O}_{(\mathbb{P}^2)^\vee}(c)) \cong \mathbb{P}^{\frac{1}{2}c(c+3)}$ denote the set of curves in $(\mathbb{P}^2)^\vee$ of degree c, with coordinates given by the coefficients of the polynomials. Let $W_{c,g,d} \subset \mathbb{P}^{\frac{1}{2}c(c+3)}$ denote the equiclassical family of curves of degree c, genus g, and class d. On the affine chart where the coefficient of w^c is 1, consider the map

$$\Phi\colon W_{c,g,d} \to \mathbb{C}[w]_{\leq c-1} \cong \mathbb{C}^c, \quad \Phi(G) := G_+(w) - w^c,$$

where

$$G_+(w) = G(-1,-i,w).$$

To fix the coefficients of G_+ is equivalent to fixing its roots, counted with multiplicity. We call the divisor of roots of G_+ the *focal divisor*. For a real curve this divisor is conjugate to the corresponding divisor of roots of G_-, and it records the real foci counted with multiplicity.

Definition 3.2. Let $W_{c,g,d}(\mathbb{R}) \subset W_{c,g,d}$ be the subset of real polynomials. The *focal map* is the restriction

$$\Phi\colon W_{c,g,d}(\mathbb{R}) \to \mathbb{R}^{2c},$$

where $\mathbb{C}[w]_{\leq c-1}$ is regarded as a real vector space.

If G is real, then $G_-(w) := G(-1,i,w)$ is the complex conjugate of $G_+(w)$. Thus each root of $G_+(w)$ gives one real focus, counted with the same multiplicity as a root of the focal divisor. A fiber of Φ therefore consists of dual curves whose corresponding real curves have fixed focal divisor; when all roots are simple, this is the same as fixing the set of real foci.

It will be important below that the real differential of Φ is controlled not by one isotropic line alone, but by the union of the two conjugate isotropic lines

$$Y = L_+ \cup L_-, \qquad L_+ = \{u+iv=0\}, \quad L_- = \{u-iv=0\}.$$

Indeed, for a real infinitesimal deformation H of G, the condition that the first variation of G_+ vanishes is equivalent to the simultaneous vanishing of H on L_+ and on L_-. Equivalently, after complexification of the real tangent map, one is restricting to the reducible conic

$$Y = \{u^2+v^2=0\}.$$

This elementary point is the source of the shift by two, rather than by one, in the adjoint bundles appearing below.

Proposition 3.3. *Let $W \subseteq W_{c,g,d}(\mathbb{R})$ be an irreducible component. The locus in W of curves with prescribed focal divisor is a fiber of Φ. Hence, at a smooth point $G \in W$, the Zariski tangent space to the local confocal locus is*

$$\ker(d\Phi_G).$$

If W has the expected real dimension $c + d - g + 1$ and $d\Phi_G$ has maximal rank, then

$$\operatorname{rank}(d\Phi_G) = \min(2c, c + d - g + 1)$$

and the expected local dimension of the confocal locus through G is

$$\max(0, d - g - c + 1).$$

In particular, positive-dimensional local confocal deformations are expected only when

$$c \leq d - g.$$

Proof. Two real curves have the same focal divisor exactly when their dual equations have the same polynomial $G_+(w)$, up to the normalization used above, because the roots of $G_+(w)$ are precisely the numbers $x + iy$ corresponding to the real foci $(x : y : 1)$, counted with multiplicity. Thus the locus of curves with prescribed focal divisor is a fiber of Φ.

At a smooth point the tangent space to the fiber is the kernel of the differential. If the differential has maximal rank, its rank is the smaller of the dimension of the source and the dimension $2c$ of the target. Since the expected dimension of $W_{c,g,d}$ is $c+d-g+1$, the asserted formula follows. □

We now introduce the adjoint line bundles which control this differential. The first one is a class adjoint bundle. The second, obtained from it by subtracting one hyperplane section, is the bundle relevant for confocal deformations.

We shall use the following standard deformation-theoretic convention. At a point $D = \{G = 0\}$ of an equiclassical family, the phrase *the usual equiclassical tangent-space description holds* means that the complex Zariski tangent space to the equiclassical stratum is represented by

$$T_D W_{\mathbb{C}} \simeq H^0(\mathcal{I}_{Z^{\mathrm{ec}}}(c))/\langle G \rangle,$$

where Z^{ec} is the equiclassical scheme of D. On the affine chart where the coefficient of w^c is normalized to be 1, this is represented by sections of $H^0(\mathcal{I}_{Z^{\mathrm{ec}}}(c))$ with zero w^c-coefficient.

Definition 3.4. Let $D \subset (\mathbb{P}^2)^\vee$ be a reduced curve of degree c, let $\nu : \widetilde{D} \to D$ be the normalization, and let Z^{ec} be an equiclassical scheme of D for which the usual tangent-space description holds. Assume also that the equiclassical adjoint sequence holds for D. The *class adjoint line bundle* $\mathcal{A}_D$ is the line bundle on $\widetilde{D}$ defined by

$$0 \to \mathcal{O}_{(\mathbb{P}^2)^\vee}(-1) \to \mathcal{I}_{Z^{\mathrm{ec}}}(c-1) \to \nu_*\mathcal{A}_D \to 0.$$

Equivalently, $\mathcal{A}_D$ is obtained by pulling the torsion-free sheaf $\mathcal{I}_{Z^{\mathrm{ec}}}(c-1)\otimes\mathcal{O}_D$ to the normalization and removing torsion; its push-forward is the last term in the displayed exact sequence.

The *focal adjoint line bundle* is

$$\mathcal{B}_D := \mathcal{A}_D \otimes \nu^*\mathcal{O}_D(-1).$$

Equivalently, it is determined by the shifted adjoint sequence

$$0 \to \mathcal{O}_{(\mathbb{P}^2)^\vee}(-2) \to \mathcal{I}_{Z^{\mathrm{ec}}}(c-2) \to \nu_*\mathcal{B}_D \to 0.$$

Remark 3.5. In the nodal-cuspidal case the definition is completely explicit. If $p_i', p_i'' \in \widetilde{D}$ are the two preimages of the node p_i, and if $q_j \in \widetilde{D}$ is the preimage of the cusp q_j, then

$$\mathcal{A}_D = \mathcal{O}_{\widetilde{D}}\left(\nu^*\mathcal{O}_D(c-1) - \sum_i (p_i' + p_i'') - 3\sum_j q_j\right),$$

and

$$\mathcal{B}_D = \mathcal{O}_{\widetilde{D}}\left(\nu^*\mathcal{O}_D(c-2) - \sum_i (p_i' + p_i'') - 3\sum_j q_j\right).$$

The correction divisor has degree $2\delta + 3\kappa$, which is the Plücker drop in the class for nodes and cusps. Thus

$$\deg \mathcal{A}_D = c(c-1) - 2\delta - 3\kappa = d, \qquad \deg \mathcal{B}_D = d - c.$$

This is why one should not identify the symbol $\deg Z^{\mathrm{ec}}$ with the ordinary scheme-theoretic length of Z^{ec} in degree computations: the relevant number here is the adjoint divisor degree on the normalization.

Proposition 3.6. *Let $D = \{G = 0\} \subset (\mathbb{P}^2)^\vee$ be a real reduced curve of degree c, genus g, and class d, corresponding to a smooth point of an equiclassical component $W \subset W_{c,g,d}(\mathbb{R})$. Assume that the usual equiclassical tangent-space description holds at D, that the equiclassical adjoint sequence holds, and that the equiclassical scheme of D is disjoint from the reducible conic*

$$Y = \{u^2 + v^2 = 0\} = L_+ \cup L_-.$$

Then the real tangent kernel of the focal map is naturally identified with the real part of

$$H^0(\widetilde{D}, \mathcal{B}_D).$$

Consequently,

$$\dim_{\mathbb{R}} \ker(d\Phi_G) = h^0(\widetilde{D}, \mathcal{B}_D)$$

and, if W is smooth of expected dimension at G,

$$\operatorname{rank}(d\Phi_G) = c + d - g + 1 - h^0(\widetilde{D}, \mathcal{B}_D).$$

Proof. Work on the affine chart in which the coefficient of w^c in G is equal to 1. By the tangent-space hypothesis, an infinitesimal real equiclassical deformation is represented by a real homogeneous polynomial H of degree c satisfying the linear equiclassical conditions, i.e. by a section of $H^0(\mathcal{I}_{Z^{\mathrm{ec}}}(c))$ with zero w^c-coefficient.

The condition that H lie in the kernel of $d\Phi_G$ is $H|_{L_+} = 0$. Since H has real coefficients, this is equivalent to $H|_{L_-} = 0$ as well. Hence H vanishes on $Y = L_+ \cup L_-$, whose equation is $u^2 + v^2 = 0$. Therefore

$$H = (u^2 + v^2)Q$$

for some homogeneous polynomial Q of degree $c-2$. Because Z^{ec} is disjoint from Y, the condition $H \in H^0(\mathcal{I}_{Z^{\mathrm{ec}}}(c))$ is equivalent to

$$Q \in H^0(\mathcal{I}_{Z^{\mathrm{ec}}}(c-2)).$$

Conversely, every such Q gives an infinitesimal deformation in the kernel. Thus the kernel is identified with the real part of $H^0(\mathcal{I}_{Z^{\mathrm{ec}}}(c-2))$.

By the shifted adjoint sequence in Definition 3.4, and since $H^0(\mathcal{O}_{(\mathbb{P}^2)^\vee}(-2)) = 0$, we have

$$H^0(\mathcal{I}_{Z^{\mathrm{ec}}}(c-2)) \cong H^0(\widetilde{D}, \mathcal{B}_D).$$

The dimension and rank formulas follow. □

Theorem 3.7. *Let $W \subseteq W_{c,0,d}(\mathbb{R})$ be an irreducible component of the real equiclassical family of rational curves in $(\mathbb{P}^2)^\vee$ of degree c and class d. Assume that W is smooth of the expected dimension $c+d+1$ at a general point, that the usual equiclassical tangent-space description holds there, and that a general point of W corresponds to a nodal-cuspidal curve whose singular scheme is disjoint from $Y = \{u^2 + v^2 = 0\}$. Then, for a general smooth point $G \in W$,*

$$\dim_{\mathbb{R}} \ker(d\Phi_G) = \max(0, d-c+1)$$

and

$$\operatorname{rank}(d\Phi_G) = \min(2c, c+d+1).$$

Equivalently, the expected local dimension of the confocal locus through G is

$$\max(0, d-c+1).$$

Proof. Let $D = \{G = 0\}$ and let $\nu : \widetilde{D} \to D$ be the normalization. Since D is rational, $\widetilde{D} \cong \mathbb{P}^1$. By Proposition 3.6, the kernel of the focal differential is the real part of $H^0(\widetilde{D}, \mathcal{B}_D)$.

In the nodal-cuspidal case, Remark 3.5 gives

$$\deg \mathcal{B}_D = d - c.$$

Therefore

$$\mathcal{B}_D \cong \mathcal{O}_{\mathbb{P}^1}(d-c),$$

and hence

$$h^0(\widetilde{D}, \mathcal{B}_D) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-c)) = \max(0, d-c+1).$$

Since the expected dimension of $W_{c,0,d}$ is $c+d+1$, the rank formula follows from Proposition 3.6:

$$\operatorname{rank}(d\Phi_G) = c+d+1-\max(0, d-c+1) = \min(2c, c+d+1).$$

□

Remark 3.8. The maximal-rank conclusion in Theorem 3.7 means maximal rank in the ordinary sense: the rank is the smaller of the source dimension and the target dimension. Thus it is not always equal to $2c$. For example, if $c > d+1$, then the source has dimension $c+d+1 < 2c$, so rank $2c$ is impossible.

The preceding discussion shows that the cohomological object relevant to confocal deformations is $\mathcal{B}_D$, not $\mathcal{A}_D$ itself. The bundle $\mathcal{A}_D$ records the class adjunction, while the focal map asks for vanishing along the two conjugate isotropic lines and therefore subtracts one further hyperplane section.

Conjecture 3.9. *Let W be an irreducible component of the real equiclassical family of curves in $(\mathbb{P}^2)^\vee$ of degree c, genus g, and class d, and let G be a general smooth point of W. Assume that the usual equiclassical tangent-space description and adjoint sequences hold at $D = \{G = 0\}$, and that the focal adjoint bundle satisfies $\deg \mathcal{B}_D = d-c$; this holds, for instance, in the nodal-cuspidal case. Then*

$$h^0(\widetilde{D}, \mathcal{B}_D) = \max(0, d-g-c+1).$$

Consequently the focal map has maximal rank at G:

$$\operatorname{rank}(d\Phi_G) = \min(2c, c+d-g+1).$$

Proposition 3.10. *Assume $\deg \mathcal{B}_D = d-c$, and set $b = d-c$. By Riemann–Roch, Conjecture 3.9 is equivalent to the following Brill–Noether type alternative:*

1. *if $b \le g-1$, then $H^0(\widetilde{D}, \mathcal{B}_D) = 0$;*
2. *if $b \ge g$, then $H^1(\widetilde{D}, \mathcal{B}_D) = 0$.*

In particular, the conjectural maximal-rank statement is automatic whenever $b < 0$, and the nonspeciality part is automatic whenever $b \ge 2g-1$.

Proof. Riemann–Roch gives

$$h^0(\mathcal{B}_D) - h^1(\mathcal{B}_D) = \deg \mathcal{B}_D - g + 1 = d-c-g+1 = b-g+1.$$

If $b \le g-1$, then $b-g+1 \le 0$, and the expected value in Conjecture 3.9 is 0; this is exactly the assertion $H^0(\mathcal{B}_D) = 0$. If $b \ge g$, then $b-g+1 > 0$, and the expected value is $b-g+1$; by Riemann–Roch this is equivalent to $H^1(\mathcal{B}_D) = 0$.

If $b < 0$, then a line bundle of negative degree has no nonzero sections, so the first alternative holds automatically. If $b \ge 2g-1$, then $\deg(K_{\widetilde{D}} \otimes \mathcal{B}_D^{-1}) = 2g-2-b < 0$, hence $H^1(\mathcal{B}_D) = 0$ by Serre duality. □

Remark 3.11. The nonspeciality of the class adjoint bundle $\mathcal{A}_D$ alone is not sufficient for maximal rank of the focal map. It controls restriction to a single hyperplane section. The real focal condition fixes the restriction to both conjugate isotropic lines, and this is why the correct bundle is

$$\mathcal{B}_D = \mathcal{A}_D \otimes \nu^* \mathcal{O}_D(-1).$$

4. Known examples of confocal families

In each example below we indicate explicitly whether prescribed real foci, or more precisely a prescribed focal divisor when multiplicities occur, determine a positive-dimensional confocal family, a finite set of curves, or a unique curve. Throughout, the dimension is the dimension of the parameter space of real curves with the prescribed focal data.

4.1. **Maximal-class rational curves.** Let $c = 2(d-1)$ general points in $\mathbb{R}^2$ be given. We ask for rational curves of degree d and class c having these points as real foci. A simple dimension count gives the answer.

The space of rational plane curves of degree d has dimension

$$\frac{1}{2}d(d+3) - \frac{1}{2}(d-1)(d-2) = 3d - 1.$$

Prescribing $c = 2(d-1)$ real foci imposes

$$2c = 4(d-1)$$

real conditions. Hence the expected dimension is

$$(3d-1) - 4(d-1) = 3 - d.$$

Therefore:

- if $d = 2$, the expected dimension is 1, so one obtains a one-dimensional confocal family;
- if $d = 3$, the expected dimension is 0, so one obtains a finite set of curves for general prescribed foci;
- if $d \geq 4$, the expected dimension is negative, so for general prescribed foci no such rational curve exists.

In particular, among maximal-class rational curves, nontrivial confocal families occur only for conics. Rational curves with many cusps can have smaller class and are covered by Theorem 3.7.

4.2. **Curves with given foci of minimal class.** The following proposition is essentially stated in [9, Ch. 5, §1, p. 69].

Proposition 4.1. *Let $c \geq 2$, let $P_j = (x_j : y_j : 1) \in \mathbb{P}^2(\mathbb{R})$, $j = 1, \dots, c$, be distinct points, and put $z_j = x_j + iy_j$. Then there exists a real plane curve of class c whose real foci are precisely $P_1, \dots, P_c$. Moreover, class c is the* *minimal possible. More precisely, the dual equations of curves of class c with these prescribed foci form an affine space of dimension* $\binom{c}{2}$*, namely*

$$G(u,v,w)=\prod_{j=1}^{c}(x_ju+y_jv+w)+(u^2+v^2)Q_{c-2}(u,v,w),$$

where Q_{c-2} *is an arbitrary real homogeneous polynomial of degree* $c-2$*.*

Proof. Set

$$G_0(u,v,w):=\prod_{j=1}^{c}(x_ju+y_jv+w).$$

Then

$$G_0(-1,-i,w)=\prod_{j=1}^{c}(w-x_j-iy_j)=\prod_{j=1}^{c}(w-z_j),$$

so $G_0=0$ has the prescribed foci. If

$$G(u,v,w)=G_0(u,v,w)+(u^2+v^2)Q_{c-2}(u,v,w),$$

then G is still a real homogeneous polynomial of degree c, and since $u^2+v^2=0$ on the isotropic lines $u\pm iv=0$, we get

$$G(-1,-i,w)=G_0(-1,-i,w)=\prod_{j=1}^{c}(w-z_j).$$

Hence every such G defines a real curve of class c with real foci $P_1,\dots,P_c$. For general Q_{c-2}, the curve $G=0$ is irreducible and smooth in the dual plane, hence its dual is an irreducible real curve of class c with these foci. The space of choices of Q_{c-2} has (affine) dimension

$$\dim H^0((\mathbb{P}^2)^\vee,\mathcal{O}_{(\mathbb{P}^2)^\vee}(c-2))=\frac{1}{2}c(c-1)=\binom{c}{2}.$$

Finally, if a curve has class c', then its focal polynomial $G(-1,-i,w)$ has degree c', so it has at most c' real foci counted with multiplicity. Therefore c distinct prescribed foci force $c'\geq c$, and the above construction shows that $c'=c$ is attainable. ☐

Thus, for arbitrary distinct prescribed real foci, curves of minimal class always exist, and they form a confocal family of dimension $\binom{c}{2}$. In particular, for $c\geq 2$ the foci do not determine a unique curve of class c with the given foci.

Example 4.2 (Conics with two prescribed real foci)**.** Let $P_j=(x_j:y_j:1)$, $j=1,2$, be two points in $\mathbb{P}^2(\mathbb{R})$. Then the real conics with real foci P_1 and P_2 form a one-dimensional confocal family. In particular, the foci do *not* determine a unique conic.

Let

$$G(u,v,w)=w^2+\alpha_1wv+\alpha_2wu+\alpha_3v^2+\alpha_4vu+\alpha_5u^2$$

be the equation of the dual conic. The condition that P_1 and P_2 be the real foci is

$$G(-1,-i,w) = (w - x_1 - iy_1)(w - x_2 - iy_2).$$

Expanding both sides and comparing coefficients gives

$$\alpha_1 = y_1 + y_2, \qquad \alpha_2 = x_1 + x_2, \qquad \alpha_4 = x_1 y_2 + x_2 y_1,$$

and

$$-\alpha_3 + \alpha_5 = x_1 x_2 - y_1 y_2.$$

Equivalently,

$$\alpha_3 = \alpha_5 - x_1 x_2 + y_1 y_2.$$

Hence four independent real conditions are imposed on the five parameters $\alpha_1, \dots, \alpha_5$, leaving exactly one free real parameter.

Therefore the dimension is

$$5 - 4 = 1.$$

So the prescribed foci determine a one-parameter confocal family of real conics.

By choosing $x_1 = 1$, $x_2 = -1$, $y_1 = y_2 = 0$, we recover the example given by Emch in [7, 5., p. 160].

Example 4.3 (Curves of class 3 with three prescribed real foci)**.** Consider real curves of class 3, given by the equation of the dual curve

$$\begin{aligned} G(u,v,w) = w^3 + \alpha_1 w^2 v + \alpha_2 w^2 u + \alpha_3 w v^2 + \alpha_4 w v u + \alpha_5 w u^2 \\ + \alpha_6 v^3 + \alpha_7 v^2 u + \alpha_8 v u^2 + \alpha_9 u^3. \end{aligned}$$

Fix three real foci $P_j = (x_j : y_j : 1)$, $j = 1, 2, 3$. Then the real curves of class 3 with these prescribed real foci form a three-dimensional confocal family. Again, the foci do *not* determine a unique curve.

Indeed, the focal condition is

$$G(-1,-i,w) = \prod_{j=1}^{3} (w - x_j - iy_j).$$

Now

$$\begin{aligned} G(-1,-i,w) = w^3 - (i\alpha_1 + \alpha_2) w^2 + (-\alpha_3 + i\alpha_4 + \alpha_5) w \\ + (i\alpha_6 + \alpha_7 - i\alpha_8 - \alpha_9). \end{aligned}$$

Comparing coefficients with

$$\prod_{j=1}^{3} (w - x_j - iy_j)$$

gives three complex equations, hence six real linear conditions on the nine real parameters $\alpha_1, \dots, \alpha_9$. More explicitly,

$$\alpha_1 = y_1 + y_2 + y_3, \qquad \alpha_2 = x_1 + x_2 + x_3,$$

while the combinations

$$\alpha_3 - \alpha_5, \qquad \alpha_4, \qquad \alpha_6 - \alpha_8, \qquad \alpha_7 - \alpha_9$$

are also determined by the prescribed foci.

Thus exactly three real parameters remain free, so the dimension is

$$9 - 6 = 3.$$

If the curve $G(u, v, w) = 0$ is nonsingular, then $G \in W_{3,1,6}$, and the corresponding dual curves are sextics with nine cusps. If $G(u, v, w) = 0$ is nodal, then $G \in W_{3,0,4}$, and there is a 2-dimensional set of corresponding confocal tri-cuspidal quartics. If $G(u, v, w) = 0$ is cuspidal, then $G \in W_{3,0,3}$, and there is a 1-dimensional set of corresponding confocal cuspidal cubics. According to Roberts [20, p. 154], in this last case there are 36 different one-dimensional families.

4.3. **Siebeck curves.** Siebeck [24] showed that if $f \in \mathbb{C}[z]$ is a cubic polynomial whose roots z_1, z_2, z_3 are not collinear (viewed as points in $\mathbb{R}^2$), then the roots of $\partial f/\partial z$ are the foci of the unique conic tangent to the sides of the triangle with vertices z_1, z_2, z_3 at their midpoints.

Thus, once the triangle (z_1, z_2, z_3) is fixed, the corresponding *Siebeck conic* is *unique*; equivalently, the parameter space has dimension 0 for this fixed tangency problem.

However, this should not be confused with uniqueness from the foci alone. For two prescribed real foci, the space of all real conics with those foci is one-dimensional, as in the conic example above. The Siebeck construction therefore produces a distinguished subclass of that one-dimensional confocal family, but the foci by themselves do not determine a unique conic.

So, in the language of this paper:

- for fixed triangle data, there is a unique Siebeck conic (dimension 0);
- for fixed foci alone, there is a one-dimensional confocal family of conics.

Linfield [17, Thm. 1, p. 247] extended Siebeck's result to arbitrary degree, see also [4, Thm. 5.1, p. 235]. Let $f \in \mathbb{C}[z]$ be a polynomial of degree n with roots $z_j = x_j + iy_j$, $j = 1, \ldots, n$, and let

$$H(u, v, w) := \prod_{j=1}^{n} (x_j u + y_j v + w).$$

Then $H = 0$ is the union of the lines in $(\mathbb{P}^2)^\vee$ dual to the points z_j, and the polar curve of $H = 0$ with respect to $(0 : 0 : 1) \in (\mathbb{P}^2)^\vee$ is given by

$$\frac{\partial H}{\partial w} = 0.$$

The curve $\partial H/\partial w = 0$ is dual to a curve of class $n - 1$, whose foci are the roots of $\partial f/\partial z = 0$.

Example 4.4 (Siebeck curves of class $n-1$)**.** Casas-Alvero [3,4] developed this point of view further. A special case of [4, Thm. 6.1, p. 236] is the following. Let $f \in \mathbb{C}[z]$ be a polynomial of degree n with distinct roots $z_1, \dots, z_n$, and let $p_{j,k}$ be the midpoint of the segment joining z_j and z_k in $\mathbb{R}^2$. Then there is a unique curve S, the *Siebeck curve*, of class $n-1$, tangent to each line $z_j z_k$ at $p_{j,k}$. Its foci are the roots of $\partial f/\partial z = 0$.

Again, the important point is that the curve is unique *after the points* $z_1, \dots, z_n$ *have been fixed*, or equivalently after all the tangency conditions have been prescribed. For this incidence/tangency problem the parameter space therefore has dimension 0.

By contrast, if one fixes only the $n-1$ foci and considers *all* real curves of class $n-1$ with those foci, then by the above results the ambient confocal family has dimension $\binom{n-1}{2}$.

Thus the Siebeck curve is not uniquely determined by the foci alone in general; rather, the Siebeck construction selects a distinguished member (or distinguished subclass) inside that larger confocal family by imposing additional tangency conditions.

So, in this example:

- for fixed roots $z_1, \dots, z_n$ (equivalently, fixed tangency data), there is a unique Siebeck curve (dimension 0);
- for fixed foci alone, the full space of curves of class $n-1$ has dimension $\binom{n-1}{2}$.

4.4. **Poncelet curves.** Hunziker et al. [11] study algebraic curves arising as envelopes of polygons supported on the unit circle $\mathbb{T} \subset \mathbb{C} = \mathbb{R}^2$. In [11, Def. 3.2, p. 15] they define a family of n-Poncelet polygons to be a family of n-gons $\mathcal{P}(z)$, $z \in \mathbb{T}$, inscribed in $\mathbb{T}$, such that z is a vertex of $\mathcal{P}(z)$ and such that, whenever $w \in \mathbb{T}$ is also a vertex of $\mathcal{P}(z)$, one has $\mathcal{P}(w) = \mathcal{P}(z)$.

An *n-Poncelet curve* is then a closed curve in the unit disc $\mathbb{D}$ which envelopes such a family of n-Poncelet polygons. In particular, the curve determines the corresponding family of polygons uniquely.

The authors also define *complete* Poncelet curves [11, Def. 3.11, p. 21]. A complete n-Poncelet curve *of minimal class* gives a real algebraic curve C of class $n-1$, all of whose real foci lie in $\mathbb{D}$. In this case there is a bijection between configurations of $n-1$ points in $\mathbb{D}$ and complete n-Poncelet curves, and C can be reconstructed from its real foci.

Therefore, in the minimal-class complete case, prescribed real foci determine a *unique* Poncelet curve. Equivalently, the parameter space has dimension 0. So this is *not* a positive-dimensional confocal family.

The authors also ask whether the assumption that the curve has class $n-1$ is in fact superfluous [11, Rmk. 4.2, p. 29]. Thus, beyond the minimal-class complete case, uniqueness from the foci is not asserted.

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