ON SINGULARITIES OF SMOOTH MAPS TO A SPACE WITH A FIXED CONE

B. Z. Shapiro

Department of Mathematics, University of Stockholm,
S-10691, Sweden, shapiro@matematik.su.se

Abstract. We consider germs of smooth maps $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, c)$, $c \in \mathcal{C}$ where $\mathcal{C}$ is the standard nondegenerate cone of some signature and classify their singularities under the actions of two natural groups of diffeomorphisms preserving $\mathcal{C}$. Occuring singularities are subdivided into 3 classes: regular, semiregular and irregular. In the regular case the classification of singularities is reduced to the classification of the usual singularities of germs of functions. We present the list of simple semiregular singularities and also analyze some irregular singularities.

§0. Preliminaries and results

The singularities of maps $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, s), s \in \mathcal{S}$ to a target space with some fixed stratified variety $\mathcal{S}$ were considered by several authors, see e.g. [AVG],[A1], [L],[Si]. In the case when $\mathcal{S}$ is a hyperplane they are called *boundary singularities* and were investigated in details in [A1]. In particular, it was shown that the simple boundary singularities correspond to the A-, B-, C-, D-, E-series and F_4 in the classification of simple Lie algebras. A much more general class of actions (including all actions on source and target with a stratified $\mathcal{S}$ used in this note) was studied by J. Damon [Da1-2]. He proved the existence of the versal unfoldings and finite determinacy of germs of maps in this situation. (Therefore, the normal forms we present can be achieved in formal, analytic and smooth categories.) In what follows we will always assume that the dimension of the source is less than the dimension of the target.

Main Notation.

Let $s \in \mathcal{S}$ be a point on a stratified variety $\mathcal{S}$ and let $\mathcal{A}_S$ denote *the product of the group of local diffeomorphisms* $(\mathbf{R}^l, 0) \to (\mathbf{R}^l, 0)$ *of the source by the group of local diffeomorphisms* $(\mathbf{R}^n, s) \to (\mathbf{R}^n, s)$ *of the target preserving* $\mathcal{S}$.

By $\mathcal{K}$ we denote *the group of contact transformations*, i.e. elements of $\mathcal{K}$ are diffeomorphisms of $(\mathbf{R}^l \times \mathbf{R}^n, 0)$ preserving a) projection on $\mathbf{R}^l$ (inducing diffeomorphisms of $\mathbf{R}^l$) and b) the subspace $(\mathbf{R}^l \times 0)$. Thus two germs f_1 and $f_2 : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, 0)$ are $\mathcal{K}$-equivalent if there exists a diffeomorphism of the source and a germ $M : (\mathbf{R}^l, 0) \to GL(\mathbf{R}^n)$ such that $f_1(x) = M(x) f_2(g(x))$. (Another standard notation for $\mathcal{K}$-equivalence introduced by J. Martinet and used in [AVG] is V-equivalence.)

Let $\mathcal{K}_S$ denote the subgroup of $\mathcal{K}$ consisting of all diffeomorphisms $H \in \mathcal{K}$ such that $H(\mathbf{R}^l \times \mathcal{S}) \subset \mathbf{R}^l \times \mathcal{S}$.

1991 *Mathematics Subject Classification.* Primary 58C27, Secondary 14B10.

Key words and phrases. Singularities of smooth maps, versal unfoldings, bifurcation diagrams.

Let $\mathcal{O}_l$ denote the ring of germs of analytic functions on $(\mathbf{R}^l, 0)$ and $\mathfrak{m}_l \subset \mathcal{O}_l$ denote the minimal ideal of functions vanishing at the origin. The space of germs of maps $(\mathbf{R}^l, 0) \to \mathbf{R}^n$ is denoted by $\mathcal{O}_l^{(n)}$ and the space of germs of maps $(\mathbf{R}^l, 0) \to (\mathbf{R}^n, 0)$ sending the origin of the source to the origin of the target is denoted by $\mathfrak{M}_l^n$. Obviously, $\mathcal{O}_l^{(n)} = \mathcal{O}_l \times \cdots \times \mathcal{O}_l$ (n factors) and $\mathfrak{M}_l^n = \mathfrak{m}_l \times \cdots \times \mathfrak{m}_l$.

Some standard notions of the singularity theory.

An arbitrary germ $\Psi(x, \lambda) = (\Psi_\lambda(x), \lambda) : (\mathbf{R}^l \times \mathbf{R}^r, 0 \times 0) \to (\mathbf{R}^n \times \mathbf{R}^r, s \times 0)$ such that $\Psi(x, 0) = f(x)$ is called *an unfolding of the map* $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, s)$. The additional space of parameters $\mathbf{R}^r$ is called *the base of the unfolding.* If additionally, for all λ one has $\Psi(0, \lambda) = s \times \lambda$ then such a Ψ is called an *origin-preserving* unfolding of f.

Let us consider the action of a group G (G equals $\mathcal{A}_\mathcal{S}$ or $\mathcal{K}_\mathcal{S}$) on the space of germs $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, s)$. Two unfoldings of f with the same base $\Psi' : (\mathbf{R}^l \times \mathbf{R}^r, 0 \times 0) \to (\mathbf{R}^n \times \mathbf{R}^r, s \times 0)$ and $\Psi'' : (\mathbf{R}^l \times \mathbf{R}^r, 0 \times 0) \to (\mathbf{R}^n \times \mathbf{R}^r, s \times 0)$ are called G-equivalent if the exists a germ of smooth map $g : (\mathbf{R}^r, 0) \to (G, e)$ where e is the identical diffeomorphism such that $\Psi'(x, \lambda) = g(\lambda)\Psi''(x, \lambda)$. Consider a smooth map $\Theta : (\mathbf{R}^{r_1}, 0) \to (\mathbf{R}^{r_2}, 0)$. The deformation induced by Θ from $\Psi_2 : (\mathbf{R}^l \times \mathbf{R}^{r_2}, 0 \times 0) \to (\mathbf{R}^n \times \mathbf{R}^{r_2}, s \times 0)$ is defined as $\Theta^*\Psi_2 = \Psi_2 \circ \Theta : (\mathbf{R}^l \times \mathbf{R}^{r_1}, 0 \times 0) \to (\mathbf{R}^n \times \mathbf{R}^{r_1}, s \times 0)$. A G-versal unfolding of a map $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, s)$ is an unfolding Φ such that any other unfolding is G-equivalent to some unfolding induced from Φ by an appropriate map of bases, see [AVG],p.142. If the dimension of the base of Φ is the minimal possible then Φ is called *a miniversal unfolding.* (We will always work with miniversal unfoldings and omit the prefix 'mini'.) *A bifurcation diagram* is a subset $\mathrm{Bif} \subset \Lambda$ of the base Λ such that for any $\lambda \in \mathrm{Bif}$ the corresponding map $f_\lambda \in \Phi$ is not in general position w.r.t. $\mathcal{S}$, i.e. either f_λ is not an immersion or the image $f_\lambda(\mathbf{R}^l)$ is nontransversal to $\mathcal{S}$. Two unfoldings of a germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, \mathcal{S})$ are called *equivalent* if each of them can be induced from the other by an appropriate map of their bases. Bifurcation diagrams $\mathrm{Bif}_1 \subset \Lambda_1$ and $\mathrm{Bif}_2 \subset \Lambda_2$ of two unfoldings with the bases Λ_1 and Λ_2 resp. are called *coinciding* if there exists a diffeomorphism of the pairs $(\Lambda_1, \mathrm{Bif}_1)$ and $(\Lambda_2, \mathrm{Bif}_2)$. Exactly the same definitions as above work if we restrict our considerations to the class of origin-preserving unfoldings. Corresponding versal unfoldings will be called *origin-preserving* or *restricted* versal unfoldings. Recall that two germs of functions on the same number of variables are called *equivalent* if there exists a local diffeomorphism sending one to the other. Two functions on a different number of variables are called *stably equivalent* if they become equivalent after addition of nondegenerate quadratic forms of extra variables.

By *the modality* of a singularity of a germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, s)$ (under the action of a chosen group G) we call the minimal number m of parameters such that a sufficiently small neighborhood of the orbit of f can be covered by a finite number of m-parameter families of orbits.

A singularity of a germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, \mathcal{C})$ is called *simple* (under the action of the chosen group) if its modality is zero.

A germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, \mathcal{C})$ is called *stable* if for any germ $\tilde{f}$ close to f there exists a point (x, y) close to $(0, 0)$ such that $\tilde{f}$ considered as $\tilde{f} : (\mathbf{R}^l, x) \to (\mathbf{R}^n, y)$ is $\mathcal{A}$-equivalent to f.

The main definition. A germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, s)$, $s \in \mathcal{S}$ is called *regular* if the following 2 conditions are satified

a) f is an embedding; b) s belongs to one of the top-dimensional strata of $\mathcal{S}$. (In this case $\mathcal{S}$ can be replaced by a germ of a smooth submanifold.)

If only a) is satisfied the germ f is called *semiregular* and finally if a) is violated such an

f is called *irregular*.

0.1. REGULAR SINGULARITIES.

REMARK. If f is (semi)regular then its $\mathcal{A}_\mathcal{S}$- and $\mathcal{K}_\mathcal{S}$-versal unfoldings are equivalent (see lemma 1.1) and in this case we just call either of them *a versal unfolding*. Analogously, its restricted $\mathcal{A}_\mathcal{S}$- and $\mathcal{K}_\mathcal{S}$-versal unfoldings are equivalent and we call either of them *a restricted versal unfolding*.

Let $\mathcal{S} \subset \mathbf{R}^n$ be a germ of a smooth manifold of codimension k and $s \in \mathcal{S}$ be some point. Let us fix k functions $h_1, \dots, h_k$ defining $\mathcal{S}$ as a germ of complete intersection $\mathcal{S} : \{h_i = 0; i = 1, \dots, k\}$ in some neighborhood of s. For any germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, s)$, $s \in \mathcal{S}$ we call by the *induced germ* $f_\mathcal{S}$ the germ $h \circ f : (\mathbf{R}^l, 0) \to \mathbf{R}^n, s)$, i.e. the pullback of the functions $h_1, \dots, h_k$ by $f(\mathbf{R}^l)$ in the neighborhood of s.

PROPOSITION A. A versal unfolding and the bifurcation diagram of a regular germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, \mathcal{S})$ is equivalent to a $\mathcal{K}$-versal unfolding and bifurcation diagram of its induced germ $f_\mathcal{S}$.

REMARK. As was pointed put by the referee Proposition A also holds for $\mathcal{K}_\mathcal{S}$-unfoldings without the requirement that f is an embedding, see [Da 3] but is virtually always false if s is not a regular point of $\mathcal{S}$.

If $S : \{Q = 0\}$ is a germ of hypersurface in $\mathbf{R}^n$ and $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, s)$ is a map then $Q_f : (\mathbf{R}^l, 0) \to (\mathbf{R}, 0)$ will denote *the germ of the induced function* $Q \circ f = f^*Q$.

COROLLARY. If $\mathcal{S} : \{Q = 0\}$ is a germ of a hypersurface then a versal deformation of a germ of a regular map $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, s)$ is equivalent to a $\mathcal{K}$-versal deformation of the induced fuction $Q_f : (\mathbf{R}^l, 0) \to (\mathbf{R}, 0)$. Therefore, if $\mathcal{S}$ is a hypersurface then the classification of singularities of regular maps coincides with the $\mathcal{K}$-classification of germs of functions. In particular, they have the same list of simple singularities.

COROLLARY B. If $\mathcal{C}$ is a nondegenerate cone then for any fixed k the list of bifurcation diagrams in versal unfoldings of regular maps $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, \mathcal{C})$ occuring in generic k-parameter families of maps stabilize as soon as $n > k + l$. This means that for any such map regular f_1 to the space of dimension $n_1 > k + l$ there exists a regular map f_2 to the space of dimension $n_2 \le k + l$ with the coinciding bifurcation diagram.

0.2. SEMIREGULAR SINGULARITIES.

PROPOSITION C. A versal unfolding of a semiregular germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, c), c \in \mathcal{C}$ can be obtained by extending its reduced versal unfolding by a $(n - l)$-dimensional space of parallel shifts of f in the directions transversal to the image $f(R^l)$ at the origin of the target.

PROPOSITION D. Taking the induced functions one defines a mapping from origin-preserving unfoldings of f to unfoldings of Q_f in $\mathfrak{m}_l^2$ with the following properties. Let $\Phi(x, \lambda) = (\Phi_\lambda(x), \lambda)$ be some origin-preserving unfolding then

a) If $\Phi_{\lambda_1}(x)$ and $\Phi_{\lambda_2}(x)$ lie in a single $\mathcal{A}_\mathcal{C}$-orbit then $Q_{\Phi_{\lambda_1}}$ and $Q_{\Phi_{\lambda_2}}$ lie in a single $\mathcal{K}$-orbit.

b) any unfolding of Q_f in $\mathfrak{m}_l^2$ can be induced from an origin-preserving unfolding of f.

In particular, the deformation of Q_f induced from any reduced versal unfolding $\Phi_{red}(x, \lambda)$ of a semiregular germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, \mathcal{C})$ is equivalent to reduced $\mathcal{K}$-versal unfolding of Q_f. (Recall that in its turn any reduced $\mathcal{K}$-versal unfolding of Q_f is equivalent to $Q_f + \sum \lambda_i e_i$ where $e_i \in \mathfrak{m}_l^2$ are representatives of any basis of the quotient module $\mathfrak{m}_l^2 / \mathfrak{m}_l (\frac{\partial Q_f}{\partial x_1}, \dots, \frac{\partial Q_f}{\partial x_l}, Q)$.)

COROLLARY E. The modality of a semiregular germ is not less than the $\mathcal{K}$-modality of the induced function Q_f. Therefore, the necessary condition for a semiregular germ $f : (\mathbf{R}^l, 0) \to$

$(\mathbf{R}^n, \mathcal{C})$ to be simple is that the induced Q_f is stably equivalent to one of the germs of the $\mathcal{K}$-simple singularities, i.e. belongs to one of the A-, D- or E-series.

REMARK F. A $(n-l)$-dimensional family of parallel shifts of a semiregular f which is included in its versal unfolding in addition to its reduced versal unfolding (see Proposition C) induces the subdeformation of Q_f which can be normalized as

$$Q_f + \sum_{i=1}^{cr} \lambda_i x_i + \sum_{j=cr+1}^{n-l} \pm\lambda_j^2,$$

where x_i, $i = 1, \dots, cr$ are coordinates in the kernel of the quadratic part of Q_f.

Therefore, bifurcation diagrams of a versal unfolding and of a reduced versal unfolding are given as the zero sets of the stably equivalent functions.

THEOREM G. A germ of a semiregular map $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, c), c \in \mathcal{C},\ l < n$ such that the quadratic part of Q_f has corank 1 and $Q_f = \alpha x^k + \dots,\ \alpha \neq 0$ along the kernel can be reduced by the $\mathcal{A}_{\mathcal{C}}$-action to the normal form $f_k : (x_1, x_1^k, x_2, \dots, x_l, 0, \dots, 0)$, where $Q = \pm(y_1y_2 + \sum_{i=3}^n \pm y_i^2)$ and $\mathcal{C} : \{Q = 0\}$. Its reduced versal unfolding depends on $k-1$ parameters and can be chosen as

$$\Phi_{red}(x_1, \dots, x_l, \lambda_1, \dots, \lambda_{k-1}) : \{y_1 = x_1, y_2 = x_1^k + \lambda_1 x_1^{k-1} + \dots + \lambda_{k-1}x_1, y_3 = x_2, \dots,$$
$$y_{l+1} = x_l, y_{l+2} = 0, \dots, y_n = 0\}.$$

The bifurcation diagram of this unfolding has two irreducible components. The first component consists of all sets $\bar\lambda$ such that the induced function $Q_{f_{\bar\lambda}}$ at the origin has a more complicated singularity than Morse singularity. The second component consists of all sets $\bar\lambda$ such that $Q_{f_{\bar\lambda}}$ has only singular points different from the origin. Thus the considered bifurcation diagram coincides with the bifurcation diagram of the singularity B_{k-1}, i.e. consists of all $(k-1)$-tuples $(\lambda_1, \dots, \lambda_{k-1})$ such that $x_1^{k-1} + \lambda_1 x_1^k + \dots + \lambda_{k-1}$ as a polynomial in x_1 has a multiple or zero root. Recall that B_k is a boundary singularity described for the first time in [A1].

COROLLARY H. Under the same assumptions as in Theorem G, a versal unfolding of f_k depends on $k+n-l-1$ parameters and can be chosen in the form

$$\Phi(x_1, \dots, x_l, \lambda_1, \dots, \lambda_{k+n-l-1}) : \{y_1 = x_1, y_2 = x_1^k + \lambda_1 x_1^{k-1} + \dots + \lambda_k, y_3 = x_2, \dots,$$
$$y_{l+1} = x_l, y_{l+2} = \lambda_{k+1}, \dots, y_n = \lambda_{k+n-l-1}\}.$$

The bifurcation diagram of this unfolding consists of all $(k+n-l-1)$-tuples $(\lambda_1, \dots, \lambda_{k+n-l-1})$ such that the hypersurface $x_1^{k+1} + \lambda_1 x_1^k + \dots + \lambda_k x_1 \pm x_2^2 \pm \dots \pm x_l^2 \pm \lambda_{k+1}^2 \pm \dots \pm \lambda_{k+n-l-1}^2 = 0$ is singular. If $l = k-1$ then the bifurcation diagram coincides with that of the singularity B_k and if $l = k-2$ with that of D_{k+1}.

By a *trivial* (r, s)*-extension* of a germ $f : (\mathbf{R}^p, 0) \to (\mathbf{R}^q, 0)$ we mean the map $\bar f : (\mathbf{R}^p \times \mathbf{R}^r \times \mathbf{R}^s, 0) \to (\mathbf{R}^q \times \mathbf{R}^r \times \mathbf{R}^s, 0)$ which is equal to $(f, id, 0)$.

THEOREM I. The normal forms of simple f are the trivial (r, s)-extensions of the following normal forms. (Below $Q = \pm(y_1y_2 + \sum_{j=3}^n \pm y_j^2)$ for the A-series and $Q = \pm(y_1y_2 + y_3y_4 + \sum_{j=5}^n \pm y_j^2)$ in the rest of the cases.)

1) $Q_f \cong A_k$; $\quad f = (x_1, x_1^k)$, see its reduced versal unfolding in theorem G;

2) $Q_f \cong D_{k+1}^\pm$; ($\pm$ forms are different only if k is odd)

the normal form: $f = (x_1, x_1^{k-1}, x_2, \pm x_1x_2)$, its reduced versal unfolding is

$$\Phi_{red}(x_1, x_2, \lambda_1, \dots, \lambda_k) = \{x_1, x_1^{k-1} + \lambda_1 x_1^{k-2} + \dots + \lambda_{k-2}x_1, x_2, \pm x_1x_2 + \lambda_{k-1}x_1 + \lambda_k x_2\};$$

3) $Q_f \cong E_6^{\pm}$; $f = (x_1, x_1^2, x_2, \pm x_2^3)$, its reduced versal unfolding is

$$\Phi_{red}(x_1, x_2, \lambda_1, \dots, \lambda_5) = \{x_1, x_1^2 + \lambda_1 x_1, x_2, \pm x_2^3 + \lambda_2 x_2^2 + \lambda_3 x_1x_2 + \lambda_4 x_1 + \lambda_5 x_2\};$$

4) $Q_f \cong E_7$;
the normal form: $f = (x_1, x_1^2, x_2, x_1x_2^2)$, its reduced versal unfolding is

$$\Phi_{red}(x_1, x_2, \lambda_1, \dots, \lambda_6) = \{x_1, x_1^2 + \lambda_1 x_1, x_2, x_1x_2^2 + \lambda_2 x_1^2 + \lambda_3 x_2^2 + \lambda_4 x_1x_2 + \lambda_5 x_1 + \lambda_6 x_2\};$$

5) $Q_f \cong E_8$; $f = (x_1, x_1^2, x_2, x_2^4)$, its reduced versal unfolding is

$$\Phi_{red}(x_1, x_2, \lambda_1, \dots, \lambda_7) = \{x_1, x_1^2 + \lambda_1 x_1, x_2, x_2^4 + \lambda_2 x_2^3 + \lambda_3 x_2^2 + \lambda_4 x_2 + \lambda_5 x_1x_2^2 + \lambda_6 x_1x_2 + \lambda_7 x_1\}.$$

The adjacency of simple semiregular singularities coincides with the usual adjacency of the $A - D - E$-series, see Fig.1, compare [A3].

Remark. In all the cases 1)-5) the number of parameters in a reduced versal unfolding of f equals $\mu - 1$, where μ is the Milnor number of Q_f and the induced deformation of Q_f is equivalent to $Q_f + \sum \lambda_i e_i$, where $e_i \in \mathfrak{m}_l^2$ are representatives of any basis of the quotient module $\mathfrak{m}_l^2/\mathfrak{m}_l(\frac{\partial Q_f}{\partial x_1}, \dots, \frac{\partial Q_f}{\partial x_l}) = \mathfrak{m}_l^2/\mathfrak{m}_l(\frac{\partial Q_f}{\partial x_1}, \dots, \frac{\partial Q_f}{\partial x_l}, Q_f)$, compare [A3]. The last identity holds in all these cases since Q_f is quasihomogeneous. The bifurcation diagram is reducible and consists of two components one of which is a cylinder over the usual cone in $\mathbf{R}^3$ in the cases 2-5 and is a smooth in the case 1. This component correponds to the case when Q_f has at the origin a more complicated singularity than just Morse singularity.

Remark. When $l = n - 1$ the only type of singularities which occurs is $Q_f \cong A_k$. Other simple singularities are realized only if $n - l$ is at least 2 while in the regular case (i.e. when $f(0)$ is a smooth point of the cone) one can realize all the simple boundary singularities. Yet another difference is that in the semiregular case not all the simple boundary singularities are realized due to the fact that Q is a nondegenerate quadratic form in the ambient space.

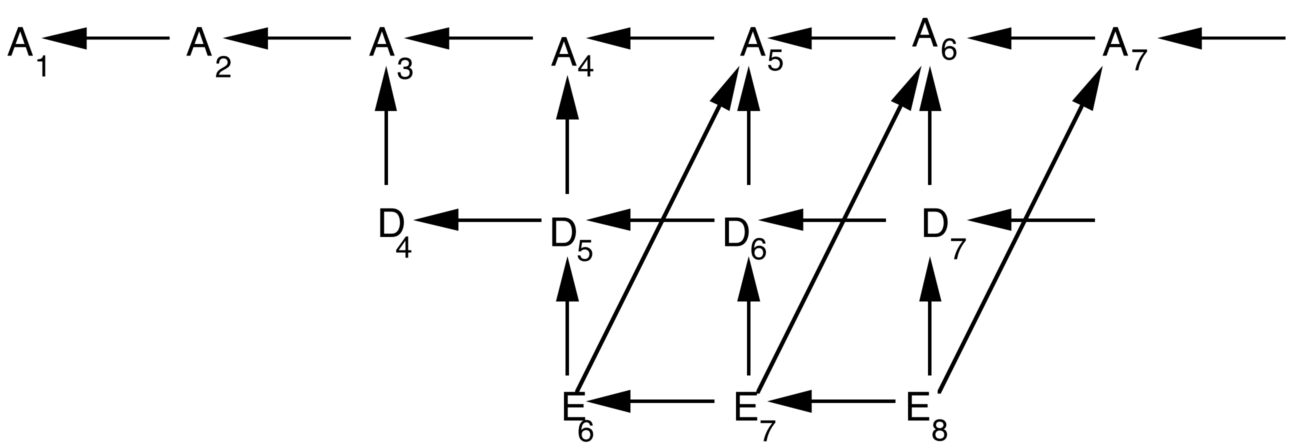

Fig.1. Adjacency of complex simple semiregular singularities.

Proposition B′. Consider the space $\mathfrak{M}_l^n$ of all germs of maps sending the origin to the vertex of the cone. For any fixed k the list of versal unfoldings and bifurcation diagrams of a semiregular germs $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, \mathcal{C})$ occuring in generic k-parameter families of germs from $\mathfrak{M}_l^n$ stabilize as soon as $n \geq k + l$, compare B.

0.3. Irregular singularities.

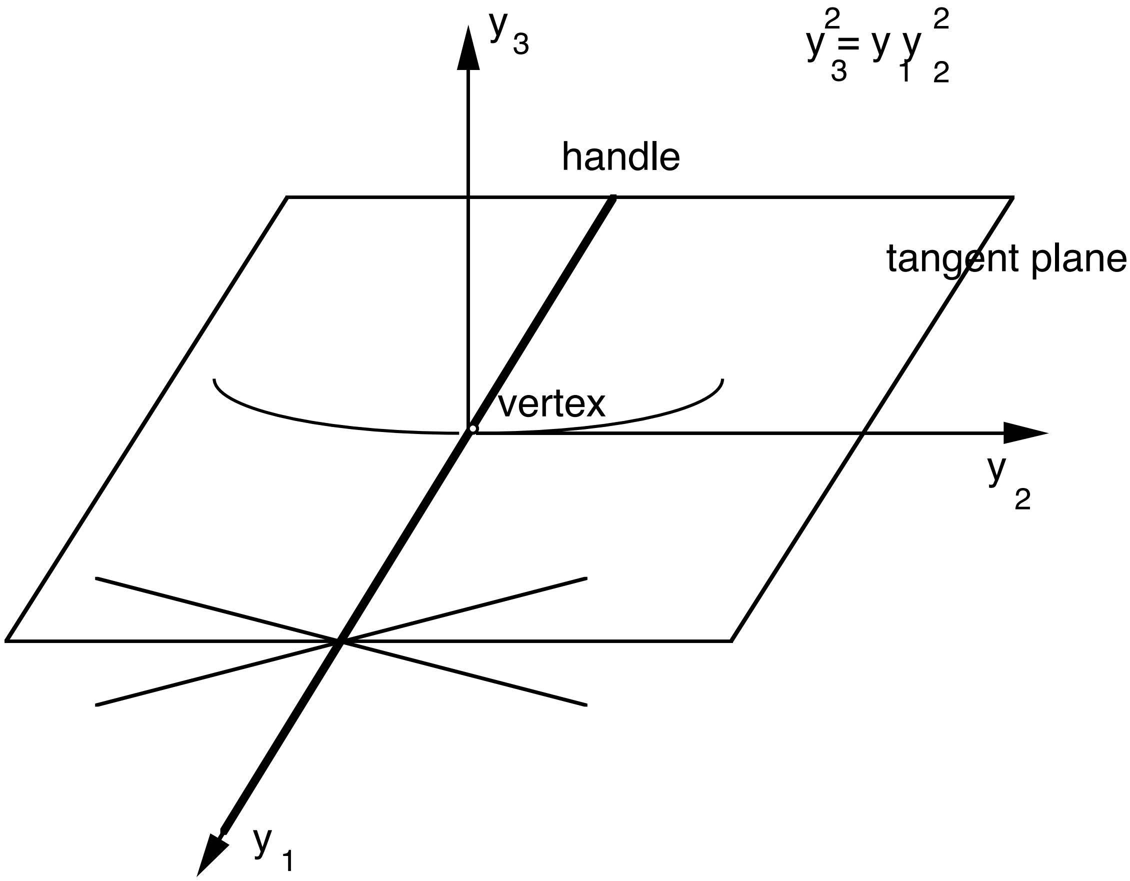

Fig. 2. The standard Whitney umbrella.

The standard Whitney umbrella is a hypersurface in $\mathbf{R}^3$ given by $y_3 = x_1x_2, y_2 = x_2, y_1 = x_1^2$ in the parametric form or by the equation $y_3^2 = y_1y_2^2$. *A Whitney umbrella* is a hypersurface diffeomorphic to the standard one. A Whitney umbrella defines a cross in the tangent plane at its vertex consisting of the tangent line to its handle and the tangent space to a Whitney umbrella considered as the image of a map. (On Fig.2. these lines are y_1- and y_2-axis resp.)

The first nontrivial irregular germ $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, \mathcal{C})$ occurs when $f(\mathbf{R}^2)$ is a standard Whitney umbrella and $f(0)$ is a smooth point of the cone $\mathcal{C}$. In this case we can substitute $\mathcal{C}$ by $\mathbf{R}^2$. Moreover instead of normalizing a Whitney umbrella in a space with a hyperplane $\mathbf{R}^2$ one can normalize an imbedding of a smooth hypersurface in a space with the fixed Whitney umbrella, see lemma 3.1.

Theorem J. The above nontrivial irregular case leads to the following singularities.

a) If $\mathcal{C}$ is transversal to the tangent plane at the vertex of the Whitney umbrella then the Whitney umbrella and a germ of $\mathcal{C}$ in the neighborhood of $f(0)$ can be normalized as $\{y_3^2 = y_1y_2^2, y_1 = y_2\}$.

b) There are 2 cases of codimension 1 when the tangent plane to $\mathcal{C}$ is given by either $y_1 = 0$ or $y_2 = 0$, see Fig.2. In both cases one gets families of singularities with the bifurcation diagrams coincides with that of the singularity B_k for some $k \geq 2$.

c) in the most complicated case $y_3 = 0$ any map is equivalent to one of the following $\Theta_k : (x_2, x_1, x_2^k)$ which has a versal deformation $\Phi_k(x_1, x_2, \lambda_0, \dots, \lambda_k) = \{y_1 = x_2, y_2 =$

$x_1, y_3 = \lambda_0 x_1 + x_2^k + \lambda_k x_2^{k-1} + \cdots + \lambda_1\}$ and its bifurcation diagram consists of three irreducible components, namely, i) $\lambda_1 = 0$, ii) $p(x_2) = x_2^k + \lambda_k x_2^{k-1} + \cdots + \lambda_1$ has a multiple zero and iii) $p(\lambda_0^2) = 0$. These components correspond to nontransversality of a smooth hypersurface to the vertex, the handle and to the smooth part of the Whitney umbrella resp.

Recently D. Mond has informed the author that the same problem in the case when the smooth germ and the germ of Whitney umbrella meet transversally was studied by him in [Mo].

We consider the real case which is more complicated than the complex one. All the results hold in the complex case as well if one drops the signs in theorem I. Some preliminary results in this direction were obtained by the author in 1989. The original motivation to consider the special case of the cone came from the hyperbolic systems with variable coefficients, compare [A2]. Sincere thanks are due to V. I. Arnold for his constant support and encouragement and to V. V. Goryunov for the assistance and explanation of the papers [Da1-2]. The considered problem is closely related to the more complicated results by V. V. Goryunov on simple projections, see [G1-2] although (to the best of the author's knowledge) the above results do not follow from [G1-2]. The author is very obliged to the referee for the constructive criticism which enabled to improve substantially the quality of the original version. The author is very greatful to IHES for the hospitality in January 94 which allowed him not only to accomplish the present article but also to enjoy the beauty of Paris.

§1. Regular case

1.1. Lemma, see [Da2]. Let $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, \mathcal{S})$ be a germ of a map then:

a) a $\mathcal{A}_S$-versal unfolding $\Phi(x, \lambda)$ of f is equivalent to $f + \lambda_i e_i$ where $e_i \in \mathcal{O}_l^n$ are representatives of any basis of the quotient module

$$\mathcal{O}_l^n / \{\mathcal{O}_l(\partial f/\partial x_1, \ldots, \partial f/\partial x_l) + f^*\mathcal{O}_n(v_1 \circ f, \ldots, v_r \circ f)\}. \tag{1}$$

Here
v_i, $(i = 1, \ldots r)$ is a basis of the module of vector fields tangent to $\mathcal{S}$, i.e. preserving the ideal of $\mathcal{S}$;
$v_i \circ f$ is the restriction of the vector field v_i to the image $f(\mathbf{R}^l)$.

The denominator in the r.h.s. presents the so-called extended tangent space $T_e\mathcal{A}_S(f)$ to the action of the group of $\mathcal{A}_S$-unfoldings on f calculated at the 0 values of parameters, see [Da2].

b) analogously, a $\mathcal{K}_S$-versal unfolding $\widetilde{\Phi}(x, \lambda)$ of f is equivalent to $f + \lambda_i e_i$ where $e_i \in \mathcal{O}_l^n$ are representatives of any basis of the quotient module

$$\mathcal{O}_l^n / \{\mathcal{O}_l(\partial f/\partial x_1, \ldots, \partial f/\partial x_l, v_1 \circ f, \ldots, v_r \circ f)\}, \tag{2}$$

Analogously, the denominator in the r.h.s. presents the extended tangent space $T_e\mathcal{K}_S(f)$ to the action of the group of $\mathcal{K}_S$-unfoldings on f calculated at the 0 values of parameters.

c) a reduced $\mathcal{A}_S$-versal unfolding $\Phi_{red}(x, \lambda)$ of the map f is equivalent to $f + \lambda_i e_i$ where $e_i \in \mathcal{M}_l^n$ are representatives of any basis of the quotient module

$$\mathfrak{M}_l^n / \{\mathfrak{m}_l(\partial f/\partial x_1, \ldots, \partial f/\partial x_l) + f^*\mathcal{O}_n(\tilde{v}_1 \circ f, \ldots, \tilde{v}_p \circ f)\}, \tag{3}$$

where $\tilde{v}_1, \ldots, \tilde{v}_p$ is a basis of the module of vector fields tangent to $\mathcal{S}$ and preserving the origin in the target;

The denominator in the r.h.s. presents the usual tangent space $T\mathcal{A}_S(f)$ to the $\mathcal{A}_S$-action on f.

d) analogously, a reduced $\mathcal{K}_S$-versal unfolding $\tilde{\Phi}_{red}(x,\lambda)$ is equivalent to $f+\lambda_i e_i$ where $e_i \in \mathcal{M}_l^n$ are representatives of any basis of the quotient module

$$\mathfrak{M}_l^n/\{\mathfrak{m}_l(\partial f/\partial x_1,\dots,\partial f/\partial x_l)+\mathcal{O}_l(\tilde{v}_1\circ f,\dots,\tilde{v}_p\circ f)\}. \tag{4}$$

The denominator in the r.h.s. presents the usual tangent space $T\mathcal{K}_S(f)$ to the $\mathcal{K}_S$-action on f.

The proof of this statement is standard and analogous to the proofs of the corresponding statements about the $\mathcal{R}$-, $\mathcal{RL}$- and $\mathcal{K}$-equivalences, see [AVG], p.122.

1.2. Proof of Proposition A. We start with the following reformulation. Let $f:(\mathbf{R}^l,0)\to(\mathbf{R}^n,\mathbf{R}^{n-k})$ be a germ of a regular map. (Recall that in the regular case any stratified $\mathcal{S}$ can be substituted by $\mathbf{R}^{n-k}$.) Denote by Φ_f a complement to the extended tangent space $T_e\mathcal{A}_{\mathbf{R}^{n-k}}(f)$; denote by $\tilde{\Phi}_f$ a complement to the extended tangent space $T_e\mathcal{K}_{\mathbf{R}^{n-k}}(f)$ and, finally, denote by $\tilde{\Phi}_{\pi\circ f}$ a complement to the extended tangent space $T_e\mathcal{K}(\pi\circ f)$, where π is the projection of $\mathbf{R}^n$ along $\mathbf{R}^{n-k}$ on $\mathbf{R}^k$ and the composition $\pi\circ f$ maps $(\mathbf{R}^l,0)$ to $(\mathbf{R}^k,0)$. These complements Φ_f, $\tilde{\Phi}_f$ and $\tilde{\Phi}_{\pi\circ f}$ are versal unfoldings of the germs f and $\pi\circ f$ resp. relative to the corresponding group of diffeomorphisms $\mathcal{A}_{\mathbf{R}^{n-k}}$, $\mathcal{K}_{\mathbf{R}^{n-k}}$ and $\mathcal{K}$, see [Da2].

Proposition A'. Versal unfoldings Φ_f, $\tilde{\Phi}_f$ and $\tilde{\Phi}_{\pi\circ f}$ are equivalent, i.e. each of them can be induced from any other, see [AVG], p.147 and above.

Proof. The equivalence of Φ_f and $\tilde{\Phi}_f$ follows directly from the formulas (1) – (2) and the fact that $f^*\mathcal{O}_n$ is isomorphic to $\mathcal{O}_l$ since f is an embedding. Let us show the equivalence of $\tilde{\Phi}_f$ and $\tilde{\Phi}_{\pi\circ f}$, i.e. that they can be induced from each other. Note that any map from $\tilde{\Phi}_f$ is defined by the inverse image of $\mathbf{R}^{n-k}$ in the space $\mathbf{R}^l$ w.r.t. the group of all diffeomorphisms of $\mathbf{R}^l$ (under the assumption that the inverse image of $\mathbf{R}^{n-k}$ in $\mathbf{R}^l$ has positive codimension). Analogously, any map from $\tilde{\Phi}_{\pi\circ f}$ is defined by the inverse image of the origin in $\mathbf{R}^l$ w.r.t. the same group. Thus we must show that for any map from $\tilde{\Phi}_f$ there exists a map from $\tilde{\Phi}_{\pi\circ f}$ with the diffeomorphic inverse image of the origin and, conversely, for any map from $\tilde{\Phi}_{\pi\circ f}$ there exists a map from $\tilde{\Phi}_f$ with the diffeomorphic inverse image of $\mathbf{R}^{n-k}$. The last statement is obvious in one direction. And conversely, let us for any $h\in\tilde{\Phi}_{\pi\circ f}$ construct a map g the inverse image of $\mathbf{R}^{n-k}$ of which coincides with the inverse image of the origin for h. Let $f_1,\dots,f_{n-k}$ be coordinate functions of the map f which are 'forgotten' by the projection π. Then, obviously, one can take the map $g=\{f_1,\dots,f_{n-k};h\}$. □

Remark. As was pointed out by the referee a more general 'invariance of $\mathcal{K}_V$-equivalence under suspension' was proved in [Da3].

Corollary. Versal unfoldings Φ_f and $\tilde{\Phi}_f$ of a regular germ $f:(\mathbf{R}^l,0)\to(\mathbf{R}^n,\mathcal{C})$ are equivalent to a $\mathcal{K}$-versal unfolding of the function Q_f, where Q is the quadratic form defining the cone $\mathcal{C}$ and Q_f is the pullback of Q in the space $\mathbf{R}^l$.

Proof. Apply the previous proposition in the neigborhhood of a smooth (by the definition of regularity) point on the cone.

Remark. If f is the germ of a smooth curve with the order k of tangency to a germ of a smooth hypersurface then its $\mathcal{K}_S$-versal unfolding of the is equivalent to a $\mathcal{K}$-versal unfolding of the map $x^k:(\mathbf{R},0)\to(\mathbf{R},0)$. (In this case $\mathcal{K}$-versal and $\mathcal{R}$-versal unfoldings are equivalent.)

1.3. Proof of Corollary B. Corollary B follows directly from the above corollary and the following count of parameters.

If f is a regular germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, C)$ occuring in a generic k-parameter family then the corank cr of the quadratic part of Q_f satisfies the inequality $cr(cr-1)/2 \le k$ and thus Q_f is stably equivalent to a function on at most $\xi(k) = [\frac{1+\sqrt{(1+8k)}}{2}]$ variables. Indeed the codimension of the stratum of all quadratic forms on $\mathbf{R}^l$ of corank cr equals to $\frac{cr(cr-1)}{2}$ and by weak transversality a generic $(k-1)$-parameter family of quadratic parts of the zero levels of functions can intersect such a stratum only if $k > \frac{cr(cr-1)}{2}$. The parametrized Morse lemma provides that such a germ Q_f is stably equivalent to a germ of function on cr variables. The assumption $cr < \xi(k) = \frac{1+\sqrt{1+8k}}{2}$ completes the proof.

§2. Semiregular singularities

2.1. Proof of Proposition C.

The tangent space $T\mathcal{A}_\mathcal{C}$ of a semiregular germ f lies in the extended tangent space $T_e\mathcal{A}_\mathcal{C}(f)$. Moreover, the latter is obtained from the former by adding the l-dimensional vector space of parallel shifts of coordinates in the source since all vector fields preserving $\mathcal{C}$ vanish at the vertex of the cone and the linear part of f is nondegenerate, compare denominators in the formulas (1)-(3). Thus the $\mathcal{A}_\mathcal{C}$-versal unfolding of f differs from its reduced $\mathcal{A}_\mathcal{C}$-versal unfolding only by some subspace of parallel shifts of the target. The deformation induced by the l-dimensional space of those shifts of the target which preserve the tangent space to $f(\mathbf{R}^l)$ at the origin is cancelled by the l-dimensional space of the parallel shifts of the source. Therefore, the $(n-l)$-dimensional quotient space belongs to the versal unfolding. □

2.2. Proof of proposition D. We start with the following simple statement.

Consider the group $O_{m,n}$ of all linear transformations preserving some nondegenerate quadratic form Q in $\mathbf{R}^n$ the index of which (i.e. the number of negative squares) is equal to m. The group $O_{m,n}$ acts in the obvious way on the Grassmanian $G_{l,n}$.

2.2.1. Lemma. An orbit of $O_{m,n}$-action on the Grassmanian $G_{l,n}$ consists of all l-dimensional subspaces L with a given corank and index of the restriction of $Q \restriction L$. Moreover, any such subspace $L \subset \mathbf{R}^n$ of dimension l with the corank cr and the index α there always exists a basis $(y_1, \ldots, y_n)$ such that $Q = \sum_{i=1}^{cr} y_{2i-1}y_{2i} - \sum_{j=2cr+1}^{m+cr} y_j^2 + \sum_{k=m+cr+1}^{n} y_k^2$ where the subspace L is spanned by the coordinates $y_1, y_3, \ldots, y_{2cr-1}, y_{2cr+1}, y_{2cr+2}, \ldots, y_{2cr+\alpha}$ and $y_{m+cr+1}, \ldots, y_{m+l-\alpha}$.

Proof. Use the reduction of the degenerate quadratic form Q_L to its normal form. □

2.2.2. Theorem. Let $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, 0)$ (0 is the vertex of $\mathcal{C}$) be a semiregular map such that the corank of the quadratic part of Q_f equals cr. Then the map f can be reduced to the form $(x_1, \phi_1(x_1, \ldots, x_l), x_2, \phi_2(x_1, \ldots, x_l), \ldots, x_{cr}, \phi_{cr}(x_1 \ldots, x_l), x_{cr+1}, \ldots, x_l, 0, \ldots, 0)$, where $\phi_1, \ldots, \phi_{cr} \in \mathfrak{m}_l^2$ in an appropriate system of coordinates $(x_1, \ldots, x_l)$ and $(y_1, \ldots, y_n)$ in which the cone $\mathcal{C}$ is given by $Q = y_1y_2 + \cdots + y_{2cr-1}y_{2cr} + \sum_{j=2cr+1}^{n} \pm y_j^2 = 0$. Moreover, a reduced versal unfolding of f can be chosen as $(x_1, \Phi_1(x_1, \ldots, x_l, \bar\lambda), x_2, \Phi_2(x_1, \ldots x_l, \bar\lambda), \ldots, x_{cr}, \Phi_{cr}(x_1, \ldots, x_l, \bar\lambda), x_{cr+1}, \ldots, x_l, 0, \ldots, 0)$, where $\Phi_1, \ldots, \Phi_{cr} \in \mathfrak{m}$ for all values of parameters $\bar\lambda$.

Proof of theorem 2.2.2 is based on Mather's homotopy method in its usual form, i.e. if $f_t,\ t \in [0,1]$ is a family of maps such that $\frac{\partial f_t}{\partial t} \in T\mathcal{A}_\mathcal{C}(f_t)$ for all $t \in [0,1]$ then f_0 is $\mathcal{A}_\mathcal{C}$-equivalent to f_1. In order to present the tangent spaces $T\mathcal{A}_\mathcal{C}(f_t)$ more explicitly, using lemma 1.1 we need the following proposition.

PROPOSITION. A basis of the module of vector fields tangent to a homogeneous surface H with an isolated singularity at the origin can be chosen in the following way, see [L]. One of the generators is the standard linear field $(y_1\frac{\partial}{\partial y_1},\dots,y_n\frac{\partial}{\partial y_n})$ while the rest are all (2×2)-determinants of the $(2\times n)$-matrix $\begin{pmatrix} \frac{\partial}{\partial y_1}, & \dots, & \frac{\partial}{\partial y_n} \\ P_{y_1}, & \dots, & P_{y_n} \end{pmatrix}$, where P is the polynomial defining the hypersurface and $P_{y_i}=\frac{\partial P}{\partial y_i}$.

REMARK. Applying this statement to $\mathcal{C}$ given by $Q=y_1y_2+\dots+y_{2cr-1}y_{2cr}+\sum_{2cr+1}^n \pm y_j^2$ one gets the basis of vector fields tangent to $\mathcal{C}$ consisting of the Euler vector field: $\sum_{i=1}^n y_i\partial/\partial y_i$ and 3 groups of generators:
a) $y_{\tilde{j}}\partial/\partial y_i - y_{\tilde{i}}\partial/\partial y_j,\ i<j\le 2cr$ where $\tilde{i}=i-1$ if i is even and $\tilde{i}=i+1$ if i is odd (the same rule applies to $\tilde{j}$ and j);
b) $\pm 2y_j\partial/\partial y_i - y_{\tilde{i}}\partial/\partial y_j;\ i\le 2cr, 2cr<j\le n$;
c) $\pm y_j\partial/\partial y_i \mp y_i\partial/\partial y_j,\ 2cr<i<j\le n$.

Proof. Applying lemma 2.2.1. to the image of the nondegenerate linear part of f one reduces it by a linear transformation to $(x_1,0,x_2,0,\dots,x_{cr},0,x_{cr+1},x_{cr+2},\dots,x_l,0,\dots,0)$ and, therefore, reduces f itself to $(x_1,\phi_1,x_2,\phi_2,\dots,x_{cr},\phi_{cr},x_{cr+1},x_{cr+2},\dots,x_l,\phi_{cr+1},\dots,\phi_{n-l})$, where all $\phi_j\in\mathfrak{m}_l^2$. Let us get rid of $\phi_{cr+1},\dots,\phi_{n-l}$ by a $\mathcal{C}$-preserving diffeomorphism. First we remove all terms in $\phi_{cr+1},\dots,\phi_{n-l})$ divisible by any x_j where $j\le cr$. Namely, for any $cr+1\le k\le n-l$ and $j\le cr$ there exists a vector field of the form $(0,0,\dots,\pm\phi_k,0,\dots,0,x_j,0,\dots,0)$ with $\pm\phi_k$ standing at the position $2j$ and x_j standing at the position $cr+k$. According to [AVG] using the homotopy method with this vector field we can remove all terms in ϕ_k divisible by x_j. In such a way we reduce f to the form $(x_1,\phi_1,x_2,\phi_2,\dots,x_{cr},\phi_{cr},x_{cr+1},x_{cr+2},\dots,x_l,$ $\tilde{\phi}_{cr+1},\dots,\tilde{\phi}_{n-l})$ where all $\tilde{\phi}_k$ depend only on $x_{cr+1},\dots,x_l$. Here we use Mather's homotopy method again. Denote by $f_t,\ t\in[0,1]$ the family $(x_1,\phi_1,x_2,\phi_2,\dots,x_{cr},\phi_{cr},x_{cr+1},x_{cr+2},\dots,x_l,$ $t\tilde{\phi}_{cr+1},\dots,t\tilde{\phi}_{n-l})$. Then $\frac{\partial f_t}{\partial t}=(0,\dots,0,\tilde{\phi}_{cr+1},\dots,\tilde{\phi}_{n-l})$. Denote by M_{l-cr} the $\mathcal{O}_{l-cr}$-submodule consisting of all maps of the form $(0,\dots,0,\zeta_{cr+1},\dots,\zeta_{n-l}),\ \zeta_i\in\mathfrak{m}_{l-cr}^2(x_{cr+1},\dots,x_l)$. Let us show using the above basis that M_{l-cr} belongs to $T\mathcal{A}_{\mathcal{C}}(f_t)$ for all t and thus f_0 is $\mathcal{A}_{\mathcal{C}}$-equivalent to f_1. By lemma 1.1 the tangent space to $T\mathcal{A}_{\mathcal{C}}(f_t)$ is the $\mathcal{O}_l$-module $\{\mathfrak{m}_l(\partial f_t/\partial x_1,\dots,\partial f_t/\partial x_l)+\mathcal{O}_l(\tilde{v}_1\circ f_t,\dots,\tilde{v}_p\circ f_t)\}$ where $\tilde{v}_i$ belongs to the above basis. As above among the vector fields of the basis there exist all vector fields of the form $(0,0,\dots,\tilde{\phi}_k,$ $0,\dots,0,x_j,0,\dots,0)$, $cr+1\le k\le n-l$ and $cr+1\le j\le l$ where $\tilde{\phi}_k$ stands at the position $cr+j$ and x_j stands at the position $cr+k$. This means that M_{l-cr} is contained in the tangent space to all f_t and the homotopy method gives the necessary reduction. □

Using lemma 2.2.2 one gets Proposition D directly. Indeed, the reduced versal unfolding of a semiregular f consists of the semiregular maps and, therefore, their induced functions belong to $\mathfrak{m}_l^2$. Taking the induced functions one maps the whole orbit $\mathcal{A}_{\mathcal{C}}f$ onto the whole orbit $\mathcal{K}Q_f$. Indeed, one can cover the whole $\mathcal{R}Q_f$-orbit by changing coordinates in the source and one can multiply the induced function Q_f by an arbitrary nonvanishing function by using the diffeomorphisms of the target which preserve the cone $\mathcal{C}$. Since the tangent space $T\mathcal{K}Q_f$ to the orbit $\mathcal{K}Q_f$ at Q_f is identified with $\mathfrak{m}_l\{\frac{\partial Q_f}{\partial x_1},\dots,\frac{\partial Q_f}{\partial x_n},Q\}$. We now show that any unfolding $Q_\lambda=Q_f+\epsilon(x_1,\dots,x_l,\lambda_1,\dots,\lambda_k)$ in the class $\mathfrak{m}_l^2$ can be covered by an appropriate origin-preserving unfolding $\Psi_\lambda=f(x_1,\dots,x_l)+\psi(x_1,\dots,x_l,\lambda_1,\dots,\lambda_k)$. By lemma 2.2.2 one has Q_λ can be reduced to $Q_\lambda=\sum_{i=1}^{cr}x_i\phi_i(x_1,\dots,x_l,\lambda_1,\dots,\lambda_k)+\sum_{j=cr+1}^{l}\pm x_j^2,\ \phi_i\in\mathfrak{m}_l$ for all values of parameters $\lambda_1,\dots,\lambda_k$. The difference $Q_\lambda-Q_f=\epsilon(x_1,\dots,x_l,\lambda_1,\dots,\lambda_k)$ can be expanded as $\sum_{1\le i\le j\le l}x_ix_j\epsilon_{i,j}(x_1,\dots,x_l,\lambda_1,\dots,\lambda_k)$. To obtain $\epsilon(x_1,\dots,x_l,\lambda_1,\dots,\lambda_k)$

we first deform the original map $f = (x_1, \phi_1, \dots, x_{cr}, \phi_{cr}, x_{cr+1}, \dots, x_l, 0 \dots, 0)$ by adding $\sum \epsilon_{i,j} x_j$ to ϕ_i for all $i \le cr$. It is left to compensate $\tilde{\epsilon} = \sum_{cr+1 \le i \le j \le l} x_i x_j \epsilon_{i,j}(x_1, \dots, x_l, \lambda_1, \dots, \lambda_k)$ by deforming $(x_{cr+1}, \dots, x_l, 0, \dots, 0)$. Since $\tilde{\epsilon}$ is a small deformation of a nondegenerate quadratic form $\sum_{cr+1}^{l} \pm x_j^2$ then the parametrized Morse lemma provides the existence of the necessary deformation. □

Proof of Corollary E. It follows immediately from the fact that taking induced functions one maps the orbit $\mathcal{A}_{\mathcal{C}} f$ submersively onto $\mathcal{K} Q_f$, proposition D and the fact that the $\mathcal{K}$-modality of any function ϕ with a singularity at the origin equals the $\mathcal{K}$-modality of ϕ in the space $\mathfrak{m}_l^2$. The last statement is proved along the same lines as the corresponding statement for the $\mathcal{R}$-equivalence, see [Ga].

Proof of Remark F. By the proposition C and lemma 2.2.1 the family of parallel shifts can be written in the form $(x_1, \phi_1 + \lambda_1, x_2, \phi_2 + \lambda_2, \dots, x_{cr}, \phi_{cr} + \lambda_{cr}, x_{cr+1}, \dots, x_l, \lambda_{cr+1}, \dots, \lambda_{n-l})$, where *cr* is the corank of Q_f. This directly gives the necessary answer.

2.3. Proof of theorem G. We start with the easiest case $f : (\mathbf{R}, 0) \to (\mathbf{R}^n, C)$, where f is the germ of a smooth curve passing through the origin tangent to some ruling of the cone $\mathcal{C} : \{Q = y_1 y_2 \pm y_3^2 \pm \cdots \pm y_n^2 = 0\}$;

By lemma 2.2.2 it suffices to consider the case of plane curves images of which lie in the plane spanned by y_1 and y_2. Let us denote by $f_k(x)$ the parameterized curve $(y_1 = x, y_2 = x^k, y_i = 0,\ i \ge 3)$ and show that any other germ of a smooth curve passing through the origin is diffemorphic to one of those. The extended tangent space $T_e \mathcal{A}_C(f_k)$ of f_k is presented as

$$T_e\mathcal{A}_C(f_k) = \mathcal{O}_x \begin{pmatrix} 1 \\ kx^{k-1} \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \mathcal{O}_x \left\{ \begin{pmatrix} x \\ x^k \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ -x^k \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -x^k \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -x \\ 0 \\ \vdots \end{pmatrix}, \dots \right\}$$

for all $3 \le i \le n$.

It is convenient to choose the reduced versal unfolding $\Phi_{red}(x, \lambda_1, \dots, \lambda_{k-1}) : \{y_1 = x, y_2 = x^k + \lambda_1 x^{k-1} + \cdots + \lambda_{k-1} x, y_3 = 0, \dots, y_n = 0\}$ using (3)-(4). The inverse image of the cone in the extended space of parameters $(x, \lambda_1, \dots, \lambda_{k-1})$ has the form $x^{k+1} + \lambda_1 x^k + \cdots + \lambda_{k-1} x^2 = 0$. The bifurcation diagram in the space of parameters $(\lambda_1, \dots, \lambda_{k-1})$ is the hypersurface of the singularities of projection or, in other words, the set of all $(k-1)$-tuples $(\lambda_1, \dots, \lambda_{k-1})$ for which $x^{k-1} + \lambda_1 x^k + \cdots + \lambda_{k-1}$ as a polynomial in the variable x has a multiple or zero root. Since for any h consisting of terms of degree $> k$ the tangent space $T\mathcal{A}_C(f_k + h)$ contains all monomials of degree $\ge k$ it follows that the $\mathcal{A}_C$-orbit of f_k contains all $f_k + h$ (probably after multiplying Q by -1) according to [Da2]. Therefore, these cases do not require separate consideration.

Let us now mention the necessary changes to adjust the above proof to the case of a semiregular map $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, \mathcal{C})$ such that the quadratic part of the restriction Q_f of $Q = y_1 y_2 \pm y_3^2 \pm \cdots \pm y_n^2$ defining $\mathcal{C}$ has a one-dimensional kernel. One can assume without loss of generality that the 1-jet of f is $(x_1, 0, x_2, \dots, x_l, 0, \dots, 0)$. The quadratic part of Q_f has the form $\pm x_2^2 \pm \cdots \pm x_l^2$ by lemma 2.2.1. After analogous considerations of the 1-jets of the vector fields one gets that the only family of germs to be considered is $(x_1, x_1^k, x_2, \dots, x_l, 0, \dots, 0)$. Now the consideration of the corresponding vector fields gives that its reduced versal unfolding can be chosen in the form $(x_1, x_1^k + \lambda_1 x_1^{k-1} + \dots + \lambda_{k-1}x_1, x_2, \dots, x_l, 0, \dots, 0)$. The restriction of the cone $\mathcal{C}$ has the form $x_1(x_1^k + \lambda_1 x_1^{k-1} + \dots + \lambda_{k-1}x_1) \pm x_2^2 \pm \dots \pm x_l^2 = 0$, i.e. is stably equivalent to the same restriction as in the previous case.

Proof of Corollary H. The $\mathcal{A}_\mathcal{C}$-versal unfolding of the map $(x_1, x_1^k, x_2, \dots, x_l, 0, \dots, 0)$ can be chosen as $(x_1, x_1^k + \lambda_1 x_1^{k-1} + \dots + \lambda_k, x_2, \dots, x_l, \lambda_{k+1}, \dots, \lambda_{n+k-l-1})$ by proposition C. The rest follows immediately. On Fig.3 one can see the monomials included in this versal unfolding of f_k; the arrow shows competing monomials, i.e. either of them but not both must be included in the versal unfolding. When $l = n-1$ then, rather obviously, the above versal unfolding and its bifurcation diagram are equivalent to that of the singularity B_k. Let us show that when $l = n-2$ the bifurcation diagram coincides with the bifurcation diagram for the singularity D_{k+1}. This follows from the form of the standard versal unfolding of D_{k+1}, see [AVG]. Namely, taking the standard versal unfolding of D_{k+1} as $\Phi(x_1, x_2, \lambda_1, \dots, \lambda_{k+1} = x_1^k + x_1x_2^2 + \lambda_1 x_1^{k-1} + \dots + \lambda_k + \lambda_{k+1}x$ one gets that its bifurcation diagram is defined by the system

$$\begin{cases} \Phi = 0 \\ \frac{\partial \Phi}{\partial x_1} = x_2^2 + kx_1^{k-1} + (k-1)\lambda_1 x_1^{k-2} + \dots + \lambda_{k-1} = 0 \\ \frac{\partial \Phi}{\partial x_2} = 2x_1x_2 + 2\lambda_{k+1} = 0. \end{cases}$$

Using the expression $x_1 = \frac{-\lambda_{k+1}}{x_2}$ from the last equation one gets that the first two equation are equivalent to the condition that the polynomial $x_1^{k+1} + \lambda_1 x_1^k + \dots + \lambda_k x_1 - \lambda_{k+1}^2$ has a multiple zero. Unfortunately the direct relation between the above semiregular singularity and D_{k+1} is still unclear.

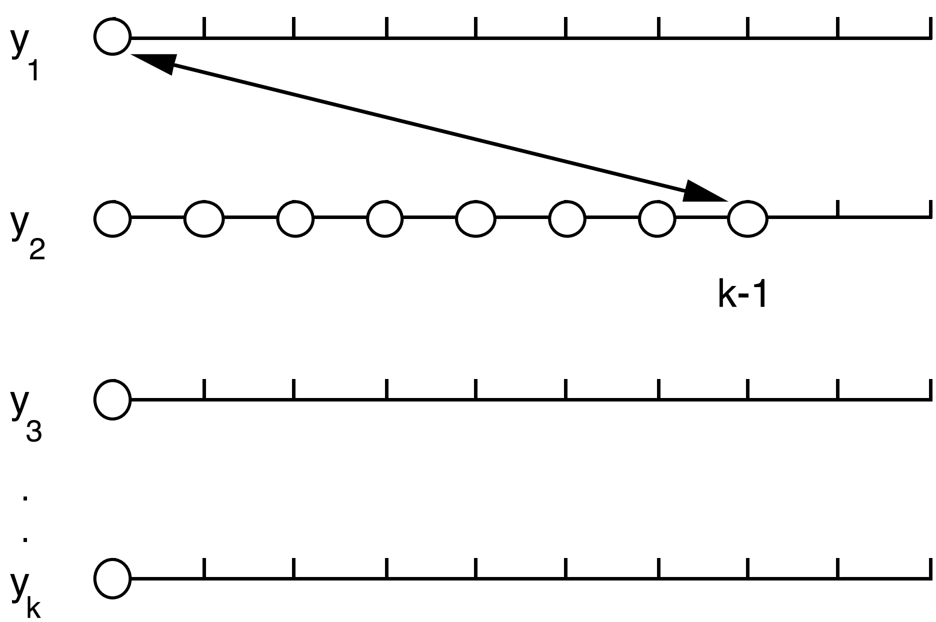

Fig.3. Circled monomials are included in the versal unfolding of f_k

2.4. Proof of theorem I.

Using corollary E one gets that the corank of the induced function Q_f of a simple semiregular f is at most 2 since all simple singularities of functions have the corank ≤ 2. The case of corank 1 is covered by theorem G. In order to classify the simple singularities of corank 2 it suffices to consider only the case $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^4, \mathcal{C})$ where $\mathcal{C}$ given by $Q = y_1y_2 + y_3y_4$ and the 1-jet of f equal to $(x_1, 0, x_2, 0)$ by lemmas 2.2.1 and 2.2.2. Any semiregular map satisfying the above conditions will be called *an adjusted* $2 \to 4$*-map.* Consideration of simple adjusted $2 \to 4$-maps splits into a series a lemmas. Since we are only interested in the cone $\mathcal{C}$ we allow to multiply its defining form Q by -1 while finding the normal forms of f.

2.4.1. Lemma. Any adjusted $2 \to 4$-map can be reduced to one of the following two forms:
 i) $(x_1, x_1^k, x_2,\ x_2g(x_1, x_2))$, $k \geq 2$ and $g \in \mathfrak{m}_2$;
 ii) $(x_1, 0, x_2, h(x_1, x_2))$, $h \in \mathfrak{m}_2^2$.

Proof. Obviously, any adjusted $2 \to 4$-map f reduces to $(x_1, \phi_1(x_1, x_2), x_2, \phi_2(x_1, x_2))$ where $\phi_1, \phi_2 \in \mathfrak{m}$. The vector field $V = (0, x_2, 0, -x_1)$ is tangent to the $\mathcal{A}_\mathcal{C}$-orbit of any such f, see 2.3. Therefore one can remove either of all the terms of ϕ_1 which are divisible by x_2 or of all the terms of ϕ_2 divisible by x_1. Namely, we apply the homotopy method in the form: if f_t satisfies $\frac{\partial f_t}{\partial t} \in T\mathcal{A}_C(f_t)$ for all $0 \leq t \leq 1$ then f_0 is $\mathcal{A}_C$-equivalent to f_1. Since $v \in T\mathcal{A}_C(f_t)$ all $f_t = f + t(0, hx_2, 0, -hx_1)$ then f is equivalent to $f + (x_1, \phi_1 + hx_2, x_2, \phi_2 - hx_1)$. Thus, if either ϕ_1 itself is divisible by x_2 or ϕ_2 is divisible by x_1 one can remove of the corresponding function completely and obtain the case ii) up to renumbering of components. Let $\alpha x_1^{k_1}$ be the smallest power of x_1 in $\phi_1(x_1, x_2)$ and $\beta x_1^{l_1}$ be the smallest power of x_1 in $\phi_2(x_1, x_2)$. Let us show that using the homotopy method and renumbering of components f can be reduced to $(x_1, x_1^k, x_2, \tilde{\phi}_2(x_1, x_2))$, where $k = \min(k_1, l_1)$. After some straightforward transformations of the basis of the tangent vector fields one gets only two fields which preserve the first and the third components and affect powers of x_1 of the second component, namely, $(0, \phi_1, 0, \phi_2)$ and $(0, \phi_2, 0, x_1\frac{\partial \phi_2}{\partial x_2})$. Using them we can remove all powers of x_1 of degree greater than $\min(k_1, l_1)$ and multiplying the first component by a constant and dividing the second component by the same constant we get that the only power of x_1 included in ϕ_1 is x_1^k (possibly after multiplication of Q by -1). Finally, we remove of all powers of x_2 in ϕ_1 as described before. Now we arrive at the form $(x_1, x_1^k, x_2, \tilde{\phi}_2(x_1, x_2))$, where $\tilde{\phi}_2(x_1, x_2)$ is not divisible by x_1. Let us assume that αx_1^l, $l \geq k$ is the smallest power of x_1 in $\tilde{\phi}_2(x_1, x_2)$. In this situation considering the basis of tangent vector fields one gets the vector field $(0, 0, 0, (k+1)x_1^k + x_2\frac{\partial \tilde{\phi}_2}{\partial x_1})$. We again apply the homotopy method using the last vector field and remove all powers of x_1 in $\tilde{\phi}_2$ and get that the fourth component is divisible by x_2. The statement is proved.

2.4.2. Proposition. In the case i) of lemma 2.4.1 an adjusted $2 \to 4$-map is simple if and only if one of the following possibilities holds.

If $k \geq 4$ then in order to be simple g must have a nondegenerate linear part $\alpha x_1 + \beta x_2$, $\alpha^2 + \beta^2 \neq 0$. If $\alpha \neq 0$ then g reduces either to $g(x_1, x_2) = \pm x_1$ which gives

1) $Q_f \cong D_{k+2}^{\pm}, k \geq 3,\ f = (x_1, x_1^k, x_2, \pm x_1x_2)$. If k is even then the $\pm$ forms coincide and one can drop signs of the last component.

If $\alpha = 0$ then g reduces to $g(x_1, x_2) = \pm x_2$ which after renumeration of components coincides with the case $k = 2$ below.

If $k = 2$ and g has a nonvanishing linear part then g reduces to $g(x_1, x_2) = \pm x_1$ which leads to

2) $Q_f \cong D_4^{\pm},\ f = (x_1, x_1^2, x_2, \pm x_1x_2)$.

If $k=2$ and g is a simple singularity with vanishing linear part and nonvanishing quadratic part $\alpha x_2^2+\beta x_1x_2+\gamma x_1^2,\ \alpha^2+\beta^2+\gamma^2\neq 0$ then one of the following options is possible.

If $\alpha\neq 0$ then g reduces to $\pm x_2^2$ which gives

3) $Q_f\cong E_6^{\pm},\ f=(x_1,x_1^2,x_2,\pm x_2^3)$.

If $\alpha=0,\ \beta\neq 0$ then g reduces to x_1x_2 which gives

4) $Q_f\cong E_7,\ f=(x_1,x_1^2,x_2,x_1x_2^2)$.

If $\alpha=\beta=0$ then using the homotopy method with the vector field V from lemma 2.4.1 one can get rid of the whole quadratic part of g and obtain either the case 5) of this lemma or the case ii) of lemma 2.4.1.

If $k=2$ and $g\in\mathfrak{m}_2^3$ then in order to be simple g must have a nontrivial cubic part containing αx_2^3 , $\alpha\neq 0$ and in this case it reduces to $g=x_2^3$ which gives

5) $Q_f\cong E_8,\ f=(x_1,x_1^2,x_2,x_2^4)$.

Proof. Case 1). If $k\geq 3$ then the induced function $Q_f=x_1^{k+1}+x_2^2g,\ g\in\mathfrak{m}_2$. Since any simple germ of function has a nontrivial cubic form, (see e.g. [A3]) then g has a nonvanishing linear part $\alpha x_1+\beta x_2,\ \alpha^2+\beta^2\neq 0$. Let us present $g=\alpha x_1+\beta x_2+g_2$, where $g_2\in\mathfrak{m}_2^2$. Working with the basis one gets 4 vector fields with the first three components vanishing and the following nontrivial quadratic parts of the last component, $(0,0,0,k\alpha x_1x_2+(k+1)\beta x_2^2+\dots)$; $(0,0,0,\alpha x_2^2+\dots)$; $(0,0,0,2\alpha x_1x_2+3\beta x_2^2+\dots)$ and $(0,0,0,2\alpha x_1^2+3\beta x_1x_2+\dots)$, where $\dots$ denotes all terms of order at least 3. If $\alpha\neq 0$ then using the second field we can remove of βx_2^2 in the fourth component of f, i.e. of βx_2 in g. Finally, by multiplying the third and the fourth components of f by an appropriate constant and its inverse we force $\alpha=\pm 1$. If k is even then changing the sign of x_1 one can force the last component to be x_1x_2. If $\alpha=0$ then we can normalize the coefficient β by making $\beta=\pm 1$. In this case one gets $f=(x_1,x_1^k,x_2,\pm x_2^2+\dots)$ which coincides with the case 2) below up to renumbering of components.

Case 2) is similar to the case 1).

Case 3). If $k=2$ one gets $Q_f=x_1^3+x_2^2g$. According to the classification of simple germs of functions we conclude that g has at least a nontrivial 3-jet. Let us first assume that its 2–jet is nonvanishing, i.e. $g=\alpha x_2^2+\beta x_1x_2+\gamma x_1^2+g_3,\ \alpha^2+\beta^2+\gamma^2\neq 0$. Working with the basis of vector fields one gets the following 4 fields affecting only the fourth component of f: $(0,0,0,\gamma x_1^2x_2+\beta x_1x_2^2+\alpha x_2^3+\dots)$; $(0,0,0,3x_1^2+2\gamma x_1x_2^2+\beta x_2^3+\dots)$; $(0,0,0,2\gamma x_1^2x_2+3\beta x_1x_2^2+4\alpha x_2^3+\dots)$ and $(0,0,0,2\gamma x_1^3+3\beta x_1^2x_2+4\alpha x_1x_2^2+\dots)$. We denote them by $v_1,\dots,v_4$ resp. Since v_2 has the term $3x_1^2$ one can always get rid of the term γx_1^2 in g. Assuming that $\gamma=0$ one gets $v_4-\beta v_2=4\alpha x_1x_2^2+\dots$, where $\dots$ denotes all terms of order at least 4. If $\alpha\neq 0$ one can get rid of the term βx_1x_2 in g and moreover normalize it to $g=\pm x_2^2+g_3$.

Case 4). If $\alpha=0$ and $\beta\neq 0$ then g reduces to $g=\beta x_1x_2+g_3$. Multiplying x_2 by an appropriate constant one gets $g=x_1x_2+g_3$.

Case 5). Let us assume that $f=(x_1,x_1^2,x_2,x_2(P_3+g_4))$, where P_3 denotes the cubic part. One concludes that if Q_f is a simple germ of function with the 3-jet equal to x_1^3 then its 5-jet must contain $x_1^3+\alpha x_2^5+\dots,\ \alpha\neq 0$ and therefore $P_3=\alpha x_2^3+\dots$, see [A3], p.13. Let us check that in this case P_3 reduces to x_2^3. At first we force $\alpha=1$ by the usual trick. Let $P_3=x_2^3+\beta x_1x_2^2+\gamma x_1^2x_2+\delta x_2^3$. The appropriate basis of vector fields is $(0,0,0,3x_2^4+2\beta x_1x_2^3+\gamma x_1^2x_2^2+\dots)$; $(0,0,0,3x_1^2+\beta x_2^4+2\gamma x_1x_2^3+3\delta x_1^2x_2^2+\dots)$; $(0,0,0,5x_1x_2^3+4\beta x_1^2x_2^2+3\gamma x_1^3x_2+2\delta x_1^4+\dots)$.

Using v_2 one removes $\gamma x_1x_2^2$ and δx_1^3 in g. Taking $\gamma=\delta=0$ one gets $v_4-\frac{4}{3}x_2^2v_2=5x_1x_2^3+\dots$ and removes the term $\beta x_1x_2^2$ in g. Thus, $g=x_2^3+g_4$.

In order to finish the proof of simplicity it suffices to check that all the jets of f presented in the formulation of lemma 2.4.2 are sufficient and describe their adjacency. Sufficiency will be discussed in the separate statement 2.4.5 and adjacency is postponed until 2.4.6. □

2.4.3. Lemma. In the case ii) of lemma 2.4.1 an adjusted $2 \to 4$-map is simple if and only if the function h has a nontrivial quadratic part $h = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 + h_3,\ \alpha^2 + \beta^2 + \gamma^2 \neq 0$ and one of the following possibilities holds.

If $\alpha \neq 0$ then h reduces either to $x_1^2 \pm x_2^2$ which gives

a) $Q_f \cong D_4^{\pm},\ f = (x_1, 0, x_2, x_1^2 \pm x_2^2)$

or h reduces to x_1^2 which gives

b) $Q_f \cong D_{k+2}^{\pm},\ f = (x_1, 0, x_2, x_1^2 \pm x_2^k),\ k \geq 3$. Here the $\pm$-forms are different only if k is even.

If $\alpha = 0$ then h reduces to $x_1 x_2$ which gives

c) $Q_f \cong D_{2k}^{\pm}, k \geq 3,\ f = (x_1, 0, x_2, x_1 x_2 + x_1^k)$.

If $\alpha = \beta = 0$ then h reduces to x_2^2 which gives

d) $Q_f \cong E_7,\ f = (x_1, 0, x_2, x_2^2 + x_1^3)$.

Proof. Considering the basis of the module of tangent vector fields one gets the following 4-tuple of vector fields affecting only the fourth component $(0,0,0,\beta x_1 x_2 + 2\gamma x_2^2 + \dots)$; $(0,0,0,2\alpha x_1^2 + \beta x_1 x_2 + \dots)$; $(0,0,0,2\alpha x_1 x_2 + \beta x_2^2 + \dots)$ and $(0,0,0,\alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 + \dots)$. After some obvious linear transformations one gets $(0,0,0,\dots)$; $(0,0,0,2\alpha x_1^2 + \beta x_1 x_2 + \dots)$; $(0,0,0,2\alpha x_1 x_2 + \beta x_2^2 + \dots)$ and $(0,0,0,(2\gamma - \frac{\beta}{2\alpha})x_2^2 + \dots)$ denoted by $v_1, \dots, v_4$ resp. Case a). If $\alpha \neq 0$ then one normalizes it by making $\alpha = 1$ (possibly after multiplying Q by -1). Then using v_3 one cancels $\beta x_1 x_2$ in h. We arrive at $h = x_1^2 + \gamma x_2^2 + h_3$. If $\gamma \neq 0$ then using the last field we can normalize it by making $\gamma = \pm 1$. Thus, $h = x_1^2 \pm x_2^2 + h_3$. In 2.4.5. we will discuss the sufficiency of the presented 2-jet. Case b). If $\gamma = 0$ then $h = x_1^2 + h_3$. Using the fields v_2 and v_3 one can remove of all terms in h_3 divisible by $x_1 x_2$. Therefore $h = x_1^2 + \delta x_2^{k-1} + \dots$. Analogous arguments show that we can remove of all the terms denoted by $\dots$ and obtain $h = x_1^2 \pm x_2^{k-1}$. Case c). If $\alpha = 0$ and $\beta \neq 0$ then using v_3 one removes γx_2^2 in h and normalizes $\beta = 1$. Now using v_2 and v_3 one removes all the terms in h_3 divisible by $x_1 x_2$ and by x_2^2. Thus, $h = x_1 x_2 + \alpha x_1^k \dots$. The same arguments show that h normalizes to $h = x_1 x_2 + x_1^k$. Case d). If $\alpha = \beta = 0$ then $h = \gamma x_2^2 + h_3$. The constant γ normalizes to 1 and the arguments analogous to the case 4 of lemma 2.4.2 shows that if f is simple then h_3 contains γx_1^3 and reduces to $h_3 = x_1^3 + h_4$.

2.4.4. Different normal forms.

Lemmata 2.4.1-2.4.3 give us the unique normal forms for the singularities A_k, E_6 and E_8 and provide the following list of normal forms for $D_{k+1}^{\pm}$ and E_7.

$Q_f \cong D_{k+1}^{\pm}$; ($\pm$ forms are different only if k is odd)

a) the normal form: $f = (x_1, x_1^{k-1}, x_2, \pm x_1 x_2)$, its reduced versal unfolding is

$$\Phi_{red}(x_1, x_2, \lambda_1, \dots, \lambda_k) = \{x_1, x_1^{k-1} + \lambda_1 x_1^{k-2} + \dots + \lambda_{k-2} x_1, x_2, \pm x_1 x_2 + \lambda_{k-1} x_1 + \lambda_k x_2\};$$

b) the second normal form: $f = (x_1, 0, x_2, x_1^2 \pm x_2^{k-1})$, its reduced versal unfolding is

$$\tilde{\Phi}_{red}(x_1, x_2, \lambda_1, \dots, \lambda_k) = \{x_1, \lambda_1 x_1, x_2, x_1^2 \pm x_2^{k-1} + \lambda_2 x_1 + \lambda_3 x_2 + \dots + \lambda_k x_2^{k-2}\};$$

c) if $k \geq 3$ then there exists the third normal form: $D_{2k}^{+} : f = (x_1, 0, x_2, x_1 x_2 + x_1^k)$, its reduced versal unfolding is

$$\tilde{\Phi}_{red}(x_1, x_2, \lambda_1, \dots, \lambda_{2k-1}) = (x_1, \lambda_1 x_1 + \lambda_2 x_1^2 + \dots + \lambda_{k-1} x_1^{k-1}, x_2,$$
$$x_1 x_2 + x_1^k + \lambda_k x_2 + \lambda_{k+1} x_1 + \lambda_{k+2} x_1^2 + \dots + \lambda_{2k-1} x_1^{k-1}).$$

$Q_f \cong E_7$;

a) the normal form: $f = (x_1, x_1^2, x_2, x_1x_2^2)$, its reduced versal unfolding is

$$\Phi_{red}(x_1, x_2, \lambda_1, \dots, \lambda_6) = \{x_1, x_1^2 + \lambda_1 x_1, x_2, x_1x_2^2 + \lambda_2 x_1^2 + \lambda_3 x_2^2 + \lambda_4 x_1 x_2 + \lambda_5 x_1 + \lambda_6 x_2\};$$

b) the second normal form: $f = (x_1, 0, x_2, x_1^3 + x_2^2)$, its reduced versal unfolding is

$$\tilde{\Phi}_{red}(x_1, x_2, \lambda_1, \dots, \lambda_6) = \{x_1, \lambda_1 x_1 + \lambda_2 x_1^2, x_2, x_2^2 + x_1^3 + \lambda_3 x_1 + \lambda_4 x_1^2 + \lambda_5 x_2 + \lambda_6 x_1 x_2\};$$

PROPOSITION. In each of the above subcases all the normal forms present 1 orbit.

Proof. In order to show that the first and the second normal forms belong to the same orbit in all the subcases one should notice the following. Any change of coordinates of the form $\tilde{y}_1 = y_1, \tilde{y}_2 = y_2 + y_3 h, \tilde{y}_3 = y_3, \tilde{y}_4 = y_4 - y_1 h$, where h is an arbitrary smooth function preserves the cone $\mathcal{C} : \{Q = y_1y_2 + y_3y_4 = 0\}$. For an adjusted $2 \to 4$-map f that means that if one adds $x_2 h$ to the second coordinate and simultaneously subtracts $x_1 h$ from the fourth coordinate then one obtains another map belonging to the orbit of f. This explains why the first and the second normal forms belong to the same orbit. It is left to show that the third normal form belongs to the same orbit as the second normal form of D_{2k}^+, i.e. why $(x_1, 0, x_2, x_1^2 - x_2^{2k-2}) \cong (x_1, 0, x_2, x_1x_2 + x_1^k)$. The second normal form is equivalent to $(x_1, x_2^2 + x_2 x_1^{k-1}, x_2, 0)$ or after renumbering of coordinates to $(x_1, 0, x_2, x_1^2 + x_1 x_2^{k-1})$. Making the coordinate change $\tilde{x}_1 = x_1 - x_2^{k-1}$ one transforms the last map into $(x_1 + x_2^{k-1}, 0, x_2, x_1 + 2x_1 x_2^{k-1})$. Multiplying x_2 by a constant one gets $(x_1 + \alpha x_2^{k-1}, 0, x_2, x_1 + x_1 x_2^{k-1})$. Finally, any change of coordinates of the form $\tilde{y}_1 = y_1 + h y_3, \tilde{y}_2 = y_2, \tilde{y}_3 = y_3, \tilde{y}_4 = y_4 - h y_2$ preserves the cone. This transformation applied to $(x_1 + \alpha x_2^{k-1}, 0, x_2, x_1 + x_1 x_2^{k-1})$ with $h = -\alpha x_2^{k-2}$ gives $(x_1, 0, x_2, x_1 + x_1 x_2^{k-1})$. □

2.4.5. CRITERION OF SUFFICIENCY OF A GIVEN JET.

Recall that the k-jet is called *sufficient* under the action of a chosen group if any perturbation of f by any terms of degree greater than k belongs to the orbit of f. Let us denote by $\mathcal{M}_l^n(j)$ the $\mathcal{O}_l$-module of all germs $(\mathbf{R}^l, 0) \to (\mathbf{R}^n, 0)$ all components of which have degree $\geq j$. By general results of [Da2] one can apply Mazer's homotopy method to check stability of a given jet. Thus sufficiency of the k-jet of a germ f (under the action of $\mathcal{A}_\mathcal{C}$) is equivalent to the fact that the $\mathcal{A}_\mathcal{C}$-tangent space to any $f + \phi$, $\phi \in \mathcal{M}_l^n(j)$ contains the whole module $\mathcal{M}_l^n(j)$. The last condition for semiregular germs f is equivalent to the following statement.

CRITERION OF SUFFICIENCY. The k-jet ofa semiregular germ $f : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, 0)$, (0 is the vertex of the cone $\mathcal{C}$) is $\mathcal{A}_\mathcal{C}$-sufficient if and only if the $\mathcal{O}_l$-module $\{\mathfrak{m}_l^2(\partial f/\partial x_1, \dots, \partial f/\partial x_l) + \mathfrak{m}_l(\tilde{v}_1 \circ f, \dots, \tilde{v}_p \circ f)\}$ contains the submodule $\mathcal{M}_l^n(k+1)$, see notation in lemma 1.1.

PROOF. Let us sketch the proof of sufficiency of the above condition (the necessity is almost obvious). Indeed, let us assume that $\{\mathfrak{m}_l^2(\partial f/\partial x_1, \dots, \partial f/\partial x_l) + \mathfrak{m}_l(\tilde{v}_1 \circ f, \dots, \tilde{v}_p \circ f)\}$ contains the submodule $\mathcal{M}_l^n(k+1)$. Then since the vector fields $v_1, \dots, v_p$ are linear the same is true for all $f + \phi$, $\phi \in \mathcal{M}_l^n(k+1)$. Therefore for all ϕ the tangent space $T\mathcal{A}_\mathcal{C}(f + \phi)$ contains $\mathcal{M}_l^n(k+1)$. Thus, for the family $f_t = f + t\phi$, $t \in [0, 1]$ the velocity vector $\frac{\partial f_t}{\partial t}$ belongs to $T\mathcal{A}_\mathcal{C}(f_t)$ and therefore f is $\mathcal{A}_\mathcal{C}$-equivalent to $f + \phi$. □

Using this criterion one can check that all the normal forms in Theorem I and section 2.4.4 are sufficient. Let us illustrate this in the most complicated case $Q_f \cong D_{2k}^+ : (x_1, 0, x_2, x_1x_2 + x_1^k)$, $k \geq 3$, see 2.4.4. We must show that the k-jet is sufficient. The corresponding vector fields are $(1, 0, 0, x_2 + kx_1^{k-1})$; $(0, 0, 1, x_1)$; $(x_1, 0, x_2, x_1x_2 + x_1^k)$; $(x_1, 0, 0, 0)$; $(x_1x_2 + x_1^k, 0, 0, 0)$;

$(x_2,0,0,0)$; $(0,x_2+kx_1^{k-1},-x_1,0)$; $(0,x_2,0,-x_1)$ and $(0,0,-x_2,x_1x_2+x_1^k)$. After some simplifications one gets $(1,0,0,x_2+kx_1^{k-1})$; $(0,0,1,x_1)$; $(0,0,0,x_1^k)$; $(0,0,0,x_1x_2)$; $(0,x_1^k,0,2x_1^2)$; $(0,x_2,0,-x_1)$ and $(0,0,0,x_2^2)$. We must represent any $(k+1)$-jet, $k\geq 3$ by $\mathfrak{m}_2^2\{(1,0,0,x_2+kx_1^{k-1});(0,0,1,x_1)\}+\mathfrak{m}_2\{(0,0,0,x_1^k);\ (0,0,0,x_1x_2);\ (0,x_1^k,0,2x_1^2);\ (0,x_2,0,-x_1);\ (0,0,0,x_2^2)\}$. Any perturbation of degree $k+1$ of the first and the third components is cancelled by $\mathfrak{m}_2^2\{(1,0,0,x_2+kx_1^{k-1});(0,0,1,x_1)\}$. Any perturbation of degree $k+1$ of the second coordinate is removed by $\mathfrak{m}_2\{(0,0,0,x_1^k);\ (0,0,0,x_1x_2)\}$. Finally, any perturbation of degree $k+1$ of the fourth coordinate is contained in $\mathfrak{m}_2^2\{(0,x_1^k,0,2x_1^2);\ (0,x_2,0,-x_1);\ (0,0,0,x_2^2)\}$. □

2.4.6. Adjacency of simple semiregular germs.

Proposition. The adjacency of the simple semiregular germs coincides with the adjacency of the corresponding induced singularities (and, therefore, is presented on Fig.1).

Proof. The statement follows from the analysis of the reduced versal unfoldings given in Theorem I together with lemmas 2.4.2-2.4.3. For $Q_f\cong A_k$ the statement is obvious. For $Q_f\cong D_{k+1}^{\pm}$ we will analyze the reduced versal unfolding of the first normal form. Nonvanishing λ_{k-1} or λ_k give nontrivial quadratic part of Q_f and thus lead to $A_l,\ l\leq k$. Nonvanishing $\lambda_1,\dots,\lambda_{k-2}$ lead to $D_l^{\pm},\ l\leq k$. In the case 3) $Q_f\cong E_6^{\pm}$ nonvanishing λ_1,λ_4 or λ_5 lead to A_l. Nonvanishing λ_2 or λ_3 give D_4 or D_5 according to lemma 2.4.2. Analogous arguments for E_7 and E_8 finish the proof. Recall that reduced versal unfoldings of the simple semiregular singularities are equivalent as versal unfoldings to $Q_f+\sum\lambda_ie_i$, where $e_i\in\mathfrak{m}_l^2$ are representatives of a basis of $\mathfrak{m}_l^2/\mathfrak{m}_l(\frac{\partial Q_f}{\partial x_1},\dots,\frac{\partial Q_f}{\partial x_l})$ and thus reduced versal unfoldings are different from the usual versal unfoldings of the induced function Q_f. In particular, the bifucation diagram of the former contains two irreducible components while the bifurcation diagram of the latter is irreducible. □

2.5. Proposition B$'$.

2.5.1. Lemma. Fixing a nondegenerate quadratic form Q in $\mathbf{R}^n$ with the number of negative squares equal to m let us consider the stratification of Grassmanian $G_{l,n}$ into strata S_{cr}, where S_{cr} consists of all l-planes such that the corank of the restriction of Q to any of these planes equals $cr\leq\min(l,n,m-n)$. Then codim $S_{cr}=cr(cr-1)/2$.

This lemma is proved by the same argument as the analogous statement for the quadratic forms.

Proof of B'. Any semiregular $f:(\mathbf{R}^l,0)\to(\mathbf{R}^n,C)$ occuring in a family of semiregular maps with k parameters has the corank cr of the quadratic part of Q_f satisfying $cr\leq\xi(k)=[\frac{1+\sqrt{1+8k}}{2}]$ by lemma 2.5.1. Therefore, any such f can be reduced to $(x_1,\phi_1,x_2,\phi_2,\dots,x_{\xi(k)},\phi_{\xi(k)},x_{\xi(k)+1},\dots,x_l,0,\dots,0)$ and its reduced $\mathcal{A}_C$-unfolding deforms only $\phi_1,\dots,\phi_{\xi(k)}$. Thus all such semiregular singularities are eqivalent to semiregular singularities on at most $\xi(k)$ variables. If the number of variables is restricted then the stabilization is obvious.

§3. Irregular singularities

Let us start proof of theorem J with the following statement, see [Da3].

3.1. Lemma. Let $f:\mathbf{R}^l\to\mathbf{R}^n$ be a germ of embedding and $\phi:\mathbf{R}^k\to\mathbf{R}^n$ be a germs of a stable map (see the definition of the stability of germs in [AVG], page 115). Let $\Phi_{im(f)}(\phi)$ and $\Phi_{im(\phi)}(f)$ be an $\mathcal{A}_{im(f)}$-versal unfolding of ϕ and an $\mathcal{A}_{im(\phi)}$-versal unfolding of f resp. Then $\Phi_{im(f)}(\phi)$ is equivalent to $\Phi_{im(\phi)}(f)$ as versal unfoldings.

Remark. Apparently the same statement holds if both f and g are stable germs.

3.2. Proof of theorem J.

Proof. Now let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, C)$ be a germ of Whitney umbrella tangent to the cone $\mathcal{C}$ at some point different from the origin. In this case one can obviously substitute $\mathcal{C}$ by a germ of a smooth hypersurface. Thus we can normalize a germ of smooth hypersurface in the presence of the standard Whitney umbrella $\{y_3^2 = y_1 y_2^2\}$ using the lemma 3.1. The natural basis of vector fields tangent to the standard Whitney umbrella is $v_1 = (0, y_2, y_3)$, $v_2 = (2y_1, 0, y_3)$, $v_3 = (2y_3, 0, y_1^2)$, $v_4 = (0, y_3, y_1 y_2)$. At first we enumerate all cases of nontransversality of the tangent plane to a smooth germ w.r.t. the standard Whitney umbrella, i.e. all orbits of positive codimension of the action of 1-jets of vector fields preserving the Whitney umbrella on the 1-jets of germs of smooth hypersurfaces.

One has to consider the following 3 types of 1-jets of f: a) $(\alpha x_1 + \beta x_2, x_2, x_1)$, b) $(x_2, \alpha x_1, x_1)$ and c) $(x_2, x_1, 0)$.

Case a). The 1-jets of vector fields have the form $(\alpha, 0, 1)$; $(\beta, 1, 0)$; $(0, x_2, x_1)$; $(2\alpha x_1 + 2\beta x_2, 0, x_1)$; $(x_1, 0, 0)$; $(0, x_1, 0)$, or after reduction $(\alpha, 0, 1)$; $(\beta, 1, 0)$; $(\alpha x_1 + \beta x_2, 0, 0)$; $(\beta x_2, 0, 0)$; $(x_1, 0, 0)$; $(\beta x_1, 0, 0)$.

It splits into 2 subcases:

a') The typical case of codimension 0 when $\beta \neq 0$; in this case the tangent plane can be reduced to the form $y_1 = y_2 = x_2, y_3 = x_1$.

a'') The special case of codimension 1 when $\beta = 0$; in this case the tangent hyperplane is $y_1 = 0$ and the 1-jet of f is reduced to the form $y_1 = 0, y_2 = x_2, y_3 = x_1$.

Case b). For the 1-jet $(x_2, \alpha x_1, x_1)$ one gets (after some obvious simplifications) the following 1-jets of vector fields $(0, \alpha, 1)$; $(1, 0, 0)$; $(0, 0, x_1)$; $(0, x_1, 0)$.

Therefore the initial jet of the map f can be always reduced to the form $(x_2, 0, x_1)$.

Let us now consider the subcases of positive codimension. The subcase a') is generic. In the subcase a'') the 1-jet of f is $(0, x_2, x_1)$ and the 1-jets of vector fields have the form $(0, 1, 0)$; $(0, 0, 1)$; $(x_1, 0, 0)$.

This means that one can restrict consideration to the germs of maps of the form $(g(x_2), x_2, x_1)$ according to results of [AVG] p.180.

Let us consider the series of maps $F_k : (x_2^k, x_2, x_1)$. The corresponding basis of vector fields has the form $(0, 0, 1)$; $(kx_2^{k-1}, 1, 0)$; $(0, x_2, x_1)$; $(2x_2^k, 0, x_1)$; $(2x_1, 0, 0)$; $(0, x_1, 0)$. The versal unfolding of F_k is

$$\Phi_k(x_1, x_2, \lambda_1, \dots, \lambda_k) : \{y_1 = x_2^k + \lambda_1 x_2^{k-1} + \dots + \lambda_1, y_2 = x_2, y_3 = x_1\}.$$

The inverse image of the Whitney umbrella is given by the formula $x_1^2 - x_2^2(x_2^k + \lambda_k x_2^{k-1} + \dots + \lambda_1)^2 = x_1^2 - x_2^2 p(x_2) = 0$.

For a generic set of λ the inverse image of Whitney umbrella has a point of transversal selfintersection at the origin and is smooth at other points. Violation of genericity occurs when $p(x_2)$ has the zero or multiple root. Thus one gets the bifurcation diagram of the singularity B_k.

Case b). The 1-jet of the map f is $(x_2, 0 x_1)$. The corresponding 1-jets of the vector fields are $(1, 0, 0)$; $(0, 0, 1)$; $(0, 0, x_2)$; $(2x_1, 0, x_2)$; $(2x_2, 0, 0)$; $(0, x_2, 0)$ or after reduction $(1, 0, 0)$; $(0, 0, 1)$; $(0, x_2, 0)$.

According to the general technique it suffices to consider the family of germs $\Psi_k : (x_2, x_1^k, x_1)$. The corresponding 1-jets of the vector fields are $(1, kx_1^{k-1}, 0)$; $(0, 0, 1)$; $(0, x_1^k, x_2)$; $(2x_1, 0, x_2)$; $(2x_2, 0, x_1^{2k})$; $(0, x_2, x_1^{k+1})$ or after reduction $(1, kx_1^{k-1}, 0)$; $(0, 0, 1)$; $(0, x_1^k, 0)$; $(0, x_2, 0)$.

The versal unfolding is $\Phi_k(x_1, x_2, \lambda_1, \ldots, \lambda_k) = \{y_1 = x_1, y_2 = x_1^k + \lambda_k x_1^{k-1} + \cdots + \lambda_k, y_3 = x_2\}$. The inverse image of the Whitney umbrella is the curve given by $x_2^2 - x_1(x_1^k + \lambda_k x_1^{k-1} + \cdots + \lambda_1)^2 = x_2^2 - x_1 p^2(x_1) = 0$.

This curve has transversal selfintersections which lie on the x_1-axis which correspond to the simple zeros of the polynomial p and is tangent to the x_2-axis at the origin. Its degeneracies are caused either by a multiple root of p or if p vanishes at the origin. Thus the bifurcation diagram is the same as for the singularity B_k.

Case c). One has to consider the series of maps $\Theta_k : (x_2, x_1, x_2^k)$ and the corresponding initial jet of the vector fields are $(0,1,0)$; $(1,0,kx_2^{k-1})$; $(0,x_1,x_2^k)$; $(2x_2,0,x_2^k)$; $(2x_2^k,0,x_1^2)$; $(0,x_2^k, x_1x_2)$ or after reduction $(0,1,0)$; $(1,0,kx_2^{k-1})$; $(0,0,x_2^k)$; $(0,0,x_1^2)$; $(0,0,x_1x_2)$.

The same argument as above shows that in the case c) considerations can be restricted only to the cases Θ_k.

The versal unfolding of Θ_k is given by the formula $\Phi_k(x_1, x_2, \lambda_0, \ldots, \lambda_k) : \{y_1 = x_2, y_2 = x, y_3 = \lambda_0 x_1 + x_2^k + \lambda_k x_2^{k-1} + \cdots + \lambda_1\}$. The inverse image of the Whitney umbrella is given by $(\lambda_0 x_1 + p(x_2))^2 - x_1^2 x_2 = 0$, where $p(x_2) = x_2^k + \lambda_k x_2^{k-1} + \cdots + \lambda_1$. Let us describe the cases of nongeneric position. The natural stratification of Whitney umbrella consists of its vertex, its handle and its smooth open 2-dimensional part, see Fig.1. If R denotes $\lambda_0 x_1 + p(x_2)$ then the nontransversality to the vertex implies $R = x_1 = x_2 = 0$. Therefore, it gives $\lambda_1 = 0$. The nontransversality to the handle implies $R = x_1 = 0$ and therefore it gives that p has a multiple root. Finally, it is easy to check that the nontransversality to the smooth part is equivalent to $p(\lambda_0^2) = 0$.

§4. Final remarks.

The following questions are quite natural from the point of view of the singularity theory.

1) Extend the theory of vanishing cycles and the technique of Dynkin diagrams to the considered case.

2) Compare modalities of the versal unfolding of f and its induced function Q_f.

3) Develop some method to calculate the dimension of the reduced versal unfolding of a semiregular f (at least in the quasihomogeneous case) and compare it with the Milnor number of Q_f.

4) Study the semiregular singularities in the case when $\mathcal{C}$ is a generic (quasi)homogeneous polynomial of some (multi)degree.

References

[A1] V. I. Arnold, *Critical points of functions on a manifold with boundary, the simple Lie groups B_k,C_k and F_4 and singularities of evolutes*, Russ. Math. Surv. **33** (1978), no. 5, 99–116.

[A2] V. I. Arnold, *On the surfaces defined by hyperbolic equations*, Mat.Zametki (1988), no. 1, 3–18.

[A3] V. I. Arnold, *Normal forms of functions near degenerate critical points, Weyl groups A_k, D_k,E_k and Lagrange singularities*, Funct. Anal. Appl. **6** (1972), no. 4, 3–25.

[AGLV] V. I. Arnold, V. V. Goryunov, O. V. Lyashko, V. A. Vassiliev, *Singularity theory I*, Encyclopedia Math. Sciences, Dynamical Systems VI, Springer, 1993.

[AVG] V. I. Arnold, A. N. Varchenko and S. M. Gusein-Zade, *Singularities of differentiable maps*, Birkhäuser, 1985.

[Da1] J. Damon, *Unfoldings of sections of singularities and Gorenstein surface singularities*, Columbia Univ. (1982), preprint.

[Da2] J. Damon, *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Memoires of the AMS **50** (1984), no. 306.

[Da3] J. Damon, *$\mathcal{A}$-equivalence and the equivalence of sections of images and discriminants*, Warwick proceedings, Lecture Notes in Mathematics 1462 **1** (1989), no. 306, 93–121.

[G1] V. V. Goryunov, *Singularities of projections of complete intersections*, Itogi nauki i tehniki, Sovr. problemy matematiki **22** (1983), 167–206.

[G2] V. V. Goryunov, *Simple projections*, Sibirian Math. Jour. **25** (1984), no. 1, 61–68.

[Ga] A. M. Gabrielov, *Bifurcations, Dynkin diagrams and modality of isolated singularities*, Funct. Anal. Appl. **8** (1974), no. 2, 7–12.

[L] O. V. Lyashko, *Classification of critical points of functions on varities with singular boundary*, Funct. Anal. and Appl. **17** (1983), no. 3, 28–36.

[Ma] J. Mather, *Finitely determined map germs*, Publ. Math. IHES **36** (1968), 127–156.

[Mo] D. Mond, *On the classification of germs from* $\mathbf{R}^2$ *to* $\mathbf{R}^3$, Proc. of London Math. Soc. **50** (1985), 333–369.

[Si] D. Siersma, *Singularities of functions on boundaries, corners etc*, Quart. J. Math, Oxford Ser **32** (1981), no. 125, 119–127.