SKEW-SYMMETRIC VANISHING LATTICES AND INTERSECTIONS OF SCHUBERT CELLS

B. Shapiro‡, M. Shapiro* and A. Vainshtein†[1]

‡ Department of Mathematics, University of Stockholm
S-10691, Sweden, shapiro@matematik.su.se
* Department of Mathematics, Royal Institute of Technology
S-10044, Sweden, mshapiro@math.kth.se
† Dept. of Mathematics and Dept. of Computer Science, University of Haifa
Mount Carmel, 31905 Haifa, Israel alek@mathcs11.haifa.ac.il

§1. Introduction and results

In the present paper we apply the theory of skew-symmetric vanishing lattices developed around 15 years ago by B. Wajnryb, S. Chmutov, and W. Janssen for the necessities of the singularity theory to the enumeration of connected components in the intersection of two open opposite Schubert cells in the space of complete real flags. Let us briefly recall the main topological problem considered in [SSV] and reduced there to a group-theoretical question solved below. Let N^{n+1} be the group of real unipotent uppertriangular $(n+1)\times(n+1)$ matrices and D_i be the determinant of the submatrix formed by the first i rows and the last i columns. Denote by Δ_i the divisor $\{D_i=0\}\subset N^{n+1}$ and let Δ^{n+1} be the union $\cup_{i=1}^n\Delta_i$. Consider now the complement $U^{n+1}=N^{n+1}\setminus\Delta^{n+1}$. The space U^{n+1} can be interpreted as the intersection of two open opposite Schubert cells in $SL_{n+1}(\mathbb{R})/B$.

In [SSV] we have studied the number of connected components in U^{n+1}. The main result of [SSV] can be stated as follows.

Consider the vector space $T^n=T^n(\mathbb{F}_2)$ of upper triangular $n\times n$-matrices over $\mathbb{F}_2$. We define the group $\mathfrak{G}_n$ as the subgroup of $GL(T^n)$ generated by $\mathbb{F}_2$-linear transformations g_{ij}, $1\leqslant i\leqslant j\leqslant n-1$. The generator g_{ij} acts on a matrix $M\in T^n$ as follows. Let M^{ij} denote the 2×2 submatrix of M formed by rows i and $i+1$ and columns j and $j+1$ (or its upper triangle in case $i=j$). Then g_{ij} applied to M changes M^{ij} by adding to each entry of M^{ij} the trace of M^{ij}, and does not change all the other entries of M. For example, if $i<j$, then g_{ij} changes M^{ij} as follows:

$$\begin{pmatrix} m_{ij} & m_{i,j+1} \\ m_{i+1,j} & m_{i+1,j+1}\end{pmatrix}\mapsto\begin{pmatrix} m_{i+1,j+1} & m_{i,j+1}+m_{ij}+m_{i+1,j+1} \\ m_{i+1,j}+m_{ij}+m_{i+1,j+1} & m_{ij}\end{pmatrix}.$$

All the other entries of M are preserved. The above action on T^n is called the *first $\mathfrak{G}_n$-action.*

1991 *Mathematics Subject Classification.* Primary 14M15, Secondary 14N10, 14P25.
[1]The research of this author was supported by the Rashi Foundation.

Proposition (Main Theorem of [SSV]). *The number* $\sharp_{n+1}$ *of connected components in* U^{n+1} *coincides with the number of orbits of the first* $\mathfrak{S}_n$*-action.*

Here we calculate the number of orbits of the first $\mathfrak{S}_n$-action and prove the following result (conjectured in [SSV]).

Main Theorem. *The number* $\sharp_{n+1}$ *of connected components in* U^{n+1} (*or, equivalently, the number of orbits of the first* $\mathfrak{S}_n$*-action*) *equals* 3×2^n *for all* $n \geqslant 5$.

Cases $n+1 = 2, 3, 4$ or 5 are exceptional and $\sharp_2 = 2$, $\sharp_3 = 6$, $\sharp_4 = 20$, $\sharp_5 = 52$.

The structure of the paper is as follows. In §2 we give a detailed description of the first $\mathfrak{S}_n$-action and its quotient by the subspace of invariants (called the second $\mathfrak{S}_n$-action). We formulate explicit results about the number and cardinalities of orbits of both actions. In §3 we find linear invariants of these actions. In §4 we present a number of results about the monodromy group of a skew-symmetric vanishing lattice over $\mathbb{F}_2$. Using these results we count in §5 the number of orbits of the second $\mathfrak{S}_n$-action. Finally, in §6 we explain the relation of the original (first) $\mathfrak{S}_n$-action to the monodromy group of the corresponding vanishing lattice and prove the results stated in §2. The concluding §7 contains some final remarks and speculations about the origin and further applications of the above $\mathfrak{S}_n$-action in Schubert calculus.

We want to thank Andrei Zelevinsky for a number of stimulating discussions and his warm support of the whole project. We are indebted to Torsten Ekedahl for pointing out the importance of the dual representation of the group $\mathfrak{S}_n$, to Anatoly Libgober for the key reference [Ja], and to Yakov Shlafman for writing sophisticated computer programs that enabled us to collect enough numerical data for putting forward the main conjecture. The third author expresses his gratitude to the Department of Mathematics, Royal Institute of Technology in Stockholm, for the hospitality and financial support during the final stage of the preparation of the manuscript.

§2. Two $\mathfrak{S}_n$-actions on uppertriangular matrices over $\mathbb{F}_2$ and their orbits. Main results.

2.1. The first $\mathfrak{S}_n$-action and its invariants. Let G be a group acting on a linear space $\mathcal{V}$ over $\mathbb{F}_2$. We say that $x \in \mathcal{V}$ is an *invariant* of the action if $g(x) = x$ for any $g \in G$, and that $f \in \mathcal{V}^*$ is a *dual invariant* if $(g(x), f) = (x, f)$ for any $x \in \mathcal{V}$, $g \in G$ (here $(\cdot,\cdot)$ is the standard pairing $\mathcal{V} \times \mathcal{V}^* \to \mathbb{F}_2$). Evidently, dual invariants are just the invariants of the dual action of G on $\mathcal{V}^*$.

In what follows we identify $(T^n)^*$ with the space of uppertriangular matrices over $\mathbb{F}_2$ in such a way that for any pair $M \in T^n$, $M' \in (T^n)^*$ one has $(M, M') = \sum_{1 \leqslant i \leqslant j \leqslant n} m_{ij} m'_{ij}$.

Let us define the matrices $R_i, E_i \in (T^n)^*$, $1 \leqslant i \leqslant n$, as follows: $R_i = \sum_{1 \leqslant r \leqslant i \leqslant s \leqslant n} E_{rs}$, and $E_i = \sum_{s-r=i-1} E_{rs}$, where E_{rs} are the standard matrix units.

Theorem 2.1. (i) *The subspace* $\mathcal{I}_n \subset T^n$ *of invariants of the first* $\mathfrak{S}_n$*-action is an* n*-dimensional vector space. It has a basis consisting of the matrices* $E_1, \dots, E_n$.

(ii) *The subspace* $\mathcal{D}_n \subset (T^n)^*$ *of dual invariants of the first* $\mathfrak{S}_n$*-action is an* n*-dimensional vector space. It has a basis consisting of the matrices* $R_1, \dots, R_n$.

Let $\mathcal{D}_n^\perp \subset T^n$ be the subspace orthogonal to $\mathcal{D}_n$ with respect to the standard pairing. The translation of $\mathcal{D}_n^\perp$ by an arbitrary element $M \in T^n$ we call a *slice* of the first $\mathfrak{G}_n$-action. By the definition, all the dual invariants are fixed at all the elements of a slice. An n-dimensional vector $h^S = (h_1^S, \dots, h_n^S)$ is said to be the *height* of a slice S (with respect to the basis $\{R_i\}$) if $(M, R_i) = h_i^S$ for any $M \in S$. We denote by S^h the slice at height h. A slice is called *symmetric* if its height is symmetric, that is, if $h_i^S = h_{n-i+1}^S$ for all $1 \leqslant i \leqslant n$.

Evidently, each slice is a union of certain orbits of the first $\mathfrak{G}_n$-action. The structure of slices is described by Theorem 2.2 below. For the sake of simplicity, we omit the words "of the first $\mathfrak{G}_n$-action" in the formulation and write just "orbit" and "slice".

Theorem 2.2. (i) *Let $n = 2k+1 \geqslant 5$, then*

each of 2^{k+1} symmetric slices consists of one orbit of size $2^{2k^2+k-1} - \varepsilon_k 2^{k^2+k-1}$, one orbit of size $2^{2k^2+k-1} + \varepsilon_k 2^{k^2+k-1} - 2^k$, and 2^k orbits of size 1, where $\varepsilon_k = -1$ for $k = 4t+1$ and $\varepsilon_k = 1$ otherwise;

each of $2^n - 2^{k+1}$ nonsymmetric slices consists of two orbits of size 2^{2k^2+k-1}.

(ii) *Let $n = 2k \geqslant 6$, and let $\bar{h}$ denote the vector of length n the first k entries of which are equal to $(1, 0, 1, 0, \dots)$ and the last k entries vanish, then*

each of 2^k symmetric slices consists of two orbits of size $(2^{2k(k-1)} - 1)2^{k-1}$ and 2^k orbits of size 1;

each of 2^k nonsymmetric slices S such that $h^S - \bar{h}$ is symmetric consists of one orbit of size $(2^{k(k-1)} - 1)2^{k^2-1}$ and one orbit of size $(2^{k(k-1)} + 1)2^{k^2-1}$;

each of $2^n - 2^{k+1}$ remaining nonsymmetric slices consists of two orbits of size 2^{2k^2-k-1}.

Observe that the total number of orbits in both cases equals 3×2^n, and we thus get Main Theorem.

2.2. The second $\mathfrak{G}_n$-action and its invariants. Let us introduce a $\mathfrak{G}_n$-action on T^{n-1} closely related to the first $\mathfrak{G}_n$-action on T^n, which we will call the *second $\mathfrak{G}_n$-action.* This action is induced by taking the quotient modulo the subspace $\mathcal{I}_n$ of the invariants of the first $\mathfrak{G}_n$-action.

Proposition 2.3. *The quotient space $T^n/\mathcal{I}_n$ is isomorphic to T^{n-1}.*

Proof. Indeed, consider the linear map $\Psi_n \colon T^n \to T^{n-1}$ that takes $M \in T^n$ to $N \in T^{n-1}$ with $n_{ij} = m_{ij} + m_{i+1,j+1}$ for $1 \leqslant i \leqslant j \leqslant n-1$. By Theorem 2.1(i), $\ker \Psi_n = \mathcal{I}_n$, and we are done. $\square$

We thus obtain the induced action of $\mathfrak{G}_n$ on T^{n-1}. It is easy to see that for any $M \in T^{n-1}$ the generator g_{ij} affects only the 3×3 submatrix of M centered at (i, j), namely:

$$\begin{pmatrix} m_{i-1,j-1} & m_{i-1,j} & m_{i-1,j+1} \\ m_{i,j-1} & m_{ij} & m_{i,j+1} \\ m_{i+1,j-1} & m_{i+1,j} & m_{i+1,j+1} \end{pmatrix} \mapsto \begin{pmatrix} m_{i-1,j-1} + m_{ij} & m_{i-1,j} + m_{ij} & m_{i-1,j+1} \\ m_{i,j-1} + m_{ij} & m_{ij} & m_{i,j+1} + m_{ij} \\ m_{i+1,j-1} & m_{i+1,j} + m_{ij} & m_{i+1,j+1} + m_{ij} \end{pmatrix}.$$

Here we use the convention that if the above 3×3-submatrix does not fit completely in the upper triangle, then we change only the entries that fit.

Let $0 \leqslant l \leqslant k = [n/2]$. We put $j_l = \min\{l, n-2l-2\}$, and for any j, $0 \leqslant j \leqslant j_l$, define $X_l^j = \sum\{E_{rs}\colon j < r < n-l-j,\ l+j < s < n-j,\ j-1 < s-r < n-l-j-1\}$. Finally, for $1 \leqslant i \leqslant k$ we define

$$P_i = \sum_{0 \leqslant j \leqslant j_i} X_i^j + \sum_{0 \leqslant j \leqslant j_{i-1}} X_{i-1}^j.$$

Theorem 2.4. (i) *The subspace of invariants of the second $\mathfrak{G}_n$-action is trivial.*

(ii) *The subspace $\mathfrak{D}_{n-1} \subset (T^{n-1})^*$ of dual invariants of the second $\mathfrak{G}_n$-action is a k-dimensional vector space, $k = [n/2]$. It has a basis consisting of the matrices $P_1, ..., P_k$.*

The slices of the second $\mathfrak{G}_n$-action and their heights (with respect to the basis $\{P_i\}$), which we denote η, are defined in the same way as for the first $\mathfrak{G}_n$-action. It is easy to see that Ψ_n takes any slice of the first $\mathfrak{G}_n$-action to a slice of the second $\mathfrak{G}_n$-action. Denote by $\psi_n\colon \mathbb{F}_2^n \to \mathbb{F}_2^k$ the mapping that takes h^S to $\eta^{\Psi_n(S)}$. The following statement gives an explicit description of ψ_n.

Theorem 2.5. *The heights of the slices S and $\Psi_n(S)$ with respect to the bases $\{R_i\}$ and $\{P_i\}$ satisfy the relations*

$$\eta_i^{\Psi_n(S)} = h_i^S + h_{i+1}^S + h_{n-i}^S + h_{n-i+1}^S \quad \textit{for} \quad 1 \leqslant i < k,$$
$$\eta_k^{\Psi_n(S)} = h_k^S + h_{n-k+1}^S.$$

The structure of the slices of the second $\mathfrak{G}_n$-action is described in Theorem 2.6 below. For the sake of simplicity, we omit the words "of the second $\mathfrak{G}_n$-action" in the formulation and write just "orbit" and "slice".

Theorem 2.6. (i) *Let $n = 2k+1 \geqslant 5$, then*

the slice S^0 consists of one orbit of size $2^{2k^2-1} - \varepsilon_k 2^{k^2-1}$, one orbit of size $2^{2k^2-1} + \varepsilon_k 2^{k^2-1} - 1$, and one orbit of size 1, where $\varepsilon_k = -1$ for $k = 4t+1$ and $\varepsilon_k = 1$ otherwise;

each of the other $2^k - 1$ slices is an orbit of size 2^{2k^2}.

(ii) *Let $n = 2k \geqslant 6$, and let $\bar{\eta}$ denote the vector of length k the first $k-1$ entries of which are equal to 1 and the last entry equals $k \mod 2$, then*

the slice S^0 consists of one orbit of size $2^{2k(k-1)} - 1$ and one orbit of size 1;

the slice $S^{\bar{\eta}}$ consists of one orbit of size $(2^{k(k-1)} - 1)2^{k(k-1)-1}$ and one orbit of size $(2^{k(k-1)} + 1)2^{k(k-1)-1}$;

each of the other $2^k - 2$ slices is an orbit of size $2^{2k(k-1)}$.

Observe that the total number of orbits in both cases equals $2^k + 2$.

§3. Invariants and dual invariants of the $\mathfrak{G}_n$-actions

In this section we prove Theorems 2.1, 2.4 and 2.5.

3.1. The first $\mathfrak{G}_n$-action. Recall that we have identified $(T^n)^*$ with the space of $\mathbb{F}_2$-valued uppertriangular matrices in such a way that for any pair $M \in T^n$, $M' \in (T^n)^*$ one has $(M, M') = \sum_{1 \leqslant i \leqslant j \leqslant n} m_{ij} m'_{ij}$.

Lemma 3.1. *The dual to the first $\mathfrak{S}_n$-action is given by*

$$(g_{ij}M')_{kl} = \begin{cases} m'_{i+1,j} + m'_{i,j+1} + m'_{i+1,j+1} & \text{if } k = i,\ l = j, \\ m'_{i+1,j} + m'_{i,j+1} + m'_{i,j} & \text{if } k = i+1,\ l = j+1, \\ m'_{k,l} & \text{otherwise.} \end{cases}$$

Proof. Indeed, the action of g_{ij} on T^n) affects only M^{ij}, while the action of g_{ij} on $(T^n)^*$ defined above affects only $(M')^{ij}$. Therefore, it suffices to consider the actions in the corresponding 4-dimensional spaces. The matrix of g_{ij} in coordinates m_{ij}, $m_{i,j+1}$, $m_{i+1,j}$, $m_{i+1,j+1}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

hence its dual is exactly as asserted by the lemma. $\square$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. (i) Follows immediately from the fact that $\mathcal{I}_n$ is defined by the equations $m_{ij} + m_{i+1,j+1} = 0$, $1 \leqslant i \leqslant j \leqslant n-1$.

(ii) By Lemma 3.1, $\mathcal{D}_n$ is defined by the equations $m'_{ij} + m'_{i,j+1} + m'_{i+1,j} + m'_{i+1,j+1} = 0$, $1 \leqslant i \leqslant j \leqslant n-1$ (with the same as above convention concerning the case when only a part of the submatrix fits into the uppertriangular shape). One can prove easily that these $\binom{n-1}{2}$ equations are linearly independent. Indeed, each of m'_{1j}, $1 < j < n$, enters exactly two equations, while m'_{11} only one equation. It follows immediately that none of the equations involving m'_{1j} may participate in a nontrivial linear combination. The rest of the equations correspond to the same situation in dimension $n-1$, so their linear independence follows by induction. We thus get $\dim \mathcal{D}_n = n$. It remains to show that $R_1, \ldots, R_n$ provide a basis for $\mathcal{D}_n$. Evidently, all these matrices are linearly independent and satisfy the equations defining $\mathcal{D}_n$. $\square$

3.2. The second $\mathfrak{S}_n$-action. For any matrix entry (i,j) we define its *neighbors* as those of $(i-1,j-1)$, $(i-1,j)$, $(i,j-1)$, $(i,j+1)$, $(i+1,j)$, and $(i+1,j+1)$ that fit in the uppertriangular shape. It is helpful to consider a graph $\mathfrak{H}_{n-1}$ whose vertices are all the matrix entries and edges join each entry with its neighbors. It is easy to see that $\mathfrak{H}_{n-1}$ is an equilateral triangle with the sides of length $n-2$ on the triangular lattice (see Fig. 1). One can give the following description of the second $\mathfrak{S}_n$-action in terms of $\mathfrak{H}_{n-1}$: g_{ij} acts on the space of $\mathbb{F}_2$-valued functions on $\mathfrak{H}_{n-1}$ by adding the value at (i,j) to the values at all of its neighbors.

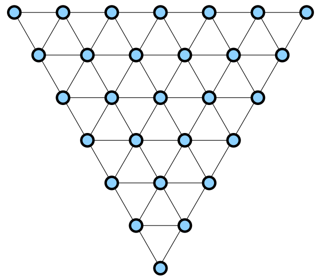

FIG.1. THE GRAPH $\mathfrak{H}_{n-1}$ IN CASE $n = 8$

Lemma 3.2. *The dual to the second $\mathfrak{G}_n$-action acts as follows: g_{ij} adds to the value at (i, j) the sum of the values at all of its neighbors.*

Proof. Similarly to the proof of Lemma 3.1, it suffices to consider the action in the corresponding 7-dimensional subspaces. The details are straightforward. $\square$

To prove Theorem 2.4, we need the following technical proposition. Consider the subspace $\mathfrak{D}^1_{n-1} \subset (T^{n-1})^*$ of all linear forms invariant under the subgroup of $\mathfrak{G}_n$ generated by $\{g_{ij} : i \geqslant 2\}$; evidently, $\mathfrak{D}_{n-1} \subseteq \mathfrak{D}^1_{n-1}$. Let $\omega : \mathfrak{D}^1_{n-1} \to \mathbb{F}_2^{n-1}$ denote the projection on the first row, and let $\mathrm{Sym}^p \subset \mathbb{F}_2^p$ denote the space of all symmetric vectors.

Lemma 3.3. (i) *The ith row of an arbitrary matrix $M \in \mathfrak{D}^1_{n-1}$ belongs to* Sym^{n-i}, $1 \leqslant i \leqslant n-1$.

(ii) $\dim \mathfrak{D}^1_{n-1} = n - 1$.

(iii) *The image of ω coincides with* Sym^{n-1}.

Proof. For $n = 3$ an immediate check shows that $\mathfrak{D}^1_2$ consists of the following four matrices:

$$\begin{pmatrix} 0 & 0 \\ & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix},$$

and so all the assertions of the lemma hold true.

Consider now an arbitrary $M \in \mathfrak{D}^1_{n-1}$. Let $a = (a_1, \dots, a_{n-1})$, $b = (b_1, \dots, b_{n-2})$, $c = (c_1, \dots, c_{n-3})$ be the first three rows of M. It follows easily from the definition of $\mathfrak{D}^1_{n-1}$ that a, b, c satisfy equations $a_i = a_1 + b_1 + b_i + b_{i-1} + c_{i-1}$, $1 \leqslant i \leqslant n-1$, where $b_0 = c_0 = 0$. Evidently, the submatrix of M obtained by deleting the first row belongs to $\mathfrak{D}^1_{n-2}$. By induction, we may assume that the rows of this matrix are symmetric, that is, $b_i = b_{n-1-i}$, $1 \leqslant i \leqslant n-2$, and $c_i = c_{n-2-i}$, $1 \leqslant i \leqslant n-3$. We thus have $a_{n-i} = a_1 + b_1 + b_{n-i} + b_{n-1-i} + c_{n-1-i} = a_i$, $1 \leqslant i \leqslant n-1$, and hence the first row of M is symmetric as well. Moreover, it is clear that the solutions of the above equations form a 1-dimensional affine subspace, and hence $\dim \mathfrak{D}^1_{n-1} = \dim \mathfrak{D}^1_{n-2} + 1 = n - 1$. To get the third statement it suffices to notice that the first two rows define M uniquely, and that the dimension of the subspace spanned by these two rows is exactly $n - 1$. $\square$

Proof of Theorem 2.4. (i) Follows immediately from the definition.

(ii) Let us define the linear mapping $\nu : \mathfrak{D}^1_{n-1} \to \mathbb{F}_2^{n-1}$ by the following rule: $\nu_i(M)$ is the sum of the neighbors of m_{1i}. By definition, $\mathfrak{D}_{n-1} = \ker \nu$, so in order to find $\dim \mathfrak{D}_{n-1}$

it suffices to determine the image of ν. As in the proof of Lemma 3.3, denote by a and b the first two rows of M. Clearly, $\nu_i = a_{i+1} + a_{i-1} + b_i + b_{i-1}$, $1 \leqslant i \leqslant n-1$, where $a_0 = a_n = b_0 = 0$. These equations define a linear mapping that takes the space Sym^{n-1} of the first rows to the image of ν. It is easy to see that the corank of this mapping is zero for $n-1 = 2k$ and one for $n-1 = 2k-1$; in other words, it equals $[n/2] - [(n-1)/2]$. Since $\dim \mathrm{Sym}^{n-1} = [n/2]$, we get that the dimension of the image of ν equals $[(n-1)/2]$. It follows immediately that $\dim \mathfrak{D}_{n-1} = n-1-[(n-1)/2] = [n/2]$.

To prove the remaining assertion we have to check the following two conditions: that P_i's are linearly independent and that they belong to $\mathfrak{D}_{n-1}$. The first of them follows from the fact that $(P_i, E_{1i}) = 1$ and $(P_i, E_{1j}) = 0$ for $1 \leqslant j \leqslant k$, $j \neq i$. To prove the second one, we introduce $\chi_i^j(n) = \{(r,s)\colon j < r < n-i-j,\ i+j < s < n-j,\ j-1 < s-r < n-i-j-1\}$, so that $X_i^j = \sum\{E_{rs}\colon (r,s) \in \chi_i^j(n)\}$, and observe that the shift $(r,s) \mapsto (r-1, s-2)$ takes $\chi_i^j(n)$ to $\chi_{i-1}^{j-1}(n-3)$. Now a straightforward induction on n proves that any matrix entry has an even number of nonzero neighbors in $\sum_{0 \leqslant j \leqslant j_i} X_i^j$, and we are done. $\square$

Now we can prove Theorem 2.5.

Proof of Theorem 2.5. It follows easily from Theorem 2.1 that the space $D^n(\mathbb{F}_2)$ of diagonal $n \times n$ matrices is transversal to the slices of the first action; moreover, Ψ^n takes $D^n(\mathbb{F}_2)$ to $D^{n-1}(\mathbb{F}_2)$. Therefore it is enough to check the assertion of the theorem only for diagonal matrices. For $M \in D^n(\mathbb{F}_2)$ and $M' = \Psi_n(M)$ Theorems 2.1 and 2.4 give

$$\begin{aligned}(M', P_i) = m'_{ii} + m'_{n-i,n-i} + m'_{n-i+1,n-i+1} = \\ (m_{ii} + m_{i+1,i+1}) + (m_{n-i,n-i} + m_{n-i+1,n-i+1}) = \\ (M, R_i) + (M, R_{i+1}) + (M, R_{n-i}) + (M, R_{n-i+1})\end{aligned}$$

for $1 \leqslant i < k$. For $n = 2k+1$ this equality holds also for $i = k$, but now $k+1 = n-k$, hence the two middle terms vanish and thus $(M', P_k) = (M, R_k) + (M, R_{n-k+1})$. Finally, for $n = 2k$ we have $(M', P_k) = m'_{kk} = (M, R_k) + (M, R_{k+1}) = (M, R_k) + (M, R_{n-k+1})$. $\square$

3.3. The second $\mathfrak{S}_n$-action: an alternative approach. In §2.2 we have introduced the second $\mathfrak{S}_n$-action as the one induced by the first $\mathfrak{S}_n$-action on the quotient of T^n modulo the subspace $\mathcal{I}_n$ of the invariants of the first $\mathfrak{S}_n$-action. Let us apply the same construction in the dual space. That is, let us consider the action induced by the dual to the first $\mathfrak{S}_n$-action on the quotient of $(T^n)^*$ modulo the subspace $\mathcal{D}_n$ of the dual invariants of the first $\mathfrak{S}_n$-action.

Proposition 3.4. *The quotient space $(T^n)^*/\mathcal{D}_n$ is isomorphic to T^{n-1}.*

Proof. Indeed, consider the linear map $\Phi_n\colon (T^n)^* \to T^{n-1}$ that takes $M \in (T^n)^*$ to $N \in T^{n-1}$ with $n_{ij} = m_{ij} + m_{i,j+1} + m_{i+1,j} + m_{i+1,j+1}$. It follows immediately from Lemma 3.1 that $\ker \Phi_n = \mathcal{D}_n$, and we are done. $\square$

The following observation gives an alternative description of the second $\mathfrak{S}_n$-action.

Lemma 3.5. *The $\mathfrak{S}_n$-action on T^{n-1} induced by Φ_n coincides with the second $\mathfrak{S}_n$-action.*

Proof. Simple exercise in linear algebra. $\square$

§4. Some results about skew-symmetric vanishing lattices

In this section we present a number of results concerning the natural action of a group generated by transvections preserving a given alternating bilinear form on a vector space over $\mathbb{F}_2$. The relation to our original problem is explained in detail in the next section and is based on the fact that the dual to the second $\mathfrak{S}_n$-action is exactly of this kind. This allows us to describe completely its orbits, as well as the orbits of the second $\mathfrak{S}_n$-action itself.

4.1. Vanishing lattices and their monodromy groups. This subsection contains all the neccessary definitions and results borrowed form [Ja]. We assume that $\mathcal{V}$ is a vector space over $\mathbb{F}_2$ equipped with an alternating bilinear form $\langle\cdot,\cdot\rangle = (\cdot, L(\cdot))$, where L is a linear map $L\colon \mathcal{V} \to \mathcal{V}^*$. A *quadratic function* q associated with $\langle\cdot,\cdot\rangle$ is an arbitrary $\mathbb{F}_2$-valued function on $\mathcal{V}$ satisfying

$$q(\lambda x + \mu y) = \lambda^2 q(x) + \mu^2 q(y) + \lambda\mu\langle x, y\rangle.$$

It is clear that a quadratic function completely determines the corresponding bilinear form. With any basis B of $\mathcal{V}$ we associate a unique quadratic function q_B by requiring it to take value 1 on all elements of B.

Let $\mathcal{K} = \ker L$ be the kernel of $\langle\cdot,\cdot\rangle$, $\kappa = \dim \mathcal{K}$, and $(e_1, f_1, ..., e_m, f_m, g_1, ..., g_\kappa)$ be a symplectic basis for $\langle\cdot,\cdot\rangle$, that is, $\langle x, y\rangle = \sum(x_i y_i' - y_i x_i')$, where $x = \sum x_i e_i + \sum x_i' f_i + \sum x_i'' g_i$ and $y = \sum y_i e_i + \sum y_i' f_i + \sum y_i'' g_i$. (Here and in what follows we occasionally use minus signs to facilitate the reading, despite it does not make much difference over $\mathbb{F}_2$.) If the restriction $q|_{\mathcal{K}}$ vanishes, then one can define the *Arf invariant* of q by $\operatorname{Arf}(q) = \sum_{i=1}^m q(e_i)q(f_i)$, see e.g. [Pf]. For fixed dimensions of $\mathcal{V}$ and $\mathcal{K}$ there exist, up to isomorphisms, at most three possibilities: (i) $q(\mathcal{K}) = 0$, $\operatorname{Arf}(q) = 1$; (ii) $q(\mathcal{K}) = 0$, $\operatorname{Arf}(q) = 0$; (iii) $q(\mathcal{K}) = \mathbb{F}_2$ (and so $\kappa \geqslant 1$).

In the first case one has $|q^{-1}(1)| = 2^{2m+\kappa-1} + 2^{m+\kappa-1}$, $|q^{-1}(0)| = 2^{2m+\kappa-1} - 2^{m+\kappa-1}$, in the second case one has $|q^{-1}(1)| = 2^{2m+\kappa-1} - 2^{m+\kappa-1}$, $|q^{-1}(0)| = 2^{2m+\kappa-1} + 2^{m+\kappa-1}$, and in the third case one has $|q^{-1}(1)| = |q^{-1}(0)| = 2^{2m+\kappa-1}$.

For any $\delta \in \mathcal{V}$ we define the *symplectic transvection* $T_\delta\colon \mathcal{V} \to \mathcal{V}$ by $T_\delta(x) = x - \langle x, \delta\rangle\delta$. (Notice that T_δ is an element of the group $\operatorname{Sp}^\sharp \mathcal{V}$ of the automorphisms of $(\mathcal{V}, \langle\cdot,\cdot\rangle)$). Given any subset $\Delta \in V$, we let $\Gamma_\Delta \subseteq \operatorname{Sp}^\sharp \mathcal{V}$ denote the subgroup generated by the transvections T_δ, $\delta \in \Delta$.

The main object of this section is a *vanishing lattice*, that is, a triple $(\mathcal{V}, \langle\cdot,\cdot\rangle, \Delta)$ satisfying the following three conditions: (i) Δ is a Γ_Δ-orbit; (ii) Δ generates $\mathcal{V}$; (iii) if $\operatorname{rank} \mathcal{V} > 1$, then there exist $\delta_1, \delta_2 \in \Delta$ such that $\langle \delta_1, \delta_2\rangle = 1$. The group Γ_Δ is called the *monodromy group* of the lattice.

We say that a basis B of $\mathcal{V}$ is *weakly distinguished* if $\Gamma_B = \Gamma_\Delta$. In this case Γ_B respects q_B, so, in particular, $q_B(\delta) = 1$ for all $\delta \in \Delta$. Bases B and B' are called *equivalent* if $B' \subset \Gamma_B \cdot B$ and $B \subset \Gamma_{B'} \cdot B'$. One can easily see that if B and B' are equivalent, then $\Gamma_B = \Gamma_{B'}$, $q_B = q_{B'}$, and if B is a weakly distinguished basis for $(\mathcal{V}, \langle\cdot,\cdot\rangle, \Delta)$, then so is B'.

Let $B = (b_1, ..., b_d)$ be a basis in $\mathcal{V}$, $d = \dim \mathcal{V}$. We define the graph $\operatorname{gr}(B)$ of B as follows: B is its vertex set, and b_i is connected by an edge with b_j if $\langle b_i, b_j\rangle = 1$. It is easy to see that $\operatorname{gr}(B)$ is connected if B is weakly distinguished.

A basis is called *special* if it is equivalent to a basis $B = (b_1, ..., b_d)$ such that for some k, $1 \leqslant k \leqslant d$, we have $\langle b_i, b_j \rangle = 0$ iff $i = j$ or $i, j \geqslant k + 1$, and *nonspecial* otherwise. A vanishing lattice admitting a special (resp. nonspecial) weakly distinguished basis and its monodromy group are called *special* (resp. *nonspecial*).

Nonspecial monodromy groups have an especially simple characterization (we are primarily interested in this case since for $n \geqslant 5$ the group $\mathfrak{G}_n$ introduced in §1 can be interpreted as a nonspecial monodromy group, see §5 for details.)

Theorem 4.1 ([Ja], Th. 3.8)**.** *Let $(\mathcal{V}, \langle\cdot,\cdot\rangle, \Delta)$ be a vanishing lattice admitting a nonspecial weakly distinguished basis B. Then Γ_B coincides with the subgroup $O^\sharp(q_B)$ of $\mathrm{Sp}^\sharp V$ consisting of all automorphisms that preserve q_B.*

It follows that the classification of nonspecial vanishing lattices reduces to the classification of the corresponding quadratic functions. We thus see that for given m and κ such that $\dim \mathcal{V} = 2m + \kappa$, $\dim \mathcal{K} = \kappa$ there exist exactly three nonspecial vanishing lattices, depending on the values of $q_B(\mathcal{K})$ and $\mathrm{Arf}(q_B)$. They are denoted by $O_1^\sharp(2m, \kappa, \mathbb{F}_2)$, $O_0^\sharp(2m, \kappa, \mathbb{F}_2)$, and $O^\sharp(2m, \kappa, \mathbb{F}_2)$, and correspond to the cases $\mathrm{Arf}(q_B) = 1$, $\mathrm{Arf}(q_B) = 0$, and $q_B(\mathcal{K}) = \mathbb{F}_2$, respectively (see [Ja, 4.2] for details).

The following statement, which can be extracted easily from [Ja, §4], provides a sufficient condition for a vanishing lattice to be nonspecial.

Lemma 4.2. *A vanishing lattice is nonspecial if it admits a weakly distinguished basis B such that $\mathrm{gr}(B)$ contains the standard Dynkin diagram of the Coxeter group E_6 as an induced subgraph.*

4.2. Orbits of nonspecial monodromy groups and of the dual actions. First of all, let us find the number of orbits of the monodromy group in the nonspecial case.

Lemma 4.3. *Let $(\mathcal{V}, \langle\cdot,\cdot\rangle, \Delta)$ be a vanishing lattice admitting a nonspecial weakly distinguished basis B. Then the number of orbits of Γ_B equals $2^\kappa + 2$, where $\kappa = \dim \mathcal{K}$. These orbits are the 2^κ points of $\mathcal{K}$ and the sets $q_B^{-1}(0) \setminus \mathcal{K}$ and $q_B^{-1}(1) \setminus \mathcal{K}$.*

Proof. Evidently, any group generated by transvections acts trivially on $\mathcal{K}$. Next, in the nonspecial case $q_B^{-1}(1)\setminus\mathcal{K}$ is an orbit of Γ_B by Theorem 3.5 of [Ja]. To prove that $q_B^{-1}(0)\setminus\mathcal{K}$ is an orbit as well, take an arbitrary pair $u, v \notin \mathcal{K}$ such that $q_B(u) = q_B(v) = 0$. It is easy to see that there exist $u', v' \notin \mathcal{K}$ such that $q_B(u') = q_B(v') = 1$ and $\langle u, u' \rangle = \langle v, v' \rangle = 1$. Define $\mathcal{D}_u = \{w \in \mathcal{V} \colon \langle w, u \rangle = \langle w, u' \rangle = 0\}$, $\mathcal{D}_v = \{w \in \mathcal{V} \colon \langle w, v \rangle = \langle w, v' \rangle = 0\}$. Evidently, $\mathcal{K}$ is a subspace of both $\mathcal{D}_u$ and $\mathcal{D}_v$, and $\dim \mathcal{D}_u = \dim \mathcal{D}_v = d - 2$. Let q_u and q_v be the restrictions of q_B to $\mathcal{D}_u$ and $\mathcal{D}_v$, respectively. If $q_B(\mathcal{K}) = \mathbb{F}_2$, then the same is true for q_u and q_v, and so the quadratic spaces $(\mathcal{D}_u, q_u)$ and $(\mathcal{D}_v, q_v)$ are isometric. Otherwise, if $q_B(\mathcal{K}) = 0$, the Arf invariants for the forms q_B, q_u, q_v are defined. Moreover, $\mathrm{Arf}(q_u) = \mathrm{Arf}(q_v) = 1 + \mathrm{Arf}(q_B)$, and hence $(\mathcal{D}_u, q_u)$ and $(\mathcal{D}_v, q_v)$ are again isometric by the Arf theorem (see [Pf]). In both cases the isometry $\mathcal{D}_u \to \mathcal{D}_v$ can be extended to an isometry of the entire $\mathcal{V}$ by letting $u \mapsto v$, $u' \mapsto v'$. By Theorem 4.1 this isometry belongs to Γ_B. $\square$

Let us consider now the dual to the action of the monodromy group of a vanishing lattice. We start from the following well-known result.

Lemma 4.4. *Let G be a finite group acting on a finite-dimensional space $\mathcal{V}$ over a finite field $\mathfrak{k}$. Then the number of orbits of the action equals the number of orbits of the dual action.*

Proof. The number of orbits of a finite group action on any finite set X can be calculated using the Frobenius formula (see e.g. [Ke]):

$$\sharp = \frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where X^g is the set of all g-stable points. But the set $\mathcal{V}^g$ of the stable points for the linear operator E_g coincides with $\ker(E_g - E)$, where E is the identity operator. Therefore, $|\mathcal{V}^g| = |\mathfrak{k}|^{\dim\ker(E_g-E)}$. The equality $\dim\ker(E_g - E) = \dim\ker(E_g - E)^* = \dim\ker(E_g^* - E)$ proves the statement. $\square$

Consider now the relation between the orbits of a nonspecial monodromy group Γ and the orbits of the dual action. Choose some basis $\{b_1,\dots,b_\kappa\}$ of $\mathcal{K}$ and a κ-tuple $\eta = \{\eta_1,\dots,\eta_\kappa\} \in \mathbb{F}_2^\kappa$. Denote by $\mathcal{A}^\eta$ the affine subspace of $\mathcal{V}^*$ of codimension κ that consists of all elements $x \in \mathcal{V}^*$ such that $(b_j, x) = \eta_j$, $1 \leqslant j \leqslant \kappa$. Evidently, $\mathcal{A}^\eta$ is invariant under the dual action of Γ for any $\eta \in \mathbb{F}_2^\kappa$. Moreover, since $\langle\cdot,\cdot\rangle$ is alternating, we see that the image $L(\mathcal{V})$ of the map $L\colon \mathcal{V} \to \mathcal{V}^*$ coincides with $\mathcal{A}^0$.

Our strategy is as follows. To study the structure of the dual action we use an additional construction. We introduce two subspaces $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{V}_2 \subset \mathcal{V}$, which are transversal to $\mathcal{K}$ (therefore $\dim\mathcal{V}_1 = \dim\mathcal{V}_2 = 2m$) and $\mathcal{V}_1 + \mathcal{V}_2 = \mathcal{V}$. To each subspace $\mathcal{V}_i$ corresponds a subgroup $\Gamma_i \subset \Gamma$ (generated by transvections w.r.t. the elements in $\mathcal{V}_i$). One can easily see that the set $\Gamma_1 \cup \Gamma_2$ generates Γ. We first study the orbits of Γ_i and of the corresponding dual action separately (which is fairly simple), and then describe their interaction. We start with the case of one subspace.

Consider a subspace $\mathcal{V}_1 \subset \mathcal{V}$ transversal to $\mathcal{K}$. The restriction of the form $\langle\cdot,\cdot\rangle$ is nondegenerate on $\mathcal{V}_1$. Let $\Gamma_1 \subset \Gamma$ be the subgroup generated by the tranvections w.r.t. elements of $\mathcal{V}_1$.

Lemma 4.5. *Let Γ be nonspecial. Then for any $\eta \in \mathbb{F}_2^\kappa$ the affine subspace $\mathcal{A}^\eta$ consists of three orbits of the dual Γ_1-action.*

Proof. Observe first that for any $\eta \in \mathbb{F}_2^\kappa$ the intersection of $\mathcal{A}^\eta$ and $\mathcal{V}_1^\perp \subset \mathcal{V}^*$ contains exactly one element, which we denote by A_1^η. Indeed, if $\{v_1,\dots,v_{2m}\}$ is an arbitrary basis of $\mathcal{V}_1$, then A_1^η is the unique solution of the equations $(b_j, A_1^\eta) = \eta_j$, $1 \leqslant j \leqslant \kappa$, $(v_j, A_1^\eta) = 0$, $1 \leqslant j \leqslant 2m$.

Next, let L_1 be the restriction of L to $\mathcal{V}_1$, and L_1^η be the composition of L_1 with the translation by A_1^η. We say that A_1^η is the *shift* corresponding to $\mathcal{A}^\eta$; observe that $A_1^0 = 0$, and thus $L_1^0 = L_1$. Evidently, L_1^η provides an affine isomorphism between $\mathcal{V}_1$ and $\mathcal{A}^\eta$. Recall that the dual action of Γ (and hence, of Γ_1) preserves $\mathcal{A}^\eta$. Moreover, since Γ_1 preserves $\mathcal{V}_1$, the element A_1^η is a fixed point of the dual to the Γ_1-action, as the unique annihilator of $\mathcal{V}_1$ in $\mathcal{A}^\eta$. Therefore, the diagram

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r, "L_1^\eta"] \arrow[d, "g_1"'] & \mathcal{A}^\eta \\ \mathcal{V}_1 \arrow[r, "L_1^\eta"] & \mathcal{A}^\eta \arrow[u, "g_1^*"'] \end{tikzcd}$$

is commutative for any $g_1 \in \Gamma_1$. Since Γ is nonspecial, the same is true for Γ_1 (by Lemma 4.2). It remains to apply Lemma 4.3 with $\kappa = 0$. $\square$

Let us now choose one more subspace $\mathcal{V}_2 \subset \mathcal{V}$ transversal to $\mathcal{K}$ such that $\mathcal{V}_1 + \mathcal{V}_2 = \mathcal{V}$ and $\Gamma_1 \cup \Gamma_2$ generates Γ. To study the orbits of the dual Γ-action we have to find which orbits of the dual Γ_1-action are glued together by the dual Γ_2-action. Let us define an affine isomorphism $I^\eta \colon \mathcal{V}_1 \to \mathcal{V}_2$ by $I^\eta = (L_2^\eta)^{-1} \circ L_1^\eta$, where L_2 is the restriction of L to $\mathcal{V}_2$.

Lemma 4.6. *Let Γ be nonspecial.*

(i) *For $\eta = 0$ the following alternative holds: the linear space $\mathcal{A}^0$ consists of two orbits of the dual Γ-action if there exist $u, v \in \mathcal{V}_1$ such that $q_B(u) = 0$, $q_B(v) = q_B(I^0(u)) = q_B(I^0(v)) = 1$, and of three orbits of the dual Γ-action otherwise.*

(ii) *For any $\eta \neq 0$ the following alternative holds: the affine subspace $\mathcal{A}^\eta$ consists of one orbit of the dual Γ-action if there exist $u, v \in \mathcal{V}_1$ such that $q_B(u) = 0$, $q_B(v) = q_B(I^\eta(u)) = q_B(I^\eta(v)) = 1$, and of two orbits of the dual Γ-action otherwise.*

Proof. The first part of this lemma follows easily from Lemma 4.3 with $\kappa = 0$ together with Lemma 4.5. To prove the second part we notice first that $I^\eta(0) \neq 0$, and then we are done by the same reasons. $\square$

§5. Orbits of the second $\mathfrak{S}_n$-action

In this section we prove Theorem 2.6 describing the orbits of the second $\mathfrak{S}_n$-action on the space of upper-triangular $(n-1) \times (n-1)$ matrices over $\mathbb{F}_2$. Observe that the basic convention in this section is different from that of §4, namely, the dual to the second $\mathfrak{S}_n$-action is the action preserving a natural alternating form, and we study the second $\mathfrak{S}_n$-action itself as the dual to its dual using Lemmas 4.4-4.6..

5.1. The dual to the second $\mathfrak{S}_n$-action. In this section we assume that $\mathcal{V} = (T^{n-1})^*$. Let us introduce a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{V}$ by

$$\langle M, N \rangle = \sum_{(i_1,j_1),(i_2,j_2)} m_{i_1,j_1} n_{i_2,j_2},$$

where $(i_1, j_1), (i_2, j_2)$ runs over all pairs of neighbors in $\mathfrak{H}_{n-1}$, see §3. Evidently, $\langle \cdot, \cdot \rangle$ is alternating.

Let B_{n-1} be the standard basis of $\mathcal{V}$ consisting of matrix units E_{rs}. Elements of B_{n-1} correspond bijectively to the vertices of $\mathfrak{H}_{n-1}$. Denote by Q the quadratic function $q_{B_{n-1}}$ associated with $\langle \cdot, \cdot \rangle$. To find the value $Q(M)$ for an arbitrary matrix $M \in \mathcal{V}$ we represent M as $M = \sum_{b_i \in B_{n-1}} \alpha_i b_i$ and define $\operatorname{supp} M = \{b_i \in B_{n-1} \colon \alpha_i = 1\}$. Let $\operatorname{gr}(M)$ denote the subgraph of $\mathfrak{H}_{n-1}$ induced by the vertices corresponding to $\operatorname{supp} M$. The following statement follows easily from the definitions.

Lemma 5.1. *The value $Q(M)$ equals the number of vertices of* $\mathrm{gr}(M)$ *plus the number of its edges.*

As a corollary, we obtain the values of Q on the dual invariants $P_1, \dots, P_k$.

Corollary 5.2. (i) *Let $n = 2k+1$, then $Q(P_i) = 0$ for $1 \leqslant i \leqslant k$.*
(ii) *Let $n = 2k$, then $Q(P_i) = 1$ for $1 \leqslant i \leqslant k-1$, and $Q(P_k) = k$.*

Proof. (i) Let us define $\widetilde{P}_i = \sum_{0 \leqslant j \leqslant j_i} X_i^j$, so that $P_i = \widetilde{P}_i + \widetilde{P}_{i-1}$ with $\widetilde{P}_k = 0$. It follows immediately from the definitions that $\mathrm{gr}(\widetilde{P}_i)$, $0 \leqslant i \leqslant k-1$, consists of $[(j_i+1)/2]$ disjoint cycles and, provided j_i is even, of a copy of $\mathfrak{H}_l$ with $l = |n-3i-2|+1$. Thus, $Q(\widetilde{P}_i)$ equals 0 for j_i odd and equals $l(2l-1)$ for j_i even. Since n is odd, we get $l = i \mod 2$; recall that $j_i = \min\{i, n-2i-2\}$, hence if j_i is even, then $j_i = i$, and thus l is even. We therefore get $Q(\widetilde{P}_i) = 0$ for $0 \leqslant i \leqslant k-1$. It remains to notice that $Q(P_i) = Q(\widetilde{P}_i) + Q(\widetilde{P}_{i-1})$, since $\widetilde{P}_i$ are dual invariants of the second $\mathfrak{S}_n$-action (see the proof of Theorem 2.4(ii)).

(ii) In the same way as in (i) we see that $Q(\widetilde{P}_i)$ equals 0 for j_i odd and equals $l(2l-1)$ for j_i even, with $l = |n-3i-2|+1$. Since n is even, we get $l = i+1 \mod 2$; in particular, $j_{k-1} = 0$ and hence $Q(P_k) = Q(\widetilde{P}_{k-1}) = k$. Let us prove that $Q(\widetilde{P}_i)$ alternate. Indeed, if $Q(\widetilde{P}_i) = 1$,then both j_i and i are even, hence $i-1$ is odd and $Q(\widetilde{P}_{i-1}) = 0$. Let now $Q(\widetilde{P}_i) = 0$, then either j_i is odd, or j_i is even and i is odd. In the first case we get $j_i = i$, hence $j_{i-1} = i-1$ is even, and thus $Q(\widetilde{P}_{i-1}) = 1$. In the second case either $j_{i-1} = j_i + 2$, or $j_{i-1} = j_i$; in both situations j_{i-1} and $i-1$ are even, and thus $Q(\widetilde{P}_{i-1}) = 1$. It remains to notice that $Q(P_i) = Q(\widetilde{P}_i) + Q(\widetilde{P}_{i-1})$. □

Let $\mathcal{K}$ be the kernel of $\langle \cdot, \cdot \rangle$. Evidently, $\mathcal{K}$ is the space of dual invariants of the second $\mathfrak{S}_n$-action, hence Theorem 2.4 gives an explicit description of $\mathcal{K}$. Applying Corollary 5.2 we get the following statement.

Corollary 5.3. *Let $n = 2k$, then $Q(\mathcal{K}) = \mathbb{F}_2$ and $|Q^{-1}(0)| = |Q^{-1}(1)| = 2^{2k^2-k-1}$, $|Q^{-1}(0) \cap \mathcal{K}| = |Q^{-1}(1) \cap \mathcal{K}| = 2^{k-1}$.*

Proof. It follows immediately from Corollary 5.2(ii) that $Q(\mathcal{K}) = \mathbb{F}_2$. The statement concerning the sizes of $|Q^{-1}(0)|$ and $|Q^{-1}(1)|$ follows easily from the general description (see §4.1) with $m = k(k-1)$ and $\kappa = k$. The last statement follows from the fact that the normal form of $Q|_{\mathcal{K}}$ in case $Q(\mathcal{K}) = \mathbb{F}_2$ is x_1^2 (see [Pf]). □

In case $n = 2k+1$ the situation is far more complicated.

Lemma 5.4. *Let $n = 2k+1$, then $Q(\mathcal{K}) = 0$. If $k = 4t+1$, then* $\mathrm{Arf}(Q) = 1$ *and $|Q^{-1}(0)| = 2^{2k^2+k-1} - 2^{k^2+k-1}$, $|Q^{-1}(1)| = 2^{2k^2+k-1} + 2^{k^2+k-1}$. Otherwise* $\mathrm{Arf}(Q) = 0$ *and $|Q^{-1}(0)| = 2^{2k^2+k-1} + 2^{k^2+k-1}$, $|Q^{-1}(1)| = 2^{2k^2+k-1} - 2^{k^2+k-1}$.*

Proof. It follows immediately from Corollary 5.2(i) that $Q(\mathcal{K}) = 0$, and hence the Arf invariant exists. Observe that by general theory of quadratic spaces (see §4.1) this means that $|Q^{-1}(0)| \neq |Q^{-1}(1)|$. Moreover, $|Q^{-1}(0)| > |Q^{-1}(1)|$ implies $\mathrm{Arf}(Q) = 0$, while $|Q^{-1}(0)| < |Q^{-1}(1)|$ implies $\mathrm{Arf}(Q) = 1$. Therefore, to find $\mathrm{Arf}(Q)$ it suffices to count the number of elements in $\mathcal{V}$ on which Q vanishes.

Let $\omega\colon \mathcal{V} \to \mathbb{F}_2^{n-1}$ denote the projection on the first row (see Lemma 3.3), and let $\mathcal{V}_a = \omega^{-1}(a)$, $a \in \mathbb{F}_2^{n-1}$. We say that $\mathcal{V}_a$ is *inessential* if $|Q^{-1}(0) \cap \mathcal{V}_a| = |Q^{-1}(1) \cap \mathcal{V}_a|$, and *essential* otherwise.

Let us choose $M_a \in \mathcal{V}_a$ such that $Q(M_a) = 0$; if $\operatorname{gr}(a)$ contains an even number of connected components, one can take $\operatorname{supp} M_a = \operatorname{supp} a$, otherwise $\operatorname{supp} M_a$ contains one more vertex, which corresponds to the matrix entry (n, n). We thus have $M_a = M_a^0 + M_a^1$, where $\operatorname{supp} M_a^0 = \operatorname{supp} a$ and $|\operatorname{supp} M_a^1| \leqslant 1$.

Let us define a function Q_a on $\mathcal{V}_0$ by $Q_a(M) = Q(M + M_a)$ for any $M \in \mathcal{V}_0$; observe that Q_a is just a shift of the restriction $Q|_{\mathcal{V}_a}$. Evidently, $Q_a(M) = Q(M) + \langle M, M_a\rangle$. Therefore, $Q_a(M + N) - Q_a(M) - Q_a(N) = \langle M, N\rangle$, which means that Q_a and Q define the same bilinear form on $\mathcal{V}_0$ (observe that $\mathcal{V}_0$ is identified naturally with $(T^{n-2})^*$). Let us evaluate Q_a on the kernel $\mathcal{K}_0$; $Q_a(\mathcal{K}_0) = \mathbb{F}_2$ would mean that Q vanishes exactly on a half of the elements of $\mathcal{V}_a$, in other words, that $\mathcal{V}_a$ is inessential, while $Q_a(\mathcal{K}_0) = 0$ would mean that $\mathcal{V}_a$ is essential.

Since both Q and Q_a define the same bilinear form on $\mathcal{V}_0$, we see that $\mathcal{K}_0$ is the space of dual invariants of the second $\mathfrak{S}_{n-1}$-action; therefore, by Theorem 2.4, it has a basis $\{P_1, \dots, P_k\}$. By Corollary 5.2(ii), $Q_a(P_i) = 1 + \langle P_i, M_a\rangle = 1 + \langle P_i, M_a^0\rangle$ for $1 \leqslant i \leqslant k-1$ and $Q_a(P_k) = k + \langle P_k, M_a^0\rangle$. Therefore, $\mathcal{V}_a$ is essential if and only if the entries of a satisfy equations $a_{k-i} + a_{k+i+1} = k + i$ for $0 \leqslant i \leqslant k-1$. It is easy to see that any solution of the above equations is represented as $a = \bar{h} + s$, where $\bar{h}$ is as defined in Theorem 2.2(ii) and $s \in \operatorname{Sym}^{n-1}$. It follows from Theorem 2.4 that for any $s \in \operatorname{Sym}^{n-1}$ there exists $S \in \mathcal{K}$ such that $\omega(S) = s$. Observe that in this case $M \mapsto M + S$ takes $\mathcal{V}_a$ to $\mathcal{V}_{a+s}$. Moreover, $Q(M + M_a + S) = Q(M + M_a)$, since $\langle M + M_a, S\rangle = 0$ follows from $S \in \mathcal{K}$ and $Q(S) = 0$ by Corollary 5.2(i). Therefore, all the essential subspaces $\mathcal{V}_a$ influence $\operatorname{Arf}(Q)$ in the same way, hence, $\operatorname{Arf}(Q; \mathcal{V}) = \operatorname{Arf}(Q_{\bar{h}}; \mathcal{V}_0)$.

To study $\operatorname{Arf}(Q_{\bar{h}})$ on $\mathcal{V}_0$ we reiterate the same process once more, that is, we decompose $\mathcal{V}_0$ into affine subspaces $\mathcal{V}_{0b}$, where b is defined by the projection ω_0 of $\mathcal{V}_0$ on the first row. We choose $M_{0b} \in \mathcal{V}_{0b}$ similarly to M_a and define a function Q_{0b} on $\mathcal{V}_{00}$ by $Q_{0b}(M) = Q_{\bar{h}}(M + M_{0b})$. As before, $\mathcal{V}_{00}$ is identified naturally with $(T^{n-3})^*$, and Q_{0b} and Q define the same bilinear form on $\mathcal{V}_{00}$. The corresponding kernel $\mathcal{K}_{00}$ is the space of dual invariants of the second $\mathfrak{S}_{n-2}$-action; its basis is $\{P_1, \dots, P_{k-1}\}$. We thus get that $\mathcal{V}_{0b}$ is essential if and only if the entries of b satisfy equations $b_{k-i} + b_{k+i} = 0$ for $1 \leqslant i \leqslant k-1$. Any solution of these equations belongs to Sym^{n-2}. As before, $M \mapsto M + S$ with $S \in \mathcal{K}_0$ and $\omega_0(S) = s$ takes $\mathcal{V}_{0b}$ to $\mathcal{V}_{0,b+s}$ and $Q_{\bar{h}}(M + M_{0b} + S) = Q_{\bar{h}}(M + M_{0b})$; identity $Q_{\bar{h}}(S) = 0$ follows readily from $Q(P_i) = \langle P_i, M_{\bar{h}}^0\rangle$, $1 \leqslant i \leqslant k-1$. We thus get $\operatorname{Arf}(Q_{\bar{h}}; \mathcal{V}_0) = \operatorname{Arf}(Q_{\bar{h}}; \mathcal{V}_{00})$.

Recall that on $\mathcal{V}_{00}$ one has $Q_{\bar{h}}(M) = Q(M) + \langle M, M_{\bar{h}}^1\rangle$. It is easy to see that for $k = 4t$ and $k = 4t + 3$ one has $M_{\bar{h}}^1 = 0$, and hence $Q_{\bar{h}} \equiv Q$ on $\mathcal{V}_{00}$. This means that the Arf invariant is constant on each triple of the form $k = 4t + 2, 4t + 3, 4t + 4$.

Let now $k = 4t + 1$ or $k = 4t + 2$, and hence $M_{\bar{h}}^1 \neq 0$. We decompose $\mathcal{V}_{00}$ into four affine subspaces $\mathcal{V}_{00}^{00}$, $\mathcal{V}_{00}^{01}$, $\mathcal{V}_{00}^{10}$, and $\mathcal{V}_{00}^{11}$. The subspace $\mathcal{V}_{00}^{ij}$ consists of the matrices having i at position $(n-1, n-1)$ and j at position $(n-1, n)$. It is easy to see that $Q_{\bar{h}} \equiv Q$ on $\mathcal{V}_{00}^{ii}$; moreover, the involution $M \mapsto M + M_{\bar{h}}^1$ reverses the value of Q (and thus of $Q_{\bar{h}}$) on these subspaces. Therefore, the subspaces $\mathcal{V}_{00}^{ii}$ are inessential. On the other hand, $Q_{\bar{h}} \equiv Q + 1$ on $\mathcal{V}_{00}^{ij}$, $i \neq j$, and the involution $M \mapsto M + M_{\bar{h}}^1$ in this case preserves the value of Q

(and thus of $Q_{\bar{h}}$). Therefore, $\mathrm{Arf}(Q_{\bar{h}}; \mathcal{V}_{00}) = 1 + \mathrm{Arf}(Q; \mathcal{V}_{00})$. This means that the Arf invariant reverses twice on each triple of the form $k = 4t, 4t+1, 4t+2$.

To complete the proof it is enough to check the value of the Arf invariant for $k = 1$.

The exact values for the sizes of orbits follow easily from the general description (see §4.1) with $m = k^2$ and $\kappa = k$. $\square$

Let us now find the relation between the dual to the second $\mathfrak{S}_n$-action and the theory of skew-symmetric vanishing lattices explained in §4.

Lemma 5.5. (i) *The triple* $(\mathcal{V}, \langle\cdot,\cdot\rangle, B_{n-1})$ *is a vanishing lattice.*

(ii) *The dual to the second* $\mathfrak{S}_n$*-action coincides with the action of* $\Gamma_{B_{n-1}}$.

(iii) *The basis* B_{n-1} *is weakly distinguished and its graph* $\mathrm{gr}(B_{n-1})$ *coincides with* $\mathfrak{H}_{n-1}$.

Proof. (i) We have to check that B_{n-1} satisfies conditions (i)–(iii) of the definition of the vanishing lattices. Condition (ii) is evident. To check to it suffices to take for δ_1 and δ_2 any pair of adjacent vertices of $\mathfrak{H}_{n-1}$. Finally, to check (i) we take an arbitrary pair b, b' of adjacent vertices of $\mathfrak{H}_{n-1}$ and find that $T_b T_{b'}(b) = b'$.

(ii) Follows immediately from Lemma 3.2.

(iii) Obvious. $\square$

Lemma 5.6. *The dual to the second* $\mathfrak{S}_n$*-action is the action of a nonspecial monodromy group for* $n \geqslant 5$.

Proof. By Lemmas 4.2 and 5.5 we have to check that the Dynkin diagram of E_6 is an induced subgraph of $\mathfrak{H}_m$ for $m \geqslant 4$. Since $\mathfrak{H}_m$ is an induced subgraph of $\mathfrak{H}_l$ for $m < l$, it suffices to find an induced subgraph corresponding to E_6 in $\mathfrak{H}_4$, see Fig. 2. $\square$

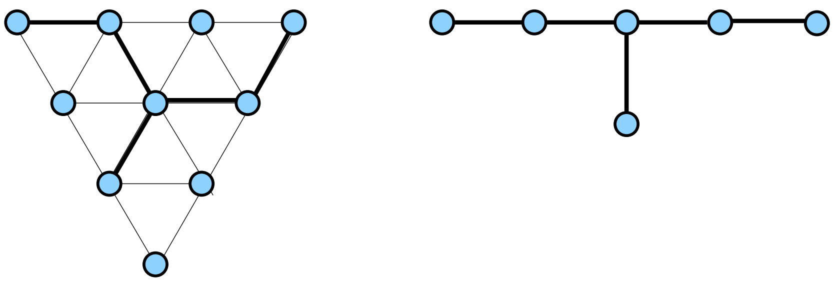

FIG. 2. THE GRAPH $\mathfrak{H}_4$ AND THE DYNKIN DIAGRAM FOR E_6

5.2. The orbits of the second $\mathfrak{S}_n$-action. First of all, let us find the number of orbits of the second $\mathfrak{S}_n$-action.

Lemma 5.7. *The number of orbits of the second* $\mathfrak{S}_n$*-action equals* $2^k + 2$ *for* $n \geqslant 5$.

Proof. Indeed, by Lemma 4.4 the number of orbits of the second $\mathfrak{S}_n$-action equals that of its dual. By Lemma 5.6 the dual to the second $\mathfrak{S}_n$-action is the action of a nonspecial monodromy group; hence, by Lemma 4.3 the number of its orbits equals $2^\kappa + 2$, where κ is the dimension of $\mathcal{K}$. Since $\mathcal{K}$ is just the space of dual invariants of the second $\mathfrak{S}_n$-action, Theorem 2.4(ii) implies $\kappa = k$. $\square$

Let us now define two subsets of the vertex set of $\mathfrak{H}_{n-1}$ as follows: Σ_1 consists of the vertices corresponding to the matrix entries $(i, n-1)$, $1 \leqslant i \leqslant k$, and Σ_2 of the vertices corresponding to $(n-i, n-i)$, $1 \leqslant i \leqslant k$, see Fig. 3.

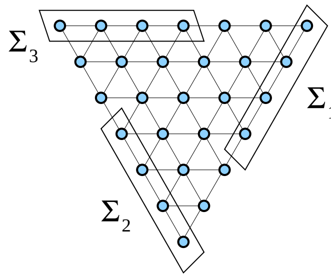

FIG. 3. THE SETS Σ_1, Σ_2, Σ_3 FOR THE CASE $n = 8$

Following the strategy described in §4, we choose the following two subspaces in $\mathcal{V} = (T^{n-1})^*$:

$$\mathcal{V}_1 = \{M \in \mathcal{V}\colon \operatorname{supp} M \cap \Sigma_1 = \varnothing\},$$
$$\mathcal{V}_2 = \{M \in \mathcal{V}\colon \operatorname{supp} M \cap \Sigma_2 = \varnothing\}.$$

Let $\mathcal{K}$ denote the kernel of $\langle \cdot, \cdot \rangle$. It follows easily from the explicit description of $\mathcal{K}$ (see Theorem 2.4(ii)) that $\operatorname{codim} \mathcal{V}_1 = \operatorname{codim} \mathcal{V}_2 = \dim \mathcal{K}$ and $\mathcal{V}_1 \cap \mathcal{K} = \mathcal{V}_2 \cap \mathcal{K} = 0$, hence $\mathcal{V}_1$ and $\mathcal{V}_2$ as above satisfy the assumptions of §4.

Fix the basis $P_1, ..., P_k$ of $\mathcal{K}$. It is easy to see that the slice S^η of the second $\mathfrak{G}_n$-action is just the affine subspace $\mathcal{A}^\eta$ as defined in §4.2. Therefore, the statements of Theorem 2.6(i) concerning the number and the sizes of orbits are yielded by the following proposition.

Lemma 5.8. *Let $n = 2k+1 \geqslant 5$.*

(i) *The linear subspace $\mathcal{A}^0 \subset \mathcal{V}^*$ consists of three orbits of the second $\mathfrak{G}_n$-action. The sizes of the orbits are $2^{2k^2-1} - \varepsilon_k 2^{k^2-1}$, $2^{2k^2-1} + \varepsilon_k 2^{k^2-1} - 1$, and 1, where $\varepsilon_k = -1$ for $k = 4t+1$ and $\varepsilon_k = 1$ otherwise.*

(ii) *For any nonzero $\eta \in \mathbb{F}_2^k$ the affine subspace $\mathcal{A}^\eta \subset \mathcal{V}^*$ is an orbit of the second $\mathfrak{G}_n$-action.*

Proof. (i) Let us first find the number of orbits contained in $\mathcal{A}^0$. By Lemma 4.6(i) it suffices to prove that $Q(M) = Q(I^0(M))$ for any $M \in \mathcal{V}_1$. Define $M^{\mathcal{K}} = I^0(M) - M$, and let $L\colon \mathcal{V} \to \mathcal{V}^*$ be the linear mapping associated with $\langle \cdot, \cdot \rangle$. Since by definition $L(M) = L(I^0(M))$ for any $M \in \mathcal{V}_1$, we have $L(M^{\mathcal{K}}) = 0$, and thus $M^{\mathcal{K}} \in \mathcal{K}$. Therefore $Q(I^0(M)) = Q(M) + Q(M^{\mathcal{K}})$, and since by Lemma 5.4 $Q(\mathcal{K}) = 0$, we are done.

It follows now from Lemmas 4.5 and 4.3 that the three orbits in question are isomorphic to $\{0\}$, $(Q^{-1}(0) \cap \mathcal{V}_1) \setminus \{0\}$ and $Q^{-1}(1) \cap \mathcal{V}_1$. Evidently, $\operatorname{Arf}(Q) = \operatorname{Arf}(Q|_{\mathcal{V}_1})$ and $(\mathcal{V}_1, Q|_{\mathcal{V}_1})$ is a nondegenerate quadratic space of dimension $2k^2$. Therefore the statement concerning the sizes of the orbits follows from Lemma 5.4 and the general description of quadratic spaces (see §4.1) with $m = k^2$ and $\kappa = 0$.

(ii) By Lemma 5.7 the total number of orbits is 2^k+2, and by part (i) of the present Lemma exactly three orbits are contained in $\mathcal{A}^0$. Therefore, each of the 2^k-1 remaining orbits coincides with the corresponding $\mathcal{A}^\eta$. $\square$

The corresponding statements of Theorem 2.6(ii) are yielded by the following proposition.

Lemma 5.9. *Let $n=2k\geqslant 6$ and $\bar\eta=(Q(P_1),\dots,Q(P_k))$.*

(i) *The affine subspace $\mathcal{A}^{\bar\eta}\subset\mathcal{V}^*$ consists of two orbits of the second $\mathfrak{G}_n$-action. The sizes of the orbits are $2^{2k(k-1)-1}-2^{k(k-1)-1}$ and $2^{2k(k-1)-1}+2^{k(k-1)-1}$.*

(ii) *The linear subspace $\mathcal{A}^0\subset\mathcal{V}^*$ consists of two orbits of the second $\mathfrak{G}_n$-action. The sizes of the orbits are $2^{2k(k-1)}-1$ and 1.*

(iii) *For any $\eta\in\mathbb{F}_2^k$, $\eta\neq 0,\bar\eta$, the affine subspace $\mathcal{A}^\eta\subset\mathcal{V}^*$ is an orbit of the second $\mathfrak{G}_n$-action.*

Proof. (i) Let $A_1^\eta\in\mathcal{V}^*$ be the shift corresponding to $\mathcal{V}_1$, as defined in the proof of Lemma 4.5, and A_2^η be the similar shift corresponding to $\mathcal{V}_2$. Observe that $A_1^\eta-A_2^\eta\in\mathcal{A}^0$, and hence there exists $H_2^\eta\in\mathcal{V}_2$ such that $L(H_2^\eta)=A_1^\eta-A_2^\eta$.

For any $M\in\mathcal{V}_1$ put $M^{\mathcal{K}}=I^\eta(M)-M-H_2^\eta$. It follows from the definition of I^η that $L(I^\eta(M))=L(M)+A_1^\eta-A_2^\eta$, hence $L(M^{\mathcal{K}})=0$, that is, $M^{\mathcal{K}}\in\mathcal{K}$.

Let us now find $Q(I^{\bar\eta}(M))$. Since $I^\eta(M)=M+H_2^\eta+M^{\mathcal{K}}$ and $\langle M,H_2^\eta\rangle=(M,A_1^\eta-A_2^\eta)$, one has $Q(I^\eta(M))=Q(M)+Q(H_2^\eta)+Q(M^{\mathcal{K}})+(M,A_1^\eta-A_2^\eta)$, where $(\cdot,\cdot)$ is the standard pairing between $\mathcal{V}$ and $\mathcal{V}^*$ defined in §2.

Since $M\in\mathcal{V}_1$, we get $(M,A_1^\eta-A_2^\eta)=(M,A_2^\eta)=|\operatorname{supp}M\cap\operatorname{supp}A_2^\eta|$. It follows easily from Theorem 2.4(ii) that $\operatorname{supp}A_2^{\bar\eta}=\{(n-i,n-i)\in\Sigma_2\colon Q(P_i)=1\}$. We thus have $(M,A_1^{\bar\eta}-A_2^{\bar\eta})=|\{(n-i,n-i)\in\Sigma_2\cap\operatorname{supp}M\colon Q(P_i)=1\}|$.

On the other hand, $M^{\mathcal{K}}=\sum\{P_i\colon(n-i,n-i)\in\Sigma_2\cap\operatorname{supp}M^{\mathcal{K}}\}$, and hence $Q(M^{\mathcal{K}})=|\{(n-i,n-i)\in\Sigma_2\cap\operatorname{supp}M^{\mathcal{K}}\colon Q(P_i)=1\}|$. Finally, $M+M^{\mathcal{K}}=I^\eta(M)-H_2^\eta\in\mathcal{V}_2$, and hence $\Sigma_2\cap\operatorname{supp}M=\Sigma_2\cap\operatorname{supp}M^{\mathcal{K}}$.

Thus, $Q(I^{\bar\eta}(M))-Q(M)=Q(H_2^{\bar\eta})$ does not depend on M, and hence u, v as in Lemma 4.6(ii) do not exist. Therefore, $\mathcal{A}^{\bar\eta}$ consists of the two orbits of the second $\mathfrak{G}_n$-action. For the same reason the trivial one-element orbit obtained from $\{0\}$ is glued to the image of $Q^{-1}(0)\setminus\{0\}$, and hence the sizes of the orbits are the sizes of $Q^{-1}(0)$ and $Q^{-1}(1)$ for the nondegenerate case. Thus $\kappa=0$ and $m=k(k-1)$, which yield the required result.

(ii), (iii) Follows immediately from Lemma 5.7 and part (i) in the same way as in Lemma 5.8(ii). To find the sizes of the orbits in $\mathcal{A}^0$ observe that $L_1(0)=L_2(0)=0\in\mathcal{A}^0$, hence the two parts that are glued together are $Q^{-1}(0)\setminus\{0\}$ and $Q^{-1}(1)$, and the result follows. $\square$

To obtain Theorem 2.6(ii) it suffices to notice that by Corollary 5.2(ii) $\bar\eta$ as defined in Lemma 5.9 coincides with $\bar\eta$ as defined in Theorem 2.6(ii).

§6. Orbits of the first $\mathfrak{G}_n$-action

In this section we prove Theorem 2.2 concerning the structure of orbits of the first G_n-action.

6.1. The structure of the slices of the first $\mathfrak{G}_n$-action. Let us introduce some notation. We write $X \sim \widetilde{X}$ if the matrices X and $\widetilde{X}$ belong to the same orbit. We do not specify in this notation which $\mathfrak{G}_n$-action is meant; this should be always clear from the context, since each space under consideration carries exactly one $\mathfrak{G}_n$-action.

Let $\Omega = \omega_1\omega_2\dots\omega_t$ denote an arbitrary word in alphabet $\{g_{ij}\}$; we write $\bar{\Omega}$ for the word $\omega_t\dots\omega_2\omega_1$. For any subset Σ of the alphabet we define $[\Omega : \Sigma]$ as the number mod 2 of occurencies of g_σ, $\sigma \in \Sigma$, in Ω. We write ΩX to denote the matrix obtained from X by applying to it the elements $\omega_1, \omega_2, \dots, \omega_t$ in this order. We say that Ω is X-nonredundant if $\{\omega_1\dots\omega_s\}X \neq \{\omega_1\dots\omega_{s-1}\}X$ for $s = 1, \dots, t$ (here $\omega_0 X = X$). Evidently, X-nonredundancy is preserved under isomorphisms and under Ψ_n.

It is easy to see that each slice S^h of the first $\mathfrak{G}_n$-action is a covering of degree 2^k over the slice $\Psi_n(S^h)$ of the second $\mathfrak{G}_n$-action. We are going to find a family of linear functionals $C^h\colon T^n \to \mathbb{F}_2$ with the following property: let $M, \widetilde{M} \in S^h$ be an arbitrary pair of matrices such that $\Psi_n(M) = \Psi_n(\widetilde{M})$, then $M \sim \widetilde{M}$ if and only if $C^h(M) = C^h(\widetilde{M})$. The existence of such a family would imply immediately that each nontrivial orbit of the first $\mathfrak{G}_n$-action is a covering over the corresponding orbit of the second $\mathfrak{G}_n$-action of degree 2^{k-1}, except for the cases when C^h is trivial; in the latter case the degree equals 2^k.

For any $M \in T^n$ we denote by $S(M)$ the set of matrices $\widetilde{M}$ such that $\Psi_n(M) = \Psi_n(\widetilde{M})$ and $M, \widetilde{M}$ belong to the same slice of the first $\mathfrak{G}_n$-action. Further, let $\mathcal{W}$ denote the k-dimensional subspace of T^n generated by the entries $(1, i)$, $1 \leqslant i \leqslant k$, and $\tau\colon T^n \to \mathcal{W}$ denote the natural projection.

Lemma 6.1. *The projection τ provides an affine isomorphism between $S(M)$ and $\mathcal{W}$ for any $M \in T^n(\mathbb{F}_2)$.*

Proof. Indeed, $S(M) = M + (\ker\Psi_n \cap \mathcal{D}_n^\perp) = M + (\mathcal{I}_n \cap \mathcal{D}_n^\perp)$. It follows from the explicit description of $\mathcal{D}_n$ (see Theorem 2.1(ii)) that $\dim(\mathcal{I}_n \cap \mathcal{D}_n^\perp) = k$ and $\ker\tau|_{\mathcal{I}_n\cap\mathcal{D}_n^\perp} = 0$. $\square$

Let us introduce, in addition to Σ_1 and Σ_2, one more subset of the vertex set of $\mathfrak{H}_{n-1}$: Σ_3, consisting of vertices corresponding to matrix entries $(1, i)$, $1 \leqslant i \leqslant k$, see Fig. 3. As in §5.2, we define the subspace $\mathcal{V}_3 \subset \mathcal{V} = (T^{n-1})^*$,

$$\mathcal{V}_3 = \{M \in \mathcal{V}\colon\ \operatorname{supp} M \cap \Sigma_3 = \varnothing\}.$$

Recall that L provides an isomorphism between $\mathcal{V}_3$ and $\mathcal{A}^0 = \mathfrak{D}_{n-1}^\perp$, and $A_1^\eta - A_2^\eta \in \mathcal{A}^0$; therefore, there exists $H_3^\eta \in \mathcal{V}_3$ such that $L(H_3^\eta) = A_1^\eta - A_2^\eta$.

Lemma 6.2. $Q(H_3^\eta) - (A_2^\eta, H_3^\eta) = 0$ *for any* $\eta \in \mathbb{F}_2^k$.

Proof. Let us give an explicit description of H_3^η. We define the matrices $H_i \in \mathcal{V}$, $1 \leqslant i \leqslant k$, in the following way. Let $1 \leqslant i < k$, then $\operatorname{supp} H_i = \{(r, s)\colon n - i \leqslant s \leqslant n - 1, s - r \leqslant n - i - 1\}$. For $n = 2k + 1$ we define H_k in the same way, and for $n = 2k$ $\operatorname{supp} H_k = \{(r, s)\colon r \geqslant k + 1\}$.

Evidently, $H_i \in \mathcal{V}_3$, $1 \leqslant i \leqslant k$. Besides, $Q(H_i) = 1$ for $1 \leqslant i < k$ and $Q(H_k) = n$. Finally, $\langle H_i, H_j\rangle = 1$ for $i \neq j$.

Observe now that $H_3^\eta = \sum\{H_i\colon (n - i, n - i) \in \operatorname{supp} A_2^\eta\}$. Therefore, for $n = 2k + 1$ we have $Q(H_3^\eta) = c + c(c - 1)/2 = c(c + 1)/2$, where $c = |\operatorname{supp} A_2^\eta|$. On the other hand, $(A_2^\eta, H_3^\eta) = 1 + 2 + \dots + c = c(c + 1)/2$. Hence $Q(H_3^\eta) - (A_2^\eta, H_3^\eta) = 0$.

Let now $n = 2k$. If $(n-k, n-k) \notin \operatorname{supp} A_2^\eta$, then the above proof applies. Otherwise $Q(H_3^\eta) = c - 1 + c(c-1)/2 = (c-1)(c+2)/2$ and $(A_2^\eta, H_3^\eta) = 1 + 2 + \dots + c - 1 + c - 1 = (c-1)(c+2)/2$, and hence $Q(H_3^\eta) - (A_2^\eta, H_3^\eta) = 0$. $\square$

Let us define a family of functionals $f^\eta \colon \mathcal{K} \to \mathbb{F}_2$, $\eta \in \mathbb{F}_2^k$, as follows: $f^\eta(X) = Q(X) + (A_2^\eta, X)$. It is easy to see that f^η is a linear functional for any η, since Q is linear on $\mathcal{K}$. The following statement stems immediately from Corollary 5.2.

Lemma 6.3. (i) *Let $n = 2k+1$, then f^η is nontrivial if and only if $\eta \neq 0$.*

(ii) *Let $n = 2k$, then f^η is nontrivial if and only if $\eta \neq \bar\eta = (1, \dots, 1, k \mod 2)$.*

Let now $\theta \colon \mathcal{V} \to \mathcal{K}$ denote the projection on $\mathcal{K}$ along $\mathcal{V}_3$. We lift f^η to the whole $\mathcal{V}$ with the help of θ by defining $F^\eta(X) = f^\eta(\theta(X))$ for any $X \in \mathcal{V}$.

It is easy to check that the standard pairing $(\cdot,\cdot)$ restricted to Σ_3 identifies $\mathcal{K}^*$ with $\mathcal{V}_3^\perp \subset \mathcal{V}^*$. Let $f^\eta = (f_1^\eta, \dots, f_k^\eta) \in \mathcal{V}_3^\perp$; we define a linear functional $c^h = (c_1^h, \dots, c_k^h) \in \mathcal{W}^*$ as follows: $c^h = \Lambda f^{\psi_n(h)}$, where $\Lambda = \sum_{1 \leqslant i \leqslant j \leqslant k} E_{ij}$, and lift it to the whole T^n with the help of τ by defining $C^h(M) = c^h(\tau(M))$ for any $M \in T^n$.

Lemma 6.4. *Let $M \in T^n$ be an arbitrary matrix in S^h, $h \in \mathbb{F}_2^n$. If $\widetilde{M} \in S(M)$ and $\widetilde{M} \sim M$, then $C^h(\widetilde{M}) = C^h(M)$.*

Proof. Indeed, let $\eta = \psi_n(h)$ and let $M' = g_{ij}M$, $M' \neq M$. Then $C^h(M') = C^h(M)$ if $(i,j) \notin \Sigma_3$ and $C^h(M') - C^h(M) = c_j^h + c_{j+1}^h = f_j^\eta$ otherwise. Hence the total variation of $C^h(M)$ along the trajectory defined by an arbitrary M-nonredundant word Ω equals $\operatorname{var}_\Omega C^h = [\Omega : \operatorname{supp} F^\eta]$. Therefore, to prove the Lemma it suffices to show that for any M-nonredundant word Ω such that $\widetilde{M} = \Omega M$ the number $[\Omega : \operatorname{supp} F^\eta]$ vanishes.

Evidently, there exists a decomposition $\Omega = \Omega_1 \Omega_2 \dots \Omega_r$ with $\Omega_i \neq \varnothing$ for $i \geqslant 2$ such that all the elements of Ω_{2i+1} belong to the subgroup $\mathfrak{G}_n^1 \subset \mathfrak{G}_n$ generated by the elements of $\mathcal{V}_1$, and all the elements of Ω_{2i} belong to the subgroup $\mathfrak{G}_n^2 \subset \mathfrak{G}_n$ generated by the elements of $\mathcal{V}_2$. By Lemma 4.5, instead of looking at the trajectory of $M_{2i} = \{\Omega_1 \dots \Omega_{2i}\}\Psi_n(M)$ under Ω_{2i+1} one can look at the trajectory of $(L_1^\eta)^{-1}(M_{2i})$ in $\mathcal{V}_1$, and instead of the trajectory of $M_{2i+1} = \Omega_{2i+1}M_{2i}$ under Ω_{2i+2}, at the trajectory of $(L_2^\eta)^{-1}(M_{2i+1})$ in $\mathcal{V}_2$. These pieces are glued together with the help of the isomorphism I^η, see Fig. 4. Let us find the total variation $\operatorname{var}_\Omega F^\eta$ of the functional F^η along the obtained trajectory.

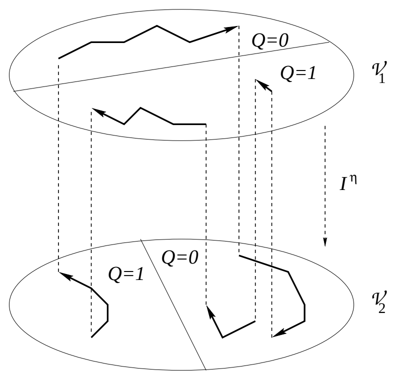

Fig. 4. Trajectory defined by Ω in the charts $\mathcal{V}_1$ and $\mathcal{V}_2$

It follows immediately from Lemma 3.2 that for any $X \in \mathcal{V}$ and $X' = g_{ij}X \neq X$ one has $F^\eta(X') - F^\eta(X) = 1$ iff $(i,j) \in \operatorname{supp} F^\eta$. Therefore, the total variation of F^η along the pieces of the trajectory that lie entirely in $\mathcal{V}_1$ or $\mathcal{V}_2$ equals $[\Omega : \operatorname{supp} F^\eta]$.

Let us find the variation of F^η under the isomorphism I^η. Similarly to the proof of Lemma 5.9, for any $X \in \mathcal{V}_1$ one has $I^\eta(X) = X + H_3^\eta + X^{\mathcal{K}}$ with $X^{\mathcal{K}} \in \mathcal{K}$. Thus, $F^\eta(I^\eta(X)) - F^\eta(X) = F^\eta(H_3^\eta) + F^\eta(X^{\mathcal{K}}) = Q(X^{\mathcal{K}}) - (A_2^\eta, X^{\mathcal{K}})$, since $\theta(H_3^\eta) = 0$.

On the other hand, $Q(I^\eta(X)) - Q(X) = Q(X^{\mathcal{K}}) + Q(H_3^\eta) + (A_1^\eta - A_2^\eta, X)$. Besides, $(A_1^\eta, X) = 0$ since $X \in \mathcal{V}_1$, $(A_2^\eta, X + H_3^\eta + X^{\mathcal{K}}) = 0$ since $I^\eta(X)) = X + H_3^\eta + X^{\mathcal{K}} \in \mathcal{V}_2$, and $Q(H_3^\eta) - (A_2^\eta, H_3^\eta) = 0$ by Lemma 6.2. Adding the last four equalities we get $Q(I^\eta(X)) - Q(X) = Q(X^{\mathcal{K}}) - (A_2^\eta, X^{\mathcal{K}})$. Therefore, $F^\eta(I^\eta(X)) - F^\eta(X) = Q(I^\eta(X)) - Q(X)$.

Recall that by Lemma 4.3 Q is constant on each orbit in $\mathcal{V}_1$ and $\mathcal{V}_2$. Therefore, $\operatorname{var}_\Omega F^\eta = \operatorname{var}_\Omega Q + [\Omega : \operatorname{supp} F^\eta]$. Since $\Psi_n(M) = \Psi_n(\widetilde{M})$, the trajectory defined by Ω is a loop, and hence $\operatorname{var}_\Omega F^\eta = \operatorname{var}_\Omega Q = 0$, thus $\operatorname{var}_\Omega C^h = [\Omega : \operatorname{supp} F^\eta] = 0$. □

Let us now prove the converse statement.

Lemma 6.5. *Let $M \in T^n \setminus \mathcal{I}_n$ be an arbitrary matrix in S^h, $h \in \mathbb{F}_2^n$. If $\widetilde{M} \in S(M)$ and $C^h(\widetilde{M}) = C^h(M)$, then $\widetilde{M} \sim M$.*

Proof. In what follows we assume that h is fixed and $\eta = \psi_n(h)$. First of all, for any $Z \in \mathcal{K}$ we choose two matrices $M_0^Z, M_1^Z \in \mathcal{V}_1$ such that

$$\theta(I^\eta(M_i^Z)) - \theta(M_i^Z) = Z, \qquad Q(M_i^Z) = i, \qquad i = 0, 1.$$

Such a pair exists for any $Z \in \mathcal{K}$, since the first condition defines a translation of $\mathcal{V}_1 \cap \mathcal{V}_2$, and Q is nonlinear on $\mathcal{V}_1 \cap \mathcal{V}_2$, and hence nonconstant on any translation of $\mathcal{V}_1 \cap \mathcal{V}_2$. Moreover, we can further assume that $M_i^Z \neq 0$ and $I^\eta(M_i^Z) \neq 0$ for $i = 0, 1$.

Let us fix an arbitrary nontrivial matrix $X \in \mathcal{V}_1$. By Lemma 4.3, X, $X' = M_{Q(X)}^Z$, and $X'' = M_{Q(X)}^0$ belong to the same $\mathfrak{S}_n^1$-orbit. Let Ω'_Z be an X-nonredundant word in $\{g_{ij}\}$, $(i,j) \notin \Sigma_1$, such that $X' = \Omega'_Z X$. We denote $Y' = I^\eta(X')$, $Y'' = I^\eta(X'')$. By the proof of Lemma 6.4, the variation of Q under I^η equals the variation of F^η. Therefore, if $f^\eta(Z) = 0$, then $Q(Y') = Q(X') = Q(X) = Q(X'') = Q(Y'')$, and hence $Y', Y'' \in \mathcal{V}_2$ belong to the same $\mathfrak{S}_n^2$-orbit. Let Ω''_Z denote a Y'-nonredundant word in $\{g_{ij}\}$, $(i,j) \notin \Sigma_2$, such that $Y'' = \Omega''_Z Y'$. Evidently, $\Omega_Z = \Omega'_Z \Omega''_Z \bar{\Omega}'_0$ is X-nonredundant and $\Omega_Z X = X$.

By Lemma 6.1, it suffices to prove that the value of any linear functional other than c^h can be changed along an appropriate Ω_Z with $f^\eta(Z) = 0$. Take any $W \in \mathcal{K}$, and let $f^W \in \mathcal{V}_3^\perp$ be the dual to W with respect to $(\cdot,\cdot)$ restricted to Σ_3; put $F^W = f^W \circ \theta$. Then, similarly to the proof of Lemma 6.4, $0 = \operatorname{var}_{\Omega_Z} F^W = [\Omega_Z : \operatorname{supp} F^W] + f^W(Z)$. Therefore, for $c^W = \Lambda f^W$ and $C^W = c^W \circ \tau$ one has $\operatorname{var}_{\Omega_Z} C^W = [\Omega_Z : \operatorname{supp} F^W] = f^W(Z)$.

Let now $W = \sum_{i=1}^k w_i P_i$. Evidently, it is enough to consider the following two types of W: (i) $w_i = 0$ for $i \neq j$, $w_j = 1$, where $\eta_j = 0$, and (ii) $w_i = 0$ for $i \neq j_1, j_2$, $w_{j_1} = w_{j_2} = 1$, where $\eta_{j_1} = \eta_{j_2} = 1$. In order to get $\operatorname{var}_{\Omega_Z} C^W = 1$ for $n = 2k+1$ one takes $Z = P_j$ in the first case and $Z = P_{j_1} + P_{j_3}$ with $\eta_{j_3} = 1$ in the second case. For $n = 2k$ one takes either $Z = P_j + P_{j_3}$ with $\eta_{j_3} = 0$ or $Z = P_{j_3}$ with $\eta_{j_3} = 1$ in the first case and $Z = P_{j_1} + P_{j_3}$ with $\eta_{j_3} = 1$ in the second case. □

Proof of Theorem 2.2. Let $n = 2k + 1$ and $\psi_n(h) \neq 0$. Then by Theorem 2.6 and Lemmas 6.3, 6.4, 6.5 the slice S^h consists of two orbits distinguished by C^h. Their sizes are equal since C^h is linear and nontrivial. For $\psi_n(h)=0$ Lemmas 6.3 and 6.5 imply that nontrivial orbits in S^h are just coverings of degree 2^k of nontrivial orbits in $\Psi_n(S^h)=\mathfrak{D}_{n-1}^{\perp}$ described in Theorem 2.6. Besides, each matrix in $\mathcal{I}_n\cap S^h$ forms itself a trivial orbit. The case $n=2k$ is treated in the same way. $\square$

6.2. The structure of the symmetric slices of the first $\mathfrak{G}_n$-action: an alternative approach. In §3.3 we gave an alternative description of the second $\mathfrak{G}_n$-action as the one induced by the dual to the first $\mathfrak{G}_n$-action on the quotient of $(T^n)^*$ modulo the subspace $\mathcal{D}_n$ of the dual invariants of the first $\mathfrak{G}_n$-action. As an immediate corollary of this description we obtain the structure of the slice $\mathcal{D}_n^{\perp}$ at height $(0,\dots,0)$.

Lemma 6.6. *The orbits of the first $\mathfrak{G}_n$-action in the slice $\mathcal{D}_n^{\perp}$ are isomorphic to the orbits of the dual to the second $\mathfrak{G}_n$-action.*

Proof. Indeed, consider the dual mapping $\Phi_n^*\colon (T^{n-1})^*\to T^n0$. It is easy to check that the image of Φ_n^* coincides with $\mathcal{D}_n^{\perp}$. Moreover, $\ker\Phi_n^*=0$, and hence Φ_n^* provides an isomorphism between $(T^{n-1}(\mathbb{F}_2))^*$ and $\mathcal{D}_n^{\perp}$. Now Lemma 3.5 implies that the first $\mathfrak{G}_n$-action on $\mathcal{D}_n^{\perp}$ is isomorphic to the dual to the second $\mathfrak{G}_n$-action. $\square$

We thus get the part of Theorem 2.2 concerning the structure of symmetric slices.

Lemma 6.7. (i) *Let $n=2k+1\geqslant 5$, then each of 2^{k+1} symmetric slices consists of one orbit of size $2^{2k^2+k-1}-\varepsilon_k 2^{k^2+k-1}$, one orbit of size $2^{2k^2+k-1}+\varepsilon_k 2^{k^2+k-1}-2^k$, and 2^k orbits of size* 1, *where $\varepsilon_k=-1$ for $k=4t+1$ and $\varepsilon_k=1$ otherwise*;

(ii) *Let $n=2k\geqslant 6$, then each of 2^k symmetric slices consists of two orbits of size $(2^{2k(k-1)}-1)2^{k-1}$ and 2^k orbits of size* 1.

Proof. The additive group $\mathcal{I}_n$ acts on T^n by translations. Evidently, any such translation takes $\mathcal{D}_n^{\perp}$ to a symmetric slice and any symmetric slice is the translation of $\mathcal{D}_n^{\perp}$ by a vector in $\mathcal{I}_n$. Besides, the action of $\mathcal{I}_n$ commutes with the first $\mathfrak{G}_n$-action, and hence takes orbits to orbits. Therefore, the orbit structure of any symmetric slice coincides with that of the slice $\mathcal{D}_n^{\perp}$.

To find the sizes of the orbits we use Lemmas 6.6, 5.6, and 4.3. Since $\kappa=k$, we immediately get 2^k orbits of size 1. The sizes of the other two orbits are obtained readily from Corollary 5.3 and Lemma 5.4. $\square$

§7. Final Remarks

7.1. The technique of the previous paper [SSV] is substantially generalized in a forthcoming paper joint with A. Zelevinsky to a wide class of intersections of pairs of Schubert cells and double Schubert cells. This generalization is based on the chamber ansatz developed for the flag varieties and semisimple groups in [BFZ, BZ, FZ]. For each reduced decomposition of an element u in a classical Coxeter group of type A, D, E we define a group which acts by symplectic transvections on the $\mathbb{Z}$-module generated by all chamber sets associated with the chosen decomposition. We prove that the $\mathbb{F}_2$-reduction of the above action counts the number of connected components in in the intersection $B\cap B_u$ of two real open Schubert cells in G/B taken in the split form. Analogous results are obtained for the intersections $G^{u,v}=BuB\cap B_-vB_-$ in a semisimple simplylaced group G.

These results lead to the following general setup. Given a connected undirected graph Γ, consider the $\mathbb{F}_2$-vector space $\mathcal{V}_\Gamma$ generated by its vertices and define an $\mathbb{F}_2$-valued bilinear form $\langle p, q\rangle = \sum\{p_i q_j : i \text{ adjacent to } j\}$. Every vertex $\delta \in \Gamma$ determines the symplectic transvection $T_\delta : \mathcal{V}_\Gamma \to \mathcal{V}_\Gamma$ sending p to $p - \langle p, \delta\rangle\delta$. Let us also choose some subset B of vertices of Γ and define the group $\mathfrak{G}_B$ generated by all T_δ, $\delta \in B$.

Problem. *Find the number of orbits of the $\mathfrak{G}_B$-action on $\mathcal{V}_\Gamma$.*

A substantial part of the results of the present paper are valid in this more general setup as well. The authors hope to solve the above Problem at least for the graphs (and their groups) arising from intersections of Schubert cells. Our preliminary considerations, supported by numerical evidence, suggest the following surprising conjecture.

Conjecture. *The number of connected components in the intersection of two open Schubert cells in $SL_{n+1}(\mathbb{R})/B$ in relative position w equals $3 \cdot 2^{n-1}$ for any generic w and $n \geqslant 5$.*

Here genericity means that the graph of bounded chambers introduced implicitly in [SSV] and studied in detail in the forthcoming paper with A. Zelevinsky contains an induced subgraph isomorphic to the Dynkin diagram of E_6.

7.2. Another very intriguing fact is that the group $\mathfrak{G}_n$ studied in the present paper is the $\mathbb{F}_2$-reduction of the monodromy group of the Verlinde algebra $su(3)_n$ introduced in [Zu] and studied recently in [GZV]. (The authors are obliged to S. Chmutov and S. M. Gusein-Zade for this observation.) Notice that the monodromy group of any $su(m)_n$ is interpreted in [GZV] as the monodromy group of the isolated singularity of a certain Newton polynomial expanded in elementary symmetric functions. This gives us a hope to both find a natural representation of Verlinde algebra $su(3)_n$ in the sections of some bundle over the intersection of open opposite Schubert cells and, more generally, to find a relation between the topology of intersections of Schubert cells and singularity theory of symmetric polynomials.

References

[BFZ] A. Berenstein, S. Fomin, and A. Zelevinsky, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math. **122** (1996), 49–149.

[BZ] A. Berenstein and A. Zelevinsky, *Total positivity in Schubert varieties*, Comm. Math. Helv. **72** (1997), 128–166.

[FZ] S. Fomin and A. Zelevinsky, *Total positivity and double Bruhat decomosition*, Preprint 199802-22-01.

[GZV] S. M. Gusein-Zade and A. Varchenko, *Verlinde algebras and the intersection form on vanishing cycles* (1997), preprint, 19 pages.

[Ja] W.A.M. Janssen, *Skew-symmetric vanishing lattices and their monodromy groups*, Math. Ann **266** (1983), 115–133.

[Ke] A. Kerber, *Algebraic Combinatorics Via Finite Group Actions*, BI & Brockhaus, Mannheim, 1991.

[Pf] A. Pfister, *Quadratic Forms with Applications to Algebraic Geometry and Topology*, Cambridge University Press, Cambridge, 1995.

[SSV] B. Shapiro, M. Shapiro, and A. Vainshtein, *On the number of connected components in the intersection of two open opposite Schubert cells in SL_n/B*, Int. Math. Res. Notes (1997), no. 10, 469–493.

[Zu] J. -B. Zuber, *Graphs and reflection groups*, Comm. Math. Phys **179** (1996), 265–294.