COMPLEXIFIED LOGARITHMIC INVERSE MOMENT PROBLEM FOR POLYGONS

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To Andrei Gabrielov, a great mathematician, earthquake predictor, bird watcher, and a friend

Abstract. We study a complexified logarithmic inverse moment problem for polygonal measures in the plane. Using the scaled harmonic moments introduced by Burman, Fröberg and Shapiro, we associate to a cyclically ordered configuration of complex vertices an algebraic fiber determined by the first $2n-2$ moments. Our main result identifies the generic complexified fiber with a finite union of affine one-dimensional families corresponding to permutations of the recovered unordered vertex set. We derive a rational generating function for the complete moment sequence, formulate a vertex-wise equipotentiality criterion in terms of slope-jump quantities γ_j, and explain the relation with the classical logarithmic inverse problem for polygons studied in earlier work of Brodsky–Strakhov and Burman–Fröberg–Shapiro. The emphasis of the present note is on the geometry of the generic complexified inverse fiber.

1. Introduction and main results

Inverse problem in logarithmic potential theory has attracted substantial attention since the publication of the fundamental paper [8], where P.S. Novikov, in particular, proved that two convex (or, more generally, star-shaped) domains in $\mathbb{C}$ with unit density cannot have the same logarithmic potential near ∞. The knowledge of the germ of a logarithmic potential of a finite compactly supported Borel measure μ at ∞ is equivalent to the knowledge of the sequence of its harmonic moments $\{m_j(\mu)\}_{j=0}^{\infty}$, where the j-th harmonic moment of μ is defined by:

$$m_j(\mu) = \int_{\mathbb{C}} z^j d\mu(z).$$

Therefore Novikov's result is the statement that two convex domains in $\mathbb{C}$ with unit density cannot have coinciding sequences of harmonic moments. It is well-known that already for non-convex domains with unit density the uniqueness in this problem no longer holds. For instance, examples of pairs of non-convex polygons with the same logarithmic potential near ∞ can be found on [2, p. 333], see Fig. 1 below. We will call two bounded domains such that the restrictions of the Lebesgue measure to these domains have the same logarithmic potential near infinity *equipotential*. Paper [2] contains also an example of two equipotential simply connected polygons discovered by A. Gabrielov. Notice that if two polygons are equipotential, then they must have the same set of vertices, see [2, Corollary 2 and Lemma 2]. Several additional restrictions on equipotential polygons can be found in [2].

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The present note should be viewed as a continuation and a refinement of the polygonal moment formalism developed in [3]. While several auxiliary constructions appearing below are related to the higher-dimensional and axial moment techniques considered there, the genuinely new point of the present paper is the analysis of the *complexified inverse fiber*. More precisely, after passing from the real polygonal inverse problem to an auxiliary complexified system in the variables (z_j, u_j), we show that the generic fiber determined by the first $2n-2$ scaled harmonic moments decomposes into finitely many affine one-dimensional families indexed by permutations of the recovered unordered vertex set. This phenomenon appears to be specific to the logarithmic polygonal inverse problem and does not seem to have been observed previously.

Example 1. Consider two 6-tuples $T = \{\pm\sqrt{3} \pm I, \pm 2I\}$ and $T' = \{\pm\frac{1\pm\sqrt{3}I}{2}, \pm 1\}$. Let $F \subset \mathbb{C}$ be the difference of the convex hull of T and the union of the set of 6 triangles obtained as the orbit of the triangle with nodes $(\sqrt{3}+I, \sqrt{3}-I, 1)$ under the rotation by $\frac{\pi}{3}$, see Fig. 1. Let $F' \subset \mathbb{C}$ be the difference of the convex hulls of T and of T'. Then F and F' have the same logarithmic potential.

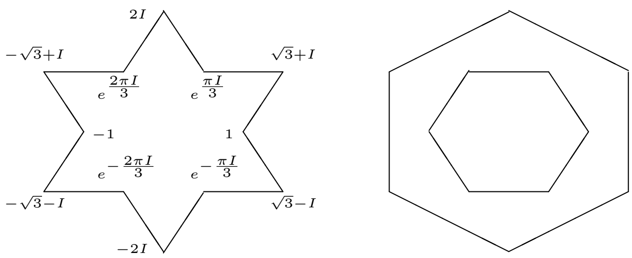

FIGURE 1. Two equipotential polygons: F on the left, F' on the right. One simply connected and one not.

In [9] the author jointly with D. Pasechnik discussed the following question.

Problem 1. Given a finite set $S \subset \mathbb{C}$, determine whether there exist two equipotential polygons whose sets of vertices coincide with S.

One can show that for generic S no pairs of equipotential polygons exist. On the other hand, the following statement holds, see [9].

Theorem A. For each $n \geq 6$, there exists a finite set $S \subset \mathbb{C}$, with $|S| = n$, admitting a pair of equipotential polygons. No such S exists if $|S| \leq 5$.

In [3] the authors considered the inverse moment problem and the relations among harmonic moment for the following class of signed measures which include the class of Lebesgue measures restricted to simply connected plane polygons.

Namely, to each sequence of cyclically ordered points $Z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ in the plane $\mathbb{R}^2 \simeq \mathbb{C}$, we associate the oriented closed polygonal curve

$$\Gamma_Z := \cup_{i=1}^{n-1} [\overrightarrow{z_i, z_{i+1}}] \cup [\overrightarrow{z_n, z_1}],$$

where each edge is oriented according to the cyclic order of its vertices. Additionally, we associate to the above Z the signed measure μ_Z as follows. Choose some vertex z_j as the base vertex and define

$$\mu_Z := \sum_{i=1}^{n} \pm \mu_{\Delta_{j,i}},$$

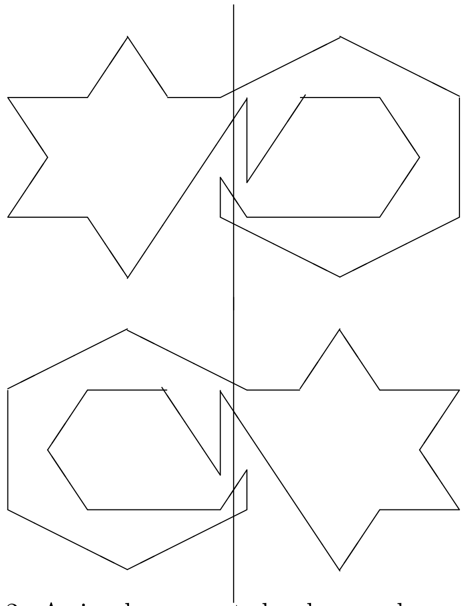

FIGURE 2. A simply connected polygon whose reflection in the y-axis is equipotential to it.

where $\Delta_{j,i}$ is the (possibly degenerate) triangle spanned by the points (z_j, z_i, z_{i+1}) and $\mu_{\Delta_{j,i}}$ is the restriction of the Lebesgue measure to this triangle. (For degenerate triangles the corresponding measure element vanishes.) Finally, for non-degenerate triangles, the sign $\pm$ is chosen as follows. Prescribe the cyclic orientation (z_j, z_i, z_{i+1}) to the vertices of $\Delta_{j,i}$. If this cyclic orientation traverses the boundary of $\Delta_{j,i}$ counterclockwise, then the sign is $+$, otherwise the sign is $-$.

Remark 1. One can easily show that the resulting measure μ_Z is independent of the choice of the base vertex z_j. Moreover if Γ_Z is a non-selfintersecting curve in $\mathbb{R}^2$, then μ_Z coincides with the restriction of the Lebesgue measure to the polygon $\mathcal{P}_Z$ bounded by Γ_Z in case when the cyclic orientation of Γ_Z orients it counterclockwise and equals minus the latter measure if the latter orientation is clockwise.

Take again $\mathbf{a} = (a_1, \dots, a_n), a_j = (x_j, y_j) \in \mathbb{C}^2$; $z_j = x_j + iy_j, \bar{z}_j = x_j - iy_j$. Obviously, z_j and $\bar{z}_j$ determine a_j since $x_j = (z_j + \bar{z}_j)/2$ and $y_j = (z_j - \bar{z}_j)/2i$. Set

$$\nu_k(\mathbf{z}, \bar{\mathbf{z}}) \overset{\text{def}}{=} \binom{k}{2} m_{k-2}(\mu_{\mathbf{a},\mathcal{T}}) \quad \text{and} \quad \bar{\nu}_k(\mathbf{z}, \bar{\mathbf{z}}) = \binom{k}{2} \bar{m}_{k-2}(\mu_{\mathbf{a},\mathcal{T}}).$$

In particular, $\nu_0 = \nu_1 = \bar{\nu}_0 = \bar{\nu}_1 = 0$ for all $(\mathbf{z}, \bar{\mathbf{z}})$.

Remark 2. The index shift $(k-2) \mapsto k$ in the latter formulas is convenient since the normalized moments ν_k and $\bar{\nu}_k$ are homogeneous of degree k with respect to the action of dilatations on $\mathbb{C}^2$. In other words, if $t\mathbf{z} = (tz_1, \dots, tz_n)$ then $\nu_k(t\mathbf{z}, t\bar{\mathbf{z}}) = t^k \nu_k(\mathbf{z}, \bar{\mathbf{z}})$.

Theorem B (see Theorem 1.6 of [3]). For any positive integer $k \geq 2$, one has

$$\begin{aligned} \nu_k(\mathbf{z}, \bar{\mathbf{z}}) &= \frac{i}{4} \sum_{j=1}^{n} (\bar{z}_j - \bar{z}_{j+1}) \frac{z_j^k - z_{j+1}^k}{z_j - z_{j+1}} \\ &= \frac{i}{4} \sum_{j=1}^{n} (\bar{z}_j - \bar{z}_{j+1})(z_j^{k-1} + z_j^{k-2} z_{j+1} + \dots + z_{j+1}^{k-1}), \quad (1.1) \end{aligned}$$

where $z_{n+1} \stackrel{\text{def}}{=} z_1$ and $\bar{z}_{n+1} \stackrel{\text{def}}{=} \bar{z}_1$.

Namely, all harmonic moments can be expressed as rational functions of the first $2n-2$ moments $\nu_2(\mathbf{z}, \bar{\mathbf{z}}), \dots, \nu_{2n-1}(\mathbf{z}, \bar{\mathbf{z}})$, and these $2n-2$ moments are algebraically independent. More precisely, denote by $\mathfrak{F}_n$ the field extension of $\mathbb{C}$ generated by the sequence of polynomials $\{\nu_j(\mathbf{z}, \bar{\mathbf{z}})\}_{j=2}^{\infty}$.

Theorem C (see Theorem 1.7 of [3]). (1) $\mathfrak{F}_n = \mathbb{C}(\nu_2, \dots, \nu_{2n-1})$, hence it is isomorphic to the field of rational functions in $2n-2$ independent complex variables.
(2) $\mathfrak{F}_n \supset \mathbb{C}(\mathbf{z})^{S_n}$, where $\mathbb{C}(\mathbf{z})^{S_n}$ is the field of symmetric rational functions.

Explicit formulas expressing harmonic moments $\nu_j(\mathbf{z}, \bar{\mathbf{z}})$ with $j \geq 2n$ via the first $2n-2$ are given by rational functions. However all their denominators are powers of one fixed polynomial $\mathfrak{D}_n$, the determinant of the matrix (1.2) given by:

$$U = \begin{pmatrix} \nu_{n-1}(\mathbf{z}, \bar{\mathbf{z}}) & \nu_{n-2}(\mathbf{z}, \bar{\mathbf{z}}) & \dots & \nu_1(\mathbf{z}, \bar{\mathbf{z}}) & \nu_0(\mathbf{z}, \bar{\mathbf{z}}) \\ \nu_n(\mathbf{z}, \bar{\mathbf{z}}) & \nu_{n-1}(\mathbf{z}, \bar{\mathbf{z}}) & \dots & \nu_2(\mathbf{z}, \bar{\mathbf{z}}) & \nu_1(\mathbf{z}, \bar{\mathbf{z}}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu_{2n-2}(\mathbf{z}, \bar{\mathbf{z}}) & \nu_{2n-3}(\mathbf{z}, \bar{\mathbf{z}}) & \dots & \nu_n(\mathbf{z}, \bar{\mathbf{z}}) & \nu_{n-1}(\mathbf{z}, \bar{\mathbf{z}}) \end{pmatrix}. \tag{1.2}$$

In fact, if one considers the *ring extension* $\mathcal{R}_n$ of $\mathbb{C}$ (as opposed to field) generated by the sequence of polynomials $\{\nu_j(\mathbf{z}, \bar{\mathbf{z}})\}_{j=2}^{\infty}$, the situation is as follows.

Theorem D (see Theorem 1.8 of [3]). (1) The ring $\mathcal{R}_n = \mathbb{C}[\nu_2, \nu_3, \dots]$ is not generated by any finite collection of harmonic moments $\nu_2, \dots, \nu_N$.
(2) For the polynomial $\mathfrak{D}_n \in \mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]$ given by the determinant of (1.2), the localization $\mathcal{R}_n|_{\mathfrak{D}_n}$ is isomorphic to $\mathbb{C}[\nu_2, \dots, \nu_{2n-1}]\left[\frac{1}{\mathfrak{D}_n}\right]$.

Notice that $\mathcal{R}_n$ does not contain the ring $\mathbb{C}[\mathbf{z}]^{S_n}$ of symmetric polynomials in the variables $z_1, \dots, z_n$ as a subring since the expression of the basic (e.g. elementary) symmetric polynomials via $\nu_2, \dots, \nu_{2n-1}$ involves division by some powers of $\mathfrak{D}_n$.

The main goal of the present paper is to study the complex algebraic fibers obtained by prescribing the first $(2n-2)$ scaled harmonic moments. The dimension and the number of components of such fibers may depend on the prescribed values. We also discuss the real part of these varieties, which is the locus relevant to the original logarithmic inverse problem.

The structure of the paper is as follows. In § 2 we recall only the Prony terminology needed later and refer to [11] for the systematic treatment of higher open and closed Prony systems arising from projections of convex polytopes. In § 3 we present our main result on the complexified inverse fiber. Finally, in § 4 we collect a few questions left for future work.

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2. Preliminaries: Prony's system and its generalizations

This section fixes notation and recalls the one-dimensional Prony language used below. The higher open and closed systems mentioned here are not used as independent new results of the present note; they are included only to explain the relation with axial moment data. For details, proofs, and the multidimensional motivation from projections of convex polytopes, see [11].

2.1. **Original Prony system.** The classical system introduced by Gaspard Clair François Marie Riche de Prony around 1795 [10] has been an object of importance and extensive study both in mathematics and other natural sciences for over two centuries. In what follows, we use its algebraic version as discussed in e.g. [4], see also references therein. Another popular version is trigonometric, but we do not use it below. The main area of application for the Prony system is signal processing.

In the classical case the Prony system looks like

$$\sum_{i=1}^{n} a_i x_i^j = m_j, \quad j = 0, \dots, 2n-1, \tag{2.1}$$

where the entries $m_0, \dots, m_{2n-1}$ (usually called the *moments*) are known, while the entries $a_1, \dots, a_n$ and $x_1, \dots, x_n$ are unknown variables. The variables $x_1, \dots, x_n$ are called the *nodes* and the variables $a_1, \dots, a_n$ are called the *amplitudes* of the Prony system. The standard interpretation of (2.1) is that if we place the charges of values $a_1, \dots, a_n$ at the points $x_1, \dots, x_n$ resp., then $\sum_{i=1}^{n} a_i x_i^j$ is the j-th moment of this configuration of charges. To solve (2.1) means to solve the inverse moment problem for a configuration of n charges knowing $2n$ first moments of this configuration.

The classical Prony method of solving (2.1) under some genericity assumptions is as follows.
Step 1. To find the nodes $(x_1, x_2, \dots, x_n)$, solve the linear system

$$M_n := \begin{pmatrix} m_0 & m_1 & \dots & m_{n-1} \\ m_1 & m_2 & \dots & m_n \\ \vdots & \vdots & \vdots & \vdots \\ m_{n-1} & m_n & \dots & m_{2n-2} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n-1} \end{pmatrix} = - \begin{pmatrix} m_n \\ m_{n+1} \\ \vdots \\ m_{2n-1} \end{pmatrix} \tag{2.2}$$

for the unknown coefficients $q_0, \dots, q_{n-1}$.
Step 2. Find all the roots of $q(x) = x^n + \sum_{j=0}^{n-1} q_j x^j$. These roots (assumed distinct) are exactly the nodes $(x_1, x_2, \dots, x_n)$.
Step 3. Substituting the recovered nodes $(x_1, x_2, \dots, x_n)$ in (2.1) solve the linear overdetermined system for the variables $(a_1, a_2, \dots, a_n)$.

2.2. **Higher open Prony systems.** The following formulation is taken from the axial moment framework of [11]. For any positive integer n and $d \le n-1$, the d-th open Prony system with n nodes is given by the system of $2n-d$ equations of the form

$$\sum_{i=1}^{n-d} a_i h_j(x_i, x_{i+1}, \dots, x_{i+d}) = \binom{j+d}{d} m_j, \text{ where } j = 0, \dots, 2n-d-1. \tag{2.3}$$

Here $a_1, \dots, a_{n-d}$ and $x_1, \dots, x_n$ are the unknown variables called the *amplitudes* and the *nodes* resp. and $m_0, \dots, m_{2n-d-1}$ are the parameters of the system called the *moments*. Observe that for $d = 0$, we get the classical Prony system (2.1).

Analogously to the classical case, to solve the Prony system, we start with determining its nodes $(x_1, \dots, x_n)$ using an appropriate Hankel matrix, comp. [7]. For a given sequence $M = (m_0, \dots, m_{2n-d-1})$, introduce the extended normalized sequence $C_M = (c_0, c_1, \dots, c_{2n-1})$, where $c_0 = c_1 = \dots = c_{d-1} = 0$ and $c_{d+i} = \frac{(i+d)!}{i!} m_i$, for $i = 0, \dots, 2n-d-1$. Now associate to the sequence C_M the $(n+1) \times (n+1)$-Hankel matrix H_M whose diagonal entries are given by the sequence C_M.

Proposition 1. In the above notation, for a generic vector M, the Hankel matrix H_M has corank equal to 1. Denote its 1-dimensional kernel by $(q_0, q_1, \dots, q_n)$ and form the polynomial $q_M(x) = q_n x^n + q_{n-1}x^{n-1} + \dots + q_0$. (Observe that $q_n \neq 0$.) Then the set of nodes $(x_1, \dots, x_n)$ coincides with the set of all roots of $q_M(x)$.

Problem 2 (Real-rootedness)**.** In the above notation, characterize those real moment vectors M for which the polynomial $q_M(x)$ has only real roots.

Remark 3 (Discussion of the real-rootedness problem). This is the first place where the higher open Prony system differs substantially from the classical Prony system. For $d = 0$, the moments come from a signed atomic measure on the line; if one assumes positivity of the amplitudes, the corresponding Hankel matrices are positive definite and the usual orthogonal-polynomial argument implies real and simple nodes. For $d > 0$ the normalized sequence C_M has the forced initial zeros $c_0 = \dots = c_{d-1} = 0$, and the Hankel matrix H_M is therefore not a standard moment matrix of a positive measure. Thus positivity of H_M alone is not the correct condition.

A natural replacement is a total-positivity condition on the finite Hankel matrix built from C_M, or equivalently on the minors defining the recurrence for q_M. The expected picture is that real-rootedness of q_M should correspond to the membership of M in the real part of the directional moment variety together with suitable sign-regularity of these Hankel minors. In this formulation the problem becomes a higher analogue of the classical Markov–Krein moment problem: one first recovers the candidate nodes as the zeros of q_M, and then asks whether the resulting divided-difference data define a nonnegative spline density supported on these nodes.

After finding the nodes (assumed distinct), we can find the amplitudes $(a_1, \dots, a_{n-d})$ by solving the linear system (2.3) for fixed $x_1, \dots, x_n$.

Conjecture 1. If $x_1, \dots, x_n$ are distinct, then the square amplitude matrix obtained from (2.3) is nonsingular. More precisely, computer experiments suggest that its determinant is, up to sign,

$$\prod_{1 \le i < j \le n,\ j-i > d} (x_i - x_j).$$

Problem 3 (Positivity of the recovered amplitudes)**.** In the above notation, assuming that $q_M(x)$ has real roots $\mathcal{X} = \{x_1 < \dots < x_n\}$, characterize those M for which the recovered amplitude vector belongs to $L^+_{d,\mathcal{X}}$.

Remark 4 (Discussion of the positivity problem). This question is logically separate from the real-rootedness of q_M. Real roots only identify the possible projected vertices; they do not guarantee that the corresponding push-forward density is produced by a convex polytope, nor even that it is nonnegative. After the nodes have been recovered, the amplitudes are obtained from a square subsystem of (2.3). Hence the positivity problem is a finite-dimensional semialgebraic problem in the coordinates of M.

One practical way to express the answer is to pass from moment coordinates to spline coordinates. The functions associated with consecutive nodes form a basis of $L_{d,\mathcal{X}}$, and the cone $L^+_{d,\mathcal{X}}$ is the subset of this spline space consisting of densities that are nonnegative and satisfy the endpoint and smoothness constraints described below in the spline constraints described below. Therefore the inequalities are expected to be the pull-backs, under the linear reconstruction map from moments to spline coefficients, of the inequalities defining the nonnegative spline cone. For $d = 0$ this reduces to the usual positivity of atomic weights; for $d = 1$ it reduces to nonnegativity of the interval densities; for $d \ge 2$ it involves positivity of a piecewise polynomial density and its B-spline coordinates.

Theorem 1. The prime ideal $L_k(d,n)$ in $\mathbb{R}[m_0, m_1, \ldots, m_k]$ is generated by the maximal minors of the Hankel matrix

$$H(m_0,\ldots,m_k) := \begin{pmatrix} c_0 & c_1 & \cdots & c_n & c_{n+1} & \cdots & c_{k+d-n} \\ c_1 & c_2 & \cdots & c_{n+1} & c_{n+2} & \cdots & c_{k+d-n+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n} & c_{2n+1} & \cdots & c_{k+d} \end{pmatrix},$$

where $c_0 = c_1 = \ldots = c_{d-1} = 0$ and $c_{i+d} = \frac{(i+d)!}{i!} m_i$ for each $i = 0, \ldots, k$.

2.3. **Higher closed Prony systems.** Following [11], the closed higher Prony system is the cyclic counterpart of (2.3). It is the form naturally suggested by polygonal measures, where the index i labels an edge or a cyclic block of consecutive vertices rather than an interval in a linearly ordered list. This cyclicity also suggests a relation with cyclic polytopes: both settings are governed by consecutive blocks of nodes, but in the closed system the first and last blocks are glued together.

For any positive integer n and $d \le n-1$, the d-th closed Prony system with n nodes is given by the system of $2n$ equations of the form

$$\begin{cases} \sum_{i=1}^{n} a_i h_j(x_i, x_{i+1}, \ldots, x_{i+d}) = 0, \text{ where } j = 0, \ldots, d-1, \\ \sum_{i=1}^{n} a_i h_j(x_i, x_{i+1}, \ldots, x_{i+d}) = \binom{j+d}{d} m_j, \text{ where } j = d, \ldots, 2n-1, \end{cases} \tag{2.4}$$

where the tuple of indices $(i, i+1, \ldots, i+d)$ is understood cyclically. (By definition, $h_0(x_i, x_{i+1}, \ldots, x_{i+d}) = 1$.)

Here $a_1, \ldots, a_n$ and $x_1, \ldots, x_n$ are the unknown variables called the *amplitudes* and the *nodes* resp. and $m_0, \ldots, m_{2n-1}$ are the parameters of the system called the *moments.*

Remark 5. The closed higher Prony system is included here because it is the natural cyclic analogue of the open system and is the form arising from polygonal measures. Its complete solution seems to require additional information beyond the usual Hankel method. In the open case the boundary knots x_1 and x_n play a distinguished role and the Hankel recurrence recovers an ordinary annihilating polynomial. In the closed case there is no boundary, and the cyclic blocks impose compatibility conditions similar in spirit to those appearing for cyclic polytopes and periodic splines. Thus a plausible strategy is to combine the Hankel-type reconstruction of the unordered nodes with additional cyclic compatibility equations for the amplitudes. This is precisely the mechanism that appears below for polygonal measures, where the recovered vertex set still has to be equipped with a cyclic ordering.

3. Main results

At first we will present an alternative condition for the equipotentiality of 2 polygons (closed curves in $\mathbb{C}^2$). Following [3], for $Z = (z_1, \ldots, z_n)$ and $\bar{Z} = (\bar{z}_1, \ldots, \bar{z}_n)$, introduce the generating function $\Psi_{Z,\bar{Z}}(w) := \sum_{k=2}^{\infty} \nu_k(\mathbf{z}, \bar{\mathbf{z}}) w^{k-2}$.

Lemma 1. In the above notation,

$$\Psi_{Z,\bar{Z}}(w) = \frac{i}{4} \sum_{j=1}^{n} \frac{z_j^2}{1 - wz_j} \left(\frac{\bar{z}_j - \bar{z}_{j+1}}{z_j - z_{j+1}} - \frac{\bar{z}_{j-1} - \bar{z}_j}{z_{j-1} - z_j} \right).$$

Proof. Indeed,

$$\Psi_{Z,\bar{Z}}(w) = \frac{i}{4} \sum_{k=2}^{\infty} w^{k-2} \sum_{j=1}^{n} \frac{\bar{z}_j - \bar{z}_{j+1}}{z_j - z_{j+1}} (z_j^k - z_{j+1}^k) = \frac{i}{4} \sum_{j=1}^{n} \frac{\bar{z}_j - \bar{z}_{j+1}}{z_j - z_{j+1}} \sum_{k=2}^{\infty} (z_j^k - z_{j+1}^k) w^{k-2} =$$

$$\frac{i}{4}\sum_{j=1}^{n}\frac{\bar z_j-\bar z_{j+1}}{z_j-z_{j+1}}\left(\frac{z_j^2}{1-wz_j}-\frac{z_{j+1}^2}{1-wz_{j+1}}\right)=\frac{i}{4}\sum_{j=1}^{n}\frac{z_j^2}{1-wz_j}\left(\frac{\bar z_j-\bar z_{j+1}}{z_j-z_{j+1}}-\frac{\bar z_{j-1}-\bar z_j}{z_{j-1}-z_j}\right).$$

□

Corollary 1. Two closed polygonal curves in $\mathbb{C}^2$ are equipotential if their unordered sequences Z coincide and values of the quantities

$$\gamma_j=\frac{\bar z_j-\bar z_{j+1}}{z_j-z_{j+1}}-\frac{\bar z_{j-1}-\bar z_j}{z_{j-1}-z_j},\ j=1,\dots,n \tag{3.1}$$

coincide at each vertex z_j. (These quantities depend on the choice of cyclic orderings.) Assuming that all z_j are distinct we obtain $(n-1)!$ cyclic orderings each giving a system of linear equations in the variables $\bar Z=(\bar z_1,\dots,\bar z_n)$.

Lemma 2. Given a cyclically ordered tuple $Z=(z_1,\dots,z_n)$ with pairwise distinct entries, the vector $\Gamma=(\gamma_1,\dots,\gamma_n)$ belongs to the linear subspace

$$V_Z=\left\{(\eta_1,\dots,\eta_n)\in\mathbb{C}^n:\sum_{j=1}^{n}\eta_j=0,\quad \sum_{j=1}^{n}z_j\eta_j=0\right\}.$$

For generic Z, this subspace has codimension 2.

Proof. Put $s_j=(\bar z_j-\bar z_{j+1})/(z_j-z_{j+1})$, with indices understood cyclically. Then $\gamma_j=s_j-s_{j-1}$. Hence

$$\sum_{j=1}^{n}\gamma_j=\sum_{j=1}^{n}(s_j-s_{j-1})=0.$$

Moreover,

$$\sum_{j=1}^{n}z_j\gamma_j=\sum_{j=1}^{n}z_js_j-\sum_{j=1}^{n}z_js_{j-1}=\sum_{j=1}^{n}(z_j-z_{j+1})s_j=\sum_{j=1}^{n}(\bar z_j-\bar z_{j+1})=0.$$

If the z_j are not all equal, the two displayed linear equations are independent; in particular this holds for pairwise distinct Z. □

Lemma 3. Let $\mathbf{a}=(a_1,\dots,a_n)$, $a_j=(x_j,y_j)\in\mathbb{C}^2$, and put $z_j=x_j+iy_j$, $\bar z_j=x_j-iy_j$. On the Zariski open set where the Toeplitz determinant in (3.6) is nonzero, the complete sequence of scaled harmonic moments $\{\nu_k(\mathbf{z},\bar{\mathbf{z}})\}_{k=2}^{\infty}$ is uniquely determined by the $(2n-2)$-tuple $\{\nu_2(\mathbf{z},\bar{\mathbf{z}}),\nu_3(\mathbf{z},\bar{\mathbf{z}}),\dots,\nu_{2n-1}(\mathbf{z},\bar{\mathbf{z}})\}$. Thus, for moment data $\Xi=(\xi_2,\xi_3,\dots,\xi_{2n-1})$ in this open set, the corresponding complex algebraic fiber $\mathcal{A}_\Xi$ is given by

$$\begin{cases}\nu_2(\mathbf{z},\bar{\mathbf{z}})=\xi_2;\\ \nu_3(\mathbf{z},\bar{\mathbf{z}})=\xi_3;\\ \vdots\\ \nu_{2n-1}(\mathbf{z},\bar{\mathbf{z}})=\xi_{2n-1}.\end{cases} \tag{3.2}$$

Any two points of the same such fiber have one and the same complete moment sequence $\{\nu_k(\mathbf{z},\bar{\mathbf{z}})\}_{k=2}^{\infty}$.

By (1.1), the system (3.2) is given by

$$\begin{cases}\frac{i}{4}\sum_{j=1}^{n}(\bar z_j-\bar z_{j+1})(z_j+z_{j+1})=\xi_2;\\ \frac{i}{4}\sum_{j=1}^{n}(\bar z_j-\bar z_{j+1})(z_j^2+z_jz_{j+1}+z_{j+1}^2)=\xi_3;\\ \quad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\\ \frac{i}{4}\sum_{j=1}^{n}(\bar z_j-\bar z_{j+1})(z_j^{2n-2}+z_j^{2n-3}z_{j+1}+\dots+z_{j+1}^{2n-2})=\xi_{2n-1}.\end{cases} \tag{3.3}$$

Remark 6. The left-hand sides of system (3.3) are homogeneous with respect to both $\mathbf{z}$ and $\bar{\mathbf{z}}$. Additionally, (3.3) is linear with respect to $\bar{\mathbf{z}}$ and invariant under the shift $z_j \mapsto z_j + K$, $j = 1, \dots, n$.

Denote $u_1 = \frac{i}{4}(\bar{z}_1 - \bar{z}_2)$, $u_2 = \frac{i}{4}(\bar{z}_2 - \bar{z}_3)$, $\dots$, $u_n = \frac{i}{4}(\bar{z}_n - \bar{z}_1)$. In these (3.3) takes the form

$$\begin{cases} \sum_{j=1}^n u_j = 0 \\ \sum_{j=1}^n u_j h_1(z_j, z_{j+1}) = \xi_2 \\ \sum_{j=1}^n u_j h_2(z_j, z_{j+1}) = \xi_3 \\ \quad \vdots \qquad \vdots \qquad \vdots \\ \sum_{j=1}^n u_j h_{2n-2}(z_j, z_{j+1}) = \xi_{2n-1}. \end{cases} \tag{3.4}$$

Let $\mathbb{C}^{2n} = \mathbb{C}^n_{\mathbf{u}} \times \mathbb{C}^n_{\mathbf{z}}$ denote the space of variables $(\mathbf{u}, \mathbf{z})$ with $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{z} = (z_1, z_2, \dots, z_n)$.

For $r \geq 0$ set

$$h_r(x, y) = x^r + x^{r-1}y + \cdots + y^r = \frac{x^{r+1} - y^{r+1}}{x - y}.$$

Lemma 4. Let $z_1, \dots, z_n$ be pairwise distinct and let M be the $(n-1) \times (n-1)$ matrix with columns indexed by $j = 1, \dots, n-1$ and rows indexed by $r = 0, \dots, n-2$,

$$M_{r+1,j} = h_r(z_j, z_{j+1}).$$

Then

$$\det M = \pm \prod_{1 \leq i < j \leq n,\ j \geq i+2} (z_j - z_i).$$

In particular, M is nonsingular whenever $z_1, \dots, z_n$ are pairwise distinct.

Proof. Write $v(t) = (t, t^2, \dots, t^{n-1})^T$. Since

$$h_r(x, y) = \frac{x^{r+1} - y^{r+1}}{x - y}, \qquad r = 0, \dots, n-2,$$

the j-th column of M is

$$\frac{v(z_j) - v(z_{j+1})}{z_j - z_{j+1}}.$$

Hence

$$\det M = \frac{\det(v(z_1) - v(z_2), \dots, v(z_{n-1}) - v(z_n))}{\prod_{j=1}^{n-1}(z_j - z_{j+1})}.$$

The determinant in the numerator is alternating in $z_1, \dots, z_n$ and vanishes when two of them coincide. Its total degree is $1 + 2 + \cdots + (n-1)$, the same as the degree of the Vandermonde product. Consequently it is a nonzero constant multiple of $\prod_{1 \leq i < j \leq n}(z_j - z_i)$. Dividing by $\prod_{j=1}^{n-1}(z_j - z_{j+1})$ gives the stated formula, up to an overall sign. The constant is nonzero, for example by comparing the coefficient of $z_1^0 z_2^1 \cdots z_n^{n-1}$ after expanding the column differences. □

Theorem 2 (generic complexified fiber)**.** Let $\Xi = (\xi_2, \dots, \xi_{2n-1})$ be moment data outside the following exceptional loci: the Toeplitz determinant in (3.6) is nonzero, the polynomial recovered from (3.5) has n distinct roots, and for every cyclic ordering of these roots the linear coefficient matrix in the variables $u_1, \dots, u_n$ has rank $n-1$. Equivalently, Ξ belongs to the complement of a natural discriminant/determinantal hypersurface in the parameter space. Then the set S_Ξ of solutions of (3.4) is a union of $n!$ lines. Each of these lines lies in the fiber $\mathbb{C}^n_{\mathbf{u}}$ over a point $\pi(\mathbf{z})$. Here $\mathbf{z}$ is some point in $\mathbb{C}_{\mathbf{z}}$, π is a permutation from S_n and $\pi(\mathbf{z})$ is the point obtained by applying π to coordinates of

$\mathbf{z}$. The line over the point $\mathbf{z} = (z_1, z_2, \dots, z_n)$ has the direction vector given by $\mathfrak{z} = (z_1 - z_2, z_2 - z_3, \dots, z_n - z_1)^T$.

Proof. The coefficients of the characteristic polynomial $\chi_{\mathbf{z}}(w) = w^n - e_1(\mathbf{z})w^{n-1} + e_2(\mathbf{z})w^{n-2} - \dots + (-1)^n e_n(\mathbf{z})$ whose (unordered) roots coincide with $\mathbf{z} = (z_1, \dots, z_n)$ can be found from the linear system of equation below, see Proof of Theorem 1.5 in [3]. This linear system has the form:

$$U \cdot E = V, \tag{3.5}$$

where U is the Toeplitz $n \times n$-matrix, given by

$$U = \begin{pmatrix} \xi_{n-1} & \xi_{n-2} & \dots & \xi_1 & \xi_0 \\ \xi_n & \xi_{n-1} & \dots & \xi_2 & \xi_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_{2n-2} & \xi_{2n-3} & \dots & \xi_n & \xi_{n-1} \end{pmatrix}, \tag{3.6}$$

and E and V are column vectors of length n, given by

$$E = (e_1(\mathbf{z}), -e_2(\mathbf{z}), \dots, (-1)^{n+1} e_n(\mathbf{z}))^T,$$

and

$$V = (\xi_n, \xi_{n+1}, \dots, \xi_{2n-1})^T.$$

(Recall that $\xi_0 = \xi_1 = 0$). Notice that E equals minus the vector of the coefficients of $\chi_{\mathbf{z}}(w)$. Assuming that U is invertible, we get $E = U^{-1}V$. This means that every $e_j(\mathbf{z})$ is expressed as a rational function of the normalized moments $\xi_2, \dots, \xi_{2n-1}$ with the fixed denominator, equal to the determinant of U.

(The same system in a somewhat different form with a Hankel matrix instead of a Toeplitz can be found as equation (28) in [5].)

Now assuming that one obtained $\chi_{\mathbf{z}}(w)$ and found the unordered collection $\mathcal{Z}$ of its roots which we assume to be distinct; let us choose one of $n!$ possible ordering. Abusing notation denote by $\mathbf{z} = (z_1, \dots, z_n)$ the chosen ordering of $\mathcal{Z}$. Consider now (3.4) as the overdetermined linear system in the variables $\mathbf{u} = (u_1, \dots, u_n)$. Our choice of $\mathbf{z}$ guarantees that (3.4) will have a solution in $\mathbf{u}$, that is the rank of the $(2n-1) \times n$-matrix H given by

$$H = \begin{pmatrix} 1 & 1 & \dots & 1 \\ h_1(z_1, z_2) & h_1(z_2, z_3) & \dots & h_1(z_n, z_1) \\ h_2(z_1, z_2) & h_2(z_2, z_3) & \dots & h_2(z_n, z_1) \\ \vdots & \vdots & \vdots & \vdots \\ h_{2n-2}(z_1, z_2) & h_{2n-2}(z_2, z_3) & \dots & h_{2n-2}(z_n, z_1) \end{pmatrix}$$

will be equal to the rank of the $(2n-1) \times (n+1)$-matrix EH obtained by appending to H the column of the right-hand sides $(0, \xi_2, \dots, \xi_{2n-1})^T$. For generic $\mathbf{z}$, the rank of H equals $n-1$. Namely, one can check that the vector $\mathfrak{z} = (z_1 - z_2, z_2 - z_3, \dots, z_n - z_1)^T$ lies in the right kernel of H. At the same time if we remove the last column of H, and take the first $n-1$ rows we obtain a square $(n-1) \times (n-1)$-matrix M given by

$$M = \begin{pmatrix} 1 & 1 & \dots & 1 \\ h_1(z_1, z_2) & h_1(z_2, z_3) & \dots & h_1(z_{n-1}, z_n) \\ h_2(z_1, z_2) & h_2(z_2, z_3) & \dots & h_2(z_{n-1}, z_n) \\ \vdots & \vdots & \vdots & \vdots \\ h_{n-2}(z_1, z_2) & h_{n-2}(z_2, z_3) & \dots & h_{n-2}(z_{n-1}, z_n) \end{pmatrix}$$

By Lemma 4, this minor is nonzero for pairwise distinct $z_1, \dots, z_n$. Hence H has rank at least $n-1$. Since the vector $\mathfrak{z} = (z_1 - z_2, z_2 - z_3, \dots, z_n - z_1)^T$ lies in the right kernel of H, the rank is exactly $n-1$. Therefore the solution set of (3.4), whenever nonempty, is an affine line of the form $\mathbf{u}^0 + t\mathfrak{z}$, $t \in \mathbb{C}$. $\square$

The affine one-dimensional families obtained above live naturally in the auxiliary complexified coordinates (z_j, u_j) rather than in the original real-analytic variables $(z_j, \bar{z}_j)$. Passing back to the genuine polygonal inverse problem therefore amounts to imposing the cyclic difference constraint

$$u_j = \frac{i}{4}(\overline{z_j} - \overline{z_{j+1}}), \qquad j = 1, \dots, n,$$

with indices understood modulo n. Thus the real locus is not obtained by the naive relation $u_j = \bar{z}_j$, but by requiring $\mathbf{u}$ to be the cyclic difference vector of the conjugate vertex tuple. Intersecting an affine complexified fiber with this antiholomorphic constraint may reduce the fiber to a finite set or may eliminate it completely. The freedom to add the same constant to all $\bar{z}_j$ is exactly the expected translational ambiguity.

3.1. **Back to the classical problem.** Here we discuss the problem of uniqueness of the logarithmic inverse moment problem for the "domains" bounded by closed curves consisting of straight segments in the complex plane. (We put domains in parenthesis since we actually mean associated plane measures.) A number of necessary conditions for two such measures to be equipotential was already formulated in [2].

We use below only the general consequence, already emphasized in [2], that equipotential polygonal domains have the same set of vertices, together with the formulation of the corresponding inverse problem in terms of harmonic moments.

The quantities γ_j appeared and were discussed in [5] and have the following geometric interpretation.

Lemma 5. In the above notation, if $z_{j-1} \neq z_j \neq z_{j+1}$, then
(i)

$$\gamma_j = e^{2i\phi_j} - e^{2i\phi_{j-1}} \tag{3.7}$$

where ϕ_ℓ is the angle between the oriented segment $[z_\ell, z_{\ell+1}]$ with the positive real axis.
(ii) given γ_j, we can uniquely find the angles $2\phi_j$ and $2\phi_{j-1}$. Thus, we can find ϕ_j and ϕ_{j-1} up to the summand π.

4. Concluding remarks and outlook

The main structural result of this note is the appearance of affine one-dimensional complexified inverse fibers for generic polygonal moment data. From the viewpoint of the classical logarithmic inverse problem, this gives a concrete algebraic description of the residual non-uniqueness which remains after complexification.

Several questions remain open. First, the description of nongeneric fibers of the map

$$(\mathbf{z}, \bar{\mathbf{z}}) \longmapsto (\nu_2(\mathbf{z}, \bar{\mathbf{z}}), \dots, \nu_{2n-1}(\mathbf{z}, \bar{\mathbf{z}}))$$

should be refined; in particular, it would be useful to stratify the parameter space by the dimension and number of irreducible components of the fibers. Second, one should characterize which complexified affine fibers meet the real polygonal locus determined by

$$u_j = \frac{i}{4}(\overline{z_j} - \overline{z_{j+1}}), \qquad j = 1, \dots, n,$$

together with the geometric constraints corresponding to simple polygonal curves. This real problem is substantially more delicate than its complexification and is the one directly relevant to equipotential plane polygons. Third, the higher open and closed Prony systems of [11] suggest a parallel inverse problem for push-forwards of Lebesgue measures on higher-dimensional polytopes. A complete solution should combine the algebraic moment relations with positivity and spline-regularity constraints.

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