TRANSPOSITION CAYLEY SPECTRA AND SUBGRAPH COUNTS

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Abstract. Let G be a finite simple graph on the vertex set $[n]$, and let $\Gamma(G)$ be the Cayley graph of the symmetric group S_n generated by the transpositions corresponding to the edges of G. We study the inverse spectral problem for the assignment $G \mapsto \Gamma(G)$ through the group-algebra operator $A_G = \sum_{\{i,j\}\in E(G)}(ij)$ and its regular spectral moments $M_k(G) = \mathrm{tr}(A_G^k)$. The moments admit two all-order forms: a universal support expansion in non-induced subgraph counts of G, and a representation-theoretic expansion over the irreducible representations of S_n. We compute the first nontrivial moments through order six. In particular, the fourth moment already contains the Coxeter triangle relation and determines the signed local invariant $6N_{K_3}(G) - N_{P_3}(G)$, rather than $N_{P_3}(G)$ alone. Consequently, for triangle-free graphs the spectrum determines the second moment of the degree sequence. We also give a simplified sixth-moment formula for triangle-free graphs, showing that in this class the spectrum determines $5N_{K_{1,3}}(G) + 2N_{P_4}(G) + 12N_{C_4}(G)$. Further structural consequences include bipartiteness and spectral symmetry of $\Gamma(G)$, a precise connectedness test, and the identification of the standard irreducible component with the shifted graph Laplacian $mI - L_G$. The paper is positioned as a theoretical spectral-moment complement to the recent CayleyPy computational programme on growth and diameter of Cayley graphs.

1. Introduction

Let G be a finite simple graph with vertex set $[n] = \{1, \dots, n\}$. Associate with every edge $\{i, j\} \in E(G)$ the transposition $(ij) \in S_n$, and put

$$T_G = \{(ij) : \{i, j\} \in E(G)\}.$$

We consider the Cayley graph

$$\Gamma(G) = \mathrm{Cay}(S_n, T_G).$$

Equivalently, $\Gamma(G)$ is the state graph of the interchange process on G before assigning transition probabilities. Its adjacency operator is left multiplication by

$$A_G = \sum_{\{i,j\}\in E(G)} (ij)$$

in the group algebra $\mathbb{C}[S_n]$.

The paper is concerned with the following inverse spectral problem.

Question 1. *If two graphs G_1 and G_2 have cospectral transposition Cayley graphs $\Gamma(G_1)$ and $\Gamma(G_2)$, must G_1 and G_2 be isomorphic? More generally, which invariants of G are determined by* $\mathrm{Spec}(\Gamma(G))$*?*

This question is motivated by several sources. Ordinary graph spectra are notoriously non-rigid; Godsil–McKay switching provides a systematic source of cospectral non-isomorphic graphs [15]. On the other hand, the Cayley graph $\Gamma(G)$ remembers not only the adjacency relations in G, but also the multiplication rules among the corresponding transpositions in S_n. Thus it is natural to ask whether passing from G to $\Gamma(G)$ produces a substantially stronger spectral invariant.

The same family of operators also appears in the study of random walks and the interchange process. Aldous' spectral gap conjecture, proved by Caputo, Liggett and Richthammer [2], asserts that the spectral gap of the interchange process on an arbitrary graph equals the spectral gap of the ordinary random walk on that graph. Cesi studied related Cayley graphs for complete multipartite transposition sets and obtained explicit representation-theoretic eigenvalue formulae in that case [3]. The present paper focuses on a different aspect: the full spectrum of $\Gamma(G)$ as a possible inverse invariant of the generating graph G.

A recent computational direction which should be mentioned explicitly is the CayleyPy project of Chervov and collaborators [4, 5, 6, 7]. That project develops efficient computational and machine-learning

2020 *Mathematics Subject Classification.* Primary 05C50; Secondary 05C25, 05C60, 20C30, 60J10.

Key words and phrases. Cayley graph, symmetric group, transposition graph, graph spectrum, inverse spectral problem, spectral moment, interchange process, CayleyPy, symmetric group representations.

tools for Cayley and Schreier graphs, especially for path-finding, diameter, growth, and large-scale experiments. In particular, [5] studies Cayley graphs of S_n generated by cyclic shifts and a transposition, proposes diameter conjectures and spectral-distribution phenomena, while [6] announces the public CayleyPy library and about two hundred conjectures, mostly concerning diameters and growth of Cayley and Schreier graphs. The relation with the present note is complementary: CayleyPy supplies a powerful experimental framework for Cayley graphs, whereas here we isolate exact algebraic moment formulae for the special family obtained from arbitrary transposition graphs.

There is also a closely related, but different, rigidity question in which one asks for the automorphism group or the abstract isomorphism type of a Cayley graph generated by transpositions. Results of Feng, Ganesan, and, in greatest generality for arbitrary transposition sets, Gijswijt–de Meijer show that for $n \geq 5$ these Cayley graphs are normal and that their automorphism groups are controlled by the automorphism groups of the underlying transposition graphs [10, 11, 12]. Those results concern graph isomorphism and automorphisms, whereas the present note asks what can be recovered from the adjacency spectrum alone.

The novelty of the paper should therefore be understood as follows. The representation-theoretic decomposition of the regular representation and the special spectra of the complete and star transposition graphs are classical or already present in the literature. The recent CayleyPy papers have also made Cayley graph computation, growth, diameter and path-finding problems highly visible and have explicitly raised spectral questions for large families of Cayley graphs [5, 6]. What seems not to have been isolated in this form is the inverse-spectral moment viewpoint for an arbitrary transposition graph G, especially the universal support expansion and the explicit low-order consequences for reconstructing graph invariants.

Our approach is based on moments. Let

$$M_k(G) = \operatorname{tr}(A_G^k)$$

where the trace is taken in the regular representation of S_n. Since only the coefficient of the identity permutation contributes to the regular trace, $M_k(G)/n!$ counts words of length k in the edge-transpositions of G whose product is the identity. This elementary observation leads to a universal expansion

$$\frac{M_k(G)}{n!} = \sum_F c_k(F) N_F(G),$$

where F ranges over finite graphs with no isolated vertices, $N_F(G)$ is the number of non-induced copies of F in G, and the coefficients $c_k(F)$ are universal integers depending only on F and k.

The same moments also have a representation-theoretic form. If V_λ runs through the irreducible representations of S_n and $f^\lambda = \dim V_\lambda$, then

$$M_k(G) = \sum_{\lambda \vdash n} f^\lambda \operatorname{tr}\left(\left(\sum_{e \in E(G)} \rho_\lambda(\tau_e)\right)^k\right).$$

These two formulae organize the same information in two different ways: the first by the supports of words in the generating graph, and the second by the irreducible representations of the symmetric group.

The central object of the present paper is not the individual moments themselves but rather the embedding

$$G \longmapsto A_G = \sum_{\{i,j\} \in E(G)} (ij) \in \mathbb{C}[S_n].$$

This construction associates to a graph a noncommutative operator in the group algebra of the symmetric group. The moments

$$M_k(G) = \operatorname{tr}(A_G^k)$$

count closed transposition words reducing to the identity permutation. The resulting support expansions are governed by the Coxeter relations in S_n, namely the cancellation relations $(ij)^2 = e$ and the braid relations

$$(ij)(jk)(ij) = (jk)(ij)(jk).$$

In this sense, the Cayley spectrum induces a finite-type graph invariant theory controlled by the Coxeter structure of the symmetric group.

The explicit calculations in Section 4 show that M_2 recovers the number of edges and that M_4 already detects the Coxeter triangle relation among the three transpositions attached to a triangle in G. More precisely, the fourth moment determines the signed local invariant $6N_{K_3}(G) - N_{P_3}(G)$. Hence, on the important subclass of triangle-free graphs, it recovers the number of adjacent edge-pairs, equivalently the second moment of the degree sequence. We also compute M_6 completely and derive a cleaner specialization for triangle-free graphs, where it determines the next local combination $5N_{K_{1,3}}(G) + 2N_{P_4}(G) + 12N_{C_4}(G)$. These formulae give concrete evidence that the transposition Cayley spectrum sees local structure of G in a way which differs sharply from the ordinary adjacency spectrum.

2. The basic operator

The central object is the element

$$A_G = \sum_{\{i,j\} \in E(G)} (ij) \in \mathbb{C}[S_n].$$

The adjacency matrix of $\Gamma(G)$ is precisely the operator of left multiplication by A_G in the regular representation of S_n.

Thus the spectrum of $\Gamma(G)$ coincides with the spectrum of A_G acting in the regular representation.

Since

$$\mathbb{C}[S_n] \simeq \bigoplus_{\lambda \vdash n} (\dim V_\lambda)\, V_\lambda,$$

one obtains

$$\mathrm{Spec}(\Gamma(G)) = \bigcup_{\lambda \vdash n} \mathrm{Spec}(\rho_\lambda(A_G)),$$

where ρ_λ denotes the irreducible representation corresponding to the partition λ.

This representation-theoretic decomposition suggests that the spectrum of $\Gamma(G)$ should contain much richer information than the ordinary adjacency spectrum of G.

There is also a simple global feature which is useful throughout the paper.

Proposition 1 (Bipartite symmetry). *For every graph G, the Cayley graph $\Gamma(G)$ is bipartite. Its two parts are the even and odd permutations in S_n. Consequently its adjacency spectrum is symmetric about zero, and*

$$M_{2r+1}(G) = 0 \qquad (r \geq 0).$$

Proof. Every generator in T_G is a transposition, hence changes the parity of a permutation. Thus every edge of $\Gamma(G)$ joins an even permutation to an odd permutation. The spectral symmetry is the standard spectral symmetry of a finite bipartite graph, and the vanishing of odd moments is equivalent to the absence of closed walks of odd length. □

3. All-order moment formulae

The low-order calculations are only the first instances of a more systematic structure. There are two useful all-order formulae. The first is purely combinatorial and expresses every moment as a universal linear combination of subgraph-counting functions. The second is representation-theoretic and expresses the same moments through the irreducible representations of S_n.

3.1. The support expansion.

A natural family of spectral invariants is given by the moments

$$M_k(G) = \mathrm{tr}(A_G^k).$$

Let $[1]B$ denote the coefficient of the identity permutation in an element $B \in \mathbb{C}[S_n]$. Since the trace of left multiplication by a non-identity element of S_n in the regular representation is zero, whereas the trace of the identity is $n!$, one has

$$M_k(G) = n!\,[1]A_G^k.$$

Equivalently,

$$\frac{M_k(G)}{n!} = \#\left\{(e_1, \ldots, e_k) \in E(G)^k : \tau_{e_1} \cdots \tau_{e_k} = 1\right\},$$

where $\tau_{\{i,j\}} = (ij)$. Thus the moments count words in the edge-transpositions whose product is trivial.

Let F be a finite simple graph with no isolated vertices. Write $E(F)$ for its edge set, and regard every edge $e = \{a, b\}$ of F as the transposition (ab) in the symmetric group on $V(F)$. For $k \geq 0$ define

$$c_k(F) = \#\left\{(e_1, \ldots, e_k) \in E(F)^k : \begin{array}{l} e_1, \ldots, e_k \text{ use every edge of } F, \\ \tau_{e_1} \cdots \tau_{e_k} = 1 \end{array}\right\}.$$

Thus $c_k(F)$ is a universal integer depending only on the isomorphism class of F.

Let $N_F(G)$ denote the number of subgraphs of G which are isomorphic to F, not necessarily induced. Then the normalized moment has the following universal expansion.

Theorem 1 (Universal support expansion). *For every simple graph G and every $k \geq 0$ one has*

$$\frac{M_k(G)}{n!} = \sum_F c_k(F) N_F(G),$$

where the sum is over all isomorphism classes of finite simple graphs F with no isolated vertices and with at most k edges.

Proof. By the coefficient-of-the-identity formula,

$$\frac{M_k(G)}{n!} = \#\{(e_1, \ldots, e_k) \in E(G)^k : \tau_{e_1} \cdots \tau_{e_k} = 1\}.$$

To each such word associate its support, namely the subgraph of G spanned by the distinct edges appearing among $e_1, \ldots, e_k$ and their endpoints. The support has no isolated vertices and at most k edges. For a fixed subgraph $H \subseteq G$ isomorphic to F, the number of identity words of length k whose support is exactly H is precisely $c_k(F)$, since the transposition relations depend only on the abstract incidence pattern of H. Summing over all possible supports gives the formula. □

This theorem is perhaps the cleanest general form of the moment method. It says that the Cayley spectrum determines, for every k, one explicit linear combination of small subgraph counts. The problem of extracting particular graph invariants from the spectrum becomes the problem of inverting, or partially triangularizing, these universal linear combinations.

Remark 1. *The constants $c_k(F)$ are algorithmically computable. For fixed k there are only finitely many graphs F with at most k edges, and one can enumerate all words of length k in the edge-transpositions of F. This gives a finite, completely explicit procedure for producing the moment formula of any prescribed order.*

Remark 2. *The support expansion gives another way to see the parity phenomenon in Proposition 1: $c_k(F) = 0$ for every odd k. The first nontrivial moments are therefore $M_2, M_4, M_6, \ldots$.*

3.2. **Representation-theoretic form.** The preceding formula is combinatorial. There is also an all-order formula directly in the representation theory of the symmetric group. Let $f^\lambda = \dim V_\lambda$, and let

$$A_G^\lambda = \rho_\lambda(A_G) = \sum_{\{i,j\} \in E(G)} \rho_\lambda((ij)) \in \mathrm{End}(V_\lambda).$$

Since the regular representation decomposes as

$$\mathbb{C}[S_n] \simeq \bigoplus_{\lambda \vdash n} (f^\lambda) V_\lambda,$$

one obtains, for every $k \geq 0$,

$$M_k(G) = \sum_{\lambda \vdash n} f^\lambda \operatorname{tr}\left((A_G^\lambda)^k\right).$$

Equivalently,

$$[1]A_G^k = \frac{1}{n!} \sum_{\lambda \vdash n} f^\lambda \operatorname{tr}\left((A_G^\lambda)^k\right).$$

This is the representation-theoretic all-order moment formula.

It is important, however, that A_G is not central unless G is the complete graph or the empty graph. Therefore the formula cannot usually be reduced to ordinary irreducible characters alone. One needs the full matrices of the transpositions in irreducible representations.

A particularly concrete version is obtained by using the Young orthogonal form. If $\mathrm{SYT}(\lambda)$ is the set of standard Young tableaux of shape λ, then

$$\begin{aligned} &\operatorname{tr}\left((A_G^\lambda)^k\right) \\ &= \sum_{T_0, T_1, \ldots, T_{k-1} \in \mathrm{SYT}(\lambda)} \sum_{e_1, \ldots, e_k \in E(G)} \rho_\lambda(\tau_{e_1})_{T_0, T_1} \rho_\lambda(\tau_{e_2})_{T_1, T_2} \cdots \rho_\lambda(\tau_{e_k})_{T_{k-1}, T_0}. \end{aligned}$$

Thus the moments can be computed by summing closed walks in the basis of standard tableaux, weighted by matrix coefficients of transpositions. This gives a direct analogue of moment formulae for ordinary graph spectra, but now with the vertices of the auxiliary walk replaced by standard Young tableaux.

Proposition 2 (Two all-order descriptions). *For every k, the same spectral invariant $M_k(G)$ admits both expressions*

$$\frac{M_k(G)}{n!} = \sum_F c_k(F) N_F(G)$$

and

$$M_k(G) = \sum_{\lambda \vdash n} f^\lambda \operatorname{tr}\left(\left(\sum_{e \in E(G)} \rho_\lambda(\tau_e)\right)^k\right).$$

The first expression organizes the answer by subgraph supports in G; the second organizes it by irreducible representations of S_n.

3.3. **The standard component and the Laplacian of G.** There is one irreducible component of the preceding representation- theoretic formula which has an especially transparent graph-theoretic meaning. Let $W = \mathbb{C}^n$ be the permutation representation of S_n, and let $\mathbf{1} \subset W$ be the one-dimensional trivial subrepresentation. Its complement

$$W_0 = \{x = (x_1, \dots, x_n) : x_1 + \cdots + x_n = 0\}$$

is the standard irreducible representation of S_n, corresponding to the partition $(n-1, 1)$.

Let L_G be the usual graph Laplacian of G, and let $m = |E(G)|$. On W, the transposition (ij) acts by interchanging the coordinates x_i and x_j. Hence

$$(ij) = I - (e_i - e_j)(e_i - e_j)^t$$

with respect to the standard coordinate basis. Summing over the edges of G gives the following elementary but useful identity.

Proposition 3 (The standard component). *On the permutation representation $W = \mathbb{C}^n$ one has*

$$\sum_{\{i,j\} \in E(G)} (ij) = mI - L_G.$$

Consequently, on the standard representation W_0 the operator $A_G^{(n-1,1)}$ has eigenvalues

$$m - \lambda_2,\ m - \lambda_3, \dots,\ m - \lambda_n,$$

where

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

are the Laplacian eigenvalues of G.

Proof. For a single edge $\{i, j\}$, the transposition (ij) fixes the orthogonal complement of the line spanned by $e_i - e_j$ and sends $e_i - e_j$ to its negative. Therefore, as an operator on W,

$$(ij) = I - (e_i - e_j)(e_i - e_j)^t.$$

Summing this identity over all edges gives

$$A_G = mI - \sum_{\{i,j\} \in E(G)} (e_i - e_j)(e_i - e_j)^t = mI - L_G.$$

The vector $(1, \dots, 1)$ is an eigenvector with eigenvalue m, and the restriction to its complement is exactly the standard representation. This proves the assertion. ☐

Remark 3. *Proposition 3 should be interpreted with a minor caveat. The full Cayley spectrum records the union of the spectra of all irreducible components, with multiplicities, but it does not label which eigenvalues come from which component. Thus the proposition shows that the* refined *irreducible spectrum contains the Laplacian spectrum of G. It also explains why the transposition Cayley spectrum is naturally expected to be stronger than the ordinary adjacency or Laplacian spectrum: the latter appears already in the single component $(n-1, 1)$, while all other irreducible representations provide additional constraints.*

3.4. **Finite-type consequences of the moment expansion.** The universal support expansion has the following immediate consequence.

Corollary 1 (Moments are finite-type graph invariants)**.** *For every fixed k, the normalized moment $M_k(G)/n!$ belongs to the finite-dimensional span of the subgraph-counting functions $N_F(G)$ with at most k edges. In particular, for fixed k it depends only on the numbers of small subgraphs of G.*

Thus the inverse problem can be refined as follows. For each $r \geq 1$, one should study the linear span of

$$G \longmapsto \frac{M_2(G)}{n!}, \frac{M_4(G)}{n!}, \dots, \frac{M_{2r}(G)}{n!}$$

inside the vector space spanned by subgraph-counting functions $N_F(G)$ with at most $2r$ edges. The concrete question is which individual subgraph counts can be recovered from these spectral linear combinations.

The formula for M_4 below shows that the edge count and the signed combination $6N_{K_3} - N_{P_3}$ are recoverable. Thus the fourth moment already sees the Coxeter relation among the three transpositions attached to a triangle. The formula for M_6 shows explicitly how the next local support types enter the moment expansion. This leads to the following natural strengthening of the basic reconstruction problem.

Question 2. *Do the functions $M_2, M_4, \dots, M_{2r}$ determine any natural triangular system of subgraph-counting functions $N_F(G)$, at least after restricting to graphs with a fixed number of vertices or to natural subclasses such as triangle-free graphs?*

A particularly accessible subfamily consists of star supports. Since star counts are polynomial expressions in the degrees of G, this suggests the following concrete problem.

Question 3. *Do the even Cayley moments determine the power sums*

$$\sum_{v \in V(G)} d_v^q$$

of the degree sequence? Equivalently, do they determine the degree sequence of G?

The first case $q = 1$ is $2|E(G)|$ and is determined by M_2. The case $q = 2$ is determined by M_4 on triangle-free graphs, but in general the triangle term in the fourth moment creates an obstruction which must be separated by higher moments or additional representation-theoretic data.

4. First moment calculations

We now record the first moments in the support-expansion language. The calculations below are meant to serve as the first concrete data for the project.

We use the following notation. The graph P_r denotes the path on r vertices, C_r the cycle on r vertices, K_r the complete graph on r vertices, $K_{1,3}$ the claw, and $2K_2$ and $3K_2$ disjoint unions of two and three edges, respectively. We write $K_3 \sqcup K_2$ and $P_3 \sqcup K_2$ for disjoint unions. The graph Paw is the triangle with one pendant edge, and $K_4 - e$ is the complete graph on four vertices with one edge removed.

Theorem 2 (Moments up to order six)**.** *Let G be a simple graph. Then*

$$\frac{M_0(G)}{n!} = 1, \qquad \frac{M_1(G)}{n!} = 0, \qquad \frac{M_2(G)}{n!} = N_{K_2}(G), \qquad \frac{M_3(G)}{n!} = 0,$$

$$\frac{M_4(G)}{n!} = N_{K_2}(G) + 4N_{P_3}(G) + 6N_{2K_2}(G) + 12N_{K_3}(G),$$

and

$$\begin{aligned}\frac{M_6(G)}{n!} = N_{K_2}(G) &+ 20N_{P_3}(G) + 30N_{2K_2}(G) + 180N_{K_3}(G) \\ &+ 30N_{K_{1,3}}(G) + 42N_{P_4}(G) + 60N_{P_3 \sqcup K_2}(G) + 90N_{3K_2}(G) \\ &+ 72N_{C_4}(G) + 96N_{\mathrm{Paw}}(G) + 180N_{K_3 \sqcup K_2}(G) \\ &+ 132N_{K_4 - e}(G) + 96N_{K_4}(G).\end{aligned}$$

Here all subgraph counts are non-induced.

Proof. The identities for M_0 and M_1 are immediate. For M_2, a product of two edge-transpositions is the identity if and only if the two transpositions are equal. Hence $M_2(G)/n! = N_{K_2}(G)$.

All odd moments vanish because a product of an odd number of transpositions is an odd permutation and hence cannot be the identity. This gives $M_3(G) = 0$.

For M_4, the support of an identity word may have one, two, or three edges. If the support is a single edge, there is one word. If the support is a path P_3, i.e. two adjacent edges, there are four identity words:

$$aabb, \quad abba, \quad baab, \quad bbaa.$$

If the support is $2K_2$, i.e. two disjoint edges, the corresponding transpositions commute, and all six words with two occurrences of each edge are identity words. Finally, if the support is a triangle with edge transpositions a, b, c, then the Coxeter relation $aba = c$ and its cyclic variants give twelve identity words of length four using all three edges. Equivalently, for every ordering of two adjacent triangle edges one has a word of the form $abac = 1$, and the same words are counted with all cyclic choices of the distinguished edge. This gives the displayed fourth-moment formula.

The sixth moment is obtained from the universal support expansion by enumerating the possible supports of length-six identity words. There are exactly thirteen support types with nonzero coefficient. Their coefficients $c_6(F)$ are as follows:

F	K_2	P_3	$2K_2$	K_3	$K_{1,3}$	P_4	$P_3 \sqcup K_2$	$3K_2$	C_4	Paw	$K_3 \sqcup K_2$	$K_4 - e$	K_4
$c_6(F)$	1	20	30	180	30	42	60	90	72	96	180	132	96.

Substituting these coefficients into

$$\frac{M_6(G)}{n!} = \sum_F c_6(F) N_F(G)$$

gives the formula. ∎

Remark 4 (Check of the enumeration)**.** *The table in the proof was obtained by the following finite procedure. For each support graph F with at most six edges one enumerates all 6-tuples of edges of F which use every edge of F and multiplies the corresponding transpositions in $S_{V(F)}$. The displayed coefficient is the number of such words with product equal to the identity. This also provides a short independent way to reproduce the formula and to compute higher moments.*

Corollary 2 (The fourth-moment invariant)**.** *The Cayley spectrum of $\Gamma(G)$ determines*

$$|E(G)| \qquad \text{and} \qquad 6N_{K_3}(G) - N_{P_3}(G).$$

In particular, on the class of triangle-free graphs it determines $N_{P_3}(G)$ and therefore

$$\sum_{v \in V(G)} d_v^2.$$

Proof. Let $m = |E(G)|$, $P = N_{P_3}(G)$, $D = N_{2K_2}(G)$, and $T = N_{K_3}(G)$. The number m is determined by $M_2(G)$. Also

$$D = \binom{m}{2} - P,$$

because every unordered pair of distinct edges is either adjacent, hence contributes to P, or disjoint, hence contributes to D. Substituting this into the fourth-moment formula gives

$$\frac{M_4(G)}{n!} = m + 6\binom{m}{2} - 2P + 12T.$$

Thus $6T - P$ is determined by the spectrum. If G is triangle-free, then $T = 0$, so P is determined. Finally,

$$P = \sum_{v \in V(G)} \binom{d_v}{2} \qquad \text{and} \qquad \sum_v d_v^2 = 2P + 2m.$$

∎

Corollary 3 (Triangle-free sixth-moment consequence)**.** *Let G be triangle-free. Then the Cayley spectrum of $\Gamma(G)$ determines the linear combination*

$$5N_{K_{1,3}}(G) + 2N_{P_4}(G) + 12N_{C_4}(G).$$

If G has girth at least five, this reduces to the determination of

$$5N_{K_{1,3}}(G) + 2N_{P_4}(G).$$

If, in addition, the maximum degree of G is at most two, then the spectrum determines $N_{P_4}(G)$.

Proof. Assume that G is triangle-free and write

$$m = N_{K_2}(G), \quad P = N_{P_3}(G), \quad S = N_{K_{1,3}}(G), \quad R = N_{P_4}(G), \quad C = N_{C_4}(G).$$

The disconnected support counts appearing in the sixth moment can be eliminated in terms of these quantities. First

$$N_{2K_2} = \binom{m}{2} - P.$$

Next,

$$N_{P_3 \sqcup K_2} = (m-2)P - 3S - 2R.$$

Indeed, for each two-edge path $u - v - w$, the number of edges disjoint from its three vertices is $m - d_u - d_v - d_w + 2$; summing this over all P_3's and using triangle-freeness gives

$$\sum_{u-v-w} (d_u + d_v + d_w) = 3S + 4P + 2R.$$

Finally, classifying unordered triples of edges by their support gives

$$N_{3K_2} = \binom{m}{3} - N_{P_3 \sqcup K_2} - S - R.$$

Substitution into Theorem 2, with all triangle support types removed, yields

$$\begin{aligned}\frac{M_6(G)}{n!} &= m + 30\binom{m}{2} + 90\binom{m}{3} + (50 - 30m)P \\ &\quad + 30S + 12R + 72C.\end{aligned}$$

By Corollary 2, m and P are already determined on the triangle-free class. Hence $30S + 12R + 72C$, equivalently $5S + 2R + 12C$, is determined. The two stated specializations follow by putting $C = 0$ and then $S = 0$. ☐

Remark 5. *The correction term $12N_{K_3}(G)$ in the fourth moment is important. It reflects the relation $(ij)(jk)(ij) = (ik)$ inside a triangle. Thus triangle effects begin at order four, not order six. The sixth moment is still the first place where the next family of local supports, such as claws, four-vertex paths and four-cycles, enters the regular trace in a substantial way.*

5. Three exactly solvable families

We next record three classes of graphs for which the transposition Cayley spectrum can be described explicitly. These examples answer part of the general program in closed form and show that the construction is not merely a source of low-order moment identities.

5.1. The complete graph.

Let

$$C_n = \sum_{1 \le i < j \le n} (ij)$$

be the full class sum of all transpositions. This is a central element of $\mathbb{C}[S_n]$. Its spectral decomposition is the standard character-theoretic decomposition of the random-transposition walk on S_n; see, for example, Diaconis–Shahshahani and Diaconis [9, 8]. Hence, by Schur's lemma, it acts as a scalar on each irreducible representation V_λ.

Proposition 4 (Complete transposition graph)**.** *For $G = K_n$, the eigenvalue of $A_G = C_n$ on the irreducible representation V_λ is*

$$\kappa_\lambda = \sum_{(a,b)\in\lambda} (b - a),$$

the sum of the contents of all boxes of the Young diagram λ. Equivalently,

$$\kappa_\lambda = \frac{1}{2}\sum_i \lambda_i(\lambda_i - 2i + 1).$$

In the regular representation this eigenvalue occurs with multiplicity $(f^\lambda)^2$.

Proof. The element C_n is central. Its scalar on V_λ is the usual central character of the transposition class. In the Young–Jucys– Murphy realization one has

$$C_n = J_2 + J_3 + \cdots + J_n, \qquad J_r = \sum_{i<r} (ir).$$

The element J_r acts diagonally in the Young orthogonal basis, with eigenvalue equal to the content of the box containing r. Summing over r gives the sum of all contents of λ. The multiplicity $(f^\lambda)^2$ follows from the decomposition of the regular representation. □

5.2. **Matchings.** Suppose that G is a matching with m edges. Then the corresponding transpositions commute. Thus A_G is a sum of m commuting involutions, and its spectrum is completely elementary.

Proposition 5 (Matching case)**.** *If $G = mK_2$ is a matching with m edges on n vertices, then*

$$\operatorname{Spec}(\Gamma(G)) = \{\, m - 2r : \ 0 \le r \le m \,\},$$

where the eigenvalue $m - 2r$ occurs with multiplicity

$$n!\, 2^{-m} \binom{m}{r}.$$

Proof. The subgroup generated by the m disjoint transpositions is $(\mathbb{Z}/2\mathbb{Z})^m$. In a simultaneous eigenbasis for these commuting involutions, each transposition contributes an eigenvalue ± 1. If exactly r of the signs are negative, the sum is $m - 2r$, and there are $\binom{m}{r}$ such sign choices. Each character of this elementary abelian subgroup occurs in the regular representation of S_n with multiplicity $n!/2^m$. □

5.3. **Stars and a Jucys–Murphy element.** The first noncentral but still completely diagonalizable family is given by stars. This family is the classical star graph on S_n; its spectrum and eigenvalue multiplicities have been studied, for instance, by Krakovski–Mohar and by Khomyakova–Konstantinova [14, 13]. Let $G = K_{1,n-1}$ be the star with centre n. Then

$$A_G = \sum_{i=1}^{n-1} (in) = J_n,$$

the last Young–Jucys–Murphy element.

Theorem 3 (Star spectrum)**.** *Let $G = K_{1,n-1}$. Then the eigenvalues of A_G are the possible contents*

$$c(b) = \operatorname{column}(b) - \operatorname{row}(b)$$

of removable boxes b of Young diagrams $\lambda \vdash n$. More precisely, in the copy of V_λ with Young orthogonal basis, A_G is diagonal and the basis vector indexed by a standard tableau T has eigenvalue equal to the content of the box occupied by n in T. Thus the multiplicity of an integer c in the regular representation is

$$\sum_{\lambda \vdash n} f^\lambda \, \#\{\, T \in \operatorname{SYT}(\lambda) : c(\operatorname{box}_T(n)) = c \,\}.$$

Proof. After choosing the centre of the star to be n, the operator A_G is exactly the Young–Jucys–Murphy element $J_n = \sum_{i<n} (in)$. The Young–Jucys–Murphy elements are diagonal in the Young orthogonal basis, and J_n acts on a standard tableau T by the content of the box containing n. The stated multiplicity again follows by multiplying the multiplicity inside V_λ by the multiplicity f^λ of V_λ in the regular representation. □

Remark 6. *The star theorem gives a genuine solution of one nontrivial instance of the inverse problem. Among stars, the Cayley spectrum determines the number of leaves, hence the graph. It also shows why the present problem is naturally related to the Gelfand–Tsetlin and Young–Jucys–Murphy approach to the representation theory of symmetric groups.*

Corollary 4 (A solved family of reconstruction examples)**.** *The reconstruction problem has an affirmative answer inside each of the following families:*

$$\{K_n\}_{n \ge 1}, \qquad \{mK_2\}_{m \ge 0}, \qquad \{K_{1,n-1}\}_{n \ge 1}.$$

In each case the transposition Cayley spectrum determines the parameter of the family and hence the graph.

The three examples above indicate that the general problem interpolates between three regimes: the central character theory of the complete graph, the elementary Fourier theory of an abelian subgroup for matchings, and the noncentral but diagonal Young–Jucys–Murphy theory for stars. This gives a conceptual explanation for why the general operator A_G should be viewed as a graph-dependent deformation of the Young–Jucys–Murphy elements.

6. The reconstruction problem

The main conjectural direction is the following.

Conjecture 1 (Reconstruction conjecture). *If*

$$\mathrm{Spec}(\Gamma(G_1)) = \mathrm{Spec}(\Gamma(G_2)),$$

then

$$G_1 \cong G_2.$$

At present it is unclear whether this conjecture is true.

The corresponding question for ordinary graph spectra is false, since there exist many pairs of cospectral nonisomorphic graphs. However, the spectrum of $\Gamma(G)$ is expected to be substantially more rigid due to the interaction of the transpositions inside the representation theory of S_n.

6.1. **Connectedness and component-size information.** The reconstruction conjecture is difficult, but one global part of it can be settled immediately. The Cayley spectrum detects whether the original graph is connected. In fact it detects a little more: it determines the index in S_n of the subgroup generated by the edge-transpositions.

Theorem 4 (Connectedness test). *Let G be a graph on n vertices with connected components of sizes*

$$s_1, \ldots, s_r.$$

Let $m = |E(G)|$. Then the multiplicity of the eigenvalue m in the spectrum of $\Gamma(G)$ is

$$\frac{n!}{s_1! s_2! \cdots s_r!}.$$

Consequently the spectrum of $\Gamma(G)$ determines whether G is connected.

Proof. The Cayley graph $\Gamma(G)$ is m-regular. For a finite undirected m-regular graph, the multiplicity of the eigenvalue m is equal to the number of connected components of that graph. The connected components of a Cayley graph are the left cosets of the subgroup generated by its connection set. In the present case this subgroup is generated by all transpositions corresponding to edges of G.

If $C_1, \ldots, C_r$ are the connected components of G, then the edge transpositions generate

$$S_{C_1} \times S_{C_2} \times \cdots \times S_{C_r} \subseteq S_n,$$

because transpositions along the edges of a connected graph generate the full symmetric group on its vertex set. Therefore the number of connected components of $\Gamma(G)$ is the index

$$[S_n : S_{C_1} \times \cdots \times S_{C_r}] = \frac{n!}{s_1! s_2! \cdots s_r!}.$$

In particular this multiplicity is equal to 1 if and only if $r = 1$, which is exactly the connectedness of G. $\square$

Corollary 5 (Necessary conditions for Cayley cospectrality). *If $\Gamma(G_1)$ and $\Gamma(G_2)$ are cospectral, then G_1 and G_2 have the same number of vertices, the same number of edges, the same value of $6N_{K_3} - N_{P_3}$, and the same value of*

$$\prod_C |C|!,$$

where the product is over the connected components C of the graph. In particular, either both G_1, G_2 are connected or neither is connected.

Proof. The number of vertices is determined by the size $n!$ of the Cayley matrix, hence by the total multiplicity of the spectrum. The number of edges and the invariant $6N_{K_3} - N_{P_3}$ are determined by Corollary 2. The product of factorials of component sizes is determined from Theorem 4, since the multiplicity of the top eigenvalue is $n!/\prod_C |C|!$. $\square$

7. Relation with the CayleyPy programme

The CayleyPy papers suggest a useful way to sharpen the computational side of the present project. Their main focus is not the inverse problem studied here, but rather the efficient exploration of very large Cayley and Schreier graphs, shortest paths, diameters, growth vectors, random walk behaviour, and experimentally generated conjectures [4, 5, 6, 7]. Still, there is a clear intersection at the level of spectra. The CayleyPy-RL paper reports numerical spectral-distribution phenomena for the "LRX" Cayley graphs of S_n, generated by a cyclic shift and a transposition [5]. The CayleyPy-Growth paper asks, among other general questions, what can be said about spectra of Cayley graphs and provides a large experimental framework for computing growth and related statistics [6].

The present paper is narrower but more algebraic. We do not study an arbitrary generating set of S_n; instead the generating set is the edge set of an arbitrary graph G, viewed as a set of transpositions. This restriction makes it possible to express the regular spectral moments of $\Gamma(G)$ as universal linear combinations of small subgraph counts in G. Thus the two approaches should be regarded as complementary. The CayleyPy framework is a natural tool for testing the reconstruction question, for producing non-isomorphic candidates with equal low-order moments, and for comparing growth or diameter data with the spectral moments studied here. Conversely, the support expansion in this note supplies exact invariants which could be used as test cases or symbolic benchmarks for CayleyPy computations.

8. Consequences and remaining problems

The results above give a finite-type spectral theory for the first Cayley moments. They also indicate several precise ways in which the paper can be pushed further.

8.1. **Triangle separation.** The fourth moment determines $6N_{K_3}(G) - N_{P_3}(G)$. Thus triangle relations are already visible in the Cayley spectrum at order four. However, this invariant mixes triangles with adjacent edge-pairs, so M_2, M_4, M_6 do not by themselves isolate $N_{K_3}(G)$ from the formula alone. A sharp next goal is therefore the following.

Question 4 (Triangle problem). *Is the number of triangles $N_{K_3}(G)$ determined by the full Cayley spectrum of $\Gamma(G)$? More modestly, is it determined by finitely many even moments $M_2, M_4, \dots, M_{2r}$?*

A positive answer would be the first genuinely nonlinear reconstruction result separating the triangle contribution from the adjacent-edge-pair contribution in the fourth moment.

8.2. **Degree sequence problem.** On triangle-free graphs the fourth moment determines $\sum_v d_v^2$. More generally, supports which are stars should control higher power sums of the degrees, but the general case requires separating star contributions from cycle and clique contributions. This leads to the following concrete conjecture.

Conjecture 2 (Degree-power conjecture). *For each $q \geq 1$, the Cayley spectrum determines the power sum*

$$p_q(G) = \sum_{v \in V(G)} d_v^q.$$

Consequently, the Cayley spectrum determines the degree sequence of G.

The case $q = 1$ follows from M_2. For triangle-free graphs the case $q = 2$ follows from M_4, and Corollary 3 shows that the sixth moment determines a first combination involving the claw $K_{1,3}$. The general case remains open because clique and cycle relations mix with the star counts.

8.3. **Irreducible spectral data.** Proposition 3 shows that the refined irreducible spectrum contains the Laplacian spectrum in the standard component. Thus a useful intermediate invariant between the ordinary Cayley spectrum and the graph G is the collection

$$\left\{ \operatorname{Spec} \left(\sum_{e \in E(G)} \rho_\lambda(\tau_e) \right) : \lambda \vdash n \right\}.$$

This refined spectrum is representation-theoretically natural and may be more accessible computationally than the undifferentiated regular spectrum.

Question 5. *Does the refined irreducible spectrum reconstruct G? If not, does it at least reconstruct all subgraph counts of G?*

8.4. **Finite computational tests.** The formulae above suggest finite tests which would substantially clarify the inverse problem:

(1) compute the full and refined Cayley spectra for all graphs up to at least eight vertices, using optimized code such as the CayleyPy framework where possible;
(2) test known Godsil–McKay switched pairs and other ordinary cospectral pairs;
(3) compute M_8 and M_{10} and search for triangular combinations isolating $N_{K_3}(G)$ and then $N_{K_{1,3}}(G)$;
(4) compare the support-expansion moment invariants with the refined irreducible spectra.

9. Final remarks

The main point of this paper is that the transposition Cayley spectrum admits a tractable moment theory. The regular trace converts spectral moments into counts of identity words in edge-transpositions, while the irreducible decomposition of $\mathbb{C}[S_n]$ gives a parallel representation-theoretic formula. These two descriptions suggest that $\mathrm{Spec}(\Gamma(G))$ is a much finer invariant than the ordinary adjacency spectrum of G.

The computations through order six are only the first step, but they already show that nontrivial local data of G is visible in the Cayley spectrum. The fourth moment detects the signed combination $6N_{K_3} - N_{P_3}$, while the triangle-free sixth-moment specialization recovers the next combination $5N_{K_{1,3}} + 2N_{P_4} + 12N_{C_4}$. The most natural next goal is to determine whether higher moments, possibly together with irreducible components of the spectrum, can separate these combined local invariants and then progressively recover all small subgraph counts.

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