Bizonotopal Graphical Algebras

Anatol Kirillov* Gleb Nenashev† Boris Shapiro‡
Arkady Vaintrob§

Abstract

Zonotopal algebras (external, central, and internal) of an undirected graph G introduced by Postnikov-Shapiro and Holtz-Ron, are finite-dimensional commutative graded algebras whose Hilbert series contain a wealth of combinatorial information about G. In this paper, we associate to G a new family of algebras, which we call *bizonotopal*, because their definition involves doubling the set of edges of G. These algebras are monomial and have intricate properties related, among other things, to the combinatorics of graphical parking functions and their polytopes. Unlike the case of usual zonotopal algebras, the Hilbert series of bizonotopal algebras are not specializations of the Tutte polynomial of G. Still, we show that in the external and central cases these Hilbert series satisfy a modified deletion-contraction relation. In addition, we prove that the external bizonotopal algebra is a complete graph invariant.

1 Introduction

Let $G = (V, E)$ be a finite undirected graph, possibly with loops and multiple edges, with vertex set V and edge set E. The *external zonotopal algebra* of

*Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Huairou District, Beijing, China, kirillov@bimsa.cn

†Department of Mathematics and Computer Science, St. Petersburg State University, St. Petersburg, 199178, Russia, glebnen@gmail.com

‡Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Huairou District, Beijing, China, and Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden, shapiro@math.su.se

§Department of Mathematics, University of Oregon, Eugene, OR 97403, USA, vaintrob@uoregon.edu

G is the following commutative graded algebra $\mathscr{A}_G^e$ introduced in [PSS]. Let $\mathbf{k}$ be a field of characteristic zero. Consider the *edge algebra*

$$\mathscr{E}_G := \mathbf{k}[E]/(x_e^2 \,:\, e \in E), \tag{1.1}$$

the quotient of the polynomial algebra $\mathbf{k}[E] := \mathbf{k}[x_e \,:\, e \in E]$ in the *edge variables* x_e, $e \in E$, by the ideal generated by their squares. It is a monomial algebra whose basis monomials correspond to subsets of E. The dimension of $\mathscr{E}_G$ is equal to $2^{|E|}$ and its Hilbert series is equal to $(1+t)^{|E|}$.

Given a choice of an orientation of G, let $A_G = (a_{v,e})$ be the *oriented incidence matrix* of G with rows and columns labeled by vertices $v \in V$ and edges $e \in E$ of G, respectively, and the entries given by

$$a_{v,e} = \begin{cases} -1, & \text{if the edge } e \text{ starts at } v; \\ 1, & \text{if } e \text{ ends at } v; \\ 0, & \text{if } e \text{ is a loop or it is not incident to } v. \end{cases} \tag{1.2}$$

The algebra $\mathscr{A}_G^e$ is defined as the subalgebra of $\mathscr{E}_G$ generated by the elements

$$y_v = \sum_{e \in E} a_{v,e} x_e \in \mathscr{E}(G), \text{ for } v \in V. \tag{1.3}$$

Reversing the orientation of an edge $e \in E$ produces an isomorphic graded algebra with an isomorphism induced by changing the sign of the corresponding generator $x_e \in \mathscr{E}_G$. Thus, the isomorphism class of $\mathscr{A}_G^e$ as a graded algebra does not depend on the choice of orientation.

In [PSS] it was shown that the dimension of $\mathscr{A}_G^e$ is equal to the number of spanning forests in G and that the Hilbert series of $\mathscr{A}_G^e$ is a specialization of the Tutte polynomial of G enumerating spanning forests by their *external activity*. In particular, this Hilbert series is determined by the graphical matroid of G. Furthermore, in [N] Nenashev proved that the external zonotopal algebras of two graphs G_1 and G_2 are isomorphic if and only if the matroids of G_1 and G_2 are isomorphic.

The algebra $\mathscr{A}_G^e$ can also be presented as the quotient algebra

$$\mathscr{A}_G^e \simeq \mathbf{k}[V]/I_G^e \tag{1.4}$$

of the polynomial algebra $\mathbf{k}[V] := \mathbf{k}[z_v \,:\, v \in V]$ in vertex-labeled variables z_v by the ideal I_G^e generated by powers of linear forms

$$I_G^e := \left(\Big(\sum_{v \in S} z_v \Big)^{d_S+1} \,:\, \emptyset \neq S \subset V \right), \tag{1.5}$$

where d_S is the number of edges in G connecting vertices from S with vertices in $V - S$.

The *central* and *internal* zonotopal algebras $\mathscr{A}_G^{\mathrm{c}}$ and $\mathscr{A}_G^{\mathrm{i}}$ of G introduced in [PS, HR] can also be presented as quotients of $\mathbf{k}[V]$ by power ideals I_G^c and I_G^i defined similarly to (1.5) with the exponent $d_S + 1$ replaced by d_S and $d_S - 1$ respectively. Specifically,

$$I_G^c := \left(\sum_{v \in V} z_v, \left(\sum_{v \in S} z_v \right)^{d_S} \; : \; S \subsetneq V \right),$$

and

$$I_G^i := \left(\sum_{v \in V} z_v, \left(\sum_{v \in S} z_v \right)^{d_S - 1} \; : \; S \subsetneq V \right),$$

These graded algebras also contain meaningful combinatorial information about G, and their Hilbert series are also specializations of the Tutte polynomial of G. In particular, the dimension of the central zonotopal algebra $\mathscr{A}_G^{\mathrm{c}}$ is equal to the number of spanning trees of G.

In this paper, we introduce three new finite-dimensional commutative graded algebras related to a graph G: the *external* $\mathscr{B}_G^{\mathrm{e}}$, the *central* $\mathscr{B}_G^{\mathrm{c}}$, and the *internal* $\mathscr{B}_G^{\mathrm{i}}$ *bizonotopal* algebras, so named because they are defined using the doubled set of edges of G. These algebras are related to the usual zonotopal algebras of G but contain substantially different information about G. For example, the dimension of the highest degree component of the external bizonotopal algebra $\mathscr{B}_G^{\mathrm{e}}$ is equal to the number of spanning forests in G (i.e. the total dimension of the algebra $\mathscr{A}_G^{\mathrm{e}}$) and the dimension of the top component of the central bizonotopal algebra $\mathscr{B}_G^{\mathrm{c}}$ is equal to the number of spanning trees in G (which is equal to the total dimension of $\mathscr{A}_G^{\mathrm{c}}$). Both these invariants are contained in the Tutte polynomial of G. However, unlike the case of usual zonotopal algebras, the Hilbert series of the new algebras are not specializations of the Tutte polynomial and contain information about G not present in the graphical matroid of G. For example, we prove that the dimension of the external algebra $\mathscr{B}_G^{\mathrm{e}}$ is equal to the number of the lattice points in the convex hull of *weak parking functions* of G, which introduce and study in Section 2.4. (For the complete graph K_n with n vertices weak parking functions coincide with usual parking functions and therefore the dimension of the algebra $\mathscr{B}_{K_n}^{\mathrm{e}}$ in this case is equal to the number of lattice points in the parking function polytope studied in [AW].)

Moreover, we prove that the external bizonotopal algebra is a complete invariant of graphs without isolated vertices and that the central bizonotopal algebra distinguishes graphs with vertices of degree at least two.

In contrast with the ordinary zonotopal algebras, the bizonotopal algebras are monomial. Quite unexpectedly, the Hilbert series of the external and central algebras satisfy a version of a deletion-contraction property, similar but subtly different from the standard one satisfied by the Tutte polynomial. Furthermore, we include our algebras into a family of r-bizonotopal algebras of G which splits into an infinite sequence of *superexternal*, corresponding to $r \geq 0$ and a finite collection of *subinternal* (for $r < 0$) algebras.

The structure of the paper is as follows. In Section 2, we introduce and study external bizonotopal algebras $\mathscr{B}^{\mathrm{e}}_G$. We prove that $\mathscr{B}^{\mathrm{e}}_G$ is a complete graph invariant and that it has a monomial basis which is in bijection with the set of lattice points in the polytope of partial score vectors of G (which, as we also show in this section, coincides with the convex hull of weak G-parking functions). In Section 3, we introduce the central, and the internal bizonotopal algebras as members of a larger family of r-bizonotopal algebras (the external algebra corresponds to $r = 1$). We prove that for $r \geq 0$, the Hilbert series of these algebras satisfy a recursion, which we call the *loopy deletion-contraction relation*. We also present an exact sequence "categorifying" this relation in the external case. In Section 4, we prove a number of additional results for the central and internal algebras. In Section 5, we formulate a number of questions for further study. Finally, in the appendix (Section 6), we collect the results of computations of the Hilbert series of the external, central, and internal algebras for complete graphs with ≤ 9 vertices.

Acknowledgements. The second and third authors are grateful to the Beijing Institute for Mathematical Sciences and Applications (BIMSA) for hospitality in June–July of 2024. The work of the third author was partially supported by the Swedish Research Council grant 2021-04900.

2 External bizonotopal algebra

In this section, we define and begin to study the external bizonotopal algebra $\mathscr{B}^{\mathrm{e}}_G$ of a graph G. We prove that this algebra is monomial and show that it has a basis corresponding to partial score vectors of G or, equivalently, to lattice points in the convex hull of weak G-parking functions.

2.1 Definitions and basic properties

2.1.1 Conventions and notation

All algebras and vector spaces in this paper are over a fixed field $\mathbf{k}$ of characteristic zero. We denote by $\mathbb{Z}_{\geq 0}$ the set of non-negative integers and by $[n]$ the set $\{1, 2, \ldots, n\}$ of the first n natural numbers.

By a graph $G = (V, E)$ in this paper we understand a finite undirected graph with vertex set V and edge set E, possibly with loops and multiple edges. For a vertex $v \in V$, we denote by $\ell(v)$ the number of loops at v and by $d(v)$ the number of edges connecting v with vertices $u \neq v$.

We define the *degree of a subset* $S \subset V$ of vertices of G as the number $\boldsymbol{\kappa}_S$ of edges incident to a vertex from S, i.e.

$$\boldsymbol{\kappa}_S := \{e \in E \,:\, e \ni v \text{ for some } v \in S\} \,. \tag{2.1}$$

To simplify notation, for a singleton set $S = \{v\}$, its degree will be denoted simply as $\boldsymbol{\kappa}_v$. Note that, since each loop is counted in $\boldsymbol{\kappa}_S$ only once, we have

$$\boldsymbol{\kappa}_v = d(v) + \ell(v),$$

which in general is different from the degree of vertex v in the traditional sense.

2.1.2 Definition of $\mathscr{B}^{\mathrm{e}}_G$

For a graph $G = (V, E)$, denote by $\widehat{E}$ the set of its oriented edges (which we also call arrows), i.e. edges with all possible orientations. Let

$$s : \widehat{E} \to V$$

be the map sending an oriented edge $e \in \widehat{E}$ to its head $s(e) \in V$ and let

$$' : \widehat{E} \to \widehat{E}$$

be the involution reversing the orientation of $e \in \widehat{E}$. Thus, the vertex $s(e')$ is the target of an oriented edge $e \in \widehat{E}$. The orbits of the involution $'$ can be identified with the set E of edges of G and its fixed points correspond to the loops of G. Thus the number of oriented edges is equal to

$$|\widehat{E}| = 2|E| - \ell, \tag{2.2}$$

where

$$\ell = \sum_{v \in V} \ell(v)$$

is the total number of loops in G.

In this notation, an orientation of a graph G is simply a section

$$\omega : E \to \widehat{E}$$

of the projection map

$$\pi : \widehat{E} \to E,\ \pi(e) := \{e, e'\} \in E \tag{2.3}$$

forgetting the direction of an arrow $e \in \widehat{E}$. We will also need a slightly more general notion.

Definition 2.1. A *partial orientation* of a graph G is a choice of orientations for a subset of its edges $S \subset E$, i.e. a section $S \to \widehat{E}$ of the restriction $\pi_{|\pi^{-1}(S)}$. Equivalently, it can be viewed as a subset of arrows $\Sigma \subset \widehat{E}$ such that the restriction π_Σ of the projection (2.3) is injective.

Similar to the square-free algebra (1.1), we consider the *partial orientation algebra*

$$\widehat{\mathscr{E}}_G := \mathbf{k}[\widehat{E}]/(x_e^2, x_e x_{e'} \ :\ e \in \widehat{E}), \tag{2.4}$$

the quotient of the polynomial algebra $\mathbf{k}[\widehat{E}] := \mathbf{k}[x_e \ :\ e \in \widehat{E}]$ in *arrow variables* x_e by the ideal generated by their squares and the products $x_e x_{e'}$ corresponding to different orientations of the same edge. The following immediate proposition explains the naming of $\widehat{\mathscr{E}}_G$.

Proposition 2.2. *For each subset $\Sigma \subset \widehat{E}$, consider the monomial*

$$x_\Sigma := \prod_{e \in \Sigma} x_e \in \mathbf{k}[\widehat{E}]. \tag{2.5}$$

(1) The image of x_Σ in $\widehat{\mathscr{E}}_G$ is nonzero if and only if Σ is a partial orientation and the images of the monomials x_Σ for different partial orientation of G,

form a basis of the algebra $\widehat{\mathscr{E}}_G$.

(2) As a graded algebra, $\widehat{\mathscr{E}}_G$ *is isomorphic to the tensor product of* ℓ *copies of the algebra of dual numbers* $D := \mathbf{k}[\varepsilon]/(\varepsilon^2)$ *and* $|E| - \ell$ *copies of the three-dimensional algebra* $T := \mathbf{k}[\varepsilon, \varepsilon']/(\varepsilon^2, (\varepsilon')^2, \varepsilon\varepsilon')$, *i.e.*

$$\widehat{\mathscr{E}}_G \simeq (D^{\otimes \ell}) \otimes (T^{\otimes(|E|-\ell)}). \tag{2.6}$$

(3) The dimension of $\widehat{\mathscr{E}}_G$ *is equal to* $2^\ell 3^{|E|-\ell}$ *and its Hilbert series is equal to* $(1+t)^\ell (1+2t)^{|E|-\ell}$.

□

To each vertex $v \in V$ we associate a degree one element

$$y_v = \sum_{e \in s^{-1}(v)} x_e \ , \tag{2.7}$$

in the algebra $\widehat{\mathscr{E}}_G$, i.e. the sum of the generators x_e over all arrows exiting from v. (To avoid notational clutter, we use x_e to denote generators of $\widehat{\mathscr{E}}_G$. The difference should be clear from context.)

Definition 2.3 (The algebra $\mathscr{B}^{\mathrm{e}}_G$)**.** For a graph G, the subalgebra $\mathscr{B}^{\mathrm{e}}_G$ of the partial orientation algebra $\widehat{\mathscr{E}}_G$ generated by the elements $y_v,\ v \in V$, is called the *external bizonotopal algebra* of G.

Clearly, $\mathscr{B}^{\mathrm{e}}_G$ is a finite-dimensional graded algebra. It has various connections with ordinary zonotopal algebras. For example, each choice of an orientation $\omega : E \to \widehat{E}$ of G induces a homomorphism from $\widehat{\mathscr{E}}_G$ onto the edge algebra $\mathscr{E}_G$ (1.1) given by

$$f_\omega : \widehat{\mathscr{E}}_G \to \mathscr{E}_G,\ \ f_\omega(x_e) := \begin{cases} x_{\pi(e)}, & \text{if } \omega(s(e)) = e \\ -x_{\pi(e)}, & \text{if } \omega(s(e)) \neq e\ , \end{cases}$$

which induces a surjective homomorphism $\mathscr{B}^{\mathrm{e}}_G \twoheadrightarrow \mathscr{A}^{\mathrm{e}}_G$ onto the usual external zonotopal algebra.

Similarly, the *orientation forgetting homomorphism* induced by π

$$\widehat{\mathscr{E}}_G \to \mathscr{E}_G,\ \ x_e \mapsto x_{\pi(e)},$$

gives a projection $\mathscr{B}^{\mathrm{e}}_G \twoheadrightarrow \mathscr{A}^{+}_G$ onto the algebra $\mathscr{A}^{+}_G$ constructed similarly to $\mathscr{A}^{\mathrm{e}}_G$ with the oriented incidence matrix (1.2) replaced by the unoriented one (see [SV]). The algebra $\mathscr{A}^{+}_G$ is the external zonotopal algebra of the *even-circle matroid* of G [D, Si].

2.1.3 Basis of $\mathscr{B}_G$

Next we will describe a basis of $\mathscr{B}_G$ and identify it with the set of partial score vectors of G.

Definition 2.4 (Partial score vectors)**.** Let $G = (V, E)$ be a graph with $n = |V|$ vertices. A collection of n non-negative integers $(a_v)_{v\in V} \in \mathbb{Z}_{\geq 0}^V$ is called a *partial score vector* of G if there exists a partial orientation $\Sigma \subset \widehat{E}$ of G such that a_v is equal to the number of arrows in Σ starting at v.

Partial score vectors arise as exponents of basis elements of $\mathscr{B}_G^{\mathrm{e}}$. For a vector $\mathbf{a} = (a_v) \in \mathbb{Z}_{\geq 0}^V$, consider the monomial

$$y^{\mathbf{a}} := \prod_{v\in V} y_v^{a_v} \in \mathscr{B}_G^{\mathrm{e}}$$

in generators (2.7).

Proposition 2.5.

(1) *The monomial $y^{\mathbf{a}} \in \mathscr{B}_G^{\mathrm{e}}$ is nonzero if and only if $\mathbf{a}$ is a partial score vector.*

(2) *Elements $y^{\mathbf{a}}$ of $\mathscr{B}_G^{\mathrm{e}}$ corresponding to different partial score vectors of G are linearly independent and, therefore, form a basis of $\mathscr{B}_G^{\mathrm{e}}$.*

Proof. We start with the following simple but crucial observation. If $e \in \widehat{E}$ is an oriented edge going from vertex $v = s(e)$ to vertex $u = s(e')$, then y_v is the only generator (2.7) containing variable $x_e \in \widehat{\mathscr{E}}_G$. Therefore, a nonzero monomial x_Σ (2.5) in variables x_e corresponding to a subset $\Sigma \subset \widehat{E}$ appears in the expansion of a unique monomial $y^{\mathbf{a}}$, specifically the one where $\mathbf{a}$ is the partial score vector corresponding to the partial orientation Σ. This shows that $y^{\mathbf{a}} \neq 0$ exactly when $\mathbf{a}$ is a partial score vector. Moreover, since the sets of monomials x_Σ appearing in expansions of different nonzero elements $y^{\mathbf{a}}$ are disjoint, Proposition 2.2 implies that these elements are linearly independent. This proves (1) and (2). □

Corollary 2.6.

(1) *The dimension of the external bizonotopal algebra $\mathscr{B}_G^{e}$ of a graph G is equal to the number of partial score vectors of G.*

(2) *The dimension of the m-th graded component of $\mathscr{B}^{\mathrm{e}}_G$ is equal to the number of partial score vectors $\mathbf{a}$ of weight m, where by the **weight** of a vector $\mathbf{a} \in \mathbb{Z}^V$ we understand the sum of all its coordinates $|\mathbf{a}| := \sum_{v \in V} a_v$.*

(3) *The degree of the top component of $\mathscr{B}^{\mathrm{e}}_G$ is equal to $|E|$, the number of edges of G.*

(4) *The dimension of the top degree component of $\mathscr{B}^{\mathrm{e}}_G$ is equal to the number of spanning forests of G.*

Proof. The first statement follows from the second part of Proposition 2.5. The grading of $\mathscr{B}^{\mathrm{e}}_G$ is induced from the polynomial algebra $\mathbf{k}[\widehat{E}]$, which implies (2). Part (3) follows from the fact that a partial orientation $\Sigma \subset \widehat{E}$ is $|E|$ can have at most $|E|$ oriented edges. So basis elements of maximal degree correspond to total orientations of G and thus the dimension of the top degree component of $\mathscr{B}^{\mathrm{e}}_G$ is equal to the number of usual score vectors of G. By the result of Kleitman and Winston [KW], this number is equal to the number of spanning forests of G, thus giving (4). □

2.1.4 External bizonotopal algebras distinguish graphs

Theorem 2.7. *Let G_1 and G_2 be two graphs without isolated vertices and let $\mathscr{B}^{\mathrm{e}}_{G_1}$ and $\mathscr{B}^{\mathrm{e}}_{G_2}$ be their external bizonotopal algebras. Then the following are equivalent.*

(1) *Graphs G_1 and G_2 are isomorphic.*

(2) *$\mathbb{Z}_{\geq 0}$-graded algebras $\mathscr{B}^{\mathrm{e}}_{G_1}$ and $\mathscr{B}^{\mathrm{e}}_{G_2}$ are isomorphic.*

(3) *Algebras $\mathscr{B}^{\mathrm{e}}_{G_1}$ and $\mathscr{B}^{\mathrm{e}}_{G_2}$ are isomorphic (as non-graded algebras).*

Proof. Clearly, if graphs G_1 and G_2 are isomorphic, then the corresponding algebras are also isomorphic. Also, the equivalence of (2) and (3) is well-known (see, e.g., [BZ]). It remains to prove the implication (2)⇒(1).

First some preliminaries. For a nilpotent element u of some algebra, we denote by $\mathrm{ord}(u)$ the smallest $m \in \mathbb{Z}_{\geq 0}$ such that $u^{m+1} = 0$. If $u \in (\mathscr{B}^{\mathrm{e}}_G)^{(1)}$ is a degree 1 element of an external bizonotopal algebra of a graph $G = (V, E)$, then it follows from Proposition 2.2 that $\mathrm{ord}(u)$ is exactly the number of distinct (unoriented) edges $\bar{e} = \{e, e'\} \in E$ appearing in the expansion of u in the basis (x_e) of $\widehat{\mathscr{E}}_G$.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs without isolated vertices such that $\mathscr{B}^e_{G_1} \simeq \mathscr{B}^e_{G_2}$, then

$$|V_1| = \dim(\mathscr{B}^e_{G_1})^{(1)} = \dim(\mathscr{B}^e_{G_2})^{(1)} = |V_2|.$$

Let $(u_1, u_2, \ldots, u_n)$ be a basis of the space $(\mathscr{B}^e_{G_1})^{(1)}$ with the smallest possible sum $\sum_{i=1}^{n} \operatorname{ord}(u_i)$. Then we claim that

$$\operatorname{ord}(u_1) + \operatorname{ord}(u_2) + \ldots + \operatorname{ord}(u_n) = \dim \widehat{\mathscr{E}}^{(1)}_{G_1} = 2|E_1| - \ell_{G_1},$$

where, as before, ℓ_G denotes the number of loops in G. Indeed, each oriented edge $e \in \widehat{E}$ must appear in u_i of the basis. This, together with equation (2.2) and Proposition 2.2, gives the lower bound

$$\sum_i \operatorname{ord}(u_i) \geq 2|E_1| - \ell_{G_1}.$$

And, since for the standard basis $(y_v)_{v \in V_1}$ (2.7) of $(\mathscr{B}^e_{G_1})^{(1)}$ we have

$$\sum_{v \in V_1} \operatorname{ord}(y_v) = \sum_v \boldsymbol{\kappa}_v = 2|E_1| - \ell_{G_1},$$

the minimality of (u_i) gives the opposite inequality $\sum_i \operatorname{ord}(u_i) \leq 2|E_1| - \ell_{G_1}$.

Since $u_1, \ldots, u_n$ is a basis of $(\mathscr{B}^e_{G_1})^{(1)}$, there is an ordering $V_1 = (v_1, \ldots, v_n)$ of the vertex set of G_1 such that the diagonal coefficients $c_{i,i}$ in the expansion

$$u_i = c_{i,1} y_{v_1} + \ldots + c_{i,n} y_{v_n}$$

are nonzero. In this case we have $\operatorname{ord}(u_i) \geq \operatorname{ord}(y_{v_i})$. Since

$$\sum_i \operatorname{ord}(u_i) = \sum_i \operatorname{ord}(y_{v_i}) = 2|E_1| - \ell_{G_1},$$

we have $\operatorname{ord}(u_i) = \operatorname{ord}(y_{v_i})$ for all i. Thus the nonzero terms in the expansion of u_i in the variables x_e correspond the same unoriented edges which appear in the expansion of y_{v_i} which are precisely the edges incident to v_i. This implies that the number of edges between v_i and v_j is equal to $\operatorname{ord}(u_i) +$

$\text{ord}(u_j) - \text{ord}(u_i + \lambda u_j)$, where $\lambda \in \mathbf{k}$ is generic (i.e. avoids finitely many special values).

Knowing the number of edges in E_1 which are incident to v_i and the number of edges between v_i and v_j for each j, we can determine the number of loops at the vertex v_i. Therefore we can reconstruct graph G_1 from $\mathscr{B}^{\mathrm{e}}_{G_1}$. Similarly we can recover G_2 from $\mathscr{B}^{\mathrm{e}}_{G_2}$. Since $\mathscr{B}^{\mathrm{e}}_{G_2}$ and $\mathscr{B}^{\mathrm{e}}_{G_2}$ are isomorphic, the underlying graphs G_1 and G_2 are isomorphic as well. □

2.2 Defining relations of $\mathscr{B}^{e}_{G}$

Now we will describe the set of relations between the generators of $\mathscr{B}^{\mathrm{e}}_{G}$ given by (2.7) which, among other things, show that $\mathscr{B}^{\mathrm{e}}_{G}$ is a monomial algebra.

Partial score vectors have a convenient characterization in terms of degrees of subsets of vertices $\boldsymbol{\kappa}_S$, see (2.1).

Proposition 2.8. *A vector* $\mathbf{a} = (a_v)_{v \in V} \in \mathbb{Z}^V_{\geq 0}$ *is a partial score vector of* $G = (V, E)$ *if and only if for every subset* $S \subset V$ *we have*

$$\sum_{v \in S} a_v \leq \boldsymbol{\kappa}_S. \tag{2.8}$$

Proof. If $\mathbf{a} = (a_v)$ is a partial score vector corresponding to a partial orientation $\Sigma \subset \widehat{E}$, then (2.8) holds because every arrow $e \in \Sigma$ contributes to a unique a_v.

Conversely, given a vector $\mathbf{a} \in \mathbb{Z}^V_{\geq 0}$ satisfying (2.8), we will find a partial orientation producing $\mathbf{a}$ by applying Hall's marriage theorem. Let $\mathscr{G}$ be a bipartite graph whose vertex set consists of two parts

$$A = \{(v, i) \,:\, v \in V, \text{ such that } a_v \neq 0, \ i \in \{1, \ldots, a_v\}\} \subset V \times \mathbb{Z}_{\geq 0},$$

and $B = E$, the set of edges of G. Vertices $(v, i) \in A$ and $e \in B$ are connected by an edge in $\mathscr{G}$ if v is incident to e. Then the inequality (2.8) implies that the graph $\mathscr{G}$ satisfies the condition of Hall's marriage theorem. Therefore there exists a perfect matching $g : A \hookrightarrow B$. This matching gives rise to a partial orientation of G by orienting each edge $e = g(v, i)$ in the image $g(A)$ as exiting from vertex v and leaving all edges which are not in $g(A)$ unoriented. The score vector corresponding to this partial orientation is $\mathbf{a}$. □

We denote by $\mathfrak{M}_S$ the set of monomials in $\mathbf{k}[V] := \mathbf{k}[z_v \,:\, v \in V]$ of total degree $\boldsymbol{\kappa}_S + 1$ in variables z_v with $v \in S$. In other words,

$$\mathfrak{M}_S = \Big\{ \prod_{v \in S} z_v^{a_v} \;:\; \sum_v a_v = \boldsymbol{\kappa}_S + 1 \Big\}. \tag{2.9}$$

Theorem 2.9. *Given a graph $G = (V, E)$, let*

$$f_G : \mathbf{k}[V] \to \mathscr{B}^e_G, \;\; z_v \mapsto y_v,$$

be the surjective homomorphism sending free generators of $\mathbf{k}[V]$ to generators (2.7) of $\mathscr{B}^e_G$. Then $\operatorname{Ker}(f_G) = \mathfrak{I}_G$, where

$$\mathfrak{I}_G := \Big(\bigcup_{\emptyset \neq S \subset V} \mathfrak{M}_S \Big) \subset \mathbf{k}[V] \tag{2.10}$$

is the ideal generated by all monomials in $\mathfrak{M}_S$ for nonempty subsets $S \subset V$. In other words, the collection of monomials from $\mathfrak{M}_S$ is the set of defining relations between the generators y_v of the external zonotopal algebra $\mathscr{B}^e_G$.

Proof. First we show that for $\emptyset \neq S \subset V$, the monomial $\prod_{v \in S} z_v^{a_v} \in \mathfrak{M}_S$ vanishes in $\mathscr{B}^e_G$, i.e. it belongs to $\operatorname{Ker} f_G$. Indeed, consider the expansion of $f_G(\prod_{v \in S} z_v^{a_v}) = \prod_{v \in S} y_v^{a_v} \in \widehat{\mathscr{E}}_G$ in arrow variables x_e, the generators of the partial orientation algebra $\widehat{\mathscr{E}}_G$. Let $x_\Sigma \in \widehat{\mathscr{E}}_G$ be some monomial in this expansion. By Proposition 2.2 and equation (2.7), if $x_\Sigma \neq 0$ then $\Sigma \subset \widehat{E}$ is a partial orientation of G with all arrows $e \in \Sigma$ starting or ending in S. But $|\Sigma| = \deg x_\Sigma = \boldsymbol{\kappa}_S + 1$, which implies that the restriction $\pi_{|\Sigma}$ is not injective. Therefore, Σ is not a partial orientation and, hence, every term in the expansion of $\prod_{v \in S} y_v^{a_v}$ vanishes. This shows that $\mathfrak{I}_G \subset \operatorname{Ker} f_G$ and thus we have a surjection

$$\mathbf{k}[V]/\mathfrak{I}_G \twoheadrightarrow \mathbf{k}[V]/\operatorname{Ker} f_G \simeq \mathscr{B}^e_G.$$

To prove the opposite inclusion $\mathfrak{I}_G \supset \operatorname{Ker} f_G$, it is now enough to show that f_G is injective on the set $\mathfrak{I}^c_G$ of monomials complementary to $\mathfrak{I}_G$. If $\mu = \prod_{v \in V} z_v^{a_v} \in \mathfrak{I}^c_G$ is such a monomial, then, by (2.9), for every nonempty

$S \subset V$ we have $\sum_{v \in S} a_v \leq \boldsymbol{\kappa}_S$. By Proposition 2.8, this means that the vector of the exponents $\mathbf{a} = (a_v)_{v\in V}$ of μ form a partial score vector of G and, by Proposition 2.2, $f_G(\mu) = \prod_{v\in V} y_v^{a_v} = y^{\mathbf{a}}$ is a basis element of $\mathscr{B}_G^e$. Thus the images $f_G(\mu)$ of the monomials from $\mathfrak{I}_G^c$ are linearly independent. □

2.3 Score vector polytope

Proposition 2.8 shows that score vectors are lattice points in an integral convex polytope. Therefore, according to Proposition 2.5, these lattice points also label elements in a basis of $\mathscr{B}_G^E$. In this subsection we study the vertices (extreme points) of this polytope.

Definition 2.10. For a graph $G = (V, E)$ with $n = |V|$ vertices, we define its *score vector polytope* $\mathscr{P}_G$ as the set of points in the n-dimensional space $\mathbb{R}^V$ satisfying the inequalities (2.8):

$$\mathscr{P}_G := \{(a_v)_{v\in V} \in \mathbb{R}^V_{\geq 0} \,:\, \sum_{v\in S} a_v \leq \boldsymbol{\kappa}_S, \text{ for all } S \subset V\}. \tag{2.11}$$

The set of vertices (extreme points) of the polytope $\mathscr{P}_G$ can be characterized in several equivalent ways as described in the following theorem.

Theorem 2.11. *For a graph $G = (V, E)$ and a vector $\mathbf{a} = (a_v)_{v\in V} \in \mathbb{Z}^V_{\geq 0}$, the following conditions are equivalent:*

(1) $\mathbf{a}$ *is a vertex of the score vector polytope $\mathscr{P}_G$ of G.*

(2) $\mathbf{a}$ *is of the form $\mathbf{a}^{\Pi,m} = (a_v)_{v\in V}$, where $\Pi = (v_1, \ldots, v_n)$ is a linear ordering of V, $m \in \{0, 1, \ldots, n\}$, and*

$$a_v = \begin{cases} 0, & \text{if } v = v_i \text{ for } i \leq m; \\ \text{number of edges from } v = v_i \text{ to } \{v_1, \ldots, v_i\}, & \text{if } i > m. \end{cases} \tag{2.12}$$

(3) $\mathbf{a}$ *is of the form $\mathbf{a}^J = (a_v)$, where $J = (v_1, \ldots, v_r)$ is a linearly ordered subset of V and*

$$a_v = \begin{cases} \text{number of edges from } v_i \text{ to } V - \{v_1, \ldots, v_{i-1}\}, & \text{if } v = v_i \in J; \\ 0, & \text{if } v \notin J. \end{cases} \tag{2.13}$$

(4) **a** *is a partial score vector of* G *corresponding to a unique partial orientation of* G*.*

In the proof of this theorem, we will use a special property of the degree function $\boldsymbol{\kappa}_S$. Recall that a function $f : 2^V \to \mathbb{R}$ is called *submodular*, if

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

for all $A, B \subset V$.

Lemma 2.12. *The degree function* $\boldsymbol{\kappa}$ *of any graph* $G = (V, E)$ *is submodular.*

Proof. To show that

$$\boldsymbol{\kappa}_I + \boldsymbol{\kappa}_J \geq \boldsymbol{\kappa}_{I\cap J} + \boldsymbol{\kappa}_{I\cup J}. \tag{2.14}$$

for $I, J \subset V$, we will compare the contributions of a given edge $e \in E$ to both sides of this inequality. There are three possibilities.

(i) If e is incident to both I and J, then it contributes 2 to the left-hand side of (2.14) and 1 or 2 to the right-hand side;

(ii) If e is incident to only one of the two subsets I, J, then its contribution to each of the two parts of (2.14) is equal to 1;

(iii) If e is incident to neither I nor J, then it contributes 0 to each part.

Since in each of the three cases the inequality (2.14) holds, we conclude that $\boldsymbol{\kappa}$ is a submodular function. □

Proof of Theorem 2.11.
(1) ⇒ (2). We use induction on the number $n = |V|$ of vertices G. If $n = 1$, then G has one vertex v and $\ell = |E|$ is the number of edges which are all loops. In this case $\mathscr{P}_G$ is the segment $[0, \ell] \subset \mathbb{Z}_{\geq 0}$. Its vertices, 0 and ℓ, have the required form $\mathbf{a}^{\Pi,m}$, given by (2.12), corresponding to the trivial ordering $\Pi = (v)$ and $m = 1$ and $m = 0$, respectively.

To carry out the induction step, assume that $\mathbf{a} = (a_v) \in \mathbb{Z}^V_{\geq 0}$ is a vertex of $\mathscr{P}_G$. There are two possibilities:

(i) *for every nonempty subset* $S \subset V$*, we have* $\sum_{v\in S} a_v < \boldsymbol{\kappa}_S$*;*

(ii) *there exists a nonempty subset* $S \subset V$ *such that* $\sum_{v\in S} a_v = \boldsymbol{\kappa}_S$*.*

In the first case, if $\mathbf{a}$ has two nonzero components a_v and a_u for $u \neq v \in S$, then the vectors $\mathbf{a}' = \mathbf{a} + \mathbf{e}_v - \mathbf{e}_u$ and $\mathbf{a}'' = \mathbf{a} - \mathbf{e}_v + \mathbf{e}_u$ are distinct and belong to $\mathscr{P}_G$. (Here $\mathbf{e}_v$ denotes the v-th standard basis vector of the lattice $\mathbb{Z}^V$.) Thus $\mathbf{a} = \frac{1}{2}(\mathbf{a}' + \mathbf{a}'')$ which contradicts the assumption that $\mathbf{a}$ is a vertex of $\mathscr{P}_G$. If $\mathbf{a}$ has at most one nonzero component a_v, then $\mathbf{a} = a_v \mathbf{e}_v$. Since both vectors $\mathbf{a}$ and $\boldsymbol{\kappa}_v \mathbf{e}_v$ belong to $\mathscr{P}_G$ and, by our assumption $a_v < \boldsymbol{\kappa}_v$, we see that $\mathbf{a}$ can be a vertex of $\mathscr{P}_G$ only when $a_v = 0$, i.e. $\mathbf{a} = \mathbf{0}$. This vector has the desired form (2.12) for $m = n$ and arbitrary Π.

In the second case, we claim that there exists $u \in S$ with $a_u = \boldsymbol{\kappa}_u$. To prove this, let S be a minimal (by inclusion) subset of V with the above property. We want to show that $|S| = 1$.

First notice that if $S' \subset V$ also satisfies $\sum_{v \in S'} a_v = \boldsymbol{\kappa}_{S'}$ and $S \cap S' \neq \emptyset$, then $S \subset S'$. Indeed, if $\emptyset \neq S \cap S' \neq S$, then minimality of S implies $\boldsymbol{\kappa}_{S \cap S'} > \sum_{v \in S \cap S'} a_v$. Therefore, by Lemma 2.12 we get

$$\boldsymbol{\kappa}_{S \cup S'} \leq \boldsymbol{\kappa}_S + \boldsymbol{\kappa}_{S'} - \boldsymbol{\kappa}_{S \cap S'} < \sum_{v \in S} a_v + \sum_{v \in S'} a_v - \sum_{v \in S \cap S'} a_v = \sum_{v \in S \cup S'} a_v,$$

which contradicts the assumption that $\mathbf{a} \in \mathscr{P}_G$.

Now, if $|S| > 1$, choose $p, q \in S$, $p \neq q$. Let us show that the vectors $\mathbf{a}' = \mathbf{a} + \mathbf{e}_p - \mathbf{e}_q$ and $\mathbf{a}'' = \mathbf{a} - \mathbf{e}_p + \mathbf{e}_q$ belong to $\mathscr{P}_G$. It is enough to do this for $\mathbf{a}'$. Since S is minimal by inclusion, we know that $a_p,\ a_q > 0$, which implies that $\mathbf{a}',\ \mathbf{a}'' \in \mathbb{Z}^V_{\geq 0}$. For a subset $S' \subset V$, let $\delta_{S'} : V \to \{0, 1\}$ be the indicator function of S'. If $|S' \cap \{p, q\}| = 0$ or $|S' \cap \{p, q\}| = 2$, then $\delta_{S'}(p) = \delta_{S'}(q)$, and we have

$$\sum_{v \in S'} a'_v = \sum_{v \in S'} a_v + \delta_{S'}(p) - \delta_{S'}(q) = \sum_{v \in S'} a_v \leq \boldsymbol{\kappa}_{S'}.$$

If $|S' \cap \{p, q\}| = 1$, then $\emptyset \neq S \cap S' \neq S$ and, as we saw above, minimality of S implies that $\sum_{v \in S'} a_v < \boldsymbol{\kappa}_{S'}$. Thus we have

$$\sum_{v \in S'} a'_v = \sum_{v \in S'} a_v + \delta_{S'}(p) - \delta_{S'}(q) < \boldsymbol{\kappa}_{S'} + \delta_{S'}(p) - \delta_{S'}(q) \leq \boldsymbol{\kappa}_{S'}.$$

Therefore, $\mathbf{a}'$ and $\mathbf{a}''$ are distinct vectors in $\mathscr{P}_G$, which shows that vector $\mathbf{a} = \frac{1}{2}(\mathbf{a}' + \mathbf{a}'')$ cannot be a vertex of $\mathscr{P}_G$.

This implies that $|S| = 1$, i.e. there exists $u \in V$ such that $a_u = \boldsymbol{\kappa}_u$. Let $G' = (V', E')$ with $V' = V - \{u\}$ be the graph obtained from G by removing the vertex u and all incident to it edges from G. If we identify $\mathbb{R}^{V'}$ with the affine hyperplane $H_u := \{(x_v) \in \mathbb{R}^V : x_u = \boldsymbol{\kappa}_u\}$ in $\mathbb{R}^V$, then the score vector polytope $\mathscr{P}_{G'}$ of G' will be identified with the face $\mathscr{P}_G \cap H_u$ of the polytope $\mathscr{P}_G$. Clearly, vertices of $\mathscr{P}_G$ lying in H_u correspond to vertices of $\mathscr{P}_{G'}$. By the induction hypothesis, the vertex $\mathbf{a}' \in \mathscr{P}_{G'}$ corresponding to $\mathbf{a} \in \mathscr{P}_G$ is of the form $\mathbf{a}^{\Pi',m}$ (2.12) for some ordering $\Pi' = (v_1, \dots, v_{n-1})$ of V'. Then, appending u at the end of Π', we see that the vector $\mathbf{a}$ is also of the required form, $\mathbf{a} = \mathbf{a}^{\Pi,m}$, where $\Pi = (v_1, \dots, v_{n-1}, u)$, thus proving the induction step.

(2) $\Rightarrow$ (1). We need to check that every vector $\mathbf{a}^{\Pi,m} = (a_v)_{v\in V}$ of the form (2.12) is a vertex of $\mathscr{P}_G$. First, from (2.12) we see that $\sum_{v\in S} a_v \le \boldsymbol{\kappa}_S$, for every $S \subset V$, i.e. $\mathbf{a}^{\Pi,m} \in \mathscr{P}_G$.

Now assume that $\mathbf{a}^{\Pi,m}$ is not a vertex of $\mathscr{P}_G$, i.e. it belongs to the convex hull of some vertices $\mathbf{b}_1, \dots, \mathbf{b}_p \in \mathscr{P}_G$. Let $\preceq$ be the lexicographic order on $\mathbb{R}^V_{\ge 0}$ induced by the reversed ordering $\overline{\Pi} = (v_n, \dots, v_2, v_1)$ of V. That is $\mathbf{x} = (x_{v_1}, \dots, x_{v_n}) \preceq \mathbf{y} = (y_{v_1}, \dots, y_{v_n})$ if $\mathbf{x} - \mathbf{y} = (z_{v_1}, \dots, z_{v_\ell}, 0, \dots, 0)$, with $z_{v_\ell} < 0$, for some $\ell \in [n]$. Since $\mathbf{a}^{\Pi,m}$ belongs to the convex hull of $\mathbf{b}_1, \dots, \mathbf{b}_p$, at least one of these vectors must be strictly greater than $\mathbf{a}^{\Pi,m}$ with respect to $\preceq$. From (2.12) we know that $a_{v_i} = 0$ for $i \le m$, which implies that the first m coordinates of each of $\mathbf{b}_j \in \mathbb{R}^V_{\ge 0}$ are also 0. This shows that there is a vector

$$\mathbf{b} = (0, \dots, 0, b_{v_{m+1}}, \dots, b_{v_\ell}, a_{v_{\ell+1}}, \dots, a_{v_n}) \in \mathscr{P}_G, \tag{2.15}$$

with $b_{v_\ell} > a_{v_\ell}$ for some $\ell \in \{m+1, \dots, n\}$. Then, for $S = \{v_\ell, \dots, v_n\}$ we have

$$\sum_{v\in S} b_v > \sum_{v\in S} a_v = \sum_{i=\ell}^{n} a_{v_i} = \boldsymbol{\kappa}_S.$$

Hence, $\mathbf{b}$ does not satisfy (2.11) which gives a contradiction with (2.15).

(2) $\Leftrightarrow$ (3). If a vector $\mathbf{a} = \mathbf{a}^{\Pi,m}$ is of the form (2.12) with an ordering $\Pi = (v_1, \dots, v_n)$ and $m \in \{0, 1, \dots, n\}$, then $\mathbf{a}$ can be presented in the form (2.13), $\mathbf{a} = \mathbf{a}^J$ by taking $J = (v_n, v_{n-1}, \dots, v_{n-m})$. (In particular, $J = \emptyset$, if $m = n$). Conversely, if $\mathbf{a} = \mathbf{a}^J$ for an ordered subset $J = (v_1, \ldots, v_r)$, then $\mathbf{a} = \mathbf{a}^{\Pi,m}$, with $m = n - |J|$ and the ordering $\Pi = (v_r, v_{r-1}, \ldots)$ obtained by reversing J and appending to it the complement $V - J$ in any order.

(2) ⇒ (4). A vector $\mathbf{a}^{\Pi,m}$ of the form (2.12) is the score vector of the partial orientation in which an edge $e \in E$ between vertices v_i and v_j with $i \geq j$ is oriented from v_i to v_j, if $i > m$, and is unoriented otherwise. The uniqueness of such partial orientation follows from the observation that the component a_{v_i} of $\mathbf{a}^{\Pi,m}$, for $i = m+1, \ldots, n$, is equal to $\boldsymbol{\kappa}_{\{v_i, v_{i+1}, \ldots, v_n\}} - \boldsymbol{\kappa}_{\{v_{n-i+1}, \ldots, v_n\}}$.

(4) ⇒ (3). Let $\mathbf{a} = (a_v)_{v \in V} \in \mathscr{P}_G$ be a partial score vector that corresponds to a unique partial orientation $\Sigma \subset \widehat{E}$. To construct an ordered subset $J \subset V$ such that $\mathbf{a} = \mathbf{a}^J$, we will use two special properties of Σ.

First, we claim that if an arrow $e \in \widehat{E}$ belongs to Σ, then every edge incident to the head $v = s(e) \in V$ of e must also be oriented in Σ (i.e. for every $f \in s^{-1}(v) \subset \widehat{E}$ either f or f' is in Σ). Indeed, if neither f nor f' are in Σ, then after replacing e by f we will obtain a new partial orientation $\Sigma' = (\Sigma \cup \{f\}) - \{e\}$ which gives the same partial score vector $\mathbf{a}$.

Second, we claim that Σ contains no oriented cycles other than loops. Indeed, if non-loop arrows $e_1, e_2, \ldots, e_p \in \Sigma$ form an oriented cycle, then replacing them in Σ with oppositely oriented arrows $e'_1, \ldots, e'_p$, will not change the partial score vector.

Now, let $J \subset V$ be the set of all vertices $v \in V$ with $a_v > 0$. If $J = \emptyset$, then $\mathbf{a} = \mathbf{0} = \mathbf{a}^J$. Thus we can assume that J is nonempty. From the first property it follows that every edge in G incident to some vertex from J is oriented in Σ. Since Σ has no oriented cycles and $J \neq \emptyset$, there must be a source vertex $v_1 \in J$ (i.e. such that all edges adjacent to v_1 are outgoing. There are no oriented cycle between vertices in $J - \{v_1\}$, therefore we can choose $v_2 \in J - \{v_1\}$ such that all edges incident to v_2 are outgoing except those incident to v_1. Continuing in this way, we will obtain a sequence of vertices $v_i \in J$, $i = 1, \ldots, |J|$, such that $v_i \in J - \{v_1, \ldots, v_{i-1}\}$ and all edges incident to v_i are outgoing except the edges incident to the subset $\{v_1, v_2, \ldots, v_{i-1}\}$. This gives a linear ordering of J such that $\mathbf{a}^J = \mathbf{a}$. □

In general it is difficult to find the exact number of vertices of the score vector polytope $\mathscr{P}_G$. However we have the following upper bound.

Corollary 2.13. *For a graph G with n vertices, the polytope $\mathscr{P}_G$ has at most $\lfloor e \cdot n! \rfloor$ vertices.*

If G is a simple graph, then the number of vertices of $\mathscr{P}_G$ does not exceed $\lfloor (e-1)\cdot n! \rfloor$. This bound is exact only if G is the complete graph K_n.

Proof. From part (3) of Theorem 2.11 it follows that the number N_G of vertices in $\mathscr{P}_G$ is less than or equal to the number of ordered subsets of V, i.e.

$$N_G \leq \sum_{S \subset V} (|S|)! = \sum_{m=0}^{n} \binom{n}{m} m! = \sum_{m=0}^{n} \frac{n!}{(n-m)!} = n! \sum_{i=0}^{n} \frac{1}{i!} = \lfloor e \cdot n! \rfloor.$$

If G is a simple graph, then any ordering $J = (v_1, \ldots, v_{n-1}, v_n)$ of the full set V and its truncation $J' = (v_1, \ldots, v_{n-1})$ give the same vertex $\mathbf{a}^J = \mathbf{a}^{J'}$ of $\mathscr{P}_G$. Therefore, in this case, we can drop the last term in the above sum and obtain a better estimate

$$N_G \leq n! \sum_{m=0}^{n-1} \frac{1}{(n-m)!} = n! \sum_{i=1}^{n} \frac{1}{i!} = \lfloor (e-1) \cdot n! \rfloor. \tag{2.16}$$

If $G = K_n$ is a complete graph with $V = [n] = \{1, \ldots, n\}$, then the vertex of $\mathscr{P}_G$ corresponding by (2.13) to the ordered subset $J = (1, \ldots, m) \subset [n]$ is given by

$$\mathbf{a}^J = (n-1, n-2, \ldots, n-m, 0, \ldots, 0)$$

and all other vertices are obtained from it by permutations of $[n]$. This vertex has m distinct nonzero entries and, therefore, each of $n!/(n-m)!$ choices of nonempty ordered m-element subsets of V gives a distinct vertex of $\mathscr{P}_G$. Thus for the complete graph, (2.16) becomes an equality:

$$N_{K_n} = \sum_{i=1}^{n} \frac{n!}{i!} = \lfloor (e-1) \cdot n! \rfloor.$$

Finally, for a non-complete simple graph G with n vertices, choose two vertices $v_1, v_2 \in V$ not connected by an edge. Then the ordered subsets $J_1 = (v_1, v_2)$ and $J_2 = (v_2, v_1)$ give the same vertex $\mathbf{a}^{J_1} = \mathbf{a}^{J_2}$ of $\mathscr{P}_G$. Therefore, in this case, the inequality (2.16) is strict. ∎

2.4 Weak parking functions of a graph

Here we will give yet another characterization of the score vector polytope $\mathscr{P}_G$ of a graph $G = (V, E)$ and, thus, another description of the monomial basis of the external bizonotopal algebra $\mathscr{B}^e_G$.

Recall the notion of a parking function of a graph introduced by Postnikov and Shapiro in [PS].

Definition 2.14. Let $G = (V, E)$ be a graph. For a subset of vertices $S \subset V$ and $v \in S$, denote by $d_S(v)$ the number of edges in E connecting v with vertices in $V - S$.

A *G-parking function*, relative to a distinguished vertex $q \in V$, is a function $f : V - \{q\} \to \mathbb{Z}_{\geq 0}$ such that for each nonempty subset $S \subset V - \{q\}$, there exists a vertex $v \in S$ with $f(v) < d_S(v)$.

The above definition does not take into account loops of G. To change this, we introduce a modification of the concept of a G-parking function which does not require a choice of a distinguished vertex and, as we will see below, is related to the score vector polytope $\mathscr{P}_G$.

Definition 2.15. For a subset of vertices $S \subset V$ of a graph $G = (V, E)$ and a vertex $v \in S$, we denote by $\widehat{d}_S(v)$ the number of edges in E with one end at v and the other in $(V - S) \cup \{v\}$. In other words,

$$\widehat{d}_S(v) = d_S(v) + \ell(v),$$

where $\ell(v)$ is the number of loops at v. In particular, $d_V(v) = \ell(v)$.

A *weak G-parking function* is a function $f : V \to \mathbb{Z}_{\geq 0}$ such that for each nonempty subset $S \subset V$, there exists a vertex $v \in S$ with $f(v) \leq \widehat{d}_S(v)$.

In the definition of a G-parking function we had to exclude the distinguished vertex because otherwise the inequality $f(v) < d_S(v)$ would be impossible to satisfy for $S = V$. For weak parking functions this problem does not arise. However, we can view weak parking functions as parking functions for a special graph.

Definition 2.16. The *delooped cone* of a graph $G = (V, E)$ is the graph C_G obtained by adding to G a new vertex, called the *apex*, connected by edges to every vertex of G, and replacing each loop by an edge connected to the apex. More precisely, if L is the set of loops in G, then

$$C_G = (V_C, E_C),$$

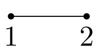
where

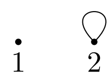
$$V_G = V \sqcup \{v_0\}$$

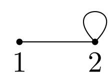
and

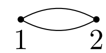
$$E_C = (E - L) \sqcup \{(v, v_0) \,:\, v \in V\} \sqcup \{(s(\ell), v_0) \,:\, \ell \in L\}.$$

If G is a loopless graph then C_G is just the usual cone graph of G.

Example 2.17. We will illustrate these notions for the following graphs
$G_1 =$ 

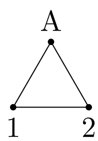
, $G_2 =$ 

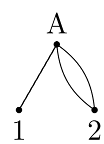
, $G_3 =$ 

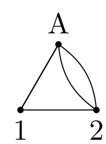
, and $G_4 =$ 

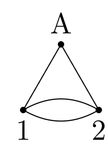
.

Their delooped cones are

$C_{G_1} =$ 

, $C_{G_2} =$ 

, $C_{G_3} =$ 

, and $C_{G_4} =$ 

and the lists $(f(1), f(2))$ of values of all their weak parking functions are given in the following table.

G_1	G_2	G_3	G_4
(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,1)	(0,1)	(0,1)
(1,0)		(0,2)	(0,2)
		(1,0)	(1,0)
		(1,1)	(2,0)

We present below some basic properties of weak parking functions.

Theorem 2.18. *Let $G = (V, E)$ be a graph.*

(i) Weak parking functions of G are precisely usual parking functions of the delooped cone C_G, relative to the apex v_0.

(ii) If f is a weak parking function of G, then

$$f(v) \leq \boldsymbol{\kappa}_v = \widehat{d}_v(v)$$

for all $v \in V$.

(iii) *If f is a weak parking function of G and $g : V \to \mathbb{Z}_{\geq 0}$ is any function such that $g(v) \leq f(v)$ for all $v \in V$, then g is also a weak parking function.*

(iv) *Let $\Pi = (v_1, \ldots, v_n)$ be a linear ordering of the set of vertices V. The function*

$$f^{\Pi} : V \to \mathbb{Z}_{\geq 0},\ \ f^{\Pi}(v_i) := \widehat{d}_{\{v_i,\ldots,v_n\}}(v_i), \tag{2.17}$$

assigning to a vertex $v = v_i \in V$ the number of edges from v to vertices $u = v_j$, with $j \leq i$, is a weak parking function.

(v) *For every weak parking function f, there exists a linear ordering Π of V such that $f(v) \leq f^{\Pi}(v)$ for all $v \in V$.*

(vi) *For every weak parking function f, we have $\sum_{v \in V} f(v) \leq |E|$.*

(vii) *A weak parking function f is maximal with respect to the point-wise order if and only if*

$$\sum_{v \in V} f(v) = |E|$$

or, equivalently, when $f = f^{\Pi}$ for some linear ordering Π of V.

Proof. Statements (i)-(iii) follow immediately from the above definitions.

To prove (iv), take a subset $\emptyset \neq S \subset V$ and consider $v = v_m \in S$, where $m := \min\{i : v_i \in S\}$. Then $S \subset S_m := \{v_m, \ldots, v_n\}$ and therefore $f^{\Pi}(v) = \widehat{d}_{S_m}(v) \leq \widehat{d}_S(v)$, which shows that f^{Π} is a weak parking function.

To show (v), let f be a weak parking function and construct a linear ordering $\Pi = (v_1, \ldots, v_n)$ of V as follows. Start with a vertex $v_1 \in V$ such that $f(v_1) \leq \widehat{d}_V(v_1) = \ell(v_1)$. Proceeding inductively, if we already have an ordered collection $(v_1, \ldots, v_m)$ with $m < n$, we take v_{m+1} to be a vertex $v \in S = V - \{v_1, \ldots, v_m\} \neq \emptyset$ such that $f(v) \leq \widehat{d}_S(v)$. From this construction it is clear that the resulting ordering Π satisfies $f(v) \leq f^{\Pi}(v)$ for all $v \in V$.

Statement (vi) now follows from (v), since, by definition of the function f^{Π}, it satisfies $\sum_{v \in V} f^{\Pi}(v) = |E|$.

Finally, (vii) follows from (v) and (vi). □

Similar to the case of usual G-parking functions, the number of weak parking functions of a loopless graph has a combinatorial interpretation.

Corollary 2.19. *The number of weak parking functions a graph G without loops is equal to the number of rooted spanning forests in G (i.e. spanning forests with a distinguished vertex in every component).*

Proof. For a loopless graph G its delooped cone C_G coincides with the usual cone of G. Therefore, by part (i) of the above theorem, the number of weak parking functions of G is equal to the number of graph parking functions of C_G relative to its apex. By [PS, Theorem 2.1], the latter number is equal to the number of spanning trees in C_G. Removing the apex v_0 from a spanning tree of C_G turns it into a spanning forest of G with one marked vertex in every component, which clearly gives a bijection between spanning trees of C_G and rooted spanning forests of G. □

Now we will discuss the connection between weak parking functions and partial score vectors.

Definition 2.20. Given a weak parking function $f : V \to \mathbb{Z}_{\geq 0}$ of a graph G, the vector

$$\mathbf{f} := (f(v))_{v \in V} \in \mathbb{Z}_{\geq 0}^V$$

is called the *parking vector* of G corresponding to f.

Theorem 2.21. *Let $G = (V, E)$ be a graph.*

(i) Every parking vector $\mathbf{f}$ of G is a partial score vector.

(ii) Parking vectors of G are precisely the partial score vectors of G coming from ***acyclic*** *partial orientations.*

(iii) Parking vectors corresponding to maximal (with respect to the component-wise order) weak parking functions of G are the partial score vectors coming from acyclic ***total*** *orientations.*

(iv) Every vertex $\mathbf{a} = (a_v)_{v \in V}$ of the score vector polytope $\mathscr{P}_G$ of a graph G is a parking vector, i.e. $f : V \to \mathbb{Z}_{\geq 0} : v \mapsto a_v$ is a weak parking function.

(v) The score vector polytope $\mathscr{P}_G$ of G is the convex hull of the set of all parking vectors.

Proof. (i) Let f be a weak parking function. To construct a partial orientation $\Sigma \subset \widehat{E}$ whose score vector is equal to the parking vector $\mathbf{f} = (f(v))_{v \in V}$, consider a linear ordering $\Pi = (v_1, \dots, v_n)$ of V such that

$$f(v_i) \leq \widehat{d}_{V - \{v_1, \dots, v_{i-1}\}}(v_i).$$

Existence of such ordering is guaranteed by part (v) of Theorem 2.18. Therefore, for each $i \leq n = |V|$, we can find a subset $E_i \subset E$ with $|E_i| = f(v_i)$ edges connecting vertex v_i with vertices in $\{v_1, \dots, v_i\}$. By orienting each edge in E_i out of v_i, we obtain a subset $\widehat{E}_i$ of $\widehat{E}$. Since the $E_i \cap E_j = \emptyset$ for $i \neq j$, we conclude that the disjoint union $\Sigma = \bigsqcup_i \widehat{E}_i$ is a partial orientation whose score vector is equal to $\mathbf{f}$.

(ii) The partial orientation constructed in the proof of part (i) from a weak parking function f is acyclic because its arrows can only go from v_i to v_j with $j \leq i$.

Conversely, let $\mathbf{a} = (a_v)_{v \in V} \in \mathscr{P}_G$ be a partial score vector corresponding to an acyclic partial orientation $\Sigma \subset \widehat{E}$. Since Σ is acyclic, it induces a partial order $\preceq_\Sigma$ on V, namely $u \preceq_\Sigma v$ when there is an oriented path from v to u formed by arrows in Σ. Let Π be a linear ordering extending $\preceq_\Sigma$ and let f^Π be the corresponding weak parking function given by (2.17). Then $a_v \leq |\{u \in V \,:\, u \preceq_\Sigma v\}| \leq f^\Pi(v)$ for all $v \in V$. By part (iii) of Theorem 2.18, we conclude that $\mathbf{a}$ is a parking vector.

(iii) If $f : V \to \mathbb{Z}_{\geq 0}$ is a weak parking function whose vector $\mathbf{f} = (f(v))$ is the score vector of a partial orientation Σ, then $\sum_{v \in V} f(v) = |\Sigma|$. By part (vii) of Theorem 2.18, f is maximal exactly when $\sum_{v \in V} f(v) = |E|$ or, equivalently, when $|\Sigma| = |E|$ i.e. when Σ is a total orientation.

(iv) If $\mathbf{a} = (a_v)_{v \in V}$ is a vertex of $\mathscr{P}_G$, then by part (4) of Theorem 2.11 it corresponds to a unique partial orientation $\Sigma \subset \widehat{E}$. If Σ had an oriented cycle $(e_1, e_2, \dots, e_p)$, then, by reversing orientations of all the arrows e_i, $i = 1, \dots, p$ we would obtain a different partial orientation $\Sigma' = (\Sigma - \{e_1, \dots, e_p\}) \cup \{e_1', \dots, e_p'\}$ giving the same partial score vector $\mathbf{a}$, which contradicts uniqueness of Σ. Therefore, orientation Σ is acyclic and, by (ii), $\mathbf{a}$ is a parking vector.

(v) Since $\mathscr{P}_G$ is a convex polytope, it is a convex hull of the set of its vertices. By (iv), each vertex of $\mathscr{P}_G$ is a parking vector and, by (i), every parking vector of G belongs to $\mathscr{P}_G$. This shows that $\mathscr{P}_G$ is the convex hull of the set of parking vectors. □

The following result is an immediate consequence of part (v) of the above theorem and Proposition 2.8.

Corollary 2.22. *The dimension of the external bizonotopal algebra $\mathscr{B}^{e}_G$ of a graph G is equal to the number of lattice points in the convex hull of the set of parking vectors of G.*

□

In particular, for a complete graph K_n on n vertices, $\dim \mathscr{B}_{K_n}$ is equal to the number of lattice points in the *parking functions polytope* $\mathscr{P}_n$ which was studied in several recent papers (cf. [AW] and [HLVM]).

3 r-bizonotopal algebras and loopy deletion-contraction

3.1 Definition of r-bizonotopal algebras

Here we consider bizonotopal analogs of central and internal zonotopal algebras, comp. [HR]. They are members of a more general family which we call r-bizonotopal and introduce below.

Definition 3.1. Let $\delta_G := \min_{v\in V} \boldsymbol{\kappa}_v$ be the smallest number of edges incident to a vertex of G. (Recall that loops only counted once in $\boldsymbol{\kappa}_v$, and therefore, in general, δ_G is not the same as the minimal degree of a vertex in G.) Choose $r \in \mathbb{Z}$ such that $r \geq -\delta_G$.

Similarly to (2.9), for a subset $S \subset V$, consider the set of monomials in the polynomial ring $\mathbf{k}[V] := \mathbf{k}[z_v \,:\, v \in V]$ given by

$$\mathfrak{M}^{(r)}_S = \Big\{ \prod_{v\in S} z_v^{a_v} \,:\, \sum_v a_v = \boldsymbol{\kappa}_S + r \Big\} \tag{3.1}$$

in variables corresponding to vertices $v \in S$ and of total degree $\boldsymbol{\kappa}_S + r$.

The *r-bizonotopal algebra* of G is the quotient algebra

$$\mathscr{B}^{(r)}_G := \mathbf{k}[V]/\mathfrak{I}^{(r)}_G, \tag{3.2}$$

where

$$\mathfrak{I}_G^{(r)} := \Big(\bigcup_{\emptyset \neq S \subset V} \mathfrak{M}_S^{(r)} \Big) \subset \mathbf{k}[V] \tag{3.3}$$

is the ideal generated by monomials in $\mathfrak{M}_S^{(r)}$ for all nonempty subsets $S \subset V$.

From Theorem 2.9 we see that algebra $\mathscr{B}_G^{(1)}$ is isomorphic to the external bizonotopal algebra considered above. By analogy with the usual zonotopal algebras, we call $\mathscr{B}_G^{(0)}$ and $\mathscr{B}_G^{(-1)}$ the *central* and *internal* bizonotopal algebras of G respectively, and will discuss them in more details in Sections 4.1 and 4.2.

Algebras $\mathscr{B}_G^{(r)}$ corresponding to $r > 1$ are called *superexternal* and the ones with $r < -1$ are called *subinternal*.

The next result extends Theorem 2.7 and is analogous to the main result of [NS].

Theorem 3.2. *For any $r > 1$, the superexternal algebras $\mathscr{B}_{G_1}^{r}$ and $\mathscr{B}_{G_2}^{r}$ of graphs G_1 and G_2 are isomorphic if and only if the graphs G_1 and G_2 are isomorphic.*

Proof. Almost identical to that of Theorem 2.7. □

3.2 Loopy deletion-contraction

In this subsection we will show that the Hilbert series of central, external, and superexternal algebras satisfy a certain deletion-contraction relation similar to the classical one and which allows to compute them recursively.

Definition 3.3. Given a graph $G = (V, E)$ and a non-loop edge $e \in E$, we consider two operations, *deletion* and *loopy contraction*, producing two new graphs,

- $G - e$, the graph with the same vertex set as G and the edge e deleted from the set of edges,
- G/e, the graph obtained from G by identifying the endpoints of the edge e *without deleting* it, thus turning the edge e as well as all other edges connecting the endpoints of e into loops.

Remark 3.4. Notice that our loopy contraction operation is different from the one familiar from the study of the Tutte polynomial and its relatives. It does not actually contract any edges; perhaps, “looping” or “loopification” would be a more appropriate term for this operation. We decided to keep the traditional terminology to emphasize similarities of our construction with the theory of the Tutte polynomial. As we will show elsewhere [KNSV], using the loopy deletion-contraction relation (3.4), one can define a new multivariable graph polynomial with properties similar to those of the Tutte polynomial and Stanley’s chromatic symmetric function [St], but which is not equivalent to either of them.

For a graph G, we denote by $h^r(t)$ the Hilbert series of its r-bizonotopal algebra (3.2), i.e. the generating function of the dimensions of homogeneous components of $\mathscr{B}^r_G$:

$$h^r(t) = \sum_{n\geq 0} \dim(\mathscr{B}^r_G)^{(n)} t^n.$$

Theorem 3.5. *For $r \geq 0$ and a non-loop edge $e \in E$ of a graph $G = (V, E)$, the Hilbert series $h^r(t)$ of the r-bizonotopal algebras of the graphs G, $G - e$, and G/e, satisfy the following* ***loopy deletion-contraction relation***

$$h^r_G(t) = h^r_{G/e}(t) + t \cdot h^r_{G-e}(t). \tag{3.4}$$

Proof. Let $p, q \in V$ be the endpoints of the edge e and let w be the vertex of G/e obtained by identifying p and q. Our goal is to describe and compare the non-vanishing monomials of a fixed degree in the bases of the monomial algebras $\mathscr{B}^r_G$, $\mathscr{B}^r_{G-e}$, and $\mathscr{B}^r_{G/e}$.

Let us fix the degrees d_v of all vertices $v \in V - \{p, q\}$. We only need to consider monomials

$$m = \prod_{v\in(V-\{p,q\})} z_v^{d_v}$$

which do not vanish in $\mathscr{B}^r_G$, because if such a monomial vanishes in $\mathscr{B}^r_G$ then it also vanishes in both $\mathscr{B}^r_{G-e}$ and $\mathscr{B}^r_{G/e}$.

Define the three numbers:

- $a := \min_{I\subset(V-\{p,q\})} \boldsymbol{\kappa}_{I\cup\{p\}} + r - 1 - \sum_{i\in I} d_i$;
- $b := \min_{I\subset(V-\{p,q\})} \boldsymbol{\kappa}_{I\cup\{q\}} + r - 1 - \sum_{i\in I} d_i$;

- $c := \min_{I \subset (V - \{p,q\})} \boldsymbol{\kappa}_{I \cup \{p,q\}} + r - 1 - \sum_{i \in I} d_i.$

From the definition of r-bizonotopal algebras we see that the monomial

$$\widetilde{m} = m z_p^{d_p} z_q^{d_q}$$

does not vanish in $\mathscr{B}_G^r$ if and only if

$$d_p \leq a, \ d_q \leq b, \text{ and } d_p + d_q \leq c.$$

Similarly, $\widetilde{m}$ does not vanish in $\mathscr{B}_{G-e}^r$ if and only if

$$d_p \leq a - 1, \ d_q \leq b - 1, \text{ and } d_p + d_q \leq c - 1,$$

because the values of $\boldsymbol{\kappa}_{I \cup \{p\}}$, $\boldsymbol{\kappa}_{I \cup \{q\}}$, and $\boldsymbol{\kappa}_{I \cup \{p,q\}}$ for the graph $G - e$ are one less than the corresponding values for G.

Finally, in $\mathscr{B}_{G/e}^r$, the monomial $m \cdot z_w^{d_w}$ does not vanish if and only if $d_w \leq c$.

It remains to show that for any $0 \leq c' \leq c$, the system

$$0 \leq d_p \leq a, \ 0 \leq d_q \leq b, \text{ and } d_p + d_q = c'$$

has one more integer solution than the system

$$0 \leq d_p \leq a - 1, \ 0 \leq d_q \leq b - 1, \text{ and } d_p + d_q = c' - 1.$$

Consider $J_1, J_2 \subset (V - \{p,q\})$ such that

$$a = \boldsymbol{\kappa}_{J_1 \cup \{p\}} + r - 1 - \sum_{v \in J_1} d_v$$

and

$$b = \boldsymbol{\kappa}_{J_2 \cup \{q\}} + r - 1 - \sum_{v \in J_2} d_v.$$

We have

$$\begin{aligned}
c' \leq c &\leq \boldsymbol{\kappa}_{J_1 \cup J_2 \cup \{p,q\}} + r - 1 - \sum_{v \in J_1 \cup J_2} d_v \\
&= \boldsymbol{\kappa}_{J_1 \cup J_2 \cup \{p,q\}} + r - 1 - \sum_{v \in J_1} d_v - \sum_{v \in J_2} d_v + \sum_{v \in J_1 \cap J_2} d_v \\
&= \boldsymbol{\kappa}_{J_1 \cup J_2 \cup \{p,q\}} + r - 1 - (\boldsymbol{\kappa}_{J_1 \cup \{p\}} + r - 1 - a) - (\boldsymbol{\kappa}_{J_2 \cup \{q\}} + r - 1 - b) + \sum_{v \in J_1 \cap J_2} d_v \\
&= a + b + \boldsymbol{\kappa}_{J_1 \cup J_2 \cup \{p,q\}} - \boldsymbol{\kappa}_{J_1 \cup \{p\}} - \boldsymbol{\kappa}_{J_2 \cup \{q\}} + \sum_{v \in J_1 \cap J_2} d_v - (r - 1).
\end{aligned}$$

Note that if $J_1 \cap J_2 \neq \emptyset$, then $\sum_{v \in J_1 \cap J_2} d_v - (r-1) \leq \boldsymbol{\kappa}_{J_1 \cap J_2}$, and if $J_1 \cap J_2 = \emptyset$, then $\sum_{v \in J_1 \cap J_2} d_v - (r-1) = -(r-1) = \boldsymbol{\kappa}_{\emptyset} - (r-1) \leq \boldsymbol{\kappa}_{\emptyset} + 1$.

Hence,

$$\begin{aligned} c' \leq \; & c \leq a + b + \boldsymbol{\kappa}_{J_1 \cup J_2 \cup \{p,q\}} - \boldsymbol{\kappa}_{J_1 \cup \{p\}} - \boldsymbol{\kappa}_{J_2 \cup \{q\}} + \boldsymbol{\kappa}_{J_1 \cap J_2} + 1 \\ & = a + b + \boldsymbol{\kappa}'_{J_1 \cup J_2 \cup \{p,q\}} - \boldsymbol{\kappa}'_{J_1 \cup \{p\}} - \boldsymbol{\kappa}'_{J_2 \cup \{q\}} + \boldsymbol{\kappa}'_{J_1 \cap J_2} \leq a + b, \end{aligned}$$

where $\boldsymbol{\kappa}'$ is the corresponding function for $G - e$, which is submodular by Lemma 2.12.

Now let us count the number of solutions (d_p, d_q) of the system

$$0 \leq d_p \leq a, \;\; 0 \leq d_q \leq b, \;\; d_p + d_q = c'. \tag{3.5}$$

This number is equal to the number of solutions of the system

$$0 \leq x \leq a, \;\; 0 \leq c' - x \leq b,$$

or, equivalently, of

$$\max(0, c' - b) \leq x \leq \min(a, c').$$

Therefore, the number of solutions of (3.5) is equal to

$$\min(a, c') - \max(0, c' - b) + 1, \tag{3.6}$$

(it is always positive because $c' \leq a + b$).

Similarly, the number of solutions of the system

$$0 \leq d_p \leq a - 1, \;\; 0 \leq d_q \leq b - 1, \;\; d_p + d_q = c' - 1$$

is equal to

$$\min(a-1, c'-1) - \max(0, (c'-1) - (b-1)) + 1 = \min(a, c') - \max(0, c' - b)$$

which is one less than (3.6). This concludes our proof. $\square$

Theorem 3.5 has the following consequence for the external and the central algebras (i.e. for $r = 1$ and $r = 0$, resp.)

Theorem 3.6. *Let $h_G(t)$ be the Hilbert series of either central ($r = 0$) or external ($r = 1$) bizonotopal algebra of $\mathscr{B}^r_G$ of a graph G. As a function on graphs $h_G(t)$ is uniquely characterized by the following properties:*

(i) loopy deletion-contraction*:*

$$h_G(t) = h_{G/e}(t) + t \cdot h_{G-e}(t),$$

if e is a non-loop edge of G;

(ii) multiplicativity*:*

$$h_{G_1 \sqcup G_2}(t) = h_{G_1}(t) \cdot h_{G_2}(t), \tag{3.7}$$

(iii) initial conditions*:*

$$h_{L_n}(t) = \begin{cases} 1 + t + \cdots + t^{n-1} = \dfrac{1 - t^n}{1 - t}, & \text{if } r = 0, \\ \\ 1 + t + \cdots + t^n = \dfrac{1 - t^{n+1}}{1 - t}, & \text{if } r = 1, \end{cases} \tag{3.8}$$

where L_n is an one-vertex graph with n loops.

Proof. The first part is a special case of Theorem 3.5 which establishes the loopy deletion-contraction relation (3.4) for $h^r_G(t)$ for $r \geq 0$. It is easy to see that the multiplicative property (3.7) holds for h^r_G for all $r \leq 1$. Therefore, for $r = 0, 1$, the computation of $h^r_G(t)$ reduces to finding its values on the n-loop graph L_n, which are given by (3.8). □

3.3 "Categorification" of the deletion-contraction relation

Here we give an explanation of the deletion-contraction relation (3.4) for external zonotopal algebras based on their functorial properties. This is similar to the proof of the standard deletion-contraction relation for Hilbert series of usual zonotopal algebras given in [SSV, Theorem 2.7].

For a graph $G = (V, E)$ and a non-loop edge $e \in E$ with endpoints $p, q \in V$, let $\varepsilon', \varepsilon'' \in \widehat{E}$ be the arcs corresponding to the orientations of e from p to q and from q to p respectively, let w be the vertex in G/e obtained by identifying vertices p and q, and let $\bar{e}$ be the loop based at w in G/e and obtained from e.

There are several natural maps connecting algebras related to the graphs G, $G-e$ and G/e.

- A surjective homomorphism $\widehat{\rho}_e : \widehat{\mathscr{E}}_G \twoheadrightarrow \widehat{\mathscr{E}}_{G-e}$, sending $x_{\varepsilon'}$ and $x_{\varepsilon''}$ to 0, maps $y_v \in \widehat{\mathscr{E}}_G$ to $y_v \in \widehat{\mathscr{E}}_{G-e}$, for each $v \in V$, and thus induces an epimorphism
$$\rho_e : \mathscr{B}^{\mathrm{e}}_G \twoheadrightarrow \mathscr{B}_{G-e}. \tag{3.9}$$
- The homomorphism $\widehat{\gamma}_e : \widehat{\mathscr{E}}_{G/e} \to \widehat{\mathscr{E}}_G$ given by , $\widehat{\gamma}_e(x_{\bar{e}}) = x_{\varepsilon'} + x_{\varepsilon''}$, and $\widehat{\gamma}_e(x_\epsilon) = x_\epsilon$, if $\epsilon \neq \bar{e}$ induces an algebra embedding
$$\gamma_e : \mathscr{B}_{G/e} \hookrightarrow \mathscr{B}_G : \ \gamma_e(y_w) = y_p + y_q, \ \text{ and } \gamma_e(y_v) = y_v, \ \text{ if } v \neq w. \tag{3.10}$$
- The composition of the "partial derivatives" $\dfrac{\partial}{\partial x_{\varepsilon'}}$ and $\dfrac{\partial}{\partial x_{\varepsilon''}}$ of $\widehat{\mathscr{E}}_G$ with the projection $\widehat{\rho}_e : \widehat{\mathscr{E}}_G \twoheadrightarrow \widehat{\mathscr{E}}_{G-e} \simeq \widehat{\mathscr{E}}_G/(x_{\varepsilon'}, x_{\varepsilon''})$, gives well-defined derivations $\partial_{x_{\varepsilon'}}, \partial_{x_{\varepsilon''}} : \widehat{\mathscr{E}}_G \to \widehat{\mathscr{E}}_{G-e}$. Then the map
$$\widehat{\delta}_e = \partial_{x_{\varepsilon'}} - \partial_{x_{\varepsilon''}} : \widehat{\mathscr{E}}_G \to \widehat{\mathscr{E}}_{G-e}$$
is also a derivation of degree -1. Moreover, since
$$\widehat{\delta}_e(y_v) = \begin{cases} y_v & \text{if } v = p \\ -y_v & \text{if } v = q \\ 0 & \text{if } v \neq p, q, \end{cases}$$
we see that $\widehat{\delta}_e$ sends the subalgebra $\mathscr{B}^{\mathrm{e}}_G \subset \widehat{\mathscr{E}}_G$ onto $\mathscr{B}^{\mathrm{e}}_{G-e} \subset \widehat{\mathscr{E}}_{G-e}$.

It is easy to check that the kernel of the map $\widehat{\delta}_e$ coincides with the image of the embedding $\widehat{\gamma}_e : \widehat{\mathscr{E}}_{G/e} \to \widehat{\mathscr{E}}_G$ and, if $\delta_e : \mathscr{B}^{\mathrm{e}}_G \to \mathscr{B}^{\mathrm{e}}_{G-e}$ is the restriction of $\widehat{\delta}_e$ to $\mathscr{B}^{\mathrm{e}}_G$, then $\operatorname{Ker} \delta_e = \gamma_e(\mathscr{B}^{\mathrm{e}}_{G/e})$.

Thus we established the following proposition.

Proposition 3.7. *There exists a surjective, degree -1, derivation of graded algebras*
$$\delta_e : \mathscr{B}^{\mathrm{e}}_G \twoheadrightarrow \mathscr{B}^{\mathrm{e}}_{G-e}$$
whose kernel coincides with the image $\gamma_e(\mathscr{B}^{\mathrm{e}}_{G/e}) \subset \mathscr{B}^{\mathrm{e}}_G$ of the embedding γ_e (see (3.10)).

As an immediate consequence of this result we obtain the loopy deletion-contraction relation for external bizonotopal algebras.

Corollary 3.8. *The Hilbert series of the graded algebras* $\mathscr{B}^{e}_{G}$*,* $\mathscr{B}^{e}_{G/e}$*, and* $\mathscr{B}^{e}_{G-e}$ *satisfy the relation*

$$h_G(t) = h_{G/e}(t) + t \cdot h_{G-e}(t). \tag{3.11}$$

Proof. From the above proposition it follows that there exists a short exact sequence of graded vectors spaces

$$0 \to \mathscr{B}^{e}_{G/e} \to \mathscr{B}^{e}_{G} \to \mathscr{B}^{e}_{G-e}[-1] \to 0, \tag{3.12}$$

where $[-1]$ is a degree shift functor indicating that the map $\mathscr{B}^{e}_{G} \to \mathscr{B}^{e}_{G-e}$ decreases degree by -1.

Now the relation (3.11) follows from the fact that Hilbert series are additive with respect to exact sequences and that the shift of grading by -1 multiplies the Hilbert series by t. □

4 Additional properties of central and internal bizonotopal algebras

Here we present some additional properties of the central $\mathscr{B}^{c}_{G} = \mathscr{B}^{(0)}_{G}$ and internal $\mathscr{B}^{i}_{G} = \mathscr{B}^{(-1)}_{G}$ bizonotopal algebras.

4.1 Central bizonotopal algebras

We start with a description of $\mathscr{B}^{c}_{G}$ as a subalgebra, similar to that of $\mathscr{B}^{e}_{G}$ in Definition 2.3. First we define an algebra $\widehat{\mathscr{E}}^{c}_{G}$, an analog of the partial orientation algebra $\widehat{\mathscr{E}}_{G}$ (2.4), in the central case.

For a subset of vertices $S \subset V$, let $\mathscr{M}_S \subset \widehat{\mathscr{E}}_{G}$ be the set of monomials $x_\Sigma \in \mathbf{k}[\widehat{E}]$ corresponding via (2.5) to the set of partial orientations $\Sigma \subset \widehat{E}$ such that $\pi(\Sigma) = E_S$ (edges incident to vertices S) and the edges in E_S with only one vertex in S are oriented in Σ *out* of S. Note that the degree of each monomial in $\mathscr{M}_S$ is equal to the degree $\boldsymbol{\kappa}_S = |E_S|$ of S and the total number of such monomials $|\mathscr{M}_S|$ is equal to 2^μ, where μ is the number of edges with two ends in S. Let

$$I^{c}_{G} := (\mathscr{M}_S \, : \; S \subset V) \subset \widehat{\mathscr{E}}_{G},$$

be the ideal in $\widehat{\mathscr{E}}_G$ generated by monomials from $\bigcup_{S\in V} \mathscr{M}_S$ and let $\widehat{\mathscr{E}}^c_G$ be the quotient algebra

$$\widehat{\mathscr{E}}^c_G := \widehat{\mathscr{E}}_G / I^c_G. \tag{4.1}$$

Theorem 4.1. *The central bizonotopal algebra $\mathscr{B}^c_G$ of a graph $G = (V, G)$ is isomorphic to the subalgebra of the algebra $\widehat{\mathscr{E}}^c_G$ generated by the degree one elements*

$$y_v = \sum_{e\in\widehat{E},\ s(e)=v} x_e,\ \ v \in V. \tag{4.2}$$

Proof. Arguing as in Part (i) of Lemma 2.5, we get that the subalgebra of $\widehat{\mathscr{E}}^c_G$ generated by $y_v, v \in V$ is monomial. We need to check that $\prod y_v^{a_v} = 0$ in $\widehat{\mathscr{E}}^c_G$ if and only if $\prod z_v^{a_v} = 0$ in $\mathscr{B}^c_G$.

Clearly for any $I \subset V$, we have that for all $k_i \in \mathbb{Z}_{\geq 0},\ i \in I$ such that $\sum_{i\in I} k_i = \boldsymbol{\kappa}_I$,

$$\prod_{i\in I} y_i^{a_i} = 0 \text{ in } \widehat{\mathscr{E}}^c_G$$

which are exactly the set of all relations in the algebra $\mathscr{B}^c_G$.

It remains to prove the converse. If $\prod_{v\in V} y_v^{a_v} = 0$ in $\widehat{\mathscr{E}}_G$, then by Theorem 2.9, we know that the similar relation holds for $z_v, v \in V$. Further, assume that $\prod_{v\in V} y_v^{a_v} = 0$ in $\widehat{\mathscr{E}}^c_G$, but not in $\widehat{\mathscr{E}}_G$. Then there is a monomial m in $\prod_{v\in V} y_v^{a_v}$ in expression through $x_e, x_{\bar{e}}, e \in E$ that vanishes in $\widehat{\mathscr{E}}^c_G$, but not in $\widehat{\mathscr{E}}_G$. Therefore it is divisible by some $m' \in \mathscr{M}_S$ for some S. We immediately get $\sum_{i\in S} a_i \geq \boldsymbol{\kappa}_S$ and, hence, $\prod_{v\in V} z_v^{a_v} = 0$ in $\mathscr{B}^c_G$. □

Theorem 4.2. *For a connected graph G, the Hilbert series*

$$h^c_G(t) := \sum_{k\geq 0} \dim(\mathscr{B}^c_G)^{(k)} \cdot t^k$$

has the following properties:

(1) *it is a polynomial of degree $|E| - 1$ where $|E|$ is the total number of edges in G;*

(2) $\dim(\mathscr{B}^e_G)^{(|E|-1)}$ *equals the number of spanning trees of G.*

Proof. The first part is trivial. The second part follows from Theorem 3.6, because the number of trees satisfies both the deletion-contraction and the multiplicative properties. □

4.2 Internal bizonotopal algebras

The internal bizonotopal algebra $\mathscr{B}^i_G := \mathscr{B}^{(-1)}_G$ is defined for all graphs G without isolated vertices. Similarly to the central case, we will realize $\mathscr{B}^i_G$ as a subalgebra of a certain quotient algebra $\widehat{\mathscr{E}}^i_G$ of the partial orientation algebra $\widehat{\mathscr{E}}$, namely

$$\widehat{\mathscr{E}}^i_G := \widehat{\mathscr{E}}_G / I^i_G,$$

where

$$I^i_G = \left(\mathscr{M}^-_S \ : \ S \subset V\right),$$

and $\mathscr{M}^-_S$ is the set of monomials m such that there exists $e \in \widehat{E}$ such that $x_e m \in \mathscr{M}_S$. Clearly the degrees of all monomials in $\mathscr{M}^-_S$ are equal to $\boldsymbol{\kappa}_S - 1$.

Theorem 4.3. *For a graph $G = (V, E)$ without isolated vertices, the algebra $\mathscr{B}^i_G$ is isomorphic to the subalgebra of $\widehat{\mathscr{E}}^i_G$ generated by the linear forms* (4.2)

$$y_v = \sum_{e \in \widehat{E},\ s(e)=v} x_e.$$

Proof. The proof is similar to that of Theorem 4.1. □

Theorem 4.4. *For the complete graph K_n with $n \geq 4$, the maximal degree of an element in $\mathscr{B}^i_{K_n}$ is equal to $\binom{n}{2} - 2$ and the dimension of the corresponding top degree component is equal to $\binom{n-2}{2} n^{n-4}$.*

Proof. We will linearly order the vertices of K_n, i.e. we will identify V with $[n] = \{1, 2, \ldots, n\}$.

Since $|E| = \binom{n}{2}$, any monomial $z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n}$ of degree $\binom{n}{2} - 1 = \boldsymbol{\kappa}_V - 1$ vanishes in $\mathscr{B}^i_{K_n}$. Hence we need to show that the graded component of degree $\binom{n}{2} - 2$ of the algebra $\mathscr{B}^i_{K_n}$ has dimension $\binom{n-2}{2} n^{n-4}$.

We will derive this from Lemma 4.5 after introducing some auxiliary combinatorial objects. □

Let us introduce two sets X and Y of vectors with integer coordinates. The set X consists of all vectors $(b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n_{\geq 0}$ satisfying the following conditions:

- $\sum_{i\in[n]} b_i = \binom{n}{2} - 2$;
- $\sum_{i\in I} b_i \le \boldsymbol{\kappa}_I - 2 = \binom{|I|}{2} + |I|(n-|I|) - 2$, for all $\emptyset \neq I \subset [n]$.

The set Y consists of all $(b_1, b_2, \dots, b_n) \in \mathbb{Z}^n_{\geq 0}$ satisfying the conditions

- $\sum_{i\in[n]} b_i = \binom{n}{2} - 1$;
- $\sum_{i\in I} b_i \le \boldsymbol{\kappa}_I - 1 = \binom{|I|}{2} + |I|(n-|I|) - 1$, for all $\emptyset \neq I \subset [n]$.

By the definition of the internal bizonotopal algebra, the dimension of its component of degree $\binom{n}{2} - 2$ is equal to $|X|$ which we count below.

Also, by the definition of the central bizonotopal algebra, we know that $|Y|$ is equal to the dimension of highest degree component of the algebra $\mathscr{B}^c_{K_n}$ which, by Theorem 4.2, is equal to the number of spanning trees in K_n. Hence, $|Y| = n^{n-2}$.

Clearly, if $(b_1, \dots, b_{n-1}, b_n) \in X$, then $(b_1, \dots, b_{n-1}, b_n{+}1) \in Y$. Therefore we have

$$\begin{aligned}|X| &= |Y| - |\{(b_1, b_2, \dots, b_n) \in Y :\ b_n = 0\}| \\ &\quad - |\{(b_1, b_2, \dots, b_n) \in Y :\ b_n > 0 \text{ and } \sum_{j\in J} b_j = \boldsymbol{\kappa}_J - 1 \text{ for some } J \subset [n-1]\}| \\ &= |Y| - |\{(b_1, b_2, \dots, b_n) \in Y :\ \sum_{j\in J} b_j = \boldsymbol{\kappa}_J - 1 \text{ for some } J \subset [n-1]\}|.\end{aligned}$$

Denote by Z the last set appearing in the previous formula, i.e.

$$Z := \{(b_1, b_2, \dots, b_n) \in Y \ :\ \sum_{j\in J} b_j = \boldsymbol{\kappa}_J - 1 \text{ for some } J \subset [n-1]\}.$$

We claim that for every $(b_1, b_2, \dots, b_n) \in Z$, there is a unique maximal subset $J \subset [n-1]$ such that $\sum_{j\in J} b_j = \boldsymbol{\kappa}_J - 1$. Indeed assume that there exist two different sets $J_1, J_2 \subset [n-1]$ satisfying

$$\sum_{j\in J_1} b_j = \boldsymbol{\kappa}_{J_1} - 1 \text{ and } \sum_{j\in J_2} b_j = \boldsymbol{\kappa}_{J_2} - 1$$

such that $J_1 \not\subset J_2$ and $J_2 \not\subset J_1$. If $J_1 \cap J_2 \neq \emptyset$, then we have

$$\begin{aligned}\sum_{j\in J_1\cup J_2} b_j &= \sum_{j\in J_1} b_j + \sum_{j\in J_2} b_j - \sum_{j\in J_1\cap J_2} b_j \\ &= \boldsymbol{\kappa}_{J_1} - 1 + \boldsymbol{\kappa}_{J_2} - 1 - \sum_{j\in J_1\cap J_2} b_j \\ &\geq \boldsymbol{\kappa}_{J_1} - 1 + \boldsymbol{\kappa}_{J_2} - 1 - (\boldsymbol{\kappa}_{J_1\cap J_2} - 1) \geq \boldsymbol{\kappa}_{J_1\cup J_2} - 1,\end{aligned}$$

where the last inequality follows from the submodularity of the function $\boldsymbol{\kappa}$ (see Lemma 2.12). If $J_1 \cap J_2 = \emptyset$, then

$$\sum_{j\in J_1\cup J_2} b_j = \sum_{j\in J_1} b_j + \sum_{j\in J_2} b_j = \boldsymbol{\kappa}_{J_1} - 1 + \boldsymbol{\kappa}_{J_2} - 1 \geq \boldsymbol{\kappa}_{J_1\cup J_2} - 1.$$

Therefore, $\sum\limits_{j\in J_1\cup J_2} b_j \geq \boldsymbol{\kappa}_{J_1\cup J_2} - 1$. Since, by definition of the set Y, we also have $\sum\limits_{j\in J_1\cup J_2} b_j \leq \boldsymbol{\kappa}_{J_1\cup J_2} - 1$, we see that the union $J_1 \cup J_2$ also satisfies our property, i.e. neither J_1 nor J_2 are maximal which contradicts our assumption.

Therefore we have a partition $Z = \bigsqcup\limits_{I\subset[n-1]} Z_I$ into the subsets

$$Z_I := \{(b_1, \ldots, b_n) \in Z : \sum_{i\in I} b_i = \boldsymbol{\kappa}_I - 1 \text{ and } \sum_{j\in J} b_j < \boldsymbol{\kappa}_J - 1 \text{ if } I \subsetneq J \subset [n-1]\}.$$

This leads to the following count

$$|X| = |Y| - |Z| = n^{n-2} - \sum_{I\subset[n-1]} |Z_I|.$$

Lemma 4.5. *For any subset $J \subset [n-1]$, we have*

$$|Z_J| = |J|^{|J|-2}\big(n - |J|\big)^{(n-|J|)-2},$$

which is equal to the product of the numbers of spanning trees in the complete graphs $K_{|J|}$ and $K_{[n]-|J|}$.

Proof. Let us list the elements of the subsets $J \subset [n]$ and $[n] - J$ in order, i.e. $J = \{j_1, j_2, \ldots, j_\ell\}$ and $[n] - J = \{i_1, i_2, \ldots, i_{n-\ell}\}$ with $j_1 < j_2 < \ldots < j_\ell$ and $i_1 < i_2 \ldots < i_{n-\ell}$. Define two maps $\phi_1 : \mathbb{Z}^n \to \mathbb{Z}^\ell$ and $\phi_2 : \mathbb{Z}^n \to \mathbb{Z}^{n-\ell}$ by

$$\phi_1(b) := (b_{j_1} - n + \ell, b_{j_2} - n + \ell, \ldots, , b_{j_\ell} - n + \ell),$$

and

$$\phi_2(b) := (b_{i_1}, b_{i_2}, \ldots, b_{i_{n-\ell-1}}, b_{i_{n-\ell}} - 1).$$

To prove the lemma, we will show that $b \in Z_J$ if and only if $\phi_1(b) \in Y_\ell$ and $\phi_2(b) \in Y_{n-\ell}$, where by Y_ℓ and $Y_{n-\ell}$ we denote the sets Y for complete graphs K_ℓ and $K_{n-\ell}$ respectively.

Assume that $b \in Z_J$. Then for any $I \subset [\ell]$, we have

$$\begin{aligned}\sum_{t\in I}(\phi_1(b))_t &= \sum_{t\in I}(b_{j_t} - n + \ell) = \big(\sum_{t\in I} b_{j_t}\big) + |I|(-n+\ell) \le \boldsymbol{\kappa}_I - 1 + |I|(-n+\ell) \\ &= \binom{|I|}{2} + |I|(n - |I|) - 1 + |I|(-n+\ell) = \binom{|I|}{2} + |I|(\ell - |I|) - 1.\end{aligned}$$

Hence $\phi_1(b) \in Y_\ell$.

For any subset $I \subset [n - \ell - 1]$, we have

$$\begin{aligned}\sum_{t\in I}\phi_2(b)_t &= \sum_{j\in\{i_t : t\in I\}} b_j = \sum_{j\in\{i_t, t\in I\}\sqcup J} b_j - \sum_{j\in J} b_j \\ &= \sum_{j\in\{i_t, t\in I\}\sqcup J} b_j - (\boldsymbol{\kappa}_J - 1) < (\boldsymbol{\kappa}_{\{i_t,t\in I\}\sqcup J} - 1) - (\boldsymbol{\kappa}_J - 1) \\ &= \binom{|I|}{2} + |I|(n - \ell - |I|)\end{aligned}$$

and for all $I \subset [n-\ell]$ s.t. $(n - \ell) \in I$, we have

$$\begin{aligned}\sum_{t\in I}\phi_2(b)_t &= \sum_{j\in\{i_t : t\in I\}} b_j - 1 = \sum_{j\in\{i_t, t\in I\}\sqcup J} b_j - \sum_{j\in J} b_j - 1 \\ &= \sum_{j\in\{i_t, t\in I\}\sqcup J} b_j - \boldsymbol{\kappa}_J \le \boldsymbol{\kappa}_{\{i_t,t\in I\}\sqcup J} - \boldsymbol{\kappa}_J - 1 \\ &= \binom{|I|}{2} + |I|(n - \ell - |I|) - 1.\end{aligned}$$

Hence, $\phi_2(b) \in Y_{n-\ell}$.

Let us prove the converse. Assume that $\phi_1(b) \in Y_\ell$ and $\phi_2(b) \in Y_{n-\ell}$. We need to show that for any $I \subset [n]$, one has $\sum_{i \in I} b_i \leq \binom{|I|}{2} + |I|(n - |I|) - 1$. To do this we will consider three cases.

Case 1: $I \cap J = \emptyset$. Since $\phi_2(b) \in Y_{n-\ell}$, we have

$$\begin{aligned}\sum_{i \in I} b_i &\leq \binom{|I|}{2} + |I|(n - \ell - |I|) - 1 + 1 \\ &= \binom{|I|}{2} + |I|(n - |I|) - n\ell \leq \binom{|I|}{2} + |I|(n - |I|) - 1.\end{aligned}$$

Case 2: $I \cap ([n] - J) = \emptyset$. Since $\phi_1(b) \in Y_\ell$, we have

$$\begin{aligned}\sum_{i \in I} b_i &= \sum_{i \in I}(b_i - n + \ell) + |I|(n - \ell) \\ &\leq \binom{|I|}{2} + |I|(\ell - |I|) - 1 + |I|(n - \ell) = \binom{|I|}{2} + |I|(n - |I|) - 1.\end{aligned}$$

Case 3: $I \cap J \neq \emptyset$, $I \cap ([n] - J) \neq \emptyset$. We have

$$\begin{aligned}\sum_{i \in I} b_i &= \sum_{i \in I \cap J} b_i + \sum_{i \in I \cap ([n] - J)} b_i \\ &\leq \binom{|I \cap J|}{2} + |I \cap J|(n - |I \cap J|) - 1 + \binom{|I \cap ([n] - J)|}{2} \\ &\qquad + |I \cap ([n] - J)|(n - \ell - |I \cap ([n] - J)|) - 1 + 1 \\ &= \binom{|I|}{2} + |I|(n - |I|) - 1.\end{aligned}$$

We also need to check that for $I = J$, we have the non-strict inequality, i.e. $\sum_{i \in J} b_i \leq \binom{|J|}{2} + |J|(n - |J|) - 1$. It follows from the second case.

It remains to check that for all I with $J \subsetneq I \subset [n-1]$, we have the strict inequality. Indeed this happens in the third case and we do not use b_n, so we have the same inequality, but without "+1".

Thus we showed that $b \in Z_J$ if and only if $\phi_1(b) \in Y_\ell$ and $\phi_2(b) \in Y_{n-\ell}$. It is easy to check that the map $(\phi_1, \phi_2) : \mathbb{Z}^n \to \mathbb{Z}^\ell \times \mathbb{Z}^{n-\ell}$ is a bijection. This implies that $|Z_J| = |Y_\ell| \times Y_{n-\ell}$. Since the numbers of elements in Y_ℓ and $Y_{n-\ell}$ are equal to the numbers of spanning trees in K_ℓ and $K_{n-\ell}$, respectively, we have that $|Z_J| = |J|^{|J|-2}(n-|J|)^{(n-|J|)-2}$ as required. □

End of the proof of Theorem 4.4. By the above lemma $|Z_J|$ is equal to the product of the numbers of spanning trees in K_J and in $K_{[n]-J}$. This implies that

$$|Z| = \sum_{\emptyset \neq I \subset [n-1]} |Z_I|$$

is equal to the number of spanning forests in K_n with two connected components (i.e. containing exactly $n-2$ edges). As was proved by Rényi [Re] (see also [LC, Eq. (19)] or [My, Theorem 2.2]) this number is equal to $n^{n-4} \cdot \dfrac{(n-1)(n+6)}{2}$. Thus we obtain that

$$|X| = n^{n-2} - n^{n-4} \cdot \frac{(n-1)(n+6)}{2} = n^{n-4} \cdot \frac{(n-2)(n-3)}{2} = n^{n-4} \binom{n-2}{2}.$$

□

Remark 4.6. From the proof of Theorem 4.4 we see that the dimension of the top degree component of the algebra $\mathscr{B}^i_{K_n}$ has a combinatorial interpretation as the difference of the number of spanning trees in K_n and the number of two-component spanning forests in K_n. It would be very interesting to find a combinatorial description of the dimension of the top degree component of $\mathscr{B}^i_G$ for general graphs. (Notice that in general, the difference between the numbers of spanning trees and two-component spanning forests of a graph many not even be a positive number.)

Unlike the external and the central bizonotopal algebras, the internal algebras provide rather weak graph invariants. In fact, as the following theorem shows, for their Hilbert series are the same for some large classes of graphs.

Theorem 4.7. *Let $G = (V, E)$ be a simple graph with $n \geq 4$ vertices and let $h^i_G(t)$ be the Hilbert series of its internal bizonotopal algebra $\mathscr{B}^i_G$.*

(i) If G is any 3-regular graph (in which case $n \geq 4$ and is even) then

$$h^i_G = (1+t)^n.$$

(ii) If G is a 4-regular 4-edge-connected graph (so that $n \geq 5$), then

$$h^i_G = (1+t+t^2)^n - nt^{2n-1} - t^{2n}.$$

Proof. (i) If G is a 3-regular graph, then for each $v \in V$, we have $\boldsymbol{\kappa}_v = 3$ and thus $z_v^2 = 0$ in $\mathscr{B}^i_G$. Also the monomial $\prod_{v\in V} z_v$ of degree n does not vanish in $\mathscr{B}^i_G$, because for any $I \subset V$ we have $\boldsymbol{\kappa}_I - 1 \geq \frac{3}{2}|I| - 1 \geq |I| + 1$. Hence, the ideal of relations of $\mathscr{B}^i_G$ does not have square-free monomials and therefore $h^i_G(t) = (1+t)^n$.
(ii) If G is 4-regular, then $\boldsymbol{\kappa}_v = 4$ for all $v \in V$ and we have $z_v^3 = 0$ in $\mathscr{B}^i_G$. We also have $\prod_{v\in V} z_v^2 = 0$ in $\mathscr{B}^i_G$, because $\boldsymbol{\kappa}_V = |E| = 2n$. For any $u \in V$, the product $\prod_{v\neq u} z_v^2$ does not vanish in $\mathscr{B}^i_G$, because for any proper subset $I \subsetneq V$ we have

$$\boldsymbol{\kappa}_I - 1 \geq \frac{4}{2}(|I| - 4) + 4 - 1 = 2|I| + 1.$$

Therefore $h^i_G(t) = (1+t+t^2)^n - nt^{2n-1} - t^{2n}$. □

5 Concluding remarks

Below we list several questions about bizonotopal algebras which, in our opinion, warrant further investigation.

(1.) The dimension of the external bizonotopal algebra $\mathscr{B}^e_G$ of a graph G has a nice combinatorial interpretation as the number of integer points in the polytope of weak parking functions of G. Finding a combinatorial interpretation for the dimension of the central algebra $\mathscr{B}^c_G$ is an interesting problem. (Currently we only know such an interpretation for the dimension of the top degree component of $\mathscr{B}^c_G$.)

(2.) It would be interesting to find a combinatorial interpretation of the dimension of the highest degree term of the internal bizonotopal algebra $\mathscr{B}^i_G$ for graphs other than K_n.

(3.) The Hilbert series of the internal bizonotopal algebras, unlike the central and external cases, do not satisfy the loopy deletion-contraction relation. Does it satisfy some a recursion of some other kind?

(4.) The external and central bizonotopal algebras are very strong (almost complete) graph invariants. On the other hand, we saw that the internal algebra is a rather weak invariant. It would be interesting to characterize pairs of graphs with isomorphic $\mathscr{B}^i_G$.

(5.) In a follow-up paper [KNSV] we show that using our loopy deletion-contraction relation (3.4) one can construct a new multivariate graph polynomial which we call the *loopy polynomial*. The Hilbert series of r-bizonotopal algebra are certain specializations of this loopy polynomial. It strongly resembles Stanley's Tutte symmetric function as well as several similar polynomials, but the exact relation between them is unclear at the moment. This aspect needs clarification.

(6.) Even though the Hilbert series of the bizonotopal algebras are not specializations of the Tutte polynomial, our computer calculations indicate that they all are unimodal and log-concave. Proving these properties and checking whether they hold for other specializations of the loopy polynomial seems to be a very interesting problem.

6 Appendix: Hilbert functions of bizonotopal algebras of complete graphs

Below we present the results of the computations for complete graphs K_n with $n \leq 9$ vertices of the dimensions of the bizonotopal algebras $\mathscr{B}^e_{K_n}$, $\mathscr{B}^c_{K_n}$ and $\mathscr{B}^i_{K_n}$ and their Hilbert function $h(k)$, the dimension of the kth graded component $\mathscr{B}^{(k)}$ of the corresponding algebra.

6.1 External algebras $\mathscr{B}^e_{K_n}$

K_2: dim $= 3$; $h(k)$: 1, 2;

K_3: dim $= 17$; $h(k)$: 1, 3, 6, 7;

K_4: dim $= 144$; $h(k)$: 1, 4, 10, 20, 31, 40, 38;

K_5: dim = 1623; $h(k)$: 1, 5, 15, 35, 70, 121, 185, 255, 310, 335, 291;

K_6: dim = 22804; $h(k)$: 1, 6, 21, 56, 126, 252, 456, 756, 1161, 1666, 2232, 2796, 3281, 3546, 3516, 2932;

K_7: dim = 383415; $h(k)$: 1, 7, 28, 84, 210, 462, 924, 1709, 2954, 4809, 7420, 10906, 15309, 20559, 26454, 32655, 38591, 43589, 46984, 47649, 45150, 36961;

K_8: dim = 7501422; $h(k)$: 1, 8, 36, 120, 330, 792, 1716, 3432, 6427, 11376, 19160, 30864, 47748, 71184, 102524, 142920, 193117, 253240, 322596, 399344, 480390, 561472, 637400, 701296, 746089, 765640, 748532, 691720, 561948;

K_9: dim = 167341283; $h(k)$: 1, 9, 45, 165, 495, 1287, 3003, 6435, 12870, 24301, 43677, 75177, 124485, 199035, 308187, 463287, 677520, 965493, 1342513, 1823553, 2421927, 3147723, 4005819, 4993839, 6100350, 7303545, 8570601, 9855829, 11101599, 12241305, 13203705, 13902291, 14254524, 14195199, 13575951, 12369033, 10026505.

6.2 Central algebras $\mathscr{B}^c_{K_n}$

K_2: dim = 1; $h(k)$: 1;

K_3: dim = 7; $h(k)$: 1, 3, 3;

K_4: dim = 66; $h(k)$: 1, 4, 10, 16, 19, 16;

K_5: dim = 792; $h(k)$: 1, 5, 15, 35, 65, 101, 135, 155, 155, 125;

K_6: dim = 11590; $h(k)$: 1, 6, 21, 56, 126, 246, 426, 666, 951, 1246, 1506, 1686, 1731, 1626, 1296;

K_7: dim = 200469; $h(k)$: 1, 7, 28, 84, 210, 462, 917, 1667, 2807, 4417, 6538, 9142, 12117, 15267, 18327, 20958, 22827, 23667, 23107, 21112, 16807;

K_8: dim = 90759016; $h(k)$: 1, 8, 36, 120, 330, 792, 1716, 3424, 6371, 11152, 18488, 29184, 44052, 63792, 88852, 119288, 154645, 193880, 235292, 276592, 315078, 347880, 371820, 384112, 382817, 364232, 328392, 262144;

K_9: dim = 2301604074; $h(k)$: 1, 9, 45, 165, 495, 1287, 3003, 6435, 12861, 24229, 43353, 74097, 121515, 191907, 292743, 432399, 619677, 863109, 1170073, 1545777, 1992195, 2506983, 3082599, 3705795, 4357593, 5013801, 5645313, 6219649, 6703245, 7064073, 7267815, 7285959, 7100739, 6660495, 5966613, 4782969.

6.3 Internal algebras $\mathscr{B}^i_{K_n}$

K_3: dim = 1; $h(k)$: 1;

K_4: dim = 16; $h(k)$: 1, 4, 6, 4, 1;

K_5: dim = 237; $h(k)$: 1, 5, 15, 30, 45, 51, 45, 30, 15;

K_6: dim = 3892; $h(k)$: 1, 6, 21, 56, 120, 216, 336, 456, 546, 580, 546, 456, 336, 216;

K_7: dim = 72425; $h(k)$: 1, 7, 28, 84, 210, 455, 875, 1520, 2415, 3535, 4795, 6055, 7140, 7875, 8135, 7875, 7140, 6055, 4795, 3430;

K_8: dim = 1521810; $h(k)$: 1, 8, 36, 120, 330, 792, 1708, 3368, 6147, 10480, 16808, 25488, 36688, 50288, 65808, 82384, 98813, 113688, 125588, 133288, 135954, 133288, 125588, 113688, 98533, 81488, 61440;

K_9: dim = 35794801; $h(k)$: 1, 9, 45, 165, 495, 1287, 3003, 6426, 12789, 23905, 42273, 71127, 114387, 176463, 261891, 374808, 518301, 693693, 899857, 1132677, 1384803, 1645791, 1902663, 2140866, 2345553, 2503053, 2602341, 2636263, 2602341, 2502423, 2342907, 2134062, 1881243, 1596861, 1240029.

References

[AW] A. Amanbayeva and D. Wang, The convex hull of parking functions of length n, Enumer. Combin. Appl. 2 (2022), article S2R10. 3, 24

[BZ] J. Bell and J. Zhang, An isomorphism lemma for graded rings, Proc. Amer. Math. Soc. 145 (2017), 989–994. 9

[D] M. Doob, An interrelation between line graphs, eigenvalues, and matroids, J. Combin. Theory Ser. B 15 (1973), 40–50. 7

[HLVM] M. Hanada, J. Lentrfer, and A. R. Vindas-Meléndez, Generalized parking function polytopes, Arxiv: 2212.06885. 24

[HR] O. Holtz, A. Ron, Zonotopal algebra, Adv. Math. 227 (2011), 847–894. 3, 24

[KNSV] A. Kirillov, G. Nenashev, B. Shapiro, A. Vaintrob, Loopy polynomial of undirected graphs, in preparation. 26, 40

[KW] D. J. Kleitman and K. J. Winston, Forests and score vectors, Combinatorica 1 (1981), 49–54. 9

[LC] C.J. Liu and Y. Chow, Enumeration of forests in a graph, Proc. AMS, 83 (1981), 659–662. [Eq. (19), p. 662] 38

[My] W. Myrvold, Counting k-component forests of a graph, Networks, 22 (1992), 647-652. 38

[N] G. Nenashev, Classification of external zonotopal algebras, Electron. J. Combin. 26 (2019), Pap. 1.32, 10 pp. 2

[NS] G. Nenashev, B. Shapiro, "K-theoretic" analogs of Postnikov-Shapiro algebra distinguishes graphs, Journal of Combinatorial Theory, Series A 148 (2017) 316–332. 25

[PS] A. Postnikov and B. Shapiro, Trees, parking functions, syzygies, and deformations of monomials ideals, Trans. Amer. Math. Soc. 356 (8) (2004), 3109–3142. 3, 19, 22

[PSS] A. Postnikov, B. Shapiro and M. Shapiro, Algebras of curvature forms on homogeneous manifolds. Differential topology, infinite-dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., 1999, 227–235. 2

[Re] A. Rényi, Some remarks on the theory of trees. Magyar Tud. Akad. Mat. Kutat Int. Kzl, 4 (1959), 73-85. 38

[SV] B. Shapiro, A. Vaintrob, On algebras and matroids associated to undirected graphs, arXiv:2005.06160. 7

[SSV] B. Shapiro, I. Smirnov, A. Vaintrob, Deformed graphical zonotopal algebras, arXiv:2204.11331. 29

[Si] J.M.S. Simões-Pereira, On matroids on edge sets of graphs with connected subgraphs as circuits II, Discr. Math. 12 (1975), 55–78. 7

[St] R. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, Adv. Math. 111 (1995), 166–194. 26