BOCHNER-KRALL PROBLEM AND BERGKVIST'S CONJECTURE, REVISITED

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To Salomon Bochner, a mathematical giant

Abstract. In this paper we study the asymptotic behavior of the sequence of eigenpolynomials of an arbitrary degenerate exactly solvable differential operator. (The class of exactly solvable differential operators naturally appears in the formulation of the famous Bochner-Krall problem.) We refine a conjecture of T. Bergkvist [Be] on the rate of growth of the maximal absolute value of the roots of eingenpolynomials and settle our refined conjecture in a number of special cases. We deduce the infromation about the support of the asymptotic root-counting measure for the above sequence of scaled eigenpolynomials and apply our results to draw conclusions about the Bochner-Krall problem.

1. Introduction

In 1929 S. Bochner published a short paper [Bo] related to orthogonal polynomials and the Sturm-Liouville problem. Namely, the following classification problem was stated by S. Bochner for order $N = 2$, and H. L. Krall for general order.

{BK-problem}

Problem 1 ([Bo, Kr1]). *Classify all linear differential operators of the form:*

(1) {opBK}
$$T = \sum_{i=1}^{k} Q_i(z) \frac{d^i}{dz^i},$$

with polynomial coefficients $Q_i(x)$ *satisfying:*
a) $\deg Q_i(z) \le i$;
b) *there* $\exists$ *a positive integer* $i_0 \le k$ *with* $\deg Q_{i_0}(z) = i_0$,
such that the sequence of polynomial solutions f *of the formal spectral problem*

$$Tf(z) = \lambda f(z), \quad \lambda \in \mathbb{R},$$

is orthogonal with respect to some real bilinear form.

Following the terminology used in physics, we call linear differential operators (1) satisfying the above two conditions *exactly solvable*. Observe that any exactly solvable operator has a unique eigenpolynomial of any sufficiently large degree, see e.g. [MS].

Denote by $\{p_n^T(z)\}$ the sequence of eigenpolynomials of an exactly solvable operator T. (Here $\deg p_n^T(z) = n$ and n runs from some positive integer n^T to $+\infty$.) An exactly solvable operator which solves Problem 1 will be called a *Bochner-Krall*

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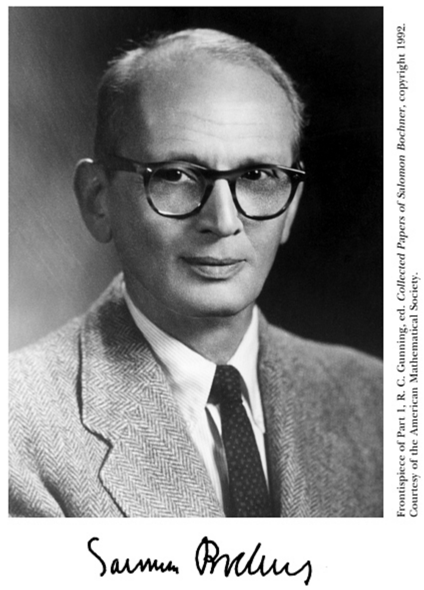


FIGURE 1. Salomon Bochner in person {fig:Bochner}

operator. If the corresponding real bilinear form comes from a positive measure supported on $\mathbb{R}$ we call such a T a *positive Bochner-Krall operator*.

Let us give some very basic information about the Bochner-Krall problem. Eleven years after Bochner stated and solved Problem 1 for order two differential operators [Bo], Krall settled the order four case [Kr2].The order two classification has four families, corresponding to the classical Hermite, Laguerre, Jacobi, and Bessel polynomials [Bo]. The order four classification has seven families: the four classical families corresponding to iterated order two operators, and three new families which are eigenfunctions of differential operators that do not factor into squares [Kr1].

The general case of Problem 1 is still open for operators of order six or more, but Kwon and Lee have found a satisfactory solution if the polynomials are required to be orthogonal with respect to a compactly supported measure, see [KL]. (A weaker result in the same direction was somewhat earlier obtained in [BRSh].)

Finally, in a recent paper [HST] jointly with E. Horozov, the authors have formulated an explicit conjecture about the algebraic version of the Bohner-Krall problem, see Conjecture 1.7 asking to describe all BK-operators whose sequence of eigenpolynomials satisfies a linear recurrence relation of order 3. If settled this conjecture will be the most important step in the solution of the original Bohner-Krall problem, see Proposition 1 in loc.cit.

One possible approach towards the Bochner-Krall problem and an important concrete problem in itself is to study the asymptotic root distribution of eigenpolynomial sequences $\{p_n^T(z)\}$ of an arbitrary exactly solvable operator T. Denoting by $\{\mu_n^T\}$ the root-counting measure of $\{p_n^T(z)\}$, one can pose the following question.

{prob:asymp}

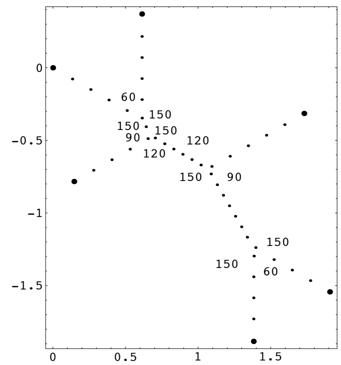


FIGURE 2. Small dots show the roots of the eigenpolynomial of degree 50 for the operator $T = Q_6(z)\frac{d^6}{dz^6}$ where $Q_6(z)$ is a sextic polynomial whose roots are the larger dots. (Numbers on the picture are the angles between the respective edges.) {fig1}

Problem 2. *For a given exactly solvable T, does the sequence $\{\mu_n^T\}$ weakly converges (after an appropriate scaling) to some compactly supported non-trivial asymptotic measure μ_T? If such μ_T exists, describe its support.*

By a non-trivial measure we mean a measure different from the unit mass located at a single point. Information about the support of the asymptotic root-counting measure μ_T of a exactly solvable operator T provides necessary conditions for T to be a Bochner-Krall operator. In particular, if the support of μ_T does not belong to $\mathbb{R}$, then T can not be a positive Bochner-Krall operator since in the latter case all roots of all p_n^T must be real and, additionally, roots of p_n^T and p_{n+1}^T must interlace.

Problem 2 was addressed and partially solved in [MS], [BR], [Be]. It turned out that exactly solvable operators split into two natural classes, *non-degenerate* and *degenerate*. Namely, an operator T as in (1) is called *non-degenerate* if $\deg Q_k = k$ and *degenerate*, otherwise. The main distinction between these two cases is that for any non-degenerate operator T, all roots of all its eigenpolynomials $p_n^T(z)$ are contained in some disk, while for any degenerate operator T, the union of all roots of its eigenpolynomials is necessarily unbounded, see [Be].

The main result about the root distribution of the eigenpolynomials of an non-degenerate exactly solvable operator T is as follows, [BR].

Theorem A. *In the above notation, assuming without loss of generality that $Q_k(z)$ is monic, the sequence $\{\mu_n^T\}$ of root-counting measures for $\{p_n^T(z)\}$ converges in the weak sense to a probability measure μ_T whose Cauchy transform $\mathcal{C}_T(z)$ satisfies a.e. in $\mathbb{C}$ the algebraic equation:*

$$\mathcal{C}_T^k(z)Q_k(z) = 1 \quad i.e., \quad \mathcal{C}_T(z) = \frac{1}{\sqrt[k]{Q_k(z)}}.$$

And conversely, there exists a unique probability measure whose Cauchy transform satisfies the latter equation a.e. in $\mathbb{C}$. Additionally its support is a curvilinear tree (i.e. a connected planar graph without cycles) lying in the convex hull of the roots of $Q_k(z)$ and the endpoints of this support are exactly all roots of $Q_k(z)$.

An example of such a support is shown in Fig. 2, see more details in [BR].

The situation with degenerate exactly solvable operators is more involved and they are the main object of study in the present paper. Namely, for any degenerate operator T, denote by r_n^T the maximal absolute value of the roots of the n-th eigenpolynomial p_n^T. As we mentioned above, $\lim_{n\to\infty} r_n^T = +\infty$. In [Be] T. Bergkvist has formulated the following conjecture about the leading term of the asymptotics of the sequence $\{r_n^T\}$ for an arbitrary degenerate exactly solvable T.

For a degenerate exactly solvable operator (1), denote by j_T the largest integer i for which $\deg Q_i = i$. Since T is a degenerate exactly solvable operator, then j_T exists and is strictly smaller than k.

{conj:Be}

Conjecture 1. *In the above notation, for any degenerate operator T, there exists a positive number $c_T > 0$ such that*

$$\lim_{n\to\infty} \frac{r_n^T}{n^{d_T}} = c_T, \tag{2}$$

where

$$d_T := \max_{i\in[j_T+1,k]} \left(\frac{i - j_T}{i - \deg Q_i} \right).$$

The geometric meaning of the above exponent d_T in terms of the Newton polygon of the operator T is shown in Fig. 3. From now on we will always use the normalisation assumption that the polynomial coefficient $Q_{j_T}(z)$ in T is monic. (Observe that by assumptions $\deg Q_{j_T}(z) = j_T$ and j_T is the maximal positive integer with the latter property.)

Under certain additional assumptions Conjecture 1 implies the following.

Corollary 1 (Follows if Conjecture 1 and some additional statements are valid)**.** *Given an arbitrary degenerate operator T as in (1), the Cauchy transform $\mathcal{C} := \mathcal{C}_T(z)$ of the asymptotic root counting measure μ_T for the sequence $\{q_n(z)\}$ of the scaled eigenpolynomials $q_n(z) := p_n(n^d z)$ satisfies a.e. in $\mathbb{C}$ the algebraic equation*

$$z^{j_T}\mathcal{C}^{j_T} + \sum_{i\in A_T} \alpha_{i,\deg Q_i} z^{\deg Q_i}\mathcal{C}^i = 1, \tag{3}$$

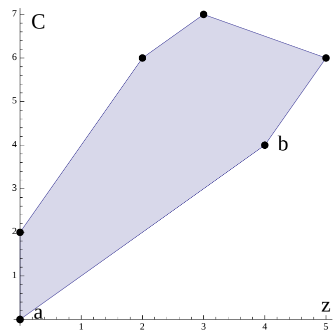
where A_T is the set consisting of all i for which the maximum $d := \max_{i\in[j_T+1,k]}\left(\frac{i-j_T}{i-\deg Q_i}\right)$
{cor:Cauchy} *is attained, i.e. $A_T = \{i : (i - j_T)/(i - \deg Q_i) = d_T\}$. (Observe that $A_T \neq \{\emptyset\}$.)*

Conjecture 1 was settled in a number of special cases already in the original paper [Be]. Additionally for an arbitrary degenerate operator T, the fact that there exists a positive constant $c > 0$ such that

$$\lim_{n\to\infty} \frac{r_n^T}{n^{d_T}} \geq c$$

was proven in a still unpublished preprint [BeBj]. In short, both Conjecture 1 and its Corollary 1 are highly plausible and are confirmed by a large number of computer experiments, see [Be]. Observe that every instance in which Conjecture 1 is settled also gives a special case supporting the main Conjecture 1 in § 3 of [BoSh].

The main goal of this paper is to present an explicit (conjectural) expression for the constant c_T and settle this refined conjecture in a number of cases which allows us to draw a number of conclusions for the Bochner-Krall problem.



{fig:slope} FIGURE 3. Newton polygon of a degenerate exactly solvable operator of order 7 and the corresponding d.

To formulate our guess, consider the left-hand side of equation (3)

$$U(\mathcal{C}, z) = z^j \mathcal{C}^j(z) + \sum_{i \in A} \alpha_{i, \deg Q_i} z^{\deg Q_i} \mathcal{C}^i(z). \tag{4}$$

Let ℓ be the maximal element in A, and $m = \deg Q_\ell$. Observe that all monomials included in $U(\mathcal{C}, z)$ lie on the straight segment connecting the monomials $z^j \mathcal{C}^j$ and $z^m \mathcal{C}^\ell$, see Fig. 3. (The latter circumstance implies that $U(\mathcal{C}, z)$ is a weighted homogeneous polynomial with the total weight $w(U) = j(\ell - m)$ and $w(z) = \ell - j$, $w(\mathcal{C}) = j - m$.) Set $V(\mathcal{C}) = U(\mathcal{C}, 1)$ and let $\zeta_1, \dots \zeta_s$ be pairwise distinct non-vanishing critical values of the univariate polynomial $V(\mathcal{C})$.

{conj:ST}

Conjecture 2. *For any degenerate exactly solvable operator T, its constant c_T appearing in* (2) *is given by condition*

(5) {eq:ct}
$$c_T^{\frac{j(\ell-m)}{\ell-j}} = \frac{1}{|\zeta_{min}|},$$

where ζ_{min} is the non-vanishing critical value of $V(\mathcal{C})$ with the minimal absolute value. (Observe that even if ζ_{min} if non-unique (5) *is well-defined.)*

Remark 1. Conjecture 2 presents the leading term in the asymptotic expansion of the root of maximal absolute value for the polynomial sequence $\{p_n^T\}$. Observe that for special families of orthogonal polynomials, the asymptotic expansion of the maximal root has been a topic of active research since the publication of [Sz], see e.g. §6.31 of loc. cit., [Kr] and references therein. Although the existence of an asymptotic expansion of r_n in the fractional powers of parameter n seems to follow from the general technique similar to the WKB-analysis, the explicit recipe how to obtain the exponents of fractional powers and the coefficients of such an expansion for a general operator T seems to be unknown at present.

The structure of the paper is as follows. In § 2, we settle Bergkvist's conjecture in a number of special cases by using Gräffe-Lobachevskii method. In § 4, we derive the consequences of Bergkvist's conjecture for the special case of exactly solvable operators such that the algebraic equation (3) has only 2 terms. In § 5, we extend some results of § 4 to general degenerate exactly solvable operators. Finally, in § 6, we state a number of relevant open problems.

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{sec:Berg}

2. Proving Bergkvist's conjecture by Gräffe-Lobachevskii method

Our approach to finding the root(s) with the maximal absolute value of a given polynomial $S(z)$ is based on the following statement called the Gräffe-Lobachevskii method, see e.g., [GrL].

{lm:GL}

Lemma 1. *Given a univariate polynomial* $S(z) = z^n + \sum_{r=0}^{n-1} \gamma_r z^r$ *whose roots (with possible repetitions) are denoted by* $z_1, \dots, z_n$, *assume that* S *has exactly one root* z_{max} *of maximal absolute value. Then*

$$\limsup_{k\to\infty} \frac{s_k}{z_{max}^k} = 1,$$

where $s_k := \sum_{j=1}^n z_j^k$.

Notice that s_t can be recurrsively expressed using the recurrence relation

$$s_k = -k\gamma_k - \sum_{j=1}^{k-1} \gamma_j s_{k-j}.$$

2.1. **Two-term case.** We start our consideration with the simplest two-term case, i.e., when

$$T = \alpha z^m \frac{d^\ell}{dz^\ell} + z^j \frac{d^j}{dz^j}.$$

2.1.1. $m = \ell - 1,\ j = 1.$

Lemma 2. *In this case,*

$$p_n^T(z) = z^n + \sum_{r=1}^n \left(\binom{n}{r} \alpha^r \prod_{j=1}^r \left(\prod_{i=0}^{\ell-2} (n-j-i) \right) \right) z^{n-r}.$$

Denote by $B(n,r,\ell) := \binom{n}{r} \alpha^r \prod_{j=1}^r \left(\prod_{i=0}^{\ell-2} (n-j-i) \right)$. We are interested in the asymptotic expansion of $B(n,r,\ell)$ w.r.t. n.

Lemma 3. *One has,*

$$B(n,r,\ell) = \frac{\alpha^r}{r!} n^{r\ell} - \frac{\ell(r+\ell-2)\alpha^r}{2Poch(1,r-1)} n^{r\ell-1} + \mathcal{O}(n^{r\ell-2}).$$

Further, we need to find asymptotic expansions of s_k w.r.t. n.

Lemma 4.

$$s_k = (-1)^k \alpha^k n^{k\ell-k+1} \frac{\ell}{(k-1)!} \prod_{t=1}^{k-2} (k\ell - t) + \mathcal{O}(n^{k\ell-k}). \tag{6}$$

According to the general scheme of Gräffe-Lobachevsky method, we need to find the limit $\limsup_{k\to\infty} \sqrt[k]{s_k}$. Expecting (6), we arrive at the result

$$\limsup_{k\to\infty} \sqrt[k]{s_k} = -\frac{\ell^\ell}{(\ell-1)^{\ell-1}} \alpha n^{\ell-1}.$$

2.1.2. $m = \ell - 2,\ j = 1$. Arguing similarly, we get that $B(n,r,\ell) = 0$ for r odd and $s_k = 0$ for k odd, while

$$B(n,r,\ell) = \prod_{t=1}^{r/2} \frac{1}{2^t \Gamma(t+1)} \left(\prod_{i=r-2t}^{\ell+2t-3} (n-i) \right).$$

Using the latter formula, one gets for even k

$$s_k = (-\alpha)^{k/2} n^{k\ell/2-k/2+1} \frac{\ell}{\Gamma(k/2)} \prod_{t=1}^{k/2-2} \left(\frac{k}{2}\ell - t \right) + \mathcal{O}(...).$$

Taking the limit when $k \to \infty$, we get tho natural subsequences: $k = 4t$ and $k = 4t+2$. Combining, we get

$$\begin{cases} \limsup_{t\to\infty} s_{4t} = \frac{\sqrt{\alpha}\cdot \ell^{\ell/2}}{(\ell-1)^{(\ell-1)/2}} \cdot n^{(\ell-1)/2} \\ \limsup_{t\to\infty} s_{4t+2} = \frac{-\sqrt{\alpha}\cdot \ell^{\ell/2}}{(\ell-1)^{(\ell-1)/2}} \cdot n^{(\ell-1)/2}. \end{cases}$$

CHECK!!!

2.1.3. $m \le \ell - 2,\ j = 1$. Similar calculations show that $B(n,r,\ell) = 0$ for all r not of the form $(\ell-m)j$, $j \in \mathbb{Z}$ and $s_k = 0$ for k not of the form $(\ell-m)j$, $j \in \mathbb{Z}$. If k is of the form $(\ell-m)i$, one should still split such k into $\ell - m$subsequence depending on the remainder of quotient $k/(\ell-m)$. These $(\ell-m)$ subsequences converge to different values of the roots of order $(\ell-m)$ in the next formula.

$$\limsup_{k\to\infty} s_k = (-1)^{\ell-m} \frac{\sqrt[\ell-m]{\alpha} \cdot \ell^{\ell/(\ell-m)}}{(\ell-1)^{(\ell-1)/(\ell-m)}} \cdot n^{(\ell-1)/(\ell-m)}.$$

WHICH VALUE OF THE ROOT ONE SHOULD TAKE???

2.2. **General case.** Consider the operator

$$T := \alpha_{\ell,m} z^m \frac{d^\ell}{dz^\ell} + z^j \frac{d^j}{dz^j}$$

and find its eigenpolynomial solutions, i.e. polynomial solutions of

$$Ty + \lambda y = 0.$$

We can write a monic eigenpolynomial p_n^T of degree n as

(7) {eq:pnt}
$$p_n^T(z) = z^n + \sum_{r=1}^{[n/(\ell-m)]} \gamma_r z^{n-(\ell-m)r},$$

where

$$\gamma_r = \frac{(m-\ell)^{m(r-1)}}{(\ell-m)^r} \frac{\Gamma(n+1)\alpha_{\ell,m}^r}{\Gamma(r+1)\Gamma(n-m+(m-\ell)r+1)} \frac{\prod_{s=1}^m \left(\frac{\ell-n-s}{\ell-m}\right)_{r-1}}{\prod_{s=1}^r A_s^{(\ell-m,j)}(n)}.$$

The Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$, coefficients $A_s^{(k,j)}$ are $(j-1)$th degree polynomials in n:

$$\begin{aligned}
A_s^{(k,1)}(n) &= 1\\
A_s^{(k,2)}(n) &= 2n-(ks+1)\\
A_s^{(k,3)}(n) &= 3n^2-3(ks+2)n+(ks+1)(ks+2)\\
A_s^{(k,4)}(n) &= 4n^3-6(ks+3)n^2+2(2k^2s^2+9ks+11)n-\\
&\quad (ks+1)(ks+2)(ks+3)\\
A_s^{(k,5)}(n) &= 5n^4-10(ks+4)n^3+5(2k^2s^2+12ks+21)n^2-\\
&\quad 5(ks+4)(k^2s^2+4ks+5)n+(ks+1)(ks+2)(ks+3)(ks+4)\\
A_s^{(k,6)}(n) &= 6n^5-15(ks+5)n^4+10(2k^2s^2+15ks+34)n^3-\\
&\quad 15(ks+5)(k^2s^2+5ks+9)n^2+\\
&\quad (6k^4s^4+75k^3s^3+340k^2s^2+675ks+548)n-\\
&\quad (ks+1)(ks+2)(ks+3)(ks+4)(ks+5)\\
&\quad \dots
\end{aligned}$$

We havn't find the general form $A_s^{(k,j)}$ yet, but know that

$$\begin{aligned}
A_s^{(k,j)}(n) \;=\; & \tbinom{j}{1}\,\mathbf{n^{j-1}}-\\
& \tbinom{j}{2}\,(ks+j-1)\mathbf{n^{j-2}}+\\
& \tbinom{j}{3}\,(k^2s^2+3(j-1)/2\,ks+(3j-1)(j-2)/4)\mathbf{n^{j-3}}-\\
& \tbinom{j}{4}\,(ks+j-1)(k^2s^2+(j-1)ks+j(j-3)/2)\mathbf{n^{j-4}}+\\
& \tbinom{j}{5}\,\big(k^4s^4+\tfrac{5}{2}(j-1)k^3s^3+\tfrac{5}{6}(j-2)(3j-1)k^2s^2+\\
& \quad \tfrac{5}{4}(j-3)(j-1)j\,ks+\tfrac{1}{48}(j-4)\left(15j^3-30j^2+5j+2\right)\big)\,\mathbf{n^{j-5}}-\\
& \tbinom{j}{6}\,(ks+j-1)\left(k^4s^4+2(j-1)k^3s^3+\tfrac{1}{4}\left(7j^2-19j+2\right)k^2s^2+\right.\\
& \quad \left.\tfrac{1}{4}(j-1)\left(3j^2-11j-2\right)ks+\tfrac{1}{16}(j-5)j\left(3j^2-7j-2\right)\right)\mathbf{n^{j-6}}+\\
& \quad \dots\\
+\;\; & (-1)^{j-1}\textstyle\prod_{t=1}^{j-1}(ks+t).
\end{aligned}$$

3. Properties of eigenpolynomials for $T = z^m\frac{d^\ell}{dz^\ell}+z^j\frac{d^j}{dz^j}$

Let as above $p_n(z) := p_n^{(m,\ell,j)}(z)$ denote the sequence of eigenpolynomials of $T = z^m\frac{d^\ell}{dz^\ell}+z^j\frac{d^j}{dz^j}$. Formula (7) with $\alpha=1$ gives an explicit expression for $p_n(z)$. In fact, each sequence $\{p_n^{(m,\ell,j)}(z)\}_{n=0}^\infty$ naturally splits in $\ell-m$ subsequences $\{p_{k(\ell-m)+i}^{(m,\ell,j)}(z)\}_{k=0}^\infty$ where $i=0,\dots,\ell-m-1$. By formula (7),

$$p_{k(\ell-m)+i}(z) = z^iQ_k^{[i]}(z^{\ell-m}),$$

where $Q_k^{[i]}(\tau)$ is a monic polynomial of degree k in τ. Thus, we get $(\ell-m)$ sequences $\{Q_k^{[i]}(\tau)\},\ i=0,1,\dots,\ell-m-1$ polynomial sequences with positive coefficients.

{th:hyp}

Theorem 5. *In the above notation, each polynomial* $Q_k^{[i]}(\tau)$ *is real-rooted.*

In order to settle Theorem 5, we will need more explicit formulas for the coefficients of $p_n^{(m,\ell,j)}(z)$.

{prop:coeff}

Proposition 6. *In the above notation with* $\ell > j \geq 1$ *and* $\ell > m \geq 0$,

$$p_n^{(m,\ell,j)}(z) = z^n + \sum_{r=1}^{[n/(\ell-m)]} \gamma_r z^{n-(\ell-m)r}, \tag{8}$$

{eq:coeffs}

where the coefficients γ_r : *are given by*

$$\gamma_r := \gamma_r^{(n,m,\ell,j)} = \frac{\prod_{i=0}^{r-1}((n-i(\ell-m))_\ell}{\prod_{i=1}^{r}((n)_j - (n-i(\ell-m))_j)}.$$

Special cases: Recall that $(a)_b$, where b is a positive integer denotes the falling factorial $(a)_b := a(a-1)\dots(a-b+1)$.
For $m = \ell - 1$ and $j = 1$,

$$\gamma_{n-\ell+2}^{(n,\ell-1,\ell,1)} = \gamma_{n-\ell+3}^{(n,\ell-1,\ell,1)} = \dots = \gamma_n^{(n,\ell-1,\ell,1)} = 0; \quad \gamma_r^{(n,\ell-1,\ell,1)} = \frac{(n)_\ell(n-1)_\ell \cdots (n-r+1)_\ell}{r!};\ r = 1, \dots, n-\ell+1.$$

In particular, for $r = 1, \dots, n-\ell$,

$$\gamma_r^{(n,\ell,\ell+1,1)} = \gamma_r^{(n,\ell-1,\ell,1)} r! \binom{n-\ell}{r}.$$

For $m = \ell - 1$ and $j = 2$,

$$\gamma_{n-\ell+2}^{(n,\ell-1,\ell,2)} = \gamma_{n-\ell+3}^{(n,\ell-1,\ell,2)} = \dots = \gamma_n^{(n,\ell-1,\ell,2)} = 0; \quad \gamma_r^{(n,\ell-1,\ell,2)} = \frac{(n)_\ell(n-1)_\ell \cdots (n-r+1)_\ell}{r!(2n-2)_r};\ r = 1, \dots, n-\ell+1.$$

In particular, for $r = 1, \dots, n-\ell$,

$$\gamma_r^{(n,\ell,\ell+1,2)} = \gamma_r^{(n,\ell-1,\ell,2)} r! \binom{n-\ell}{r},$$

which is the same relation as for $m = \ell - 1$ and $j = 1$.
For $m = \ell - 1$ and any $j < \ell$,

$$\gamma_{n-\ell+j}^{(n,\ell-1,\ell,j)} = \gamma_{n-\ell+3}^{(n,\ell-1,\ell,j)} = \dots = \gamma_n^{(n,\ell-1,\ell,j)} = 0; \quad \gamma_r^{(n,\ell-1,\ell,j)} = \frac{(n)_\ell(n-1)_\ell \cdots (n-r+1)_\ell}{\prod_{i=1}^{r}((n)_j - (n-i)_j)};\ r = 1, \dots, n-\ell+1.$$

One still has the same relation, for $r = 1, \dots, n-\ell$,

$$\gamma_r^{(n,\ell,\ell+1,j)} = \gamma_r^{(n,\ell-1,\ell,j)} r! \binom{n-\ell}{r},$$

as for $m = \ell - 1$ and $j = 1$.
For $m = \ell - 2$ and any $j < \ell$,

$$\gamma_{n-\ell+j}^{(n,\ell-2,\ell,j)} = \gamma_{n-\ell+3}^{(n,\ell-2,\ell,j)} = \dots = \gamma_n^{(n,\ell-2,\ell,j)} = 0; CHECK \quad \gamma_r^{(n,\ell-2,\ell,j)} = \frac{(n)_\ell(n-2)_\ell \cdots (n-2r+2)_\ell}{\prod_{i=1}^{r}((n)_j - (n-2i)_j)};\ r = 1, .$$

For $r = 1, \dots, n-\ell$,

$$\gamma_r^{(n,\ell-1,\ell+1,j)} = \gamma_r^{(n,\ell-2,\ell,j)}(n-\ell)(n-2-\ell)\cdots(n-2r+2-\ell),$$

Proof of Theorem 5. Let us start with the case $T = z^{\ell-1}\frac{d^\ell}{dz^\ell} + z\frac{d}{dz}$. Then, there is just a single family

$$p_n(z) := p_n^{(\ell-1,\ell,1)} = z^n + \sum_{r=1}^{n-\ell+1} \frac{(n)_\ell(n-1)_\ell \cdots (n-r+1)_\ell}{r!} z^{n-r}.$$

(To avoid triviality, one has to assume that $n \geq \ell$. For $n < \ell$, $p_n(z) = z^n$.)

Petter's idea! For $r = 1, \dots, n-\ell$, one has a relation

$$\gamma_r^{(n,\ell,\ell+1,1)} = \gamma_r^{(n,\ell-1,\ell,1)} r! \binom{n-\ell}{r}.$$

For fixed $n \geq 2$, we will prove the real-rootedness of $p_n(\ell, z)$ by induction on $\ell = 2, 3, \dots, n$. Note that for $\ell = 2$, $p_n^{(1,2,1)}(z)$ is the monic n-th Laguerre polynomial which is known to have all negative roots. We will use the Schur-Szegö product $\star$ to present $p_n^{(\ell,\ell+1,1)}(z)$ as

$$p_n^{(\ell,\ell+1,1)}(z) = p_n^{(\ell-1,\ell,1)}(z) \star Q_n^{(\ell)}(z),$$

where

$$Q_n^{(\ell)}(z) = \sum_{r=0}^{n-\ell} r! \binom{n-\ell}{r} \binom{n}{r} z^{n-r}.$$

One needs to show that for any $n \geq \ell$, $Q_n^{(\ell)}(z)$ has all real roots. ???

For $m = \ell - 1$ and $j = 2$, we have

$$p_n^{(\ell,\ell+1,2)}(z) = p_n^{(\ell-1,\ell,2)}(z) \star Q_n^{(\ell)}(z),$$

where $Q_n^{(\ell)}(z)$ are exactly the same polynomials as above. We need however to check that the initial polynomial $p_n^{(2,3,2)}$, $n \geq 3$ is hyperbolic. ???

For $m = \ell - 1$ and an arbitrary $j \geq 2$, we still

$$p_n^{(\ell,\ell+1,j)}(z) = p_n^{(\ell-1,\ell,j)}(z) \star Q_n^{(\ell)}(z),$$

where $Q_n^{(\ell)}(z)$ are exactly the same polynomials as above. We need however to check that the initial polynomial $p_n^{(2,3,j)}$, $n \geq j+1$ is hyperbolic. ???

□

{sec:two-term}

4. Support of the limiting measure in the two-term case

In this and the next sections, assuming the validity of (3) we draw several conclusions about the support and the density of the limiting root-counting measure μ_T for a degenerate exactly solvable operator T which is subject to some restriction.

In this section we consider the special case of degenerate exactly solvable operators T for which the set A_T contains a single element $\ell > j := j_T$. In this case, equation (3) becomes a simple two-term equation

$$\alpha z^m \mathcal{C}^\ell + z^j \mathcal{C}^j = 1, \tag{9}$$

where $\alpha \in \mathbb{C}^\star$, $0 \leq m < \ell > j \geq 1$. To simplify our notation, we suppress the index T in the formulas.

Remark 2. Observe that, in general, ℓ might be smaller than k which is the order of T. In this case $d := d_T = \frac{\ell-j}{\ell-m} > 0$, where $j := j_T$.

{lm:br3term}

Lemma 7. *The set of branching points different from the origin for the projection of the algebraic curve given by* (9) *onto the* z*-plane is given by the equation:*

$$z^{\frac{j(\ell-m)}{\ell-j}} = \frac{\ell}{\ell-j}\left(\frac{-j}{\alpha\ell}\right)^{\frac{\ell-j}{j}}. \tag{10}$$

Proof. The system of equations defining the required branching points is given by

$$\begin{cases} \alpha z^m \mathcal{C}^\ell + z^j \mathcal{C}^j = 1 \\ \alpha \ell z^m \mathcal{C}^{\ell-1} + j z^j \mathcal{C}^{j-1} = 0. \end{cases}$$

Since $z=0$ does not satisfy the first equation we can factor out $z^j\mathcal{C}^{j-1}$ from the second equation and obtain

$$\alpha\ell z^{m-j}\mathcal{C}^{\ell-j} + j = 0 \quad \Leftrightarrow \quad \mathcal{C}^{\ell-j} = \frac{-j}{\alpha\ell} z^{j-m} \quad \Leftrightarrow \quad \mathcal{C} = \left(\frac{-j}{\alpha\ell}\right)^{\frac{1}{\ell-j}} z^{\frac{j-m}{\ell-j}}.$$

Substituting the latter expression for $\mathcal{C}$ in the first equation, we get

$$\alpha z^m \left(\frac{-j}{\alpha l}\right)^{\frac{\ell}{\ell-j}} z^{\frac{\ell(j-m)}{\ell-j}} + z^j \left(\frac{-j}{\alpha\ell}\right)^{\frac{j}{\ell-j}} z^{\frac{j(j-m)}{\ell-j}} = 1.$$

Both terms in the left-hand side of the latter equation have the same powers of z which gives after some manipulations

$$\frac{\ell-j}{\ell}\left(\frac{-j}{\alpha\ell}\right)^{\frac{j}{\ell-j}} z^{\frac{j(\ell-m)}{\ell-j}} = 1 \quad \Leftrightarrow \quad z^{\frac{j(\ell-m)}{\ell-j}} = \frac{\ell}{\ell-j}\left(\frac{-j}{\alpha\ell}\right)^{\frac{\ell-j}{j}}.$$

□

Below we present more information about the branching points.

{lm:more}

Lemma 8. *For an arbitrary equation* (9)*,*

(i) *the set of branching points consists of those satisfying* (10) *and the origin if* $m>0$ *(if* $m=0$ *the origin is not a branching point);*
(ii) *the total multiplicity of all its branching points equals* $j\ell + m(\ell-j-1)$*;*
(iii) *the multiplicity of each non-vanishing branching point (i.e. satisfying* (10)*) equals* $GCD[j,\ell]$*, the total number of these points equals* $\frac{j(\ell-m)}{GCD[j,\ell]}$*;*
(iv) *the multiplicity of the origin equals* $m(\ell-1)$*;*

Proof. BLA □

The main result of this section is as follows.

{th:3term}

Theorem 9. *For any complex number* $\alpha \in \mathbb{C}^\star$ *and any triple of integers* $0 \le m < \ell > j \ge 1$*, there exists a unique probability measure* μ *whose Cauchy transform satisfies equation* (9) *a.e. in* $\mathbb{C}$*. Its support consists of a number of straight segments connecting the origin to some of the branching points of* (9)*. Namely, these segments connect the origin with each* $\frac{j}{\mathrm{GCD}[j,\ell]}$*-th branching point, see Fig. 4 and 5.*

{cor:legs}

Corollary 2. *The total number of legs in the support of the measure satisfying equation* (9) *equals* $\ell - m$*.*

Proof. STOR BLA □

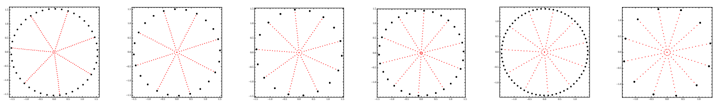

FIGURE 4. Measure μ for $j = 6$, $m = 0$ and $\ell = 7, 8, \dots, 12$. {figNice}

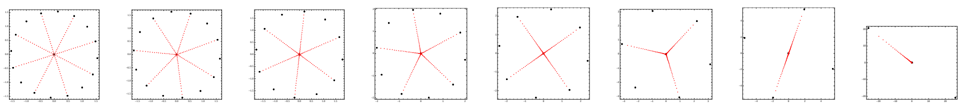

FIGURE 5. Measure μ for $j = 6$, $\ell = 9$ and $m = 1, 2, \dots, 8$. {figNice2}

(WHAT ABOUT EXCEPTIONAL CASES OF THE HORIZONTAL AND VERTICAL INTERVALS?)

We call the case of 2 legs, i.e. $\ell - m = 2$ *the Hermite-like case* and the case of 1 leg, i.e. $\ell - m = 1$ *the Laguerre-like case.*

{lm:cases}

Proposition 10. (i) For the equation

$$-z^{\ell-1}\mathcal{C}^{\ell} + z^j\mathcal{C}^j = 1,$$

the support of the asymptotic root-counting measure coincides with $[0, z_{cr}]$ where z_{cr} =???. Moreover the density of the asymptotic root-counting measure is given by ????

(ii) For the equation

$$-z^{\ell-2}\mathcal{C}^{\ell} + z^j\mathcal{C}^j = 1,$$

the support of the asymptotic root-counting measure coincides with $[-z_{cr}, z_{cr}]$ where z_{cr} =???. Moreover the density of the asymptotic root-counting measure is given by ????

Observe that any non-vanishing value of α can be attain by appropriate rescaling of the variable z. The choice $\alpha = -1$ seems to be convenient in several aspects.

Proof. BLA □

Problem 3. *If $l > 2$ can such densities occur for the scaled orthogonal polynomials?*

{sec:general}

5. SUPPORT OF THE LIMITING MEASURE IN THE GENERAL CASE

We start by calculating the cardinality of A_T and generalizing Lemma 7. Namely, suppressing the index T everywhere, consider the left-hand side of equation (3)

$$U(\mathcal{C}, z) = z^j\mathcal{C}^j(z) + \sum_{i\in A} \alpha_{i,\deg Q_i} z^{\deg Q_i}\mathcal{C}^i(z). \tag{11}$$

Let ℓ be the maximal element in A, and $m = \deg Q_\ell$. Then all monomials included in $U(\mathcal{C}, z)$ lie on the straight segment connecting the lattice points (j, j) and (m, ℓ) (i.e. connecting the monomials $z^j\mathcal{C}^j$ and $z^m\mathcal{C}^\ell$), see Fig. 2.

{lm:A}

Lemma 11. *The cardinality of A_T equals $GCD[m-j, \ell-j]$. In particular, if $m-j$ and $\ell - j$ are coprime, then we are necessarily in the two-term situation.*

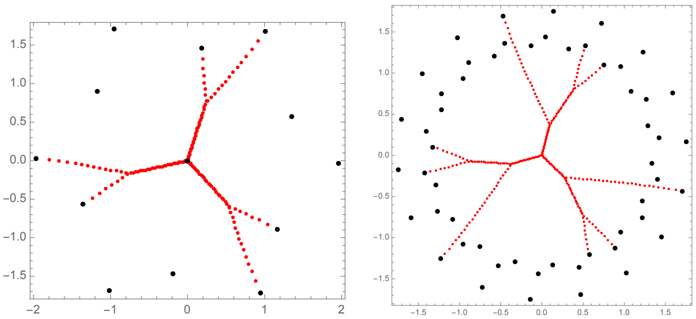

FIGURE 6. Measures for $T=(1+5I)x^2\frac{d^8}{dx^8}+(3+I)x^3\frac{d^6}{dx^6}+x^4\frac{d^4}{dx^4}$ and $T=-(10-8I)\frac{d^9}{dx^9}-\frac{13-15I}{2}x^2\frac{d^8}{dx^8}+\frac{1+5I}{2}x^4\frac{d^7}{dx^7}+x^6\frac{d^6}{dx^6}$. {figNice3}

Proof. Obviously, the set of integer points on the line connecting the origin to the point $(m-j,l-j)$ is a lattice generated by the vector $\frac{(m-j,\ell-j)}{GCD[m-j,\ell-j]}$. The number of integer points (except the origin) on the closed interval between the origin and the point $(m-j,\ell-j)$ equals $GCD[m-j,\ell-j]$. Each such point after a shift by (j,j) becomes an integer point on the required interval between (j,j) and (m,ℓ). □

Observe that $U(\mathcal{C},z)$ is weighted homogeneous with the total weight $w(U)=j(\ell-m)$ and $w(z)=\ell-j$, $w(\mathcal{C})=j-m$. Set $V(\mathcal{C})=U(\mathcal{C},1)$ and let $\zeta_1,\dots\zeta_s$ be pairwise distinct non-vanishing critical values of the univariate polynomial $V(\mathcal{C})$. {lm:general}

Lemma 12. *The set of non-vanishing branching points of the projection of the algebraic curve given by* (3) *(which coincides with $U(\mathcal{C},z)=1$) on the z-plane lie on s circles centered at the origin. Location of the branching points is given by the equations*

$$z^{\frac{j(\ell-m)}{\ell-j}}=\frac{1}{\zeta_i}\quad i=1,\dots,s. \tag{12}$$ {eq:brgeneral}

In particular, the total number of branching points equals s times the least positive integer of the form $\frac{j(\ell-m)}{\ell-j}\cdot\kappa$ where κ is an integer.

Observe that (12) coincides with (10) for $U(\mathcal{C},z)=\alpha z^m\mathcal{C}^\ell+z^j\mathcal{C}^j$.

Proof. Indeed, the set of branching points under consideration is determined by the system

$$\begin{cases}U(\mathcal{C},z)=1\\ U'_{\mathcal{C}}(\mathcal{C},z)=0.\end{cases} \tag{13}$$

Since $U(\mathcal{C},z)$ is weighted homogeneous, then $U'_{\mathcal{C}}(\mathcal{C},z)$ is also weighted homogeneous with the same weight of variables and the total weight $w(U')=j(\ell-1)+m(1-j)$. Using this fact and substituting $\mathcal{C}=\tau z^{\frac{w(\mathcal{C})}{w(z)}}=\tau z^{\frac{j-m}{\ell-j}}$ in both equations, we obtain

$$\begin{cases}U(\mathcal{C},z)=z^{\frac{j(\ell-m)}{\ell-j}}V(\tau)=1\\ U'_{\mathcal{C}}(\mathcal{C},z)=z^{\frac{j(\ell-1)+m(1-j)}{\ell-j}}V'(\tau)=0,\end{cases}$$

which is exactly (12). □

We need more detailed information about the univariate polynomial $V(\mathcal{C})$.

{lm:poly}

Lemma 13. *$V(\mathcal{C}) = \mathcal{C}^j \widetilde{V}(\mathcal{C})$, where $\widetilde{V}(\mathcal{C})$ is a polynomial of degree $\ell - j$ whose constant term equals 1. Moreover, besides the constant term and the leading term, $\widetilde{V}(\mathcal{C})$ contains only monomials whose degrees are multiples of $\frac{\ell-j}{GCD[m-j,\ell-j]}$.*

Proof. Follows immediately from the argument in the proof of Lemma 11. □

Let $\mathcal{L}(m,j,l)$ be the affine space of monic polynomials of the form $\mathcal{C}^j\widetilde{V}(\mathcal{C})$ satisfying the assumptions of Lemma 13. We say that a r distinct complex numbers form a *cyclic set* if they solve the equation $z^r = B$ for some non-vanishing B.

{lm:extra}

Lemma 14. *A generic polynomial $V(\mathcal{C}) \in \mathcal{L}(m,j,l)$,*

(i) *has $\ell - j$ distinct non-vanishing critical points which form $GCD[m-j,\ell-j]$ cyclic sets of cardinality $\frac{\ell-j}{GCD[m-j,\ell-j]}$. (Multiplicity of the critical point at the origin equals $j-1$.)*

(ii) *has $\frac{\ell-j}{\kappa}$ distinct non-vanishing critical values where $\kappa = GCD[j, \frac{\ell-j}{GCD[m-j,\ell-j]}]$. Moreover these critical values come in $GCD[m-j,\ell-j]$ cyclic sets of cardinality $\frac{\ell-j}{\kappa\cdot GCD[m-j,\ell-j]}$. (In particular, if j is prime then the number of distinct non-vanishing critical values of $V(\mathcal{C})$ equals $\ell - j$.)*

Proof. Seems obvious. □

Corollary 3. *For a generic $U(\mathcal{C}, z)$, the set of non-vanishing branching points of the projection of the algebraic curve given by (3) (which coincides with $U(\mathcal{C}, z) = 1$) on the z-plane .... WHAT DO I MEAN BY THAT?*

Proof. We need to find out how many solutions has the equation

$$z^{\frac{j(\ell-m)}{\ell-j}} = \mathcal{S},$$

where $\mathcal{S}$ is a cyclic set of cardinality $\frac{\ell-j}{\kappa\cdot GCD[m-j,\ell-j]}$. Indeed, ... □

{th:MAIN}

The main result of this section is as follows

Theorem 15. *For any equation (3), the support of the positive measure μ satisfying (3) is a (curvilinear) rooted tree with root at the origin. Its number of leaves (vertices of valency 1) is at least $\frac{\ell-m}{GCD[m-j,\ell-j]}$ and at most $\ell - m$. (In fact, the least number of leaves equals $\frac{(\ell-m)\nu}{GCD[m-j,\ell-j]}$, $\nu \geq 1$ which comes from the monomial with the least degree in $\mathcal{C}$ except for $z^j\mathcal{C}^j$ in $U(\mathcal{C}, z)$.) The symmetry group of the support is (at least) $\mathbb{Z}_?$.*

Proof. Very STOR BLA... □

Corollary 4. *If $\frac{\ell-m}{GCD[m-j,\ell-j]} > 2$ then this can't be a Bochner-Krall situation. This gives two natural series for the remaining cases: a) Hermite-like when $\widetilde{m} = \widetilde{\ell} - 2$, $\widetilde{\ell} > j$ where $\widetilde{m} = \frac{m}{GCD[m-j.\ell-j]}$ and $\widetilde{\ell} = \frac{\ell}{GCD[m-j.\ell-j]}$ and the Laguerre-like where in the above notation $\widetilde{m} = \widetilde{\ell} - 1$, $\widetilde{l} > j$.*

{sec:final}

6. Final remarks

{pr:HPO}

Observe that at present the following natural question is still open.

Problem 4. *Characterize the class of exactly solvable operators T such that every eigenpolynomial p_n^T has all real zeros.*

Apparently Problem 4 is closely related to the notion of hyperbolicity preserving operators, see e.g. [BB]. In particular, one can easily see that any hyperbolicity preserving exactly solvable operator solves Problem 4, but the converse is not true.

If we additionally assume that the positive measure defining the bilinear functional belongs to the so-called Nevai class, see [Ne] then it is known that the limiting root-counting measure of any sequence of orthogonal polynomials in the Nevai class has the arcsine distribution on some finite interval $[a, b]$. The latter fact implies that up to a constant the leading coefficient of the corresponding positive BK-operator must be of the form $(x-a)^l(x-b)^l$, where the order k of the operator equals $2l$, see [BRSh]. (More details about this case can be found in [KL].)

7. Odds and ends

Lemma 16. *The n-th monic eigenpolynomial p_n^T is given by*

$$p_n^T = z^n + \sum_{r=1}^{[\ell/(\ell-m)]} \gamma_r z^{n-(\ell-m)r},$$

where $\gamma_r = ...$

General Q_k:

$$z^m y^{(k)}(z) + \alpha_{j,j} z^j y^{(j)}(z) + \lambda y(z) = 0,$$

where $m = 0, \dots, k-1$, $k > j$

$$S_n^{(m,k,j)}(z) = z^n + \sum_{r=1}^{[n/k]} \binom{n}{kr} \frac{\Gamma(kr)}{k^{r-1}\Gamma(r)} \frac{(m-k)^{m(r-1)} \prod_{\ell=1}^{m} \left(\frac{k-\ell-n}{k-m}\right)_{r-1} z^{n-kr}}{\prod_{\ell=1}^{r} \left(\alpha_{j,j} A_\ell^{(k-m,j)}(n)\right)}$$

$$\begin{aligned}
A_\ell^{(k,1)}(n) &= 1\\
A_\ell^{(k,2)}(n) &= 2n-(k\ell+1)\\
A_\ell^{(k,3)}(n) &= 3n^2-3(k\ell+2)n+(k\ell+1)(k\ell+2)\\
A_\ell^{(k,4)}(n) &= 4n^3-6(k\ell+3)n^2+2(2k^2\ell^2+9k\ell+11)n-(k\ell+1)(k\ell+2)(k\ell+3)\\
A_\ell^{(k,5)}(n) &= 5n^4-10(k\ell+4)n^3+5(2k^2\ell^2+12k\ell+21)n^2-\\
&\quad 5(k\ell+4)(k^2\ell^2+4k\ell+5)n+(k\ell+1)(k\ell+2)(k\ell+3)(k\ell+4)\\
A_\ell^{(k,6)}(n) &= 6n^5-15(k\ell+5)n^4+10(2k^2\ell^2+15k\ell+34)n^3-\\
&\quad 15(k\ell+5)(k^2\ell^2+5k\ell+9)n^2+\\
&\quad (6k^4\ell^4+75k^3\ell^3+340k^2\ell^2+675k\ell+548)n-\\
&\quad (k\ell+1)(k\ell+2)(k\ell+3)(k\ell+4)(k\ell+5)\\
&\quad \dots
\end{aligned}$$

$$\begin{aligned}
A_\ell^{(k,j)}(n) \quad = \quad & \tbinom{n}{1}\,\mathbf{n^{j-1}}- \\
& \tbinom{n}{2}\,(k\ell+j-1)\mathbf{n^{j-2}}+ \\
& \tbinom{n}{3}\,(k^2\ell^2+3(j-1)/2\,k\ell+(3j-1)(j-2)/4)\mathbf{n^{j-3}}- \\
& \tbinom{n}{4}\,(k\ell+j-1)(k^2\ell^2+(j-1)k\ell+j(j-3)/2)\mathbf{n^{j-4}}+ \\
& \tbinom{n}{5}\,\big(k^4\ell^4+\tfrac{5}{2}(j-1)k^3\ell^3+\tfrac{5}{6}(j-2)(3j-1)k^2\ell^2+ \\
& \qquad \tfrac{5}{4}(j-3)(j-1)j\,k\ell+\tfrac{1}{48}(j-4)\left(15j^3-30j^2+5j+2\right)\big)\,\mathbf{n^{j-5}}- \\
& \tbinom{n}{6}\,(k\ell+j-1)\left(k^4\ell^4+2(j-1)k^3\ell^3+\tfrac{1}{4}\left(7j^2-19j+2\right)k^2\ell^2+\right. \\
& \qquad \left.\tfrac{1}{4}(j-1)\left(3j^2-11j-2\right)k\ell+\tfrac{1}{16}(j-5)j\left(3j^2-7j-2\right)\right)\mathbf{n^{j-6}}+ \\
& \qquad \cdots \\
+ \quad & (-1)^{j-1}\textstyle\prod_{s=1}^{j-1}(k\ell+s)\mathbf{n^0}.
\end{aligned}$$

$$z_{max} \sim \frac{(-1)^{\frac{1}{k-m}}\,k^{\frac{k}{j(k-m)}}}{j^{\frac{1}{k-m}}\,(k-j)^{\frac{k-j}{j(k-m)}}}\,\frac{n^{\frac{k-j}{k-m}}}{\sqrt[k-m]{\alpha_{j,j}}}, \qquad n \to \infty.$$

MILOS SHOULD FILL IN THE DETAILS!

$$p_n(z) = z^n + \sum_{r=1}^{n} \left(\binom{n}{r} \prod_{j=1}^{r} \prod_{i=0}^{\ell-2} (n-j-i) \right) z^{n-r},$$

where $\ell \geq 2$ is a positive integer.

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