QUADRATIC DIFFERENTIALS AND SIGNED MEASURES

By

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Abstract. In this paper, motivated by the classical notion of a Strebel quadratic differential on a compact Riemann surface without boundary, we introduce several more general classes of quadratic differentials (called non-chaotic, gradient, and positive gradient) which possess certain properties of Strebel differentials and often appear in applications. We discuss the relation between gradient differentials and special signed measures supported on their set of critical trajectories. We provide a characterization of gradient differentials for which there exists a positive measure in the latter class.

1 Introduction

The theory of quadratic differentials was pioneered in the late 1930's by O. Teichmüller as a useful tool to study conformal and quasi-conformal maps. Since then it has been substantially extended and found numerous applications. (For a general survey on quadratic differentials, consult [11, 24, 26].) One important class of quadratic differentials with especially nice properties was introduced by J. A. Jenkins and K. Strebel in the 1950's; these differentials are called **Strebel** or **Jenkins–Strebel**, see [5, 11, 26] and §3 below.

In several areas where quadratic differentials naturally appear—such as potential theory, asymptotics of orthogonal polynomials, WKB-methods in spectral theory of Schrödinger equations in the complex domain etc.—one often encounters quadratic differentials which are not Strebel, but share some of their properties; see, e.g., [1, 4, 15, 14, 21, 23] and references therein. A large class of examples of such non-Strebel differentials that have many interesting properties is provided by the polynomial quadratic differentials on the complex plane, with critical trajectories connecting pairs of their zeros.

Motivated by the above examples we present below several natural classes of quadratic differentials containing the class of Strebel differentials and possessing certain nice properties. The most general class we introduce is characterized by the property that the closure of any horizontal trajectory of such a differential is

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nowhere dense (we refer to such quadratic differentials as **non-chaotic**). Further, we introduce a natural subclass of non-chaotic differentials which we call **gradient**. They are characterized by the existence of a continuous function that is equal (up to an additive constant) to $\pm\Im\sqrt{\Psi}$, the imaginary part of (a branch of) the square root of the differential at its points of smoothness. Finally, we discuss a relevant notion of positivity for such gradient differentials.

The main result of the present paper is Theorem 3, which gives necessary and sufficient conditions for positivity of a gradient differential in terms of its so-called Reeb graph and the fat graphs associated to its edges; see §5.

The structure of the paper is as follows. In §2 we recall basic facts about quadratic and Strebel differentials. In §3 we introduce and discuss non-chaotic differentials and describe some of their properties. In §4 we define and characterize gradient differentials. In §5 we study positive gradient differentials. Finally, in Appendix I we briefly recall our earlier motivating results which relate some relevant classes of quadratic differentials to the Heine–Stieltjes theory; see [23].

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2 Crash course on quadratic differentials

2.1 Basic notions. Recall the following key definitions from [11] and [26].

Definition 1. A (meromorphic) quadratic differential Ψ on a compact orientable Riemann surface Y without boundary is a (meromorphic) section of the tensor square $(T^*_{\mathbb{C}}Y)^{\otimes 2}$ of the holomorphic cotangent bundle $T^*_{\mathbb{C}}Y$. The zeros and the poles of Ψ form the set of **critical points** of Ψ denoted by Cr_Ψ. Non-critical points of Ψ are called **regular**. Zeros and simple poles are called **finite critical points** while poles of order at least 2 are called **infinite critical points**. The set of finite critical points of Ψ will be denoted by Cr^f_Ψ.

The next statement can be found in, e.g., Lemma 3.2 of [11].

Lemma 1. *The Euler characteristic of* $(T^*_{\mathbb{C}}Y)^{\otimes 2}$ *equals* $-2\chi(Y)$, *where* $\chi(Y)$ *is the topological Euler characteristic of the underlying Riemannian surface* Y. *Therefore, the difference between the number of poles and zeros (counted with* *multiplicity) of a meromorphic differential* Ψ *on* Y *of order* k *equals* $2\chi(Y)$. *In particular, the number of poles minus the number of zeros of any quadratic differential* Ψ *on* $\mathbb{CP}^1$ *equals* 4. *Such examples can be found in, e.g., [4, Chap. 3].*

Obviously, if Ψ is locally represented in two overlapping charts by $f(z)dz^2$ and by $\tilde{f}(\tilde{z})d\tilde{z}^2$ resp. with a transition function $\tilde{z}(z)$, then $f(z) = \tilde{f}(\tilde{z})\,(d\tilde{z}/dz)^2$. Any quadratic differential induces a metric on its Riemann surface Y punctured at the poles of Ψ, whose length element in local coordinates is given by

$$|dw| = |f(z)|^{\frac{1}{2}}|dz|.$$

The metric $|dw| = |f(z)|^{\frac{1}{2}}|dz|$ on Y canonically associated to Ψ is closely related to two distinguished line fields spanned by the vectors $\xi \in T_zY$ such that $f(z)\xi^2$ is either positive or negative. The integral curves of the field given by $f(z)\xi^2 > 0$ are called **horizontal trajectories** of Ψ, while the integral curves of the other field given by $f(z)\xi^2 < 0$ are called **vertical trajectories** of Ψ. Trajectories of Ψ can be naturally parameterized by their arclength. In fact, in a neighborhood of a regular point z_0 on Y, one can introduce a local coordinate w called **canonical** (despite being defined up to a sign) which is given by

$$w(z) := \int_{z_0}^{z} \sqrt{f(\xi)}d\xi.$$

Obviously, in this coordinate the quadratic differential itself is given by $dw^2 = f(z)dz^2$ implying that horizontal trajectories on Y correspond to horizontal straight lines in the w-plane.

Since we will only consider horizontal trajectories of quadratic differentials, we will refer to them simply as **trajectories**.

Definition 2. A trajectory of a meromorphic quadratic differential Ψ is called **critical**, if its closure is one-dimensional and a **finite** critical point of Ψ is an end of that trajectory, i.e., it belongs to the closure of the trajectory. For a given meromorphic differential Ψ, denote by $K_\Psi \subset Y$ the closure of the union of critical trajectories of Ψ.

By Jenkins' Basic Structure Theorem [11, Theorem 3.5, pp. 38–39], up to a few exceptions,[1] the set $Y \setminus (K_\Psi \cup Cr_\Psi)$ consists of a finite number of the so-called circle, ring, strip and end domains. (For the detailed definitions and information we refer to loc. cit.) The names **circle**, **ring** and **strip** domain are reflecting the shapes of their images under the analytic continuation of the mapping given by the canonical coordinate; an **end** domain (also referred to as **half-plane** domain) is mapped by the canonical coordinate onto the half-plane.

[1] Such as elliptic curves with holomorphic differentials, or the genus 0 surfaces, with quadratic differential having two poles of order two each.

The interior of K_Ψ can be non-empty, and can consist of finitely many components, each bounded by a (finite) union of critical trajectories. These components are referred to as the **density** domains.

The decomposition of $Y \setminus (K_\Psi \cup Cr_\Psi)$ into circle, ring, strip, end and density domains constitutes the so-called **domain configuration** of Ψ.

To provide more details about the domain configuration, let us give here the descriptions of circle and strip domains since they will be most important in the present text. For definitions of other domains, we refer to [11, Chap. 3].

We recall that a circle domain of Ψ is a simply connected domain $D \subset Y$ with the following properties:

(1) D contains exactly one critical point z_0 of Ψ, which is a second-order pole,
(2) the complement $D \setminus z_0$ is foliated by trajectories of $\Psi(z)$ each of which is a closed Jordan curve separating z_0 from the boundary ∂D,
(3) ∂D contains at least one finite critical point.

Similarly, a strip domain of Ψ is a simply connected domain D with the following properties:

(1) D contains no critical points of Ψ,
(2) ∂D contains exactly two boundary points z_1 and z_2 belonging to the set of infinite critical points of Ψ (these boundary points may be situated at the same point of the Riemann surface Y),
(3) the points z_1 and z_2 divide ∂D into two boundary arcs each of which contains at least one finite critical point,
(4) D is swept out by trajectories of Ψ each of which is a Jordan arc connecting points z_1 and z_2.

Remark 1. It is known that quadratic differentials on $\mathbb{CP}^1$ with at most three distinct poles do not have density domains; see Theorem 3.6 (three pole theorem) of [11]. In particular, Example 1 fits this case having just one pole: there the domain configuration consists only of strip and end domains; see, e.g., [1]. But starting with 4 distinct poles in $\mathbb{CP}^1$, the density domains become generic.

2.2 Strebel differentials.

Definition 3. A compact non-critical trajectory γ of a meromorphic Ψ is called **closed**. It is necessarily diffeomorphic to a circle.

Definition 4. A quadratic differential Ψ on a compact Riemann surface Y without boundary is called **Strebel** if the complement to the union of its closed trajectories has vanishing area (in the standard Lebesgue measure on Y).

Remark 1. In the nomenclature of Definition 2, the complement $Y \setminus (K_\Psi \cup Cr_\Psi)$ for an arbitrary Strebel differential Ψ on Y consists of (finitely many) circular and ring domains, as can be easily deduced from the results of [26, Chap. 3].

Results of §23 of [26], in particular, imply the following.

Lemma 2. *If a meromorphic quadratic differential Ψ is Strebel, then it has no poles of order greater than 2. If it has a pole of order 2, then the residue of $\sqrt{\Psi}$ at this pole is purely imaginary.*

These reasonings are summarized in the next statement. (By a **cylinder** we mean an open Riemann surface conformally equivalent either to an annulus $0 < r < |z| < R < +\infty$ or to a punctured disk $0 < |z| < R < +\infty$.)

Lemma 3. *For any Strebel differential Ψ on Y, the following hold.*

(i) *K_Ψ is the set of all non-closed horizontal trajectories of Y and $Y \setminus (K_\Psi \cup Cr_\Psi)$ is a disjoint union of finitely many cylinders.*

(ii) *The metric $|\Psi|$ restricted to any of these cylinders gives the standard metric of a cylinder with some perimeter p given by the length of the horizontal trajectories and some length l given by the length of the vertical trajectories joining the bases of the cylinder. (Notice that l can be infinite.) Moreover, each such cylinder is conformally equivalent to the annulus $e^{-l/p} < |z| < 1$, or to a punctured disc if $l = \infty$.*

Strebel differentials play an important role in the theory of univalent functions and the moduli spaces of algebraic curves. They enjoy a large number of extremal properties. Basic results on their existence and uniqueness can be found in Chap. VI of [26]; see especially Theorem 21.1.

3 Non-chaotic quadratic differentials

Definition 5. Given a meromorphic quadratic differential Ψ on a compact Riemann surface Y, we say that Ψ is **non-chaotic** if its domain configuration contains no density domains, i.e., the interior of K_Ψ is empty.

Example 1. By the "Three Pole Theorem" of Jenkins [11], any polynomial quadratic differential $\Psi = P(z)dz^2$ is non-chaotic on $\mathbb{C}P^1$. (Here $P(z)$ is a univariate polynomial.)

We now give an alternative characterization of non-chaotic differentials.

Definition 6. Given a meromorphic quadratic differential Ψ on a compact Riemann surface Y, we say that Ψ **possesses a level function** if there exists a continuous and piecewise smooth function

$$F : Y \setminus Cr_\Psi \to \mathbb{R}$$

(called the **level function** of Ψ) which is defined on the complement to the set Cr_Ψ and has the properties:

(i) F is non-constant on any open subset of $Y \setminus Cr_\Psi$,

(ii) yet F is constant on each horizontal trajectory of Ψ.

It is almost immediate that Strebel differentials possess level functions: the distance (in the Riemannian metric induced by Ψ) to $K_\Psi \cup Cr_\Psi$ serves as the level function.

Theorem 1. *A quadratic differential Ψ is non-chaotic if and only if Ψ possesses a level function.*

Proof. Assume non-chaoticity. In this case $K_\Psi \cup Cr_\Psi$ is a union of a finite number of critical trajectories and critical points, and its complement is a union of domains comprised either of compact trajectories (ring and circle domains) or of trajectories isometric to a real line (strip and end domains).

On each of these domains we can construct a function that is continuous, constant on the trajectories, but not on any open set, and which is vanishing on the boundary of the domain: on circle and end domains, one can just take the imaginary part of the canonical coordinate, on a ring or strip domain the sine of a properly rescaled imaginary part of the canonical coordinate.

Gluing together these functions (each originally defined on individual domains, but vanishing on the boundary) along K_Ψ delivers the desired continuous level function.

If Ψ is chaotic, there exists a trajectory with closure having a non-empty interior. A level function should be constant on this trajectory and continuous, hence constant on an open set, a contradiction. □

As we mentioned, any Strebel differential possesses a level function. Moreover, the following holds.

Proposition 1. *A quadratic differential Ψ is Strebel if and only if it has a level function F with finite limits at each critical point $p \in Cr_\Psi$, i.e., a level function F that can be extended by continuity from $Y \setminus Cr_\Psi$ to Y.*

Proof. As we mentioned above, a non-chaotic differential is Strebel if and only if its domain decomposition consists only of ring and circle domains. It is clear that in this case the construction of Lemma 1 yields a function continuous on all of Y. Conversely, the existence of an end or a strip domain implies that there is a one-parametric family of non-critical trajectories converging (on one end of the strip) to a critical point $C_o \in Cr_\Psi$. The union of these trajectories forms a sub-strip in a strip or an end domain. A non-constant function on such a sub-strip which is constant along the trajectories will automatically have discontinuity at C_o. □

To move further, let us recall some basic facts from complex analysis and potential theory on Riemann surfaces; see, e.g., [7].

Let Y be an (open or closed) Riemann surface and h be a real- or complex-valued smooth function on Y.

Definition 7. The **Levy form** of h (with respect to a local coordinate z) is given by

$$\mu_h := 2i\frac{\partial^2 h}{\partial z \partial \bar{z}}\, dz \wedge d\bar{z}. \tag{3.1}$$

In terms of the real and imaginary parts (x, y) of z, μ_h is given by

$$\mu_h = \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) dx \wedge dy = \Delta h dx \wedge dy.$$

If h is a smooth real-valued function, μ_h can be also thought of as a real signed measure on Y with a smooth density. In potential theory h is usually referred to as the (**logarithmic**) **potential** of the measure μ_h; see, e.g., [7, Chap. 3]. Notice that (3.1) makes sense for an arbitrary complex-valued distribution h on Y if one interprets μ_h as a 2-current on Y, i.e., a linear functional on the space of smooth compactly supported functions on Y; see, e.g., [3].

Such a current is necessarily exact since the inclusion of smooth forms into currents induces the (co)homology isomorphism. Recall that any complex-valued measure on Y is a 2-current characterized by the additional requirement that its value on a smooth compactly supported function depends only on the values of this function, i.e., on its 0-jet (and does not depend on its derivatives, i.e., on higher jets). Notice that if Y is compact and connected, then exactness of μ_h is equivalent to the vanishing of the integral of μ_h over Y.

We should remark here that the Levy form depends on the (local) metric structure defined by the (local) coordinate z, unless it is a sum of the delta-functions.

Example 2. (a) If $h = \ln|z|$ on $\mathbb{C}\mathrm{P}^1$, then $\mu_h = 2\pi(\delta(0) - \delta(\infty))$ is the signed measure supported on the 2-point set $\{0, \infty\}$. (Here $\delta(a)$ is the Dirac measure supported at a.)

(b) If $h = |\Im z|$ on $\mathbb{C}$, then μ_h is a measure supported on the real axis, namely, $\mu_h = 2\delta(y)dx \wedge dy$, i.e. twice the usual Lebesgue measure on the real line.

The easiest way to verify these examples is to use Green's formula:

$$\mu_h(D) = \int_D 2i\,\partial\bar{\partial}h = \int_{\partial D} \frac{\partial h}{\partial n} dl,$$

which provides a way to calculate $\mu_h(D)$, where D is an arbitrary compact domain in Y with a smooth boundary, $\frac{\partial h}{\partial n}$ is the derivative of h w.r.t. the outer normal, and dl is the length element of the boundary. Now

$$\int_D f\Delta h = -\int_D (df, dh) + \int_{\partial D} f\frac{\partial h}{\partial n} dl.$$

Here D is a domain bounded by a piecewise smooth loop ∂D oriented counterclockwise, $\partial h/\partial n$ is the derivative in the orthogonal direction to ∂D, while dl is the length element on ∂D.

Our next goal is, for a given non-chaotic Ψ, to find its level function F which is closely related to the metrics on Y induced by Ψ or, alternatively, whose Levi form μ_F has a small support. Such F is readily available by the following statement.

Proposition 2. *For any non-chaotic differential Ψ on Y, there exist its level functions which are piecewise harmonic and which are non-smooth on finitely many trajectories of Ψ.*

Proof. Take as F the distance to $K_\Psi \cup Cr_\Psi$ in the metric defined by Ψ. Locally, its differential coincides with $\Im\sqrt{\Psi}$, hence is harmonic. It is immediate that F is smooth outside $K_\Psi \cup Cr_\Psi$ and the (finite) union of the horizontal trajectories running in the middle of the strip and annular domains. □

Remark 2. We will call level functions constructed in Proposition 2 **piecewise harmonic**. For any piecewise harmonic level function F, its Levy form μ_F is a well-defined 2-current supported on the union of Cr_Ψ and finitely many trajectories where F is non-smooth. For example, in case of a Strebel differential Ψ, the Levy form of any piecewise harmonic level function will have point masses exactly at the double poles of Ψ.

As the simplest example of this situation consider

$$\Psi(z) = -\frac{dz^2}{z^2}$$

in which case as a (piecewise) harmonic level function one can choose $F(z) = \ln|z|$ implying that

$$\mu_F = \delta(0) - \delta(\infty),$$

where $\delta(a)$ stands for the unit point mass located at a.

4 Gradient differentials

Denote by $\bar{K}_\Psi$ the union of the critical trajectories and all finite critical points of Ψ,

$$\bar{K}_\Psi = K_\Psi \cup Cr^{\mathrm{f}}_\Psi.$$

This is a one-dimensional cell complex embedded into $Y \setminus Cr^{-}_\Psi$ (where Cr^{-}_Ψ is the union of all infinite critical points) and equipped with a metric given by $|\Re\sqrt{\Psi}|$ that turns $\bar{K}_\Psi$ into a complete metric space such that topologically open ends of the graph have an infinite length.

Consider the decomposition

$$\bar{K}_\Psi = \amalg_{\alpha\in A}\bar{K}_\alpha$$

into the set of its connected components.

Each of these components also carries the structure of a fat graph, encoded by the collection of cyclic permutations of the edges incident to a given vertex, one for each vertex of the graph. (A detailed account on the relation between fat graphs and quadratic differentials can be found in, e.g., [17].)

These cyclic permutations can be thought of as a single permutation σ_0 of the set of flags of a graph, that is, of the pairs consisting of a vertex and its incident edge; the orbits of the permutation σ_0 are in one-to-one correspondence with the vertices of the fat graph.

The other permutation of the flags of the fat graph is the involution σ_1 interchanging two flags corresponding to the same edge.

The product $\sigma_0\sigma_1$ decomposes into cycles which correspond to **boundary components** of the fat graph, in turn corresponding to the trajectories bounding the connected components of the complement $Y \setminus K_\Psi$ in the order fixed by the orientation.

We define the **Reeb graph** Rg_Ψ of a non-chaotic quadratic differential Ψ as follows.

Definition 1. The **Reeb graph** Rg_Ψ is the metric graph with possibly edges of infinite length necessarily ending at leaves. The vertices $V_\Psi = A$ of the Reeb graph are identified with the set of connected components of $\bar{K}_\Psi$. The edges E_Ψ are the spaces of non-critical trajectories (or, equivalently, the factor spaces of the connected components of $Y \setminus K_\Psi$ by the equivalence relations given by belonging to the same trajectory). The length element on the edges of the Reeb graph is given by $|\Im\sqrt{\Psi}|$.

In other words, the points in the interior of the edges of the Reeb graph correspond to non-critical trajectories of Ψ. That the resulting space is Hausdorff is an immediate corollary of the absence of the density domains.

The Reeb graph Rg_Ψ might have loops and multiple edges.

We remark that on each of the connected components of $Y\setminus K_\psi$, the **square root** of the quadratic differential is the meromorphic 1-form $\sqrt{\Psi}$ defined unambiguously, up to a sign.

The lengths of the edges are finite for the strip or ring components, and infinite for the circle or end components. The components corresponding to edges of infinite length are necessarily adjacent to the poles of order at least 2, i.e., to the infinite critical points.

We will call a level function F **natural** if on any of the connected components of $Y \setminus (K_\Psi \cup Cr_{\overline{\Psi}})$, its gradient matches the imaginary part of a branch of $\sqrt{\Psi}$:

$$dF = \pm\Im\sqrt{\Psi}.$$

A natural level function fixes the orientation on each of the edges of the Reeb graph Rg_Ψ. Together with the length elements on the edges, these orientations define a family of 1-forms on the edges, and hence a de Rham cocycle on the Reeb graph Rg_Ψ.

Theorem 2. *A non-chaotic differential Ψ admits a natural level function if and only if the edges of Rg_Ψ can be oriented in such a way that the resulting 1-cocycle on Rg_Ψ is trivial. In other words, the edges of Rg_Ψ can be oriented in such a way that the sum of the lengths of the edges in any oriented cycle in the Reeb graph, taken with the signs $\pm$ depending on whether the orientation of the cycle is consistent with the orientations of the edges or not, vanishes.*

Conversely, any such orientation defines a natural level function up to an additive constant.

Definition 8. Any non-chaotic quadratic differential satisfying the conditions of Theorem 2 will be called a **gradient**, and any of the corresponding level functions F will be called a **potential**.

Any potential of a gradient quadratic differential is constant on the components of $\bar{K}_\Psi$.

Proof of Theorem 2. The claim that a potential defines an orientation on the edges of the Reeb graph is immediate from the definition, as is the exactness of that cocycle. Conversely, the exactness of the cocycle on the Reeb graph defined by the length elements and orientations on the edges allows one to integrate it to a function on the Reeb graph, which lifts to a potential. $\square$

Lemma 4. *Levy form μ_F of any potential F of a gradient quadratic differential Ψ is supported on $K_\Psi \cup Cr_\Psi$.*

Proof. Certainly, the restriction of the potential to each of the domains in $Y \setminus (K_\Psi \cup Cr_\Psi)$ is harmonic. $\square$

Potential functions may fail to exist (for example, if the Reeb graph has a loop). But the number of classes of potential functions (identified if their difference is a constant) is obviously finite, as we can identify the potential functions with an element of the finite set of orientations of the edges of the Reeb graph. In fact, the following generalization of Theorem 4 of [23] holds:

Proposition 3. *For a gradient differential* Ψ*, the number of different potentials (considered up to an additive constant) is either* 0 *or a power of 2.*

Proof. The group `Flips` $= \mathbb{Z}_2^{E_\Psi}$ of flipping the orientations of the edges acts on the space of cochains on the Reeb graph by reflections. The collections of flips that preserve the subspace annihilating the cycles in the Reeb graph is, clearly, a subgroup in `Flips`. $\square$

Existence of a potential function imposes further restrictions on the local properties of the quadratic differential Ψ.

Let F be a potential for Ψ. We will refer to a pole of Ψ as ***F*-clean** if some punctured neighborhood of the pole does not intersect the support of the Levy measure μ_F. Equivalently, in a punctured vicinity of the pole, the potential F is harmonic and therefore smooth.

Of course, the Levy measure can be non-vanishing at a clean pole: for example, $-dz^2/z^2$ has $\log|z|$ as its potential function, with the corresponding Levy measure equal to $\pm\delta$ at 0 and ∞: both poles are clean.

Lemma 5. *The order* $r \geq 2$ *of any* F*-clean pole is even.*

Proof. Indeed, the $\mathbb{Z}_2$-bundle of orientations defined by $\pm dF$ does not admit a section in a (punctured) vicinity of a pole of Ψ of odd order. Hence dF cannot be continuous in an arbitrarily small neighborhood of such a pole. $\square$

The F-clean poles of even order exist. For a pole of even order, one can also define $\sqrt{\Psi}$ in its punctured neighborhood.

Proposition 4. *Let* F *be a potential of a non-chaotic quadratic differential* Ψ*. Then for any* F*-clean pole* z_* *of* Ψ *of even order, the residue of the* $\sqrt{\Psi}$ *at* z_* *(defined up to a sign) is purely imaginary.*

For example, the residues of $\Psi = -dz^2/z^2$ at $0, \infty$ are $\pm i$.

Proof. The statement follows immediately from the fact that for a clean pole, F is smooth in a punctured vicinity of z_*, and therefore the increment of the potential F equals the residue of $\sqrt{\Psi}$. $\square$

Lemma 6. *A gradient differential Ψ on a compact Y is uniquely defined by the Levy form μ_F of any of its potentials F.*

Proof. Two functions F_1 and F_2 (considered as 0-currents) have the same Levy forms (considered as 2-currents) only if the difference $F_1 - F_2$ has vanishing Laplacian, and hence, by compactness of Y, is a constant.

Now, if two gradient quadratic differentials Ψ_1, Ψ_2 have corresponding potentials F_1, F_2 coinciding (up to a constant) on an open subset of Y, then the (locally defined) holomorphic 1-forms $\sqrt{\Psi}_1$, $\sqrt{\Psi}_2$ have identical real parts (equal to dF_1, dF_2, respectively) on the same subset, and hence coincide everywhere. $\square$

5 Levy measures and Positivity

In this section we discuss the notion of positivity for gradient quadratic differentials. Observe that for any potential function F, its Levy form μ_F is an exact 2-current on Y, i.e., $\int_Y \mu_F = 0$. Many applications in asymptotic analysis lead to the situation when a gradient differential has a potential F whose Levy form is a signed measure whose positive part is supported on K_Ψ, and whose negative part is supported on (F-clean) poles of Ψ. (We discuss such an example in §Appendix I:.)

Definition 9. We will call a potential F **clean** if all of the poles of Ψ are clean with respect to F and **positive** if it is clean and the restriction of μ_F to K_Ψ is a positive measure. A quadratic potential admitting a positive potential will also be referred to as positive.

Equivalently, the potential F is clean if the support of its Levy measure μ_F intersects $\bar{K}_\Psi$ over a compact subset.

We remark that the notion of positivity depends only on the potential function F but not on the particular coordinate chart.

Whether or not a potential of a gradient quadratic differential Ψ is positive, depends not only on the structure of the Reeb graph of Ψ (equipped with the lengths and the widths of the edges), but also on the structure of fat graphs of the components K_α corresponding to the vertices of the Reeb graph.

Specifically, each edge e^α_β of the fat graph K_α is adjacent to either one or two boundary components (corresponding to the orbits of the two flags incident to the edge under the action of permutation $\sigma_0\sigma_1$ defining the fat graph structure). The boundary components of the fat graph K_α correspond to the edges of Rg_Ψ. We will be calling these edges of the Reeb graph **incident to the corresponding edge** e^α_β of the fat graph K_α.

Theorem 3. *The potential F of a gradient quadratic differential is positive if and only if for any edge e^α_β of any of the fat graphs K_α, the orientation of the Reeb graph defined by F has at least one of the (at most two) incident edges of the Reeb graph oriented away from K_α.*

Proof. As locally the potential is const + $\pm\Im\sqrt{\Psi}$, its Levy form vanishes at its smooth points.

At the points where F is not smooth (which can happen, by definition, only at the points of K_Ψ) it is locally given by const $\pm\ |\Im\sqrt{\Psi}|$. The non-smoothness of the potential results from discordant orientations the corresponding edges of the Reeb graph. A local computation shows that if the edges incident to e^α_β are both oriented away from K_α, the corresponding measure on the edge equals $2|\Re\sqrt{\Psi}|$; if one edge is oriented towards, and one away from, K_α, then $\mu_F = 0$ near the edge, and if incident edges are oriented towards K_α, then $\mu_F = -2|\Re\sqrt{\Psi}|$. □

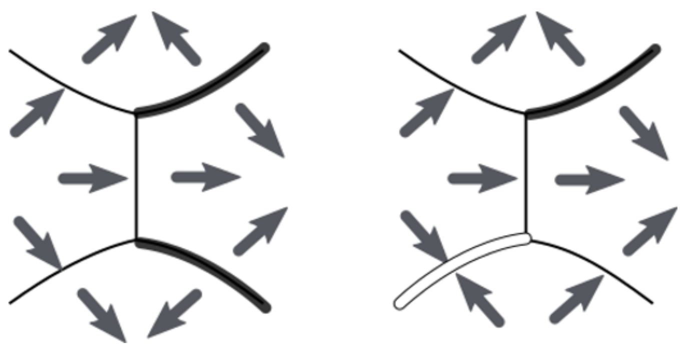

Figure 1. Illustration of how the orientation of the Reeb graph affects the positivity on a component of K_Ψ: on the left display, all edges of the fat graph component have the Reeb graph orientation pointing toward them, or "through" them; on the right display, the SW edge has both adjacent edges of the Reeb graph outgoing, resulting in locally negative mass. (Positive charges are shown as fat solid lines, negative charges as fat hollow line.)

We remark that Theorem 3 turns the computational question of the positivity of a given gradient quadratic differential Ψ into an instance of a 2-satisfiability problem [16]. Indeed, one can interpret the orientations of the edges of the Reeb graph as Boolean variables, and the absence of two outgoing edges of the Reeb graph incident to an edge of a fat graph K_α as a 2-clause. Such interpretation implies that given the fat graph structures, the positivity can be efficiently decided in time quadratic in the number of the critical points of Ψ.

Theorem 3 also allows one to construct a positive gradient quadratic differential—Strebel or not—by explicitly finding the orientations of the edges of the Reeb graph, and verifying the conditions.

Thus, in the example on Figure 2, one immediately verifies that the differential is positive.

The natural length element $|\Re\sqrt{\Psi}|$ on the edges of the fat graphs K_α's (i.e., on the critical trajectories of Ψ) defines the **widths** on the boundary components of the fat graphs K_α, or, equivalently, on the edges of the Reeb graph Rg_Ψ. (Recall that the **lengths** of the edges of the Reeb graph are defined by $|\Im\sqrt{\Psi}|$.) For the components containing poles of Ψ in their closure, the width can be infinite.

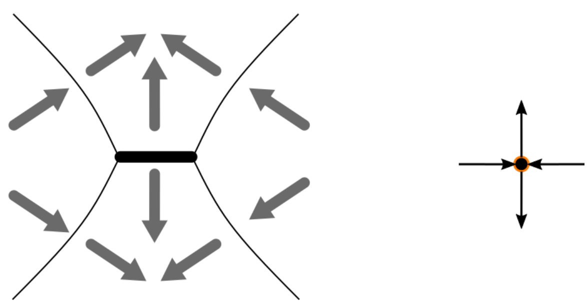

Figure 2. Gradient quadratic differential $(1-z^2)dz^2$ is positive: its set of critical trajectories is shown on the left, the support of the Levy measure being the fat edge (straddling between $z=\pm1$). The Reeb graph (and the corresponding orientations of the edges) are shown on the right. This critical differential is non-Strebel, as its (clean) pole at infinity is of order 6. The residue of $\sqrt{\Psi}$ at infinity is $1/2$.

Next result is immediate:

Lemma 7. *The total mass of μ_F supported by a component K_α equals the difference of the widths of all incoming and all outgoing components.*

Lemma 7 implies the following necessary condition of positivity:

Corollary 1. *If a potential F is positive, then for any component K_α, the total width of the incoming edges is greater than or equal to the total width of the outgoing edges.*

It is worth mentioning that the latter condition is not sufficient: just fixing the Reeb graph of a gradient quadratic differential, and widths and orientations of its edges is not enough to deduce the positivity of the corresponding potential. Indeed, isometrically shifting the glueing map along the boundaries of the domains of the

Jenkins decomposition preserves the structure of the Reeb graph, as well as all the lengths, orientations and the widths of its edges. However, such shifts can destroy positivity; see Figure 3.

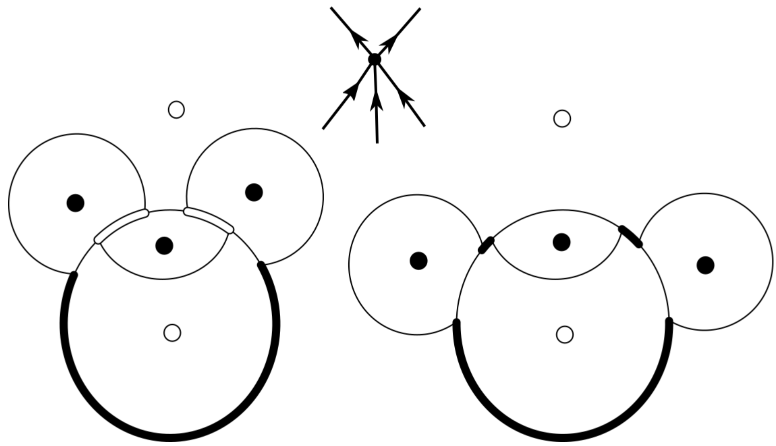

Figure 3. Singular sets and trajectories of clean quadratic differentials on $\mathbb{C}P^1$ with 5 poles of order 2, and 6 simple zeros each. The residues of $\sqrt{\Psi}$ at the poles (equal to the widths of the circle domains centered at these poles) are the same on the left and on the right displays. Yet the potential on the left has negative components of μ_F; the one on the right is positive. The Reeb graph is sketched on the top. (We keep the convention that positive masses are shown as solid lines or dots, negative, as hollow ones.)

Another constraint on the orientation of the edges of the Reeb graph required by the positivity of a potential comes from the simple poles of Ψ; see Figure 4. As the edge of the fat graph adjacent to a simple pole has the same domain on both sides, the positivity implies that the orientation of the edge of the Reeb graph should be incoming; compare Proposition 2, [23].

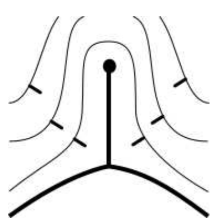

Figure 4. Positivity forces the orientation of the potential function near a simple pole of Ψ.

Appendix I: Quadratic differentials and Heine–Steiltjes theory

We have earlier encountered Strebel and gradient differentials in the study of the asymptotic properties of Van Vleck and Heine–Stieltjes polynomials and solutions of the Schrödinger equation with polynomial potential; see [9, 23, 22]. Some of these results are formulated below as a major motivation for the present study.

Given a pair of polynomials $P(z)$ and $Q(z)$ of degree m and at most $m-1$ respectively, consider the differential equation

$$P(z)S''(z) + Q(z)S'(z) + V(z)S(z) = 0. \tag{6.1}$$

The classical Heine–Stieltjes problem for equation (6.1) asks for any positive integer n, to find the set of all possible polynomials $V(z)$ of degree at most $m-2$ such that (6.1) has a polynomial solution $S(z)$ of degree n; see [8], [25]. Already E. Heine proved that for a generic equation (6.1) and any positive n, there exist $\binom{n+l-2}{l-2}$ polynomials $V(z)$ of degree $l-2$ having the corresponding polynomial solution $S(z)$ of degree n. Such polynomials $V(z)$ and $S(z)$ are referred to as **Van Vleck** and **Heine–Stieltjes** polynomials respectively. The following localization result for the zero loci of $S(z)$ and $V(z)$ was proven in [21].

Proposition 5. *For any $\epsilon > 0$, there exists a positive integer N_ϵ such that all roots of $V(z)$ and its corresponding $S(z)$ lie within an ϵ-neighborhood of $Conv_P$ if* $\deg S(z) \geq N_\epsilon$. *Here $Conv_P$ stands for the convex hull of the zero locus of the leading coefficient $P(z)$.*

Notice that in Proposition 5, the polynomial $S(z)$ is the Stieltjes polynomial corresponding to the Van Vleck polynomial $V(z)$. The above localization result implies that there exist plenty of converging subsequences $\{\tilde{V}_n(z)\}$, where $V_n(z)$ is some Van Vleck polynomial for equation (6.1) whose Stieltjes polynomial $S_n(z)$ has degree n and $\tilde{V}_n(z)$ is the monic polynomial proportional to $V_n(z)$. (Convergence is understood coefficient-wise.)

Recall that the Cauchy transform $\mathcal{C}_\nu(z)$ and the logarithmic potential $u_\nu(z)$ of a (complex-valued) measure ν supported in $\mathbb{C}$ are by definition given by

$$\mathcal{C}_\nu(z) = \int_{\mathbb{C}} \frac{d\nu(\xi)}{z-\xi} \quad \text{and} \quad u_\nu(z) = \int_{\mathbb{C}} \log|z-\xi|\,d\nu(\xi).$$

Obviously, $\mathcal{C}_\nu(z)$ is analytic outside the support of ν and has a number of important properties, e.g., that

$$\mathcal{C}_\nu(z) = \frac{\partial u_\nu(z)}{\partial z}, \qquad \nu = \frac{1}{\pi}\frac{\partial \mathcal{C}_\nu(z)}{\partial \bar{z}},$$

where the derivative is understood in the distributional sense. Detailed information about Cauchy transforms can be found in, e.g., [6].

Theorem 4. *Choose a sequence* $\{V_n(z)\}$ *of Van Vleck polynomials, where* $\deg S_n(z) = n$ *with converging sequence* $\{\tilde{V}_n(z)\} \to \tilde{V}(z)$*. Then the sequence of root-counting measures* μ_n *of* $S_n(z)$ *weakly converges to the probability measure* μ *whose Cauchy transform* $\mathcal{C}_\mu(z)$ *satisfies a.e. in* $\mathbb{C}$ *the algebraic equation*

$$\mathcal{C}_\mu^2(z) = \frac{\tilde{V}(z)}{P(z)}.$$

Moreover, the logarithmic potential $u_\mu(z)$ *of* μ *has the property that its set of level curves coincides with the set of closed trajectories of the quadratic differential* $-\frac{\tilde{V}(z)dz^2}{P(z)}$ *which is therefore Strebel.*

Theorem 4 implies further results for arbitrary rational Strebel differentials with a second-order pole at ∞. (These statements are special cases of the results in §5.)

Theorem 5 (see Theorem 4, [23])**.** *Let* $U_1(z)$ *and* $U_2(z)$ *be arbitrary monic complex polynomials with* $\deg U_2 - \deg U_1 = 2$*. Then:*

(1) *The rational quadratic differential* $\Psi = -U_1(z)dz^2/U_2(z)$ *on* $\mathbb{CP}^1$ *is Strebel if and only if there exists a real and compactly supported in* $\mathbb{C}$ *measure* μ *of total mass* 1 *(i.e.,* $\int_{\mathbb{C}} d\mu = 1$*) whose Cauchy transform* $\mathcal{C}_\mu$ *satisfies a.e. in* $\mathbb{C}$ *the equation*

$$\mathcal{C}_\mu^2(z) = U_1(z)/U_2(z). \tag{6.2}$$

(2) *For any* Ψ *as in* (1) *there exists exactly* 2^{d-1} *real measures whose Cauchy transforms satisfy* (6.2) *a.e. and whose support is contained in* K_ψ*, where* d *is the total number of connected components in* $\mathbb{CP}^1 \setminus K_\Psi$ *(including the unbounded component, i.e., the one containing* ∞*). These measures are in* $1-$ 1*-correspondence with* 2^{d-1} *possible choices of the branches of* $\sqrt{U_1(z)/U_2(z)}$ *in the union of* $(d-1)$ *bounded components of* $\mathbb{CP}^1 \setminus K_\Psi$*.*

Concerning measures positive in $\mathbb{CP}^1$ in the case when Ψ has a non-simple pole at infinity, we notice first that the Reeb graph is necessarily a tree, with infinite length edges corresponding to the edge domains (adjacent to $\infty \in \mathbb{CP}^1$) and with the leaves which correspond to the components of K_Ψ containing, necessarily, simple poles of Ψ. Given that, the following statement should be quite obvious.

Theorem 6 (see Theorem 5, [23])**.** *For any Strebel differential*

$$\Psi = -U_1(z)dz^2/U_2(z)$$

on $\mathbb{CP}^1$ *(in the notation of Theorem 5) there exists at most one positive measure satisfying* (6.2) *a.e. in* $\mathbb{C}$*. Its support necessarily belongs to* K_Ψ*, and, therefore, among* 2^{d-1} *real measures described in Theorem 5 at most one is positive.*

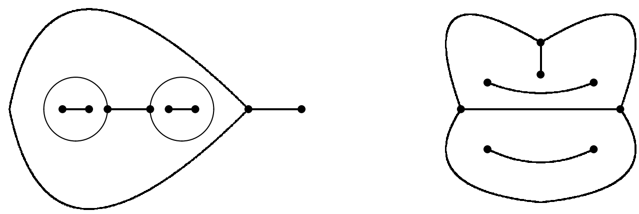

Figure 5. K_Ψ admitting and not admitting a positive measure.

Moreover, we can formulate an exact criterion of the existence of a positive measure in terms of rather simple topological properties of K_Ψ. To do this we need one more definition. Observe that in our situation K_Ψ is a planar multigraph, i.e., a planar graph with possibly multiple edges.

Definition 10. By a **simple cycle** in a planar multigraph K_Ψ, we mean any closed non-self-intersecting curve formed by the edges of K_Ψ. (Obviously, any simple cycle bounds an open domain homeomorphic to a disk which we call the **interior of the cycle**.)

Proposition 6 (see Proposition 2, [23])**.** *A Strebel differential*

$$\Psi = -U_1(z)dz^2/U_2(z)$$

admits a positive measure satisfying (6.2) *if and only if no edge of K_Ψ is attached to a simple cycle from inside. In other words, for any simple cycle in K_Ψ and any edge not in the cycle but adjacent to some vertex in the cycle this edge does not belong to its interior. The support of the positive measure coincides with the forest obtained from K_Ψ after the removal of all its simple cycles.*

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