ON ASYMPTOTIC GAUSS-LUCAS THEOREM

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Abstract. In this note we extend the Gauss-Lucas theorem on the zeros of the derivative of a univariate polynomial to the case of sequences of univariate polynomials whose almost all zeros lie in a given convex bounded domain in $\mathbb{C}$.

1. Introduction

The celebrated Gauss-Lucas theorem claims that for any univariate polynomial $P(z)$ with complex coefficients, all roots of $P'(z)$ belong to the convex hull of the roots of $P(z)$, see Theorem 6.1 of [5]. Many generalizations have been obtained over the years, see, e.g., [1, 2, 7] and references therein.

In the present note, motivated by problems in potential theory in $\mathbb{C}$, we extend the Gauss-Lucas theorem to sequences of polynomials of increasing degrees whose almost all zeros lie in a given convex bounded domain in $\mathbb{C}$. Namely, given a convex bounded domain $\Omega \subset \mathbb{C}$, let $\{p_n(z)\}_{n=0}^{\infty}$ be a sequence of univariate polynomials with the degrees $\deg p_n = m_n$ such that $\lim_{n \to \infty} m_n = +\infty$. Assume that $\lim_{n \to \infty} \frac{\#_n(\Omega)}{m_n} = 1$, where $\#_n(\Omega)$ is the number of zeros of $p_n$ lying in $\Omega$ (counted with multiplicities).

Problem 1. Following the above notation we now ask whether there exists $\lim_{n \to \infty} \frac{\#'_n(\Omega)}{m_n - 1}$, where $\#'_n(\Omega)$ denotes the number of zeros of $p_n'(z)$ lying in $\Omega$?

It turns out that the answer to Problem 1 formulated verbatim as above, is, in general, negative.

Example 1. Let $O$ be the open square $(-2, 2) \times (-4i, 0)$. If $T_n(z) := \cos(n \arccos z)$ is the $n$-th Chebyshev polynomial of the first kind, then the derivative of $(z-i)T_n(z)$ has all its zeros in the upper half plane. Therefore, if we replace $z$ by $z + ia_n$ for some sufficiently small $a_n$ (i.e., shift all zeros downward by $a_n$), then we obtain polynomials of degrees $n+1$ with $n$ zeros in $O$, but whose derivatives have no zeros.
in $O$. Choosing $a_n$ appropriately for each $n$, we get a sequence of polynomials with all but one zeros in $O$ whose derivatives have no zeros in $O$.

Strict convexity (e.g., as in the case of the open unit disk) will not be of much help either. Just replace above $z$ by $M_nz$ with some large $M_n$ and then make a vertical translation so that after all these operations the image of $[-1,1]$ becomes a tiny secant segment of the unit circle. (This example was suggested to the third author by Professor V. Totik.)

However with slightly weaker requirements Problem 1 has a positive answer.

**Theorem 1.** Given a polynomial sequence $\{p_n(z)\}$ as above and any $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{\#'_{n}(\Omega_{\epsilon})}{m_n - 1} = 1,$$

where $\#'_{n}(\Omega_{\epsilon})$ is the number of zeros of $p'_n(z)$ lying in $\Omega_{\epsilon}$, and $\Omega_{\epsilon}$ is the open $\epsilon$-neighborhood of $\Omega \subset \mathbb{C}$.

Two illustrations of Theorem 1 are given in Figure 2.

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2. **Proof**

We first prove Theorem 1 in the case when $\Omega$ is a disk. Fix $\epsilon > 0$. Let $D = \{|z| < 1\}$ be the open unit disk, $p_n(z) = \prod_{k=1}^{m_n}(z - a_k)$, $\lim m_n = \infty$. Let us factor $p_n$ as follows:

$$p_n = q_n r_n = \prod_{k=1}^{k_n}(z - a_k) \prod_{k=k_n+1}^{m_n}(z - b_k), \quad |b_k| > 1 + \epsilon, \quad |a_k| < 1 + \epsilon, \quad \text{with} \quad \lim_{n \to \infty} \frac{k_n}{m_n} = 1.$$

Denote by $\mathcal{F}_n := \{z : |z| < 1 + \epsilon : p'_n(z) = 0\}$. We want to show that

$$\lim_{n \to \infty} \frac{\mathcal{F}_n}{m_n - 1} = 1.$$
Let
\[ \hat{\mu}_n(z) := \frac{1}{m_n} \sum_{k=1}^n \frac{1}{z - \alpha_k} = \frac{1}{m_n} \frac{p_n'}{p_n}, \]
denote the Cauchy transform of the root-counting probability measure \( \mu_n \) of \( p_n \).

Note that
\[ \hat{\mu}_n(z) = \frac{1}{m_n} \left( \frac{q_n'}{q_n} + \frac{r_n'}{r_n} \right) = \frac{1}{m_n} \left( k_n \hat{\nu}_n + (m_n - k_n) \hat{\psi}_n \right), \]
where \( \nu_n \) and \( \psi_n \) are the root-counting measures of \( q_n \) and \( r_n \) respectively. All zeros of \( q_n' \) lie in the unit disk \( D \) by the Gauss-Lucas theorem. Also (1) implies that
\[ \frac{(m_n - k_n)}{m_n} ||\psi_n|| \to 0. \]

Formula (4) implies that for all \( p : 1 \leq p < 2 \), we have
\[ \left\| \frac{1}{m_n} \frac{r_n'}{r_n} \right\|_{L^p(dA)} \to 0 \]
on compact subsets of \( \mathbb{C} \). Here, \( dA = \frac{1}{\pi} dxdy \) denote the normalized area measure.

Equation (5) follows from a trivial observation. Let \( \mu \) be a Borel measure with a compact support. Then for any compact set \( K \subset \mathbb{C} \), and any \( p : 1 \leq p < 2 \), we have
\[ ||\hat{\mu}(z)||^p_{L^p(K,dA)} \leq C(p,K)||\mu||. \]

Indeed, \( \hat{\mu} = \int \frac{d\mu(\xi)}{\xi - z} \), hence
\[ \int_K ||\hat{\mu}||^p dA \leq ||\mu|| \int_{|\xi| < R} \frac{1}{|\xi|^p} dA \leq C||\mu||, \]
where \( R \) is chosen so that \( \forall \xi \in K \) the disk of radius \( R \) centered at \( \xi \) contains \( K \). The integral in (7) converges for all \( p < 2 \) and (6) follows, hence does (5).

Thus, we have from (5) the following corollary.

**Corollary 1.** For any fixed \( R > 1 + \epsilon \), and any \( p : 1 \leq p < 2 \), for almost all \( r : 1 + \epsilon < r < R \), we have
\[ \lim_{n \to \infty} \frac{1}{m_n} \int_{|z| = r} \frac{|r_n'|}{|r_n|^p} ds_r = 0, \]
where \( ds_r \) is the arclength measure on \( \{z : |z| = r\} \).

Thus, from (3), we now obtain

**Corollary 2.**
\[ \lim_{n \to \infty} \frac{1}{m_n} \int_{|z| = r} \frac{|p_n'|}{p_n} - \frac{|q_n'|}{q_n} ds_r = 0 \]
for almost all \( r : 1 + \epsilon < r < R \) and \( p : 1 \leq p < 2 \).
However

\[
\frac{1}{m_n} \left( \frac{p'_n}{p_n} - \frac{q'_n}{q_n} \right) = \frac{1}{m_n} \left( \sum_{k=k_n+1}^{m_n} \frac{1}{z-b_k} \right)
\]

by (3), and hence is analytic inside \( \{|z| < 1 + \epsilon\} \) since \( |b_k| > 1 + \epsilon \).

Therefore, from standard results on Hardy spaces \( H^p \) in the disk, cf. [3], we conclude that

\[
(9) \quad \frac{1}{m_n} \left( \frac{p'_n}{p_n} - \frac{q'_n}{q_n} \right) \to 0
\]

uniformly in the closed disk \( \overline{D} = \{|z| \leq 1 + \epsilon\} \).

Since \( \frac{1}{m_n} \frac{q'_n}{q_n} \) vanishes at \( k_n - 1 \) points in \( D \) by Gauss-Lucas theorem, invoking Hurwitz’s theorem, we conclude that

\[
(10) \quad \lim_{n \to \infty} \frac{1}{m_n} \left[ \#(z : \{z : |z| < 1 + \epsilon : p'_n(z) = 0\}) - (k_n - 1) \right] = 0.
\]

Since, by assumption, \( \lim_{n \to \infty} \frac{k_n - 1}{m_n} = \lim_{n \to \infty} \frac{k_n}{m_n} = 1 \), we arrive at

\[
\lim_{n \to \infty} \frac{3n'}{m_n} = \lim_{n \to \infty} \frac{3n'}{m_n - 1} = 1,
\]

which settles Theorem 1 in the case of a disk.

To finish the proof for the general case of an arbitrary convex domain \( \Omega \) observe that we only used some properties of a disk to get a convenient foliation of a neighborhood of the unit disk by concentric circles and when applying Gauss-Lucas theorem. Both these facts are readily available for an arbitrary bounded convex domain. Finally, the Hardy spaces of analytic functions in the disk are replaced by the Smirnov classes \( E^p \) of functions representable by Cauchy integrals with \( L^p \)-densities (with respect to arclength). In the domains with piecewise smooth boundaries, e.g., convex domains, the latter behave in the very same manner as Hardy spaces – cf. [3]. \( \square \)

REFERENCES