



On Limit Sets for Geodesics of Meromorphic Connections

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Abstract

Meromorphic connections on Riemann surfaces originate and are closely related to the classical theory of linear ordinary differential equations with meromorphic coefficients. Limiting behavior of geodesics of such connections has been studied by, e.g., Abate and Bianchi (*Math Z* 282:247–272, 2016) and Abate and Tovena (*J Differ Equ* 251(9):2612–2684, 2011) in relation with generalized Poincaré-Bendixson theorems. At present, it seems still to be unknown whether some of the theoretically possible asymptotic behaviors of such geodesics really exist. In order to fill the gap, we use the branched affine structure induced by a Fuchsian meromorphic connection to present several examples with geodesics having infinitely many self-intersections and quite peculiar ω -limit sets.

Keywords Meromorphic connections · k -differentials · Branched affine structures · Riemann surfaces

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1 Introduction

Meromorphic connections on complex analytic manifolds are both classical and modern area of mathematics interacting with a wide variety of topics including, e.g., linear ordinary and partial differential equations in several complex variables, theoretical and mathematical

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physics, differential geometry and tensor calculus, representation and singularity theories. (For detailed expositions of the general theory of meromorphic connections an interested reader might consult [6, 7]).

Motivated by the papers [1, 2, 8], in this short note we discuss one aspect of meromorphic connections on Riemann surfaces, namely the asymptotic behavior of their self-intersecting geodesics. The structure of the paper is as follows.

In Section 2, we recall some general notions and facts related to meromorphic connections with a special emphasis on Fuchsian connections on tangent bundles of Riemann surfaces since they induce branched affine structures which we shall use later. In Section 3, we introduce the notion of a hyperbolic cylinder and use it to exhibit branched affine structures on $\mathbb{C}P^1$ with geodesics accumulating on (possibly self-intersecting) limit cycles. In Section 4, we focus on branched affine structures defined by k -differentials. We provide several special examples of k -differentials on $\mathbb{C}P^1$ and on the torus. These differentials have geodesics with both infinitely many self-intersections and minimal dynamics. In spite of this, the presented geodesics are still not dense in the ambient surfaces.

2 Meromorphic Connections and Branched Affine Structures

2.1 Connections on Vector Bundles

Let $\pi : E \rightarrow X$ be a holomorphic vector bundle on a complex manifold X . A *meromorphic connection* on E is a \mathbb{C} -linear operator $\nabla : \mathcal{M}_E \rightarrow \mathcal{M}_E \otimes \mathcal{M}_X^1$, where \mathcal{M}_E is the sheaf of germs of meromorphic sections of E and \mathcal{M}_X^1 is the sheaf of germs of meromorphic 1-forms on X , satisfying the Leibniz rule

$$\nabla(fs) = s \otimes \partial(f) + f\nabla(s), \tag{2.1}$$

for all $s \in \mathcal{M}_E$ and $f \in \mathcal{M}_X$, the sheaf of meromorphic functions on X .

Let s be a meromorphic section of E , $x \in X$ and $F_x = \pi^{-1}(x)$. The $\nabla_x(s)$ is an F_x -valued one-form on $T_x X$ providing for every $\xi \in T_x X$ a vector $\nabla_x(s)(\xi) \in F_x$ considered at “the derivative of s at $s(x)$ in direction ξ ”.

Geometric Point of View The connection can be defined alternatively by a choice of a right inverse of $d_p\pi$ at every point $p \in E$, i.e., by the choice of lifting of vectors tangent to X to points of E . More exactly, let $p \in E$ and $x = \pi(p) \in X$. The inclusion $\iota : F_x = \pi^{-1}(x) \hookrightarrow E$ and the projection $\pi : E \rightarrow X$ canonically induce the exact sequence

$$0 \rightarrow T_p F_x \xrightarrow{d_p \iota} T_p E \xrightarrow{d_p \pi} T_x X \rightarrow 0, \tag{2.2}$$

however there is no canonical way to identify $T_p E$ with $T_x X \times T_p F_x$.

A choice of a right inverse $\sigma = \sigma(p) : T_x X \rightarrow T_p E$ to $d_p\pi$ extends to the exact sequence

$$0 \rightarrow T_x X \xrightarrow{\sigma} T_p E \xrightarrow{\tilde{\tau}} T_p F_x \rightarrow 0,$$

and defines an isomorphism $(d_p\pi, \tilde{\tau}) : T_p E \cong T_x X \times T_p F_x$. Combining $\tilde{\tau}$ with the canonical isomorphism $T_p F_x \cong F_x$ (here we use that F_x is a linear space) we get the map $\tau : T_p E \rightarrow F_x$ and the isomorphism

$$(d_p\pi, \tau) : T_p E \cong T_x X \times F_x.$$

The vectors in $\text{Im } d_p\iota$, $\text{Im } \sigma$ are called vertical and horizontal vectors, respectively.

Now, let s be a germ of a meromorphic section of E at point $p = (x, s(x)) \in E$ and $\gamma : (\mathbb{R}, 0) \rightarrow (X, x)$ be a germ of a curve at $x \in X$, with $\dot{\gamma}(0) = \xi \in T_x X$. Let $\Xi = \frac{d(s(\gamma))}{dt}(0) \in T_p E$ be the tangent vector at p tangent to the lifting $s(\gamma(t))$ of the curve $\gamma(t)$ to the section s , $d_p \pi(\Xi) = \xi$. We define

$$\nabla_x(s)(\xi) = \tau(\Xi).$$

If (as we always assume) the map $\sigma((x, v))(\xi) : F_x \times T_x X \rightarrow TE|_{F_x}$ depends \mathbb{C} -linearly on v, ξ and meromorphically on x then thus defined F_x -valued map ∇ satisfies the Leibnitz rule (2.1).

Expressing a Connection in Local Coordinates A local trivialization $U \times F_x, x \in U \subset X$, of the bundle E also induces an isomorphism $T_p E \cong T_x X \times F_x, p = (x, v) \in E$. Let $\{x_j\}$ be local coordinates on $U, \{e_k\}$ be a basis of F_x and s_k be the local sections $U \times \{e_k\}$. Then

$$\sigma((x, s_k))(\partial_{x_j}) = \partial_{x_j} - \sum_i \Gamma_{jk}^i(x) s_i \in T_p E, \tag{2.3}$$

where the *Christoffel symbols* $\Gamma_{jk}^i(x)$ are meromorphic functions on U . Thus

$$\nabla_x s_k(\partial_{x_j}) = \tau(\partial_{x_j}) = \partial_{x_j} - \sigma((x, s_k))(\partial_{x_j}) = \sum_i \Gamma_{jk}^i(x) s_i,$$

and therefore

$$\nabla_x s_k = \sum_j \left(\sum_i \Gamma_{jk}^i(x) s_i \right) dx_j. \tag{2.4}$$

Thus, for a section $s = \sum c_k(x) s_k$ we have

$$\nabla_x(s) = \sum_k c_k(x) \nabla_x s_k + \sum_k s_k \otimes dc_k(x). \tag{2.5}$$

Example 1 Setting $\Gamma_{jk}^i = 0$ in (2.4) one obtains the *standard connection* ∇_{st} on a local chart $U_i \times F_x$ and, in particular, on $T\mathbb{C}^n = \mathbb{C}^n \times \mathbb{C}^n$. The horizontal sections of ∇_{st} are the constant sections $U_i \times \{v\}$ or the constant vector fields, respectively.

We denote by $\Sigma = \Sigma(\nabla)$ the common *singularity locus* of all $\Gamma_{jk}^i(x)$'s. If $\Sigma = \emptyset$ then the connection is called *holomorphic*. Tautologically, any meromorphic connection is holomorphic on $X \setminus \Sigma$.

2.1.1 Horizontal Paths and Parallel Transport

A path $\tilde{\gamma}(t) = (\gamma(t), v(t)) \subset E$ is called *horizontal* if $\dot{\tilde{\gamma}}(t)$ is horizontal (i.e., $\nabla_{\gamma(t)}(\tilde{\gamma}(t))(\dot{\tilde{\gamma}}(t)) = 0$) for every t .

Let $\gamma : [0, 1] \rightarrow X \setminus \Sigma$ be a smooth path.

Lemma 1 *For any $v \in F_{\gamma(0)}$, there exists a horizontal lifting $\tilde{\gamma} : [0, 1] \rightarrow E$ of γ with $\tilde{\gamma}(0) = v$.*

Proof The induced vector bundle $\gamma^* E$ over $[0, 1]$ inherits the induced connection $\gamma^* \nabla$. The condition $\gamma^* \nabla(s) = 0$ for a section s to be horizontal becomes, in a chosen trivialization, a

system of linear ordinary differential equations. A solution of this equation with the initial condition $\tilde{\gamma}(0) = v$ is the required horizontal lifting of γ . \square

Horizontal liftings of γ provide a global trivialization $E|_{\gamma(t)} \simeq [0, 1] \times F_{\gamma(0)}$ of the restriction of E to the curve $\gamma(t)$, and ∇ becomes the standard connection in this trivialization. In particular, this trivialization defines an isomorphism $\Gamma(\gamma)_0^1 : F_{\gamma(0)} \rightarrow F_{\gamma(1)}$ called the “parallel transport of $F_{\gamma(0)}$ to $F_{\gamma(1)}$ along γ .”

2.1.2 Locally Flat Connections

A holomorphic connection ∇ is called *locally flat* if there exist local trivializations $U_i \times F$ of E such that ∇ becomes the standard connection in this chart. Equivalently, for a locally flat connection, there exists an $\mathcal{O}(U_i)$ -basis s_i of the set of holomorphic sections of E over U_i called the *basis of horizontal sections* such that $\nabla(s_i) = 0$.

More geometrically, local flatness means that the parallel transport of F_p to $F_{p'}$ along any curve γ joining p and p' depends only on the homotopy class of γ (with fixed endpoints). In particular, for a simply connected X , any locally flat holomorphic connection ∇ on E defines a global trivialization $E \simeq X \times F$, with ∇ being a standard connection in this trivialization.

Horizontal sections of a locally flat connection on E define a *local system* on X , i.e., a locally constant sheaf associating to every open simply connected set $U \subset X$ the set of all horizontal sections of ∇ over U . Vice versa, if for an atlas $\{U_i \times V\}$ of the vector bundle E , the transition functions $\psi_{ij} : U_i \cap U_j \rightarrow GL(V)$ are constant then the restriction of the constant sections from U_i and U_j to $U_i \cap U_j$ coincide. Equivalently, the restrictions of ∇_{s_i} from $U_i \times F$ and $U_j \times F$ to $U_i \cap U_j \times F$ coincide, thus they glue together to a locally flat connection on E .

For $\dim X > 1$, the equation $\nabla(s_i) = 0$ in local chart $U \times F$ becomes a system of partial differential equations on U which can have no solutions. The integrability condition for this system (i.e., the criterion of existence of a local basis of flat sections) is provided by the famous Frobenius theorem originally formulated in [5]. The Frobenius condition can be equivalently reformulated as the condition of vanishing of the curvature tensor of the connection ∇ .

For $\dim X = 1$, i.e., in the case of Riemann surfaces, the system of partial differential equations $\nabla(s_i) = 0$ for a horizontal section reduces to a system of *ordinary* differential equations. The latter is always solvable by the standard existence and uniqueness results for solutions of ordinary differential equations. In other words, for $\dim X = 1$, a local basis of horizontal sections always exists and a holomorphic connection on a vector bundle over a Riemann surface is always locally flat.

2.1.3 Tangent Bundle of a Riemann Surface

From now on we concentrate on the case of the tangent bundle TX of a Riemann surface X .

2.1.4 Geodesics

Let $\gamma(t) \subset X$ be (a germ of a) C^2 -smooth curve on X . A choice of a holomorphic connection ∇ on TX implies, by Lemma 1, the existence of a horizontal lifting $\tilde{\gamma}(t) \subset TX$ of $\gamma(t)$ starting at $p = (\gamma(0), \dot{\gamma}(0))$.

Also, independently of the choice of ∇ , there is the canonical lifting $j_1\gamma(t) = (\gamma(t), \dot{\gamma}(t)) \subset TX$ of $\gamma(t)$.

Definition 1 The curve $\gamma(t) \subset X$ is called *geodesic* if these two liftings coincide.

This means that the vector $\xi(t) = j_1^*\dot{\gamma}(t) \in T_{j_1\gamma(t)}TX$, is horizontal for all t , i.e., $\nabla_{\dot{\gamma}(t)}(\dot{\gamma}(t)) = 0$.

Choosing a local trivialization and using (2.5), we see that this condition becomes an ordinary differential equation of second order. Thus, by the existence and uniqueness theorems for solutions of ordinary differential equations, for any point $x \in X$ and any $\xi \in T_xX$, there exists a geodesic $\gamma(t)$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$.

Moreover, since the connection is holomorphic, the aforementioned ordinary differential equation of second order is real-analytic which implies that the geodesics are real-analytic as well.

Example 2 For a standard connection on $T\mathbb{C} \cong \mathbb{C} \times \mathbb{C}$, the geodesics are exactly the straight lines $\gamma(t) = at + b$, $a, b \in \mathbb{C}$. Indeed, by Example 1 the horizontal sections are just the constant sections, i.e., $\dot{\gamma}(t) \equiv a$ is constant.

Lemma 2 Given a Riemann surface X , the holomorphic connection on TX is uniquely defined by its geodesics.

Indeed, for $\xi \in T_xX$ define $\sigma(\xi) \in T_\xi TX$ as the vector tangent to the lifting $j_1\gamma$ of the geodesic satisfying the initial conditions $\gamma(0) = x$, $\dot{\gamma}(0) = \xi$. As $\dim T_xX = 1$, this completely defines the map σ and therefore the connection.

Remark 1 For $\dim X > 1$ the connection is uniquely defined by its geodesics if we assume that the connection is *torsion free*, i.e., $\Gamma^i_{jk} = \Gamma^i_{kj}$ for all i, j, k .

Affine Charts Any open and simply connected chart $\phi : U \rightarrow \mathbb{C}$, $U \subset X$, on a Riemann surface X canonically extends to a chart $\Phi = (\phi, d\phi) : TU \rightarrow T\mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ on the tangent bundle TX . By pulling back the standard connection ∇_{st} on $T\mathbb{C}$ one then defines a locally flat connection $\nabla = \Phi^*\nabla_{st}$ on TU . The pull-backs $(d\phi)^{-1}(a\partial_z)$ of the horizontal sections of ∇_{st} are the horizontal sections of $\Phi^*\nabla$ and the pull-backs $\phi^{-1}(at + b)$ of geodesics $at + b$ of ∇_{st} are the geodesics of ∇ .

We say that the chart ϕ is the *affine chart* of the connection ∇ .

Lemma 3 For any holomorphic connection ∇ on TX and any $x \in X$, there is an affine chart $\phi : U \rightarrow \mathbb{C}$ for ∇ defined in a neighborhood U of x .

Proof Let $\gamma(t)$ with $\gamma(0) = x$, $\gamma'(0) \neq 0$, be a germ of a geodesic of ∇ . By the implicit function theorem, there exists $\phi : (U, x) \rightarrow (\mathbb{C}, 0)$ such that $\phi(\gamma(t)) \equiv t$. Then ϕ is an affine chart for ∇ : the curve t is a geodesic for $\Phi_*\nabla$, so $\Phi_*\nabla = \nabla_{st}$ by Lemma 2 and Example 2. □

Lemma 4 Let $\phi_i : U_i \rightarrow \mathbb{C}$, $i = 1, 2$, be two affine charts on X . The transition map $\phi_{12} : \phi_2(U_1 \cap U_2) \rightarrow \phi_1(U_1 \cap U_2)$ has form $\phi_{12}(z) = C_1z + C_2$ on any connected component of $U_1 \cap U_2$.

Proof Indeed, ϕ maps geodesics of ∇_{st} to geodesics of ∇_{st} , i.e., $\phi_{12}(at + b) = a't + b'$ for some $a, b \in \mathbb{C}$. □

Lemma 5 *A germ $\sigma : (\mathbb{R}, 0) \rightarrow (X, p)$ of a real curve at $p = \sigma(0) \in X$ is a geodesic of a holomorphic connection ∇ on TX if and only if for some (and thus for any) affine chart ϕ of ∇ we have $\phi(\sigma(t)) = at + b$ where $a, b \in \mathbb{C}$.*

In local coordinates an affine chart looks as follows. Let ∇ be a holomorphic connection on TX and let $\tilde{\phi} : U \rightarrow \mathbb{C}$ be a local chart on X as above. Then $\tilde{\Phi} : TU \cong U \times \mathbb{C}$ is a local chart on TX and

$$(\tilde{\Phi}_* \nabla)(\partial_z) = \partial_z \otimes \eta$$

for some one-form η on $\phi(U)$. Let s be a horizontal section of ∇ . Then $\tilde{\Phi}(s) = f(z)\partial_z$ is a horizontal section of the connection $\tilde{\Phi}_* \nabla$ on $T\phi(U)$ and by (2.1)

$$0 = (\tilde{\Phi}_* \nabla)(f\partial_z) = \partial_z \otimes (df + f\eta) \quad \text{which implies} \quad f = C \exp\left(-\int \eta\right). \quad (2.6)$$

Finally, let $\psi : \phi(U) \rightarrow \mathbb{C}$ be the map such that $\frac{d\psi}{dz} = \frac{1}{f}$, i.e.,

$$\psi = C^{-1} \int \exp\left(\int \eta\right) dz + C_2, \quad (2.7)$$

and define $\phi := \psi \circ \tilde{\phi}$. Then the section $d(\phi)(s) = \partial_z$ is the horizontal section of $\Phi_* \nabla$. Thus $\Phi_* \nabla = \nabla_{st}$ and $\phi : U \rightarrow \mathbb{C}$ is an affine chart of ∇ .

2.2 Regular Connections

Let ∇ be a meromorphic connection on a vector bundle $E \rightarrow X$ over a Riemann surface X , and let $p \in \Sigma$ be a pole of ∇ .

The connection ∇ is called *regular at p* if in some chart $U \times V$ of E the horizontal sections s have *moderate growth at p* , i.e., for any germ $\gamma(t)$ of a real-analytic curve at p , there exists $N > 0$ such that $\|s(\gamma(t))\| = o(t^{-N})$ as $t \rightarrow 0$.

2.2.1 Local Theory

Here we consider only the tangent bundle TX . In this case the regularity property is equivalent to the condition that ∇ is *Fuchsian at p* , i.e., it has at most a simple pole at p .

Let p be a simple pole of ∇ and let $\alpha = \text{res}_p \eta$ be the residue of η at p . In a local chart $z : U_p \rightarrow (\mathbb{C}, 0)$ near p we have

$$\nabla(\partial_z) = \partial_z \otimes \eta, \quad \eta = \frac{\alpha}{z} g(z) dz,$$

where $g(z)$ is a holomorphic function such that $g(0) = 1$.

Lemma 6 *After a suitable biholomorphic change of variable z we can assume that $g \equiv 1$ and $\eta = \frac{\alpha}{z} dz$.*

Proof By integrating $\frac{\alpha}{z} g(z) dz = \frac{\alpha}{w} dw$ we get

$$\log w = \log z + \int g_1(z) dz,$$

where $g_1(z) = \frac{g(z)-1}{z}$ is holomorphic at 0. Thus $w = z \exp(\int g_1(z)dz) = z + \dots$ is the required change of variables. \square

Therefore by (2.6) the horizontal sections of ∇ in this chart are given by $Cz^{-\alpha}\partial_z$ and, by (2.7), the affine charts are

$$w = \phi(z) = C_1z^{-\alpha+1} + C_2 \quad \text{if } \alpha \neq 1, \quad \text{or} \tag{2.8}$$

$$w = \phi(z) = C_1 \log z + C_2 \quad \text{if } \alpha = 1. \tag{2.9}$$

For $\alpha \notin \mathbb{Z}$ and for $\alpha = -1$, these maps are branching at p which implies that there is no uni-valued affine chart in a neighborhood of p . However, one can always cover the punctured neighborhood D_o of P by open simply connected domains and define the affine charts for them. By Lemma 4, in the connected components of the intersections of these domains, the affine charts will be related by affine transformations.

For example, cover D_o by the union of two sectors $S_- = \{0 < |z| < \epsilon, |\arg z| < \pi\}$ and $S_+ = \{0 < |z| < \epsilon, 0 < \arg z < 2\pi\}$. The intersection $S_- \cap S_+ = U_+ \cup U_-$ consists of the union of two sectors, namely, $U_- = \{0 < |z| < \epsilon, \pi < \arg z < 2\pi\}$ and $U_+ = \{0 < |z| < \epsilon, 0 < \arg z < \pi\}$. Let ϕ_+, ϕ_- be uni-valued branches of the affine chart (2.8) in S_+, S_- respectively. If $\phi_- = \phi_+$ in U_+ then on U_- these two different holomorphic continuations of the same affine chart on U_- are related by the linear transition map $\phi_{+-}(w) = \phi_+ \circ \phi_-^{-1}(w)$, where

$$\phi_{+-}(w) = e^{2\pi i\alpha}w + C_2(1 - e^{2\pi i\alpha}), \quad \alpha \neq -1, \quad \text{or} \tag{2.10}$$

$$\phi_{+-}(w) = w + 2\pi iC_1, \quad \alpha = -1. \tag{2.11}$$

The multiplier $|e^{2\pi i\alpha}| = e^{-2\pi \text{Im}(\alpha)}$ is called the *dilation ratio* of ∇ at p .

2.2.2 Global Theory

Let X be a Riemann surface and ∇ be a holomorphic connection on TX . The parallel transport defines the monodromy representation $M : \pi_1(X, p_0) \rightarrow GL(T_{p_0}X) \cong \mathbb{C}^*$, the monodromy of ∇ along γ .

Let $\{U_i\}$ be a covering of X by affine charts ϕ_i , and assume that $U_i \cap U_j$ are connected. The transition maps

$$\phi_{ij} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j), \quad \phi_{ij}(z) = \phi_i \circ \phi_j^{-1}(z)$$

are affine by Lemma 4.

Lemma 7 *One can choose affine charts ψ_i such that all transition maps ψ_{ij} will be of the form*

$$\psi_{ij}(z) = a_{ij}z + b_{ij}, \quad a_{ij} \in M(\pi_1(X, p_0)) \subset \mathbb{C}^*.$$

Proof Indeed, for a simply connected X , the bundle TX can be globally trivialized by parallel transport, $TX \cong X \times \mathbb{C}$, with the section $s = X \times \{1\}$ being the horizontal section of ∇ . Let ϕ_i be an affine chart on X . The image $d\phi_i(s)$ is a constant vector field $a_i\partial_z$. Replacing ϕ_i by $a_i^{-1}\phi_i$, one can assume that $a_i = 1$ for all i . Then the transition maps are given by $\psi_{ij}(z) = z + b_{ij}$ since the differentials $d\psi_i, d\psi_j$ agree on every fiber $T_pX, p \in U_i \cap U_j$.

For a non-simply connected X , consider its universal cover $\tilde{X} \rightarrow X$. The lifts $U_{i,l}$ of U_i define a cover of \tilde{X} with the same transition maps ϕ_{ij} . Repeating the previous step for

the lifting of ∇ to the $T\tilde{X}$, choose new affine charts $\psi_{i,\iota}$ of $U_{i,\iota}$ and define the affine charts $\psi_i := \psi_{i,\iota} \circ (\pi_{U_i})^{-1}$ for some arbitrary ι . By definition, the maps $d\psi_{i,\iota} \circ d\psi_{i,\iota'}^{-1}$ present the monodromy of ∇ along some path and the claim follows. \square

Definition 1 An affine structure on a Riemann surface X branched at a discrete subset $\Sigma \subset X$ is a finite holomorphic atlas on a $X \setminus \Sigma$ whose charts $U_i \subset X$ have piecewise real-analytic boundary and transition maps are affine. We will call a Riemann surface with a branched affine surface a *branched affine surface*.

The affine structure is called *Fuchsian* if for every $p \in \Sigma$ and any affine chart $\phi : U \rightarrow \mathbb{C}$, $p \in \bar{U}$, there exists a (finite or infinite) limit $\lim_{U \ni z \rightarrow p} \phi(z)$.

Summing up, we obtain the following.

- Theorem 1** (i) *Affine charts of any Fuchsian connection ∇ on TX define a Fuchsian affine structure on X branched at the polar locus Σ .*
 (ii) *The multipliers of the affine transition maps belong to the monodromy group $M(\pi_1(X \setminus \Sigma, p_0)) \subset \mathbb{C}^*$.*
 (iii) *In particular, for $X = \mathbb{CP}^1$, the multipliers belong to the multiplicative subgroup of \mathbb{C}^* generated by $e^{2\pi i \alpha_j}$, where α_j are the residues of ∇ at its poles.*

The opposite holds as well.

Theorem 2 *An affine structure on a Riemann surface X branched at a discrete set $\Sigma \subset X$ defines a connection ∇ on $T(X \setminus \Sigma)$, with charts of the structure being the affine charts of ∇ .*

If the structure is Fuchsian, then the connection ∇ is Fuchsian as well.

Proof Using charts $\phi_i : U_i \subset X \setminus \Sigma \rightarrow \mathbb{C}$ we can pull-back the standard connection ∇_{st} on $T(\phi_i(U_i))$ to define a connection ∇_i on TU_i . Since the transition maps ϕ_{ij} are affine, the geodesics of ∇_i and ∇_j on the intersections $U_i \cap U_j$ coincide: they both are sent to the straight lines $at + b$ by ϕ_i, ϕ_j , respectively. By Lemma 2, the ∇_i and ∇_j then coincide on $T(U_i \cap U_j)$, so the connections ∇_i glue to a connection ∇ on $T(X \setminus \Sigma)$.

Assume now that the structure is regular and let D_o be a punctured disk with center at $p \in \Sigma$ covered by some affine charts $\phi_i : U_i \rightarrow \mathbb{C}$, i.e., $D_o = \cup_{i=1, \dots, n} U_i$. Shrinking D_o and U_i we can assume that U_i are sectors of D_o ordered counterclockwise.

Lemma 8 *The functions $\phi_i(z)$ have moderate growth at p (here z is a local parameter at p).*

Proof Postcomposing $\phi_i(z)$ with affine maps, we can ensure that the transition maps $\phi_{i,i+1}$ are identity for $i = 1, \dots, n - 1$ and $\phi_{n1}(w) = e^{2\pi i \alpha} w + C$. Then the functions $(z^\alpha + C') \phi_i(z)$ (or $C' \log z + \phi_i(z)$ if $\alpha \in \mathbb{Z}$) coincide on $U_i \cap U_j$ for a suitable C' and are therefore restrictions of a function ϕ holomorphic in D_o .

Clearly, the limits $\lim_{U_i \ni z \rightarrow p} \phi(z)$ are all either simultaneously infinite or simultaneously finite. Thus either $\phi(z)$ or $1/\phi(z)$ is bounded as $z \rightarrow 0$ and therefore holomorphic at p . Then $\phi(z)$ (and therefore $\phi_i(z)$ as well) has moderate growth at p . \square

The second claim now follows easily: the pull-backs of constant sections of TC (horizontal sections of ∇_{st}) by maps with moderate growth have moderate growth. \square

2.3 Branching Points

Poles of a meromorphic Fuchsian connection are the branching points of the corresponding affine structure.

Computations of Section 2.1.1 can be interpreted in the following way. Given a singularity p , the geometric model of the branched affine structure around p is represented by an angular sector with the total angle θ where the extremal rays are identified by a homothety with ratio λ (see Lemma 4). A natural local coordinate on this sector is of the form $f(z) = z^{1+\alpha}$ where $Re(\alpha) = \frac{\theta}{2\pi} - 1$ and $Im(\alpha) = -\frac{\log \lambda}{2\pi}$. The affine Schwarzian (invariant under postcompositions with affine transformations) is given by $\frac{f'''}{f''} = \frac{\alpha}{z}$ where $-\alpha$ is the residue of the Fuchsian meromorphic connection at p .

Just like the fact that the conical angle can be interpreted as the discrete analog of curvature in the Gauss-Bonnet formula, the dilation ratio around a singularity can be thought of as a discrete imaginary curvature.

2.4 Geodesics

A curve on a branched affine surface is called a *geodesic* if it is locally conjugated to a straight line in any affine chart (see Lemma 5). This class of curves coincides with the class of geodesics defined by the Fuchsian meromorphic connection on the tangent bundle corresponding to the branched affine structure under consideration.

A *regular geodesic* is a geodesic that does not meet any branching point of the affine structure.

A *saddle connection* (not to be confused with other connections we introduced earlier!) is a (possibly self-intersecting) geodesic segment whose endpoints are branching points of the branched affine structure or, equivalently, the singularities of the meromorphic connection.

Let us finally introduce the most crucial notion of this paper, namely, the ω -limit set of a geodesic. Informally, it is the set to which a geodesic accumulates. More exactly, the definition is as follows.

Definition 2 Let $\gamma : \mathbb{R}^+ \rightarrow X$ be a geodesic curve on a branched affine surface (or, equivalently, on a Riemann surface with a meromorphic connection). We define the ω -limit set of γ as $\bigcap_{t \in \mathbb{R}^+} \overline{\gamma([t, +\infty[)}$.

3 Hyperbolic Cylinders and Examples

3.1 Cylinders

Lemma 9 Every regular closed geodesic without self-intersections on a branched affine surface belongs to a 1-parameter family of homotopy equivalent closed regular geodesics.

Proof We consider a neighborhood V of a regular closed geodesic γ on a branched affine surface. Without loss of generality, we can assume V is an open topological cylinder without branching points. The monodromy along γ preserves its local direction. Therefore, the multiplier of the affine map induced by the monodromy along γ is a real positive number a . If $a = 1$, there is a neighborhood of γ inside V that is affinely equivalent to a

parallelogram of \mathbb{C} where a pair of sides is identified by a translation. If $a \neq 1$, then there is a neighborhood of γ inside V that is affinely equivalent to a trapezoid of \mathbb{C} where a pair of sides is identified by a homothety. In both cases, γ belongs to a continuous family of homotopic geodesics. \square

For any regular closed geodesic, we refer to the maximal family of geodesics it embeds into as a *cylinder*.

3.2 Hyperbolic Cylinders and Limit Cycles

A *Hopf torus* is an annulus in the complex plane \mathbb{C} whose inner and outer boundaries are identified by a homothety (see Subsection 2.2 in [4]).

A *hyperbolic cylinder* of angle θ and dilation ratio λ is an angular portion of a Hopf torus whose angle equals θ and whose dilation coefficient equals λ . Since this surface has an atlas with transition maps of the form $z \rightarrow az + b$ where $a \in \mathbb{R}_+^*$ this topological cylinder has an affine structure. Besides, since homotheties preserve directions, the foliation of a hyperbolic cylinder by the straight lines having the same slope is globally well-defined.

A hyperbolic cylinder of angle $\theta < \pi$ is affinely equivalent to a trapezoid whose identified sides are segments of \mathbb{C} between the endpoints of the two circular arcs. This implies that every regular closed geodesic whose monodromy coincides with a nontrivial homothety belongs to a hyperbolic cylinder (see Lemma 9).

In the case of a hyperbolic cylinder, the closed geodesics are the rays of the annulus. Any oriented geodesic entering such a cylinder with the same direction as a closed geodesic accumulates inside the cylinder on an attracting limit cycle (see Fig. 1). Indeed, since directions are well-defined in a hyperbolic cylinder, a geodesic cannot cross a ray of the annulus that belongs to the same direction.

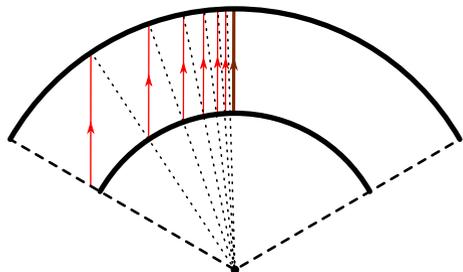
In the latter case, for a hyperbolic cylinder covering the interval $[\alpha, \beta] \subset \mathbb{S}^1$ of directions and bounded by saddle connections, a geodesic in direction β entering the cylinder by crossing the boundary of direction α will accumulate onto the saddle connection forming the boundary of direction β . We will need the following result.

Theorem 3 (Theorem 0.1 of [2]) *For an infinite geodesic γ on $\mathbb{C}\mathbb{P}^1$ without self-intersections, the ω -limit set of γ is either*

- (i) *a closed geodesic*
- or*
- (ii) *a singular limit cycle formed by saddle connections.*

Example 8.1 of [2] contains a numerical experiment illustrating Case (i) while there is no example in loc. cit. to illustrate Case (ii). However, geodesics entering a hyperbolic

Fig. 1 Hyperbolic cylinder



cylinder confirm the existence of both cases (see Fig. 1). (Theorem 2 proves that the branched affine surface containing a hyperbolic cylinder has a well-defined Fuchsian meromorphic connection where the geodesics are the same).

3.3 Self-intersecting Cylinder

In our next example, we consider a branched affine structure on $\mathbb{C}P^1$ corresponding to a meromorphic Fuchsian connection such that the residues at its poles have their real parts belonging to $\frac{1}{4}\mathbb{Z}$. Following Section 2.3, this means that directions in the affine structure are well-defined up to rotations of order four.

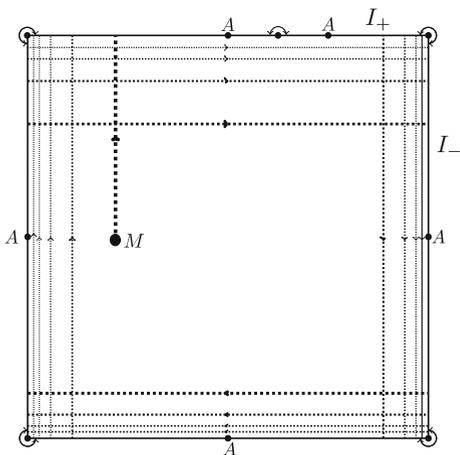
Such an affine structure admits a ramified covering of degree at most four on which all directions are well-defined. This canonical cover (discussed in Section 6 of [8]) is a generalization of the canonical cover of Section 4.1. Then, every regular closed geodesic, even if in itself has self-intersections, will have no self-intersection on the finite cover since on this cover it belongs to a cylinder (see Lemma 9). The projection of this cylinder to the original surface is a *self-intersecting cylinder*.

In Fig. 2, the surface is a square with specific identifications of the segments on its horizontal and vertical sides. Circular arrows correspond to conical singularities of angles $\frac{\pi}{2}$ for four of them and of angles π for one of them. The conical singularity between segments I_+ and I_- has a dilation ratio of $\frac{1}{2}$. The last singularity (A on the figure) has a total angle of 5π and a dilation ratio of 2. The sum of the angle defects then implies that the surface has genus zero.

Geodesic (the dotted line in Fig. 2) starting from M with a vertical direction has infinitely many self-intersections and becomes closer and closer to the sides of the square (because of the contraction of the segment I_- to the segment I_+). Its ω -limit set is a union of finitely many saddle connections. In particular, this example shows that even if a geodesic has infinitely many self-intersections, its ω -limit set might have empty interior.

The latter example presents a geodesic related to a branched affine structure. However Lemma 5 implies that this curve is also a geodesic for the Fuchsian meromorphic connection induced by the branched affine structure (see Theorem 2).

Fig. 2 Self-intersecting cylinder: the segments I_+ and I_- are identified by an affine map with coefficient $\frac{1}{2}$. Circular arrows identify adjacent segments



4 Flat Structures and k -Differentials

4.1 General Facts

For a branched affine surface such that all the linear coefficients of its (affine) transition maps are the k^{th} roots of unity, one gets a global flat structure. (Observe that in such a case the length of a curve is globally well-defined since the holonomy is unitary.) In other words, for any local coordinate f , the k -differential $(df)^k$ is globally well-defined on the whole surface. Therefore, this special kind of a branched affine structure is canonically associated with a k -differential.

Locally, the union of all the k^{th} roots of a k -differential constitute a multi-valued Abelian differential defining a flat structure. This structure consists of an atlas of complex charts whose transition maps are given by parallel translations and rotations of order k or its multiple. In particular, for such a structure, the notions of length, area and direction modulo rotations of order k are well-defined. In the flat structure defined by a k -differential, a singularity of order $a > -k$ corresponds to a conical singularity of angle $\frac{(a+k)2\pi}{k}$.

Flat geometry of k -differentials has been studied in, e.g., [3, 9, 10]. In particular, k -differentials have *canonical covers* of order k ramified at their singularities whose orders are not divisible by k . The *canonical cover* of a Riemann surface X endowed with a k -differential is a Riemann surface \tilde{X} (usually of a higher genus) endowed with a k -differential which is a global k^{th} power of a meromorphic 1-form together with a natural projection $\mu : \tilde{X} \rightarrow X$ sending one k -differential to the other. Consequently, up to a rotation of order k , there is a well-defined translation structure on \tilde{X} . Recall that a *translation structure* is an atlas of complex charts where all transition maps are translations and all singularities are conical points whose angles are integer multiple of 2π (see [11]).

Every geodesic on X lifts to \tilde{X} and thus the dynamics of a geodesic flow in the flat structure of the original k -differential splits into the dynamics on the translation surface \tilde{X} and a finite monodromy in the fiber. The following theorem classifies the invariant components of translation surfaces for the directional foliation. (This result is a special case of Proposition 5.5 of [10]).

Theorem 4 *Cutting a translation surface without boundary which has a finite total area along its saddle connections in a given direction one decomposes the surface into finitely many connected components of the following two types:*

- (i) *Flat cylinders whose vertical leaves are periodic geodesics;*
- (ii) *Minimal components for which the noncritical vertical leaves are minimal. The geodesic dynamics on each minimal component is that of an interval exchange map.*

The latter dichotomy generalizes the distinction between foliations on flat tori with rational and irrational slopes. As opposed to the hyperbolic cylinders (see Section 3), flat cylinders are obtained by identifying a pair of sides of a parallelogram by a translation.

The only part of Theorem 4 we will need later is the case of a direction in which there is no saddle connection.

Corollary 1 *In a translation surface without boundary having a finite area, in a direction without saddle connections, every leaf of the directional foliation is dense in the surface.*

Proof Invariant components in a given direction are bounded by saddle connections. In a direction without any saddle connection, either the whole surface is a minimal component or the whole surface is a flat cylinder. A flat cylinder which is not bounded by any saddle connection is of infinite area. \square

Minimality of geodesics in generic directions in the case of translation surfaces extends easily to the case of k -differentials.

Corollary 2 For a Riemann surface X endowed with a k -differential inducing a flat structure with finite area, a geodesic in a generic direction accumulates on the whole surface and its self-intersections are everywhere dense.

Proof Directions in the affine structure of X are well-defined up to a rotation of order k . Every singularity of X is conical since otherwise the flat structure would have infinite area. Besides, in every homotopy class of topological arcs between conical singularities (with possible self-intersections), there is at most one saddle connection. Therefore, directions of saddle connections are countable in the circle of directions (modulo rotation of order k). These saddle connections lift to saddle connection of the canonical cover \tilde{X} of X . Since \tilde{X} is a translation surface, the lift of a geodesic γ of X with a generic direction is a geodesic $\tilde{\gamma}$ of \tilde{X} with a direction without any saddle connection. Corollary 1 then implies that $\tilde{\gamma}$ is minimal in \tilde{X} . Since γ is the projection of $\tilde{\gamma}$, it accumulates everywhere and its self-intersection is also everywhere dense. \square

4.2 Cut-and-Paste Constructions

Branched affine structures are well-suited to perform surgeries. Cutting along saddle connections on a surface with a branched affine structure, we obtained a surface with boundary. Moreover, this boundary is geodesic for the underlying affine structure.

For two such affine surfaces with boundary X and Y , an identification of corresponding saddle connections by affine maps provides a branched affine structure on $X \cup Y$.

In the case we are interested in, X and Y have affine structures induced by k -differentials (the multipliers of the monodromy are k^{th} roots of the unity). In this case, if the boundary saddle connections are identified by a composition of translations and rotations of order k , then the affine structure on $X \cup Y$ also has a constrained monodromy. Since $X \cup Y$ (punctured at the branching points) has a complex affine atlas, it has a natural structure of a Riemann surface. Besides, since the monodromy is constrained the affine structure on $X \cup Y$ is induced by some k -differential.

Though the surgery is easy to perform and it has a clear geometric meaning, we have no control on the resulting complex structure of $X \cup Y$ (its genus and location of the singularities). In particular, there is no obvious way to give an explicit expression of the k -differential.

On a Riemann surface, a branched affine structure (like the one produced by a cut-and-paste surgery) induces a meromorphic Fuchsian connection (see Theorem 2).

4.3 First Example: Quartic Differential on $\mathbb{C}P^1$

The branched affine surface X illustrated in Fig. 3 is obtained by a cut-and-paste construction. The left part X_l and the right part X_r are branched affine surfaces with geodesic boundary.

The left part X_l is a square whose horizontal and vertical sides are identified in a specific way (the circular arrows indicate the pairs of identified segments). Corners B, C, D and E are conical singularities of angle $\frac{\pi}{2}$. The singularity A has a total angle of 4π . The boundary is made of the two sides of a slit IJ . Additionally, the slit belongs to a generic direction. Since the identifications are rotations of angle $\frac{\pi}{2}$, the branched affine structure comes from a quartic differential. A computation of the total angle defect shows that the X_l has genus zero and one boundary component.

The right part X_r is a rectangle whose sides are identified in a way indicated by the circular arrows. Corners R, S, T, U are conical singularities of angle $\frac{\pi}{2}$. The two lateral segments IJ form the boundary of this branched affine surface. I and J are conical singularities of angle 3π . One can check that X_r is also a surface of genus zero with one boundary component. Its branched affine structure is also induced by a quartic differential.

The gluing of the left and right parts along segments IJ preserves lengths and directions. Therefore, the obtained surface X is $\mathbb{C}P^1$ with a branched affine structure also induced by a quartic differential.

Proposition 1 There is a Fuchsian meromorphic connection on $\mathbb{C}P^1$ for which a geodesic γ has infinitely many self-intersections and is not everywhere dense.

Proof The example of Fig. 3 provides a branched affine surface X of genus zero. Its affine structure induces a Fuchsian meromorphic connection (see Theorem 2). We consider a geodesic γ starting at the conical singularity E of X_l and having the same direction as the slit IJ .

The key point is that γ can cross the slit IJ and enter X_r in only one direction (perpendicular to the slit). Indeed, the flat structure is given by a quartic differential so there are at most two tangent directions at each self-intersection. A branch of γ which is parallel to the slit IJ cannot cross it. Then, since X_r is a rectangle, if γ enters X_r through a side of the slit IJ , it leaves X_r directly through the other boundary segment (without self-intersections) and crosses the other side of the slit IJ . We can glue portions of $\gamma \cap X_l$ to each other and get a geodesic γ' of \tilde{X}_l (a surface without boundary obtained from X_r by erasing the slit). By our hypothesis, γ never hits a singularity since otherwise it will be a saddle connection which contradicts to the genericity hypothesis on the direction of the slit IJ . Following

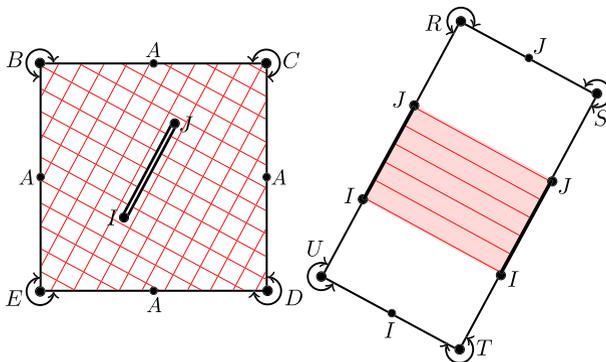


Fig. 3 Quartic differential on $\mathbb{C}P^1$

Corollary 2, γ' is everywhere dense in \tilde{X}_l and its self-intersections are also everywhere dense. Thus, γ accumulates on the union of X_l and the rectangle in X_r . Its self-intersections are dense in X_l . □

4.4 Second Example: Quartic Differential on a Torus

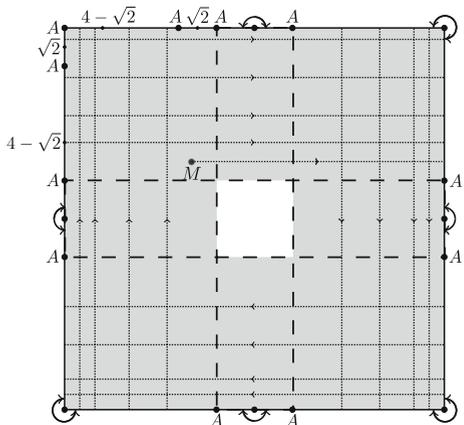
The branched affine surface X illustrated in Fig. 4 is a square with boundary segments identified in a special way. Circular arrows identify the adjacent segments. These identifications create three conical singularities of angle $\frac{\pi}{2}$ and four conical singularities of angle π . Besides, two pairs segments of lengths $\sqrt{2}$ and $4 - \sqrt{2}$ are identified. The identification maps are compositions of translations and rotations of order four which implies that the branched affine structure is induced by a quartic differential. Singularity A has a total angle of $\frac{21\pi}{2}$. The total angle defect of the singularities on the surface is zero which using the Gauss-Bonnet formula for a flat metric with conical singularities implies that the surface has genus one.

The canonical cover of this branched affine surface is a translation surface of finite area \tilde{X} (see Section 4.1). Horizontal and vertical directions of X are lifted to the same direction on \tilde{X} . Theorem 4 then gives its decomposition into invariant components. For this direction, \tilde{X} decomposes into three invariant components two of which being cylinders. Namely, there is one horizontal and one vertical cylinder in Fig. 4. They are formed by periodic geodesics, are bounded by dashed lines and their intersection is the central white square. The third invariant component is a minimal component \mathcal{M} whose dynamics is given by an interval exchange map transposing two segments of lengths $\sqrt{2}$ and $4 - \sqrt{2}$.

We consider a geodesic γ starting from the point M in the horizontal direction. The lift $\tilde{\gamma}$ of this trajectory belongs to the minimal component \mathcal{M} of \tilde{X} . Then, the ω -limit set of γ in X is the projection of \mathcal{M} on X . This projection is the complement of the small white square at the center of X . This central square is the intersection of the projections on X of the two cylinders of \tilde{X} . Every horizontal or vertical geodesic of X crossing this square is periodic.

Geodesic γ is another example of a geodesic with infinitely many self-intersections whose ω -limit set avoids some part of the surface. In fact, any other geodesic whose lift belongs to the same minimal component could have been chosen.

Fig. 4 Quartic differential on a torus



5 Final Remarks

All the previous examples can be generalized to surfaces of higher genera by using the following local surgery. Assume that we have a surface with a connection and its geodesic which is not everywhere dense. Take an arbitrarily small disk disjoint from the closure of the latter geodesic and remove a quadrilateral in this disk. Then identify the opposite sides of what was the boundary of the removed quadrilateral. This surgery provides a branched affine surface corresponding to a meromorphic Fuchsian connection (see Theorem 2 and Section 4.2). It increases the genus of the underlying surface by one, but does not impact the given geodesic. Therefore, the behavior observed in our previous examples exists in all genera.

Besides, as the sides of the quadrilateral can be given arbitrary length and directions, the monodromy of the branched affine structure can be made more generic. Such a surgery proves that examples of Sections 4.3 and 4.4 are not specific to quartic differentials.

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