AROUND A MULTIVARIATE SCHMIDT-SPITZER THEOREM

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Abstract. Given an arbitrary complex-valued infinite matrix $\mathcal{A} = (a_{ij})$, $i = 1, \dots, \infty$; $j = 1, \dots, \infty$ and a positive integer n we introduce a naturally associated polynomial basis $\mathfrak{B}_{\mathcal{A}}$ of $\mathbb{C}[x_0, \dots, x_n]$. We discuss some properties of the locus of common zeros of all polynomials in $\mathfrak{B}_A$ having a given degree m; the latter locus can be interpreted as the spectrum of the $m \times (m+n)$-submatrix of $\mathcal{A}$ formed by its m first rows and by the $(m+n)$ first columns. We initiate the study of the asymptotics of these spectra when $m \to \infty$ in the case when $\mathcal{A}$ is a banded Toeplitz matrix. In particular, we present and partially prove a conjectural multivariate analog of the well-known Schmidt-Spitzer theorem which describes the spectral asymptotics for the sequence of principal minors of an arbitrary banded Toeplitz matrix. Finally, we discuss relations between polynomial bases $\mathfrak{B}_{\mathcal{A}}$ and multivariate orthogonal polynomials.

1. Introduction

The approach of this paper is motivated by the modern interpretation of the Heine-Stieltjes multiparameter spectral problem as presented in [13] and [14]. Let us recall some relevant results in the matrix set-up.

Given integers $m > 0$ and $n \geq 0$ consider the space $Mat(m, m+n)$ of complex-valued $m \times (m+n)$-matrices. For $s = 0, \dots, n$ define the s-th unit matrix

$$\mathcal{I}_s := (\delta_{s+i-j}) \in Mat(m, m+n).$$

(In what follows the sizes of matrices can be infinite.)

Definition 1 (see [14])**.** Given a matrix $A \in Mat(m, m+n)$ define its *eigenvalue locus* $\mathcal{E}_A$ as

$$\mathcal{E}_A := \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1} \colon \operatorname{rank}\left(A - \sum_{s=0}^{n} x_s \mathcal{I}_s \right) < m \right\}.$$

For $n = 0$, $\mathcal{E}_A$ coincides with the usual set of eigenvalues of a square matrix A.

Proposition 2 (Lemma 1 of [14])**.** *For arbitrary $A \in Mat(m, m+n)$ the eigenvalue locus $\mathcal{E}_A$ consists of $\binom{m+n}{n+1}$ points counting multiplicities. In other words, counting multiplicities there exist $\binom{m+n}{n+1}$ eigenvalue tuples $(x_0, x_1, \dots, x_n)$ such that $A - \sum_{s=0}^{n} x_s \mathcal{I}_s$ has rank smaller than m.*

Remark 3. Notice that for $n > 0$, the locus $\mathcal{E}_A$ is not a complete intersection (we need more polynomials than expected to define the ideal) since it is given by the vanishing of *all* maximal minors of A. (A similar phenomenon can be observed for common zeros of multivariate Schur polynomials, since Schur polynomials are given by determinant formulas.)

2000 *Mathematics Subject Classification.* Primary 15B07; Secondary 34L20, 35P20.

Key words and phrases. asymptotic root distribution, square and rectangular Toeplitz matrices.

Notation 4. *Given an infinite matrix $\mathcal{A} = (a_{ij}),\ i = 1,\dots,\infty; j = 1,\dots,\infty,$ an integer $n \geq 0$, and an m-tuple of positive integers $I = (i_1, i_2, \dots, i_m)$ satisfying $1 \leq i_1 < i_2 < \cdots < i_m \leq m+n$, consider the submatrix A_I of $\mathcal{A} - \sum_{s=0}^{n} x_s \mathcal{I}_s$ formed by the first m rows and by the m columns indexed by I. Define*

$$P^I_{\mathcal{A}}(x_0, x_1, \dots, x_n) := \det A_I. \tag{1}$$

Evidently, $P^I_{\mathcal{A}}(x_0,\dots,x_n)$ is a maximal minor of the principal $m \times (m+n)$ submatrix of $\mathcal{A} - \sum_{s=0}^{n} x_s \mathcal{I}_s$ formed by its m first rows and $m+n$ first columns. Therefore $P^I_{\mathcal{A}}(x_0,\dots,x_n)$ is a polynomial in $x_0,\dots,x_n$.

Proposition 5. *In the above notation the following holds:*

(i) for any multiindex I with $|I| = m$, $\deg P^I_{\mathcal{A}}(x_0,\dots,x_n) = m$;
(ii) all $\binom{m+n}{m}$ polynomials $P^I_{\mathcal{A}}(x_0,...,x_n) \in \mathbb{C}[x_0,\dots,x_n]$ with $|I| = m$ are linearly independent, which implies that the totality of all polynomials $P^I_{\mathcal{A}}(x_0,...,x_n)$ is a linear basis of $\mathbb{C}[x_0,\dots,x_n]$;
(iii) the set $\mathcal{E}^{(m)}_{\mathcal{A}}$ of common zeros of all $P^I_{\mathcal{A}}(x_0,...,x_n)$ with $|I| = m$ is a finite subset of $\mathbb{C}^{n+1}$ of cardinality $\binom{m+n}{n+1}$ counting multiplicities. Note that $\mathcal{E}^{(m)}_{\mathcal{A}}$ coincide with that in Definition 1.

Remark 6. Notice that for $\binom{m+n}{m}$ randomly chosen polynomials in $\mathbb{C}[x_0, x_1 \dots, x_n]$ of degree m, the set of their common zeros is typically empty.

Proposition 5 together with our numerical experiments motivate the following question.

Given an arbitrary infinite matrix $\mathcal{A}$ as above, associate to each $\mathcal{E}^{(m)}_{\mathcal{A}}$ its "root-counting" measure $\mu^{(m)}_{\mathcal{A}}$ supported on $\mathcal{E}^{(m)}_{\mathcal{A}} \subset \mathbb{C}^{n+1}$ by assigning to every point $p \in \mathcal{E}^{(m)}_{\mathcal{A}}$ the point mass $\kappa(p)/\binom{m+n}{n+1}$ where $\kappa(p)$ is the multiplicity of p. (Obviously, $\mu^{(m)}_{\mathcal{A}}$ is a discrete probability measure.)

Main Problem. Under which assumptions on $\mathcal{A}$ does the weak limit $\mu_{\mathcal{A}} = \lim_{m\to\infty} \mu^{(m)}_{\mathcal{A}}$ exist? In case $\mu_{\mathcal{A}}$ exist, is it possible to describe the support and density of the measure?

In the classical case $n = 0$, the above problem was intensively studied by many authors. The main focus has been when $\mathcal{A}$ is either a Jacobi or a Toeplitz matrix (or their generalizations such as block-Toeplitz matrices etc.), see e.g. [4, 3, 15, 16], and the more recent development [7, 5, 6].

The main goal of this note is to present a multivariate analogue of the well-known theorem by P. Schmidt and F. Spitzer [12], who describe $\mu_{\mathcal{A}}$ for an arbitrary banded Toeplitz matrix $\mathcal{A}$ in the case $n = 0$.

Namely, let $\mathcal{A} = (c_{i-j})$, with $i, j = 1, 2, \dots$ be an infinite, banded Toeplitz matrix, where $c_i = 0$ if $i < -k$ or $i > h$. Fixing $n \geq 0$ as above, we obtain for each positive integer m the eigenvalue locus $\mathcal{E}^{(m)}_{\mathcal{A}}$ of the principal $m \times (m+n)$ submatrix $A^{(m)}$ of $\mathcal{A}$.

Define the *limit set $B_{\mathcal{A}}$ of eigenvalue loci* as

$$B_{\mathcal{A}} = \left\{ \mathbf{x} \in \mathbb{C}^{n+1} \colon \mathbf{x} = \lim_{m\to\infty} \mathbf{x}_m, \mathbf{x}_m \in \mathcal{E}^{(m)}_{\mathcal{A}} \right\},\ \mathbf{x} = (x_0,\dots,x_n). \tag{2}$$

In other words, $B_{\mathcal{A}}$ is the set of limit points of the sequence $\{\mathcal{E}^{(m)}_{\mathcal{A}}\}$. Thus $B_{\mathcal{A}}$ is the support of the limiting measure $\mu_{\mathcal{A}}$ if it exists. (For a general infinite matrix $\mathcal{A}$ as above, its limit set $B_{\mathcal{A}}$ might be empty.)

Set

$$Q(t,\mathbf{x}) = t^k \left(\sum_{j=-k}^{h} c_j t^j - \sum_{j=0}^{n} x_j t^j \right), \tag{3}$$

and let $\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x}), \ldots, \alpha_{k+h}(\mathbf{x})$ be the roots of $Q(t,\mathbf{x}) = 0$, ordered according to their absolute values, i.e. $|\alpha_i(\mathbf{x})| \leq |\alpha_{i+1}(\mathbf{x})|$ for all $0 < i < k+h$. Let $C_{\mathcal{A}}$ be the real semi-algebraic set given by the condition:

$$C_{\mathcal{A}} = \{\mathbf{x} \in \mathbb{C}^{n+1} \colon |\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})|\}. \tag{4}$$

Our main conjecture is as follows.

Conjecture 7. *For any banded Toeplitz matrix $\mathcal{A}$, if $B_{\mathcal{A}}$ is defined by* (2) *and $C_{\mathcal{A}}$ defined by* (4) *then $B_{\mathcal{A}} = C_{\mathcal{A}}$.*

By Conjecture 7 the set $B_{\mathcal{A}}$ is a real semi-algebraic $(n+1)$-dimensional subset of $\mathbb{C}^{n+1}$. In the classical case $n = 0$, Conjecture 7 is settled by P. Schmidt and F. Spitzer in [12]. Another important case when Conjecture 7 has been proved follows from some known results on multivariate Chebyshev polynomials, which is is presented in Example 8 below. Namely, when $k = 1$ and $h = n+1$ with c_{-1} and c_{n+1} non-zero, we may do an affine change of the variables and a scaling of $\mathcal{A}$. This reduces the latter case to $c_{-1} = c_{n+1} = 1$ and all other $c_i = 0$.

For these particular values, the family $\{P^I_{\mathcal{A}}(\mathbf{x})\}$ becomes the multivariate Chebyshev polynomials of the second kind, see e.g. [8, 10, 2, 17]. These polynomials also have a close connection to another well-known family of polynomials, namely the Schur polynomials.

Example 8. For the above matrices corresponding to the multivariate Chebyshev polynomials the eigenlocus $\mathcal{E}^{(m)}_{\mathcal{A}}$ can be described explicitly, see for example [9].

More precisely, the points in $\mathcal{E}^{(m)}_{\mathcal{A}}$ lie on a real, n-dimensional surface $C_{\mathcal{A}} \subset \mathbb{C}^{n+1}$ which is naturally parametrized by an $(n+1)$-dimensional torus T^{n+1}. This parametrization is given by

$$C_{\mathcal{A}} = \left\{\mathbf{x} \in \mathbb{C}^{n+1} | x_j = -e_{j+1}(\exp(i\theta_1), \ldots, \exp(i\theta_{n+1}), \exp(i\theta_{n+2}))\right\} \tag{5}$$

where $(\theta_1, \ldots, \theta_{n+1})$ lie on T^{n+1}, $\sum_{j=0}^{n+2} \theta_j = 0$, and e_j is the j-th elementary symmetric function in $n+2$ variables.

Notice that for $\mathbf{x} \in C_{\mathcal{A}}$,

$$\begin{aligned} Q(t,\mathbf{x}) &= 1 + x_0 t + x_1 t^2 + \cdots + x_n t^{n+1} + t^{n+2} \\ &= \prod_j (t + e^{i\theta_j}) \end{aligned}$$

by the Vieta formulas. Thus, for $\mathbf{x} \in C_{\mathcal{A}}$, *all* roots of Q (as a polynomial in t) have absolute value equal to 1 when the x_j are parametrized as in (5).

Furthermore, the points in $\mathcal{E}^{(m)}_{\mathcal{A}}$ are also expressed by (5), with the parameters $(\theta_1, \ldots, \theta_{n+2})$ being certain rational multiples of π, distributed in a regular lattice. The mapping from the 2-torus to the eigenlocus is illustrated in Figure 1.

Another interesting aspect of Example 8 is that all the points $\mathbf{x} = (x_0, \ldots, x_n)$ in the sets $\mathcal{E}^{(m)}_{\mathcal{A}}$ satisfy the conditions $x_j = \overline{x_{n-j}}$, $j = 0, 1, \ldots, n$, which explains why we can draw $C_{\mathcal{A}} \subset \mathbb{C}^2$ in Figure 1a as a 2-dimensional set. For larger n, $C_{\mathcal{A}}$ is an $(n+1)$-dimensional analogue of the two-dimensional deltoid, shown in Figure 1a.

For general $\mathcal{A}$, we do not have the inclusion $\mathcal{E}_{\mathcal{A}}^{(m)} \subseteq C_{\mathcal{A}}$ for arbitrary finite m. However, if $\mathcal{A}$ has an additional extra symmetry, this seems to be case, as we will see below.

Definition 9. An $m \times (m+n)$ rectangular matrix (a_{ij}) is called *centro-hermitian* if its entries satisfy $a_{ij} = \overline{a}_{m+1-i,m+n+1-j}$.

(These matrices appear in various context in linear algebra, see e.g., [11]*.)*

One can show that for any such centro-hermitian matrix, we have that $(x_0, x_1, \dots, x_n) \in \mathcal{E}_{\mathcal{A}}^{(m)}$ if and only if $(\overline{x}_n, \overline{x}_{n-1}, \dots, \overline{x}_0) \in \mathcal{E}_{\mathcal{A}}^{(m)}$. (Professor Yuan Xu kindly informed us about this fact.) In other words, for centro-hermitian matrices, eigenvalues come in "complex conjugate" pairs where by a "complex conjugation" we mean the latter anti-holomorphic involution.

However, for centro-hermitian Toeplitz matrices, all the eigenvalues seem to be "real" with respect to the above "complex conjugation". We pose the following conjecture.

Conjecture 10. *If $\mathcal{A}$ is Toeplitz and centro-hermitian, then each point $(x_0, x_1, \dots, x_n) \in \mathcal{E}_{\mathcal{A}}^{(m)}$ satisfies $x_j = \overline{x_{n-j}}$ for $j = 0, 1, \dots, n$.*

Conjecture 10 obviously holds for the case $n = 0$, as it reduces to the fact that hermitian matrices have real eigenvalues. It is also straightforward to check that Conjecture 10 is true for the Chebyshev case of Example 8 above.

We have extensive numerical evidence for this conjecture. Another strong indication supporting Conjecture 10 is that if $\mathcal{A}$ is Toeplitz and centro-hermitian, then every point $\mathbf{x} \in C_{\mathcal{A}}$ (which by Conjecture 7 is in the limit eigenlocus) satisfies the required symmetry $x_j = \overline{x_{n-j}}$ for $j = 0, 1, \dots, n$.

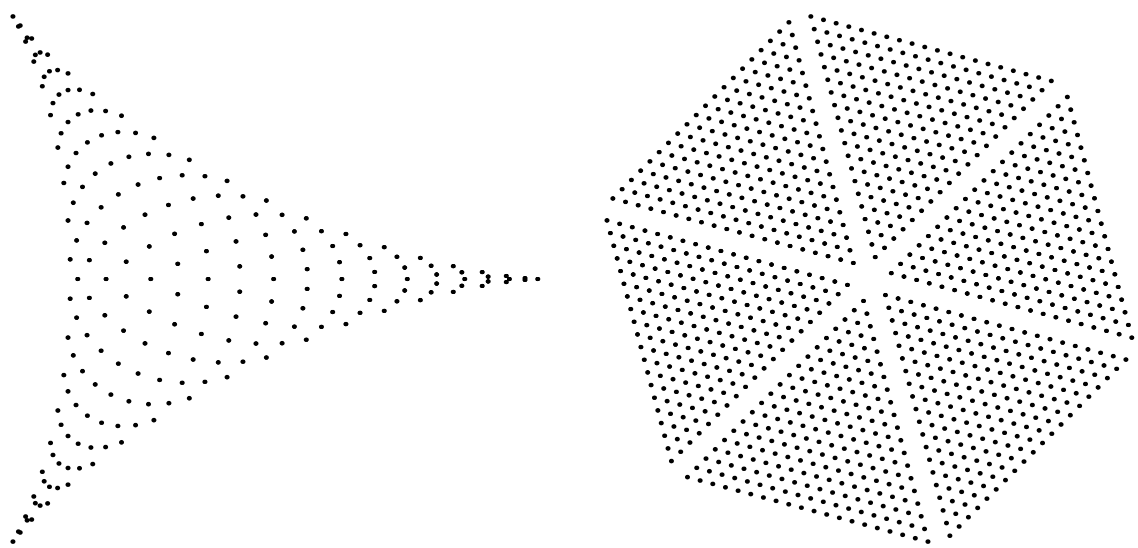

FIGURE 1. The eigenvalue locus $\mathcal{E}_2^{(20)}$ and its pull-back to T^2. The torus T^2 is covered with a hexagon, where each triangle is mapped to the eigenlocus. The 6-fold symmetry is due to the S_3-action by permutation of the arguments $\theta_1, \theta_2, \theta_3$ in (5). (Notice $\theta_1 + \theta_2 + \theta_3 = 0$ and this is the subspace which is illustrated in the figure to the right.)

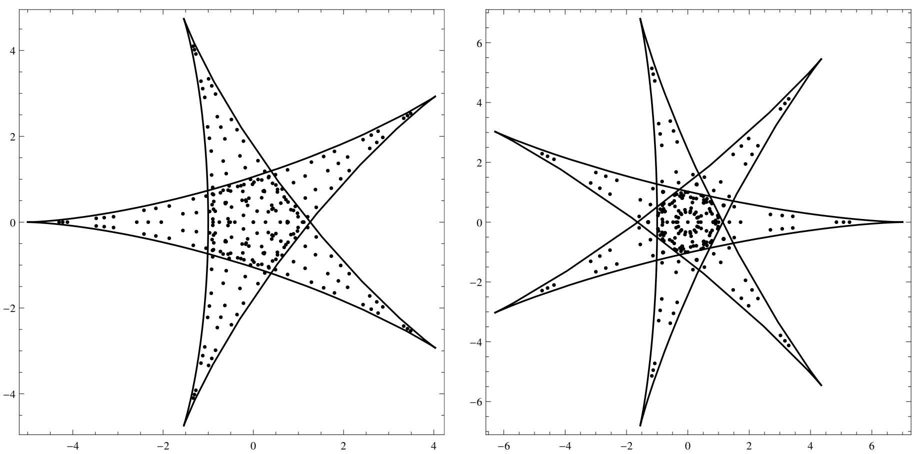

FIGURE 2. 5-edged star, when $d = 2$ and 7-edged star, when $d = 3$

The next group of examples are bivariate analogues of special univariate cases originally studied in [12], and later in [3], where they are referred to as "star-shaped curves":

Example 11. The bivariate case when $Q(t, \mathbf{x}) = 1 + t^d x_0 + t^{d+1} x_1 + t^{2d+1}$, $d \geq 1$ gives sets in $\mathbb{C}^2$ where $x_0 = \overline{x_1}$, by Conjecture 10. They correspond to Toeplitz matrices of the form

$$\begin{pmatrix} x_0 & x_1 & 1 & 0 & 0 & \cdots \\ 1 & x_0 & x_1 & 1 & 0 & \cdots \\ 0 & 1 & x_0 & x_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \quad \begin{pmatrix} x_0 & x_1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & x_0 & x_1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & x_0 & x_1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \cdots$$

The above two matrices represent $d = 1$ and $d = 2$.

Figures 2 and 3 present the distributions of $x_0 \in \mathbb{C}$, for $d = 2, 3, 4$. (Recall that $x_1 = \bar{x}_0$.) The points shown on these figures belong to $\mathcal{E}_{\mathcal{A}}^{(m)}$ for $m = 13, 14, 15$, and the curves are certain hypocycloids, parametrizing the boundary of $C_{\mathcal{A}}$. More explicitly, for a given integer $d \geq 1$ the hypocycloid boundary for $x_0 \in \mathbb{C}$ is given by

$$x_0 = (-1)^d e^{-i(d+2)\theta} \left((d+2) e^{i(2d+3)\theta} + d + 1 \right) \text{ where } \theta \in [0, 2\pi],$$

which is one of the implications of Conjecture 7.

Finally, the main result of this note is as follows.

Theorem 12. *For any banded Toeplitz matrix $\mathcal{A}$, where $B_{\mathcal{A}}$ is defined by* (2) *and $C_{\mathcal{A}}$ is defined by* (4)*, one has $B_{\mathcal{A}} \subseteq C_{\mathcal{A}}$.*

2. PROOFS

Proof of Proposition 5. We shall prove items (i) and (ii) simultaneously by calculating the leading homogeneous part of $P_{\mathcal{A}}^I(x_0, ..., x_n)$. Let us order the set of all admissible indices $I = (1 \leq i_1 < \ldots < i_m \leq m + n)$ lexicographically. We can also order lexicographically all monomials of degree m in $x_0, \ldots, x_n$. By equation (1) $P_{\mathcal{A}}^I(x_0, ..., x_n) = \det A_I$ where the columns of A_I are indexed by

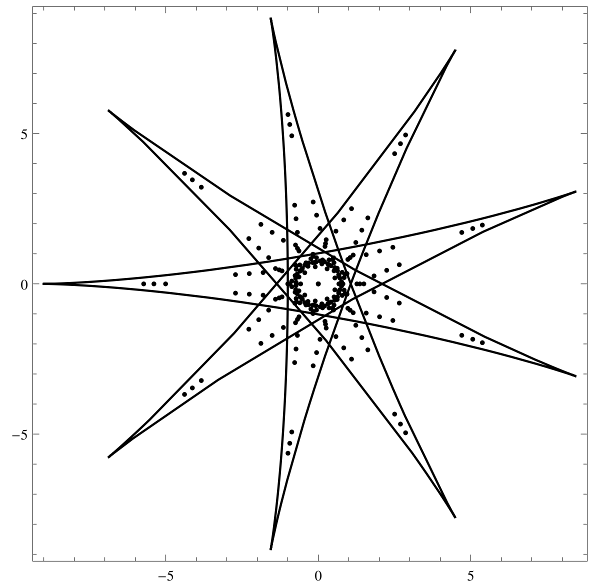

FIGURE 3. 9-edged star, when $d = 4$.

I. Let $\widetilde{P}^I_{\mathcal{A}}(x_0, ..., x_n)$ be the homogeneous part of $P^I_{\mathcal{A}}(x_0, \dots, x_n)$ of degree m. One can easily see that the product of all entries on the main diagonal of A_I contains the monomial $\mathfrak{m}_I$ of degree m given by $\mathfrak{m}_I = \prod_{j=1}^m x_{i_j - j+1}$. Moreover it is straight-forward that $\widetilde{P}^I_{\mathcal{A}}(x_0, \dots, x_n) = \mathfrak{m}_I + \dots$ where $\dots$ stands for the linear combination of monomials $\mathfrak{m}_{I'}$ of degree m obtained from other indices I' which are lexicographically smaller than I. In other words, the matrix formed by $\widetilde{P}^I_{\mathcal{A}}(x_0, \dots, x_n)$ against monomials is triangular in the lexicographic ordering with unitary main diagonal, which proves items (i) and (ii).

Item (iii) is just a reformulation of Proposition 2 above. □

Throughout the rest of the paper, we put $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{h+k})$. We will also assume that $c_h = 1$, which corresponds to a rescaling of the original matrix $\mathcal{A}$. This is equivalent to the assumption that $Q(t, \mathbf{x})$ is monic. By shifting the variables in $\mathbf{x}$, we may also assume, without loss of generality, that $c_0 = c_1 \cdots = c_n = 0$ in $\mathcal{A}$.

In the above notation, it is convenient to work with the roots of $Q(t, \mathbf{x})$. This motivates the following definitions. Let $\Gamma_j \subset \mathbb{C}^{h+k}$, $j = k, \dots, k+n$ denote the real semi-algebraic hypersurface consisting of all $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{h+k})$ such that when the α_j are ordered with increasing moduli, $|\alpha_j| = |\alpha_{j+1}|$. Similarly, let G_j be defined as the real semi-algebraic set

$$\{\mathbf{x} \in \mathbb{C}^{n+1} \colon Q(t, \mathbf{x}) = (t - \alpha_1) \cdots (t - \alpha_{h+k}) \text{ where } \boldsymbol{\alpha} \in \Gamma_j\}.$$

Then, by definition, $C_{\mathcal{A}} = \bigcap_{j=k}^{k+n} G_j$.

Proposition 13. *For any banded Toeplitz matrix $\mathcal{A}$ and any non-negative $n < h$, the set $C_{\mathcal{A}}$ defined by* (3)-(4) *is compact.*

Proof. As discussed above, we may without loss of generality assume that $c_h = 1$ and $c_0 = c_1 = \dots = c_n = 0$. Since Q may be assumed to be monic, we have $c_j = e_{h-j}(-\boldsymbol{\alpha})$ for $-k \le j < 0$ and

$n < j \le h$, and $x_j = -e_{h-j}(-\boldsymbol{\alpha})$ when $0 \le j \le n$. Thus, it suffices to show that the set of $\boldsymbol{\alpha} \in \mathbb{C}^{h+k}$ that satisfies the conditions (3)-(4), is compact. It is also evident that the set $C_{\mathcal{A}}$ is closed, so we only need to show that it is bounded. We show this fact by contradiction.

Assume we have a sequence of roots $\{\boldsymbol{\alpha}^m\}_{m=1}^{\infty}$ of (3) such that $\|\boldsymbol{\alpha}^m\| \to \infty$ where (4) is satisfied for each $\boldsymbol{\alpha}^m$. We assume that the modulus of the roots are always ordered increasingly. There are two cases to consider.

Case 1: Assume that for some $0 \le b < k$, a sequence of individual roots satisfies the condition $|\alpha_{b+1}^m| \to \infty$ but $|\alpha_j^m|$ are bounded for all m and $j \le b$. Then consider $e_{h+k-b}(\boldsymbol{\alpha})$. Since $b < k$, in our notation $e_{h+k-b}(\boldsymbol{\alpha})$ equals the coefficient c_{b-k}. Notice that e_{h+k-b} contains the term $\alpha_{b+1}\alpha_{b+2}\cdots\alpha_{h+k}$ which grows quicker than all other terms in the expansion of $e_{h+k-b}(\boldsymbol{\alpha})$. This contradicts the assumption $e_{h+k-b}(\boldsymbol{\alpha}) = c_{b-k}$.

Case 2: Assume that for some b with $k+n \le b < h+k$, we have a sequence of individual roots $|\alpha_{b+1}^m| \to \infty$ but $|\alpha_j^m|$ are bounded for all m and $j \le b$. Consider

$$e_b(\boldsymbol{\alpha}) = e_b(\alpha_1, \dots, \alpha_{h+k}) = \sum_{\sigma \in \binom{[h+k]}{b}} \frac{e_0}{\alpha_{\sigma_1}\alpha_{\sigma_2}\cdots\alpha_{\sigma_b}}.$$

This contains an expression with the denominator $\alpha_1\alpha_2\cdots\alpha_b$, i.e. the product of all bounded roots. Now, since $h+k-b$ roots among all $h+k$ roots grow in absolute value, and the product $\alpha_1 \dots \alpha_{h+k}$ equals c_h, it follows that $|\alpha_1\alpha_2\cdots\alpha_b| \to 0$ as $m \to \infty$, and this term converges to 0 quicker than any other product $\alpha_{\sigma_1}\alpha_{\sigma_2}\cdots\alpha_{\sigma_b}$. Thus, $|e_b|$ should grow. This contradicts the assumption $e_b(\boldsymbol{\alpha}) = c_{h-b}$.

Notice that under our assumptions, the above cases cover all possible ways for a sequence of roots to diverge. Since both cases yield a contradiction, it follows that any sequence of roots of (3) satisfying (4) must be bounded. Thus, $C_{\mathcal{A}}$ is compact. □

The following result is a multivariate analog of a known fact in the case $n = 0$, see [3, Prop. 11.8, Prop. 11.9].

Proposition 14. *In the notation of* (3)–(4), *for any* $\mathbf{x}$ *belonging to the boundary* $\partial C_{\mathcal{A}}$ *of* $C_{\mathcal{A}}$*, at least one of the following three conditions is satisfied:*

(i) the discriminant of $Q(t,\mathbf{x})$ *with respect to* t *vanishes, i.e.* $Q(t,\mathbf{x})$ *has (at least) a double root in* t.
(ii) $|\alpha_{k-1}(\mathbf{x})| = |\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})|$.
(iii) $|\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})| = |\alpha_{k+n+2}(\mathbf{x})|$.

Proof. We need the following two simple statements.

Lemma 15. *Let* Pol_d *be the set of all monic polynomials of degree* d *with complex coefficients. Let* $\Sigma_{p,q} \subset Pol_d$ *be the subset of polynomials satisfying*

$$|\alpha_p| = |\alpha_{p+1}| = \cdots = |\alpha_q|, \tag{6}$$

where $1 \le p < q \le d$ *and* $\alpha_1, \alpha_2, \dots, \alpha_d$ *are the roots of the polynomials ordered according to their increasing absolute values. Then* $\Sigma_{p,q}$ *is a real semi-algebraic set of codimension* $q-p$ *whose boundary is the union of three pieces:* $\Sigma_{p-1,q}$, $\Sigma_{p,q+1}$ *and the intersection of* $\Sigma_{p,q}$ *with the standard discriminant in* Pol_d*, i.e. the set of polynomials having multiple roots. (Notice that if* $p = 1$ *then* $\Sigma_{p-1,q}$ *is empty, and if* $q = d$ *then* $\Sigma_{p,q+1}$ *is empty by definition.)*

Proof. $\Sigma_{p,q}$ is obtained as the image under the Vieta map of an obvious semi-algebraic set $|\alpha_1| \leq |\alpha_2| \leq \dots \leq |\alpha_p| = |\alpha_{p+1}| = \dots = |\alpha_q| \leq |\alpha_{q+1}| \leq \dots \leq |\alpha_d|$. Notice that the Vieta map is a local diffeomorphism outside the preimage of the standard discriminant. Therefore the boundary of $\Sigma_{p,q}$ must either belong to the standard discriminant or to one of $\Sigma_{p-1,q}$ or $\Sigma_{p,q+1}$. The former is the common boundary between $\Sigma_{p,q}$ and $\Sigma_{p-1,q-1}$ and the latter is the common boundary between $\Sigma_{p,q}$ and $\Sigma_{p+1,q+1}$. □

Given a closed Whitney stratified set X (for example, semi-analytic) we say that X is *a k-dimensional stratified set without boundary* if

(i) the top-dimensional strata of X have dimension k;
(ii) for any point x lying in any stratum of dimension $k-1$, one can choose an orientation of the (germs of) k-dimensional strata of a sufficiently small neighborhood of x in X so that $\partial X = 0$.

Lemma 16. *The boundary of the intersection of any closed semi-algebraic set Γ with any closed algebraic set Θ is included in the intersection of the boundary $\partial\Gamma$ with Θ.*

Proof. Observe that any real algebraic variety X of dimension k is a stratifiable set without boundary. Indeed, the fact we are proving is local, and it suffices to prove it for generic x on $(k-1)$-dimensional strata.

Consider an embedding of X in some high-dimensional linear space, take the Whitney stratification with x on its stratum $Y \subset B$ of dimension $k-1$, and a transversal to Y of codimension $k-1$ at x.

Therefore, we may now assume that the germ of X near x is topologically a product of a germ of an algebraic curve and a germ of a smooth manifold of dimension $k-1$. Furthermore, a germ of any real algebraic curve Γ can always be oriented so that $\partial\Gamma = 0$, which follows from the existence of Puiseux series for an arbitrary branch of an algebraic curve. This argument shows that any point in the intersection $\Gamma \cap \Theta$ which does not belong to the boundary of Γ can not lie on the boundary of this intersection, which settles Lemma 16. □

Lemmas 15 and 16 immediately imply Proposition 14 since every $C_{\mathcal{A}}$ is the intersection of an appropriate $\Sigma_{p,q}$ with an appropriate affine subspace in Pol_{k+h}. □

Proof of Theorem 12. In our notation, let $D_j^m(\mathbf{x})$ be the determinant of the $m \times m$-matrix A_I with $I = \{j+1, j+2, \dots, j+m\}$ for $0 \leq j \leq n$. It is evident that $\mathcal{E}_{\mathcal{A}}^{(m)}$ is a subset of the set $\widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)}$ of solutions to the system of polynomial equations

$$D_0^m(\mathbf{x}) = D_1^m(\mathbf{x}) = \dots = D_n^m(\mathbf{x}) = 0. \tag{7}$$

We will show the stronger statement that, in notation of Theorem 12,

$$\lim_{m\to\infty} \widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)} \subseteq C_{\mathcal{A}}.$$

which follows from Proposition 18 below.

Remark 17. Although each individual $\widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)}$ (considered as a points set with multiplicities) is strictly bigger than $\mathcal{E}_{\mathcal{A}}^{(m)}$, the limits $B_{\mathcal{A}} = \lim_{m\to\infty} \mathcal{E}_{\mathcal{A}}^{(m)}$ and $\lim_{m\to\infty} \widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)}$ seem to coincide as infinite sets.

In Theorem 4 of [1] it was shown that each sequence of determinants $\{D_j^m(\mathbf{x})\}_{m=1}^{\infty}$ as above satisfies a linear recurrence relation with coefficients depending on $\mathbf{x}$. The characteristic polynomial $\chi_j(t)$ of the j-th recurrence can be factorized as

$$\chi_j(t,\mathbf{x}) = \prod_{\sigma}(t - r_{j\sigma}), \text{ where } r_{j\sigma} = (-1)^{k+j}(\alpha_{\sigma_1}\cdots\alpha_{\sigma_{k+j}})^{-1}, \tag{8}$$

and σ is a $k+j$-subset of $1,2,\ldots,k+h$.

Proposition 18. *Suppose that $\{\mathbf{x}_m\}_1^{\infty}$ is a sequence of points in $\mathbb{C}^{n+1}$ satisfying the system of equations*

$$D_j^m(\mathbf{x}_m) = 0 \text{ for } j = 0,1,\ldots,n \text{ and } m = 1,2,\ldots \tag{9}$$

and such that the limit $\lim_{m\to\infty}\mathbf{x}_m =: \mathbf{x}^*$ exists. Then for all $j = 0,\ldots,n$ we have $|\alpha_{k+j}(\mathbf{x}^*)| = |\alpha_{k+j+1}(\mathbf{x}^*)|$ when the α_i are indexed with increasing order of their modulus.

Proof. Provided that all the roots of $\chi_j(t,\mathbf{x})$ are distinct, by using a version of Widom's formula, (see [1, 4]) we have

$$D_j^m(\mathbf{x}) = \sum_{\sigma}\prod_{l\in\sigma, i\notin\sigma}\left(1 - \frac{\alpha_l(\mathbf{x})}{\alpha_i(\mathbf{x})}\right)^{-1}\cdot r_{j\sigma}(\mathbf{x})^m. \tag{10}$$

We may assume that for $\mathbf{x}^*$ and fixed j, the $r_{j\sigma}(\mathbf{x}^*)$ are ordered decreasingly with respect to their modulus (for some ordering $\sigma = 1,2,\ldots$). The goal is to prove that $|r_{j1}(\mathbf{x}^*)| = |r_{j2}(\mathbf{x}^*)|$ since this implies $|\alpha_{k+j}(\mathbf{x}^*)| = |\alpha_{k+j+1}(\mathbf{x}^*)|$. We show this fact by contradiction.

Assume that $|r_{j1}(\mathbf{x}^*)| > |r_{j2}(\mathbf{x}^*)| \geq \cdots \geq |r_{jb}(\mathbf{x}^*)|$, i.e. that the largest root is simple and has modulus strictly larger than any other root of the characteristic equation (8). By examining (10), it is evident that $r_{j1}(\mathbf{x}_m)^m$ is the dominating term for sufficiently large m, that is, $D_j^m(\mathbf{x}_m)/r_{j1}(\mathbf{x}_m)^m \to L \neq 0$ as $m \to \infty$.

By standard properties of linear recurrences, this holds even when there are multiple zeros among the smaller roots; remember that our assumption was that $r_{j1}(x_m)$ is a simple zero of (8) when m is large enough.

Hence, for sufficiently large m, $D_j^m(\mathbf{x}_m) \approx Lr_{j1}(x_m)^m$, which is non-zero for sufficiently large m. This contradicts the condition that $\mathbf{x}_m$ satisfies (9). Consequently, $|r_{j1}(\mathbf{x}^*)| = |r_{j2}(\mathbf{x}^*)|$ for $j = 0,1,\ldots,n$ and this implies Proposition 18. □

Proposition 18 implies that $\mathbf{x}$ lies in $B_{\mathcal{A}}$ only if $\mathbf{x}$ is a limit of solutions to (9), but such limit $\mathbf{x}$ must satisfy that $|\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})|$. Therefore, $B_{\mathcal{A}} \subseteq C_{\mathcal{A}}$. □

3. Further directions

1. It seems relatively easy to describe the stratified structure of $C_{\mathcal{A}}$ at least in the case of generic $\mathcal{A}$. In particular, in the Chebyshev case of Example 8 the set $C_{\mathcal{A}}$ has the same stratification as a simplex of corresponding dimension. One can also understand the stratified structure of the sets $\Sigma_{p,q}$ introduced in Lemma 15. Since each $C_{\mathcal{A}}$ is obtained from a corresponding $\Sigma_{p,q}$ by intersecting it with an affine subspace, the stratified structure of the former for generic $\mathcal{A}$ is also describable. On the other hand, our Example 11 seems to show more complicated stratified structure due to the presence of additional symmetry.

2. We say that an (infinite) complex-valued matrix $\mathcal{A}$ has a *weak univariate orthogonality property* if the sequence of characteristic polynomials of its principal minors obeys the standard 3-term recurrence relation with complex coefficients. There is a straightforward version of this notion for finite square matrices. Obviously, any Jacobi matrix has this property. However, it seems that for any $m \geq 3$ the set $WO_m \subset Mat(m,m)$ of all $m \times m$-matrices with the latter property has a bigger dimension than the set $Jac_m \subset Mat(m,m)$ of all Jacobi $m \times m$-matrices.

Problem 19. Find the dimension of WO_m.

3. Analogously, given a non-negative integer n, we say that an (infinite) complex-valued matrix $\mathcal{A}$ has a *weak n-variate orthogonality property* if the above family $\{P^I_{\mathcal{A}}(x_0, x_1, \dots, x_n)\}$ (see Definition 4) satisfies the 3-term recurrence relation (2.2) of Theorem 2.1 of [17] with complex coefficients.

There are many similarities between families $\{P^I_{\mathcal{A}}(x_0, x_1, \dots, x_n)\}$ and families of multivariate orthogonal polynomials which by one of the standard definitions of such polynomials also satisfy (2.2) of Theorem 2.1 of [17] with real coefficients.

Our computer experiments show that in this aspect the case $n > 0$ is quite different from the classical case $n = 0$. In particular, we believe that the following conjecture holds.

Conjecture 20. *Given $n > 0$, a banded matrix $\mathcal{A}$ has a weak n-variate orthogonality property if it is of the form*

$$\mathcal{A} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n+1} & 0 & 0 & 0 & \dots \\ d_{-1} & d_0 & d_1 & \dots & d_n & d_{n+1} & 0 & 0 & \dots \\ 0 & d_{-1} & d_0 & \dots & d_{n-1} & d_n & d_{n+1} & 0 & \dots \\ 0 & 0 & d_{-1} & \dots & d_{n-2} & d_{n-1} & d_n & d_{n+1} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_0, \dots, a_{n+1}, d_{-1}, \dots, d_{n+1} \in \mathbb{C}$.

Acknowledgement. The authors are sincerely grateful to Professor M. Tater who actively participated in the consideration of some initial examples related to multivariate Chebyshev polynomials for his help and to the Nuclear Physics Institute in Řež near Prague for the hospitality in March 2011. We want to thank Professor A. Gabrielov of Purdue University for his help with the proof of Proposition 14. We also thank Professor Y. Xu for helpful discussions and references.

References

[1] P. Alexandersson, *Schur polynomials, banded Toeplitz matrices and Widom's formula*, Electronic Journal of Combinatorics **19** (2012), no. 4.
[2] R. J. Beerends, *Chebyshev Polynomials in Several Variables and the Radial Part of the Laplace-Beltrami operator*, Transactions of the American Mathematical Society **328** (1991), no. 2, 779–814.
[3] A. Böttcher, S. M. Grudsky, *Spectral properties of banded Toeplitz matrices*, SIAM, Philadelphia, PA, 2005.
[4] A. Böttcher, B. Silbermann, *Introduction to large truncated Toeplitz matrices*, Springer, New York, 1998.
[5] S. Delvaux, M. Duits, *An equilibrium problem for the limiting eigenvalue distribution of rational toeplitz matrices*, SIAM Journal on Matrix Analysis and Applications **31** (2010), no. 4, 1894–1914.
[6] S. Delvaux, *Equilibrium problem for the eigenvalues of banded block toeplitz matrices*, Mathematische Nachrichten **285** (2012), no. 16, 1935–1962.
[7] M. Duits, A. B. J. Kuijlaars, *An equilibrium problem for the limiting eigenvalue distribution of banded Toeplitz matrices*, SIAM Journal on Matrix Analysis and Applications **30** (2008), no. 1, 173–196.
[8] K. B. Dunn, R. Lidel, *Multi-dimensional generalizations of the Chebyshev polynomials*, Proceedings of the Japan Academy **56(A)** (1980), 154–159.
[9] D. Gepner, *Fusion rings and geometry*, Communications in Mathematical Physics **141** (1991), no. 2, 381–411.
[10] T. Koornwinder, *Two-variable analogues of the classical orthogonal polynomials*, Theory and application of special functions, Academic Press, New York, 1975.

[11] A. Lee, *Centrohermitian and skew-centrohermitian matrices*, Linear Algebra and its Applications **29** (1980), 205–210.
[12] P. Schmidt, F. Spitzer, *The Toeplitz matrices of an arbitrary Laurent polynomial*, Mathematica Scandinavica **8** (1960), 15–38.
[13] B. Shapiro, *Algebro-geometric aspects of Heine-Stieltjes theory*, Journal of the London Mathematical Society **83** (2011), no. 1, 36–56.
[14] B. Shapiro, M. Shapiro, *On eigenvalues of rectangular matrices*, Proceedings of The Steklov Institute of Mathematics **267** (2009), 248–255.
[15] H. Widom, *Asymptotic behavior of block Toeplitz matrices and determinants*, Advances in Mathematics **13** (1974), no. 3, 284 – 322.
[16] ———, *Asymptotic behavior of block Toeplitz matrices and determinants. II*, Advances in Mathematics **21** (1976), no. 1, 1–29.
[17] Y. Xu, *Lecture notes on orthogonal polynomials of several variables*, Advances in the Theory of Special Functions and Orthogonal Polynomials **2** (2004), 135–188.

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