

AROUND GENERALISED BOCHNER-KRALL PROBLEM

LUIS MIGUEL ANGUAS, DOLORES BARRIOS ROLANÍA, BORIS SHAPIRO, AND MILOŠ TATER

ABSTRACT. Below we study which differential operators of the form $z^j \partial_z^j + z^m \partial_z^\ell$ have sequences of eigenpolynomials satisfying a finite recurrence relation.

1. INTRODUCTION

In [4], the third and the fourth authors jointly with E. Horozov suggested an algebraic high order generalisation of the classical Bochner-Krall problems and its conjectural solution, see Conjectures 1.7–1.10 in loc. cit. The goal of the present paper is a partial confirmation of these conjectures for a special but representative three-index family of linear differential operators studied in [2].

We consider the differential operator

$$(1) \quad T \equiv z^j \partial_z^j + z^m \partial_z^\ell$$

where $0 \leq m < \ell$, $1 \leq j < \ell$. It is a linear differential operator in the form

$$(2) \quad \sum_{i=1}^N a_i(z) \partial_z^i$$

where $a_i(z) = \sum_{q=0}^i a_{i,q} z^q$ is a polynomial with $\deg(a_i) \leq i$. In our case, the order is $N = \ell$ and the coefficients are defined by

$$a_{j,q} = \begin{cases} 0, & q \neq j \\ 1, & q = j, \end{cases} \quad a_{\ell,q} = \begin{cases} 0, & q \neq m \\ 1, & q = m. \end{cases}$$

Such differential operators have infinite sequences of eigenpolynomials, see e.g. [2]. In [1] the sequences $\{\delta_n^{(k)}\}$, $k = 0, 1, \dots, n$, were used to study the differential operators (2) as well as their eigenvalues and their eigenpolynomials. Using these tools we see that the operator given in (1) has the eigenvalues

$$(3) \quad \lambda_n = \delta_n^{(0)} = \sum_{i=1}^n \binom{n}{i} i! a_{i,i} = \begin{cases} 0, & n < j \\ \binom{n}{j} j!, & n \geq j, \end{cases}$$

and, for $k > 0$,

$$(4) \quad \delta_n^{(k)} = \sum_{i=k}^n \binom{n}{i} i! a_{i,i-k} = \begin{cases} 0, & n < \ell, \\ \binom{n}{\ell} \ell!, & n \geq \ell, \quad k = \ell - m, \\ 0, & n \geq \ell, \quad k \neq \ell - m. \end{cases}$$

We used the sequences $\{\delta_n^{(k)}\}$ in [1] to prove that the eigenpolynomials of certain differential operator (2) satisfy a recurrence relation.

Our main conjectural claim is the following. In this paper we prove it completely for $0 \leq m < \ell - 1$ and prove the sufficiency of $j = 1$ in the remaining case $m = \ell - 1$.

Conjecture 1. *Let*

$$T = z^j \partial_z^j + z^m \partial_z^\ell,$$

with integers $0 \leq m < \ell$ and $1 \leq j < \ell$, and let $\{p_n(z)\}_{n \geq 0}$ be the sequence of monic eigenpolynomials of T ,

$$T p_n = \lambda_n p_n, \quad \deg p_n = n.$$

Then the sequence $\{p_n\}$ satisfies a finite-term recurrence relation

$$p_{n+1}(z) = z p_n(z) + \sum_{s=0}^d \alpha_{n,n-s} p_{n-s}(z),$$

with d independent of n , if and only if $j = 1$.

Equivalently, for $j > 1$, the eigenpolynomials of T do not satisfy any finite-term recurrence relation.

In particular, we proved the following theorem.

Theorem 1. [1, Theorem 5] *Let L be the differential operator defined as*

$$(5) \quad L := \sum_{i=1}^N a_i z^{i-1} \partial_z^i + z \partial_z.$$

If $\{P_n\}$ is the sequence of eigenpolynomials and $\{\lambda_n\}$, $n \in \mathbb{N}$, the corresponding sequence of eigenvalues for L , (that is, $LP_n = \lambda_n P_n$, $n \in \mathbb{N}$,) then the following recurrence relation is satisfied,

$$(6) \quad \left. \begin{aligned} \sum_{s=1}^{N-1} \alpha_{n,n-s} P_{n-s}(z) + (\alpha_{n,n} - z) P_n(z) + P_{n+1}(z) &= 0, \quad n = 0, 1, \dots \\ P_{-N+1} = P_{-N+2} = \dots = P_{-1} &= 0, \quad P_0 = 1, \end{aligned} \right\}$$

with

$$(7) \quad \alpha_{n,n-s} = \frac{\delta_n^{(1)} \delta_{n-1}^{(1)} \dots \delta_{n-s+1}^{(1)}}{(s+1)!} \sum_{r=0}^{s+1} \binom{s+1}{r} (-1)^r \delta_{n-s+r}^{(1)}, \quad s = 0, \dots, N-1,$$

(understanding $\delta_n^{(1)} \delta_{n-1}^{(1)} \dots \delta_{n-s+1}^{(1)} = 1$ when $s = 0$).

The main goal of this paper is to obtain a similar result of Theorem 1 for the operator T of (1).

Firstly, we will determine the monic eigenpolynomials of such differential operator T . We denote these polynomials as

$$P_n(z) = z^n + b_{n,n-1} z^{n-1} + \dots + b_{n,0}.$$

From (26) in [1] we have

$$(8) \quad \sum_{k=0}^{\ell} \delta_{s+k}^{(k)} b_{n,s+k} = \lambda_n b_{n,s}, \quad s = 0, 1, \dots, n,$$

which allows us to determine the coefficients $b_{n,i}$ in terms of the sequences $\{\delta_n^{(k)}\}$. In our case, taking into account (3) and (4), (8) drives to

$$(9) \quad (\lambda_n - \lambda_s) b_{n,s} = \delta_{s+\ell-m}^{(\ell-m)} b_{n,s+\ell-m}, \quad s = 0, 1, \dots, n.$$

Here and in the sequel, we assume $b_{n,i} = 0$ when $i > n$.

We underline that any polynomial $P_n(z)$ of degree $\deg(P_n) = n < j$ is an eigenfunction of (1) because in this case $TP_n = 0$ and the corresponding eigenvalue is $\lambda_n = 0$. Then, there is not a unique sequence of eigenpolynomials for T . This fact could allow us to choose those polynomials appropriately to our interests (for example, to verify a certain recurrence relation). (This also seems to indicate the usefulness to choose $j = 1$ to eliminate trivial situations.)

On the other hand, if $n \geq \max\{m, j\}$ then, since (4), in (9) we have

$$(10) \quad \left[\binom{n}{j} - \binom{s}{j} \right] j! b_{n,s} = \begin{cases} 0, & s = 0, 1, \dots, m-1, \\ \binom{s+\ell-m}{\ell} \ell! b_{n,s+\ell-m}, & s = m, \dots, n - (\ell - m), \end{cases}$$

understanding $\binom{s}{j} = 0$ when $s < j$. From this, taking into account that $\binom{n}{j} - \binom{s}{j} \neq 0$ for $s = 0, 1, \dots, m-1$, we arrive to

$$b_{n,0} = b_{n,1} = \dots = b_{n,m-1} = 0$$

and

$$(11) \quad P_n(z) = z^n + b_{n,n-1} z^{n-1} + \dots + b_{n,m} z^m.$$

Moreover, if $s > n - (\ell - m)$ then $s + \ell - m > n$ and we have $b_{n,s+\ell-m} = 0$ in (9). Hence, $b_{n,s} = 0$ for $s = n - (\ell - m) + 1, n - (\ell - m) + 2, \dots, n - 1$. For the same reason we have $b_{n,s} = 0$ for $s = n - t(\ell - m) + 1, n - t(\ell - m) + 2, \dots, n - t(\ell - m) + (\ell - m - 1)$, $t = 1, 2, \dots$. Therefore, in (11) we have only coefficients $b_{n,n-t(\ell-m)}$. This is,

$$(12) \quad P_n(z) = z^n + b_{n,n-t(\ell-m)} z^{n-t(\ell-m)} + \dots + b_{n,n-r(\ell-m)} z^{n-r(\ell-m)},$$

where $r \in \mathbb{N}$ is such that

$$n - r(\ell - m) \geq m > n - (r+1)(\ell - m),$$

this is,

$$r = \left\lfloor \frac{n-m}{\ell-m} \right\rfloor.$$

Now, we determine the coefficients $b_{n,n-t(\ell-m)}$, $t = 1, 2, \dots, r$, in (12). Taking $s = n - t(\ell - m)$ in (10),

$$\left[\binom{n}{j} - \binom{n-t(\ell-m)}{j} \right] j! b_{n,n-t(\ell-m)} = \binom{n-t(\ell-m)}{\ell} \ell! b_{n,n-t(\ell-m)}.$$

Then, for $t = 1$,

$$b_{n,n-(\ell-m)} = \frac{\binom{n}{\ell} \ell!}{\left[\binom{n}{j} - \binom{n-(\ell-m)}{j} \right] j!}$$

and, iterating,

$$(13) \quad b_{n,n-t(\ell-m)} = \frac{\binom{n}{\ell} \binom{n-(\ell-m)}{\ell} \dots \binom{n-(t-1)(\ell-m)}{\ell} (\ell!)^t}{(j!)^t \prod_{i=1}^t \left[\binom{n}{j} - \binom{n-i(\ell-m)}{j} \right]}, \quad t = 1, 2, \dots, r.$$

Besides, in the case $\ell > n \geq \max\{m, j\}$ we have $n - (\ell - m) < m$ and $r = 0$. Then, $P_n(z) = z^n$ in (11).

Finally, if $j \leq n < m$ we see that $s + \ell - m < \ell$ and $\delta_{s+\ell-m}^{(\ell-m)} = 0$ in (4) for $s = 0, 1, \dots, n$. Then, in (9) is $b_{n,s} = 0$ for $s = 0, 1, \dots, n-1$. Hence, $P_n(z) = z^n$ also in this case.

In summary,

$$(14) \quad P_n(z) = \begin{cases} \text{arbitrarily chosen,} & \text{if } n < j \\ z^n, & \text{if } j \leq n < \ell \\ z^n + \sum_{t=1}^{\lfloor \frac{n-m}{\ell-m} \rfloor} b_{n,n-t(\ell-m)} z^{n-t(\ell-m)}, & \text{if } n \geq \ell. \end{cases}$$

2. CASE OF SYMMETRIC POLYNOMIALS

In this section we assume $m \leq \ell - 2$, this is, $0 \leq m < \ell - 1$. The case $m = \ell - 1$ will be considered in next section.

Due to the above expression of the polynomials, it is immediate to verify that the following property is fulfilled,

$$P_n(wz) = w^n P_n(z), \quad n \in \mathbb{N},$$

where $w = e^{\frac{2\pi i}{\ell-m}}$. This property means that the sequence $\{P_n(z)\}$ is $(\ell - m - 1)$ -symmetric (see [5, page 5] and [3, Lemme 5.1]).

As a consequence (see [3, Theoreme 5.1]), if these polynomials satisfy a recurrence relation with a fixed number of terms, then this recurrence relation is necessarily of the form

$$(15) \quad \begin{cases} P_{n+(\ell-m)}(z) = z P_{n+(\ell-m-1)}(z) + \gamma_n P_n(z), & n \geq 0, \\ P_n(z) = z^n, & n = 0, 1, \dots, \ell - m - 1, \end{cases}$$

where $\gamma_n \neq 0$ for any $n \geq 0$.

The main result of this section in the following.

Theorem 2. *Assume $0 \leq m < \ell - 1$. Then the sequence of eigenpolynomials $\{P_n\}$ of (1) satisfy a recurrence relation if and only if $m = 0$ and $j = 1$.*

To prove this theorem we need the following auxiliary result.

Lemma 1. *Take $0 \leq m < j = \ell - 1$. If the sequence of eigenpolynomials $\{P_n\}$ satisfy a recurrence relation with a fixed number of terms, then $\ell = 2$.*

Proof.- Since the sequence $\{P_n(z)\}$ is $(\ell - m - 1)$ -symmetric, under the conditions given in the statement we know that (15) is satisfied. Let $k = \ell - m$ and take $n = \ell$. Then,

$$P_{\ell+k}(z) = z P_{\ell+k-1}(z) + \gamma_\ell P_\ell(z),$$

for some $\gamma_\ell \in \mathbb{C}$, $\gamma_\ell \neq 0$. Using (14),

$$\begin{aligned} P_{\ell+k}(z) &= z^{\ell+k} + b_{\ell+k,\ell} z^\ell + b_{\ell+k,\ell-k} z^{\ell-k}. \\ P_{\ell+k-1}(z) &= z^{\ell+k-1} + b_{\ell+k-1,\ell-1} z^{\ell-1}. \\ P_\ell(z) &= z^\ell + b_{\ell,\ell-k} z^{\ell-k}. \end{aligned}$$

Then, comparing coefficients,

$$(16) \quad b_{\ell+k,\ell-k} = (b_{\ell+k,\ell} - b_{\ell+k-1,\ell-1}) b_{\ell,\ell-k}$$

and, taking into account (13),

$$\begin{aligned} 0 &= b_{\ell+k,\ell-k} - (b_{\ell+k,\ell} - b_{\ell+k-1,\ell-1}) b_{\ell,\ell-k} = \\ &= \frac{\binom{\ell+k}{\ell} \ell^2}{\left[\binom{\ell+k}{\ell-1} - \ell \right] \binom{\ell+k}{\ell-1}} - \left(\frac{\binom{\ell+k}{\ell}}{\binom{\ell+k}{\ell-1} - \ell} - \frac{\binom{\ell+k-1}{\ell}}{\binom{\ell+k-1}{\ell-1} - 1} \right) \ell = \frac{\binom{\ell+k}{\ell} \ell}{\left[\binom{\ell+k}{\ell-1} - \ell \right] \binom{\ell+k}{\ell-1}} \left[\ell - \binom{\ell+k}{\ell-1} \right] + \frac{\binom{\ell+k-1}{\ell}}{\binom{\ell+k-1}{\ell-1} - 1} \ell \end{aligned}$$

$$(17) \quad = -\frac{\binom{\ell+k}{\ell}\ell}{\binom{\ell+k}{\ell-1}} + \frac{\binom{\ell+k-1}{\ell}\ell}{\binom{\ell+k-1}{\ell-1}-1} = \frac{-\binom{\ell+k}{\ell}\left[\binom{\ell+k-1}{\ell-1}-1\right] + \binom{\ell+k}{\ell-1}\binom{\ell+k-1}{\ell}}{\binom{\ell+k}{\ell-1}\left[\binom{\ell+k-1}{\ell-1}-1\right]\frac{1}{\ell}}.$$

The numerator of the above expression is

$$(18) \quad 0 = \binom{\ell+k}{\ell} - \binom{\ell+k}{\ell}\binom{\ell+k-1}{\ell-1} + \binom{\ell+k}{\ell-1}\binom{\ell+k-1}{\ell} = \binom{\ell+k}{\ell}\left[1 - \frac{1}{k+1}\binom{\ell+k-1}{\ell-1}\right].$$

Hence

$$\frac{1}{k+1}\binom{\ell+k-1}{\ell-1} = \prod_{i=1}^k \frac{\ell+k-i}{2+k-i} = 1$$

Moreover, $\frac{\ell+k-i}{k+2-i} \geq 1$, $i = 1, \dots, k$, from which it follows that $\ell = 2$. \square

Proof of Theorem 2. In the first place, we prove that $m = 0$ is a necessary condition for the existence of a recurrence relation with a fixed number of terms.

We differentiate between two options for j .

Firstly, if $j \leq \ell - 2$ and such recurrence relation exists we know that (15) is satisfied. Taking $n = m - 1$, then $n + \ell - m = \ell - 1$ and, due the expression (14) for the polynomials,

$$P_{n+(\ell-m-1)}(z) = z^{\ell-2}, \quad P_{n+(\ell-m)}(z) = z^{\ell-1}.$$

Substituting in (15) we see

$$\gamma_{m-1}P_{m-1}(z) = 0.$$

Because $\gamma_n \neq 0$ for any $n \in \mathbb{N}$, we deduce $P_{m-1}(z) = 0$. However, this only is possible when $m = 0$.

Secondly, if the recurrence relation exists and $j = \ell - 1$ then since Lemma 1 necessarily $j = 1$ and $\ell = 2$. Then, because $0 \leq m < \ell - 1$, we have also $m = 0$.

In the following we prove that $j = 1$ is also a necessary condition for the existence of recurrence relation. Assume that such recurrence relation (15) exists. From the above we know $m = 0$. Moreover,

$$P_{2\ell}(z) = zP_{2\ell-1}(z) + \gamma_{\ell}(z)P_{\ell}(z),$$

where

$$P_{2\ell}(z) = z^{2\ell} + b_{2\ell,\ell}z^{\ell} + b_{2\ell,0}, \quad P_{2\ell-1}(z) = z^{2\ell-1} + b_{2\ell-1,\ell-1}z^{\ell-1}, \quad P_{\ell}(z) = z^{\ell} + b_{\ell,0}.$$

Comparing the coefficients,

$$\gamma_{\ell} = b_{2\ell,\ell} - b_{2\ell-1,\ell-1}, \quad \gamma_{\ell}b_{\ell,0} = b_{2\ell,0},$$

and

$$(19) \quad b_{\ell,0}(b_{2\ell,\ell} - b_{2\ell-1,\ell-1}) = b_{2\ell,0}.$$

Using (13) to obtain the expression of each coefficient in terms of j and ℓ :

$$b_{\ell,0} = \frac{\ell!}{\binom{\ell}{j}j!}, \quad b_{2\ell,0} = \frac{\binom{2\ell}{\ell}(\ell!)^2}{(j!)^2\left[\binom{2\ell}{j} - \binom{\ell}{j}\binom{2\ell}{j}\right]}, \quad b_{2\ell,\ell} - b_{2\ell-1,\ell-1} = \frac{\binom{2\ell}{\ell}\ell!}{\left[\binom{2\ell}{j} - \binom{\ell}{j}\right]j!} - \frac{\binom{2\ell-1}{\ell}\ell!}{\left[\binom{2\ell-1}{j} - \binom{\ell-1}{j}\right]j!}.$$

Replacing in (19) and multiplying both terms by $\left[\binom{2\ell}{j} - \binom{\ell}{j}\right]\left[\binom{2\ell-1}{j} - \binom{\ell-1}{j}\right]$,

$$\frac{1}{\binom{\ell}{j}}\left[\binom{2\ell}{\ell}\left[\binom{2\ell-1}{j} - \binom{\ell-1}{j}\right] - \binom{2\ell-1}{\ell}\left[\binom{2\ell}{j} - \binom{\ell}{j}\right]\right] = \frac{\binom{2\ell}{\ell}}{\binom{2\ell}{j}}\left[\binom{2\ell-1}{j} - \binom{\ell-1}{j}\right].$$

After some easy rearrangements

$$\left[\frac{\binom{2\ell}{\ell}}{\binom{\ell}{j}} - \frac{\binom{2\ell}{\ell}}{\binom{2\ell}{j}}\right]\left[\binom{2\ell-1}{j} - \binom{\ell-1}{j}\right] = \frac{\binom{2\ell-1}{\ell}}{\binom{\ell}{j}}\left[\binom{2\ell}{j} - \binom{\ell}{j}\right]$$

and finally,

$$\binom{2\ell}{\ell}\left[\binom{2\ell-1}{j} - \binom{\ell-1}{j}\right] = \binom{2\ell}{j}\binom{2\ell-1}{\ell}.$$

This is,

$$\frac{(2\ell-1)!/(2\ell-j)!}{(\ell-1)!/(\ell-j)!} = 1.$$

We can express the left hand side of the above equality as $\prod_{k=0}^{\ell-1} \frac{2\ell-1-k}{2\ell-j-k}$. Then we have

$$\prod_{k=0}^{\ell-1} \frac{2\ell-1-k}{2\ell-j-k} = 1.$$

Because $\frac{2\ell-1-k}{2\ell-j-k} \geq 1$ for each $k = 0, 1, \dots, \ell-1$, from this we arrive to $j = 1$.

We will prove that $m = 0, j = 1$ are sufficient conditions for the existence of a recurrence relation. In this case, (14) becomes

$$P_n(z) = \begin{cases} z^n, & \text{if } 0 \leq n < \ell \\ z^n + \sum_{t=1}^{\lfloor \frac{n}{\ell} \rfloor} b_{n,n-t\ell} z^{n-t\ell}, & \text{if } n \geq \ell. \end{cases}$$

Moreover, since $j = 1$ and $m = 0$, formula (13) simplifies to

$$(20) \quad b_{n,n-t\ell} = \frac{\binom{n}{\ell} \binom{n-\ell}{\ell} \dots \binom{n-(t-1)\ell}{\ell} (\ell!)^t}{\prod_{r=1}^t \left[\binom{n}{1} - \binom{n-r\ell}{1} \right]} = \frac{(t\ell)! \binom{n}{t\ell}}{\ell^t t!}, \quad t = 1, 2, \dots, \left\lfloor \frac{n}{\ell} \right\rfloor.$$

Indeed,

$$\prod_{r=1}^t \left[\binom{n}{1} - \binom{n-r\ell}{1} \right] = \prod_{r=1}^t r\ell = \ell^t t!.$$

We claim that the sequence $\{P_n\}$ satisfies

$$(21) \quad P_{n+\ell}(z) = zP_{n+\ell-1}(z) + \gamma_n P_n(z), \quad n \geq 0,$$

where

$$(22) \quad \gamma_n := b_{n+\ell,n} - b_{n+\ell-1,n-1},$$

understanding $b_{n+\ell-1,n-1} = 0$ when $n = 0$.

First, let $0 \leq n < \ell$. Then

$$\begin{aligned} P_n(z) &= z^n, \\ P_{n+\ell}(z) &= z^{n+\ell} + b_{n+\ell,n} z^n, \\ P_{n+\ell-1}(z) &= z^{n+\ell-1} + b_{n+\ell-1,n-1} z^{n-1} \end{aligned}$$

(with $b_{n+\ell-1,n-1} = 0$ if $n = 0$). Therefore,

$$zP_{n+\ell-1}(z) + \gamma_n P_n(z) = z^{n+\ell} + (b_{n+\ell-1,n-1} + \gamma_n) z^n = P_{n+\ell}(z),$$

by the definition of γ_n .

Now let $n = k\ell + i$, where $k \geq 1$ and $i \in \{0, 1, \dots, \ell - 1\}$. We will prove

$$(23) \quad P_{(k+1)\ell+i}(z) = zP_{(k+1)\ell+i-1}(z) + \gamma_{k\ell+i} P_{k\ell+i}(z).$$

Using (14), we have

$$\begin{aligned} P_{(k+1)\ell+i}(z) &= z^{(k+1)\ell+i} + \sum_{s=0}^k b_{(k+1)\ell+i, (k-s)\ell+i} z^{(k-s)\ell+i}, \\ P_{(k+1)\ell+i-1}(z) &= z^{(k+1)\ell+i-1} + \sum_{s=0}^k b_{(k+1)\ell+i-1, (k-s)\ell+i-1} z^{(k-s)\ell+i-1}, \end{aligned}$$

where $b_{(k+1)\ell+i-1, i-1} = 0$ if $i = 0$, and

$$P_{k\ell+i}(z) = z^{k\ell+i} + \sum_{s=1}^k b_{k\ell+i, (k-s)\ell+i} z^{(k-s)\ell+i}.$$

Hence (23) is equivalent to

$$(24) \quad b_{(k+1)\ell+i, (k-s)\ell+i} - b_{(k+1)\ell+i-1, (k-s)\ell+i-1} = \gamma_{k\ell+i} b_{k\ell+i, (k-s)\ell+i}, \quad s = 0, 1, \dots, k,$$

where for $s = 0$ we use the convention $b_{k\ell+i, k\ell+i} = 1$. In particular,

$$\gamma_{k\ell+i} = b_{(k+1)\ell+i, k\ell+i} - b_{(k+1)\ell+i-1, k\ell+i-1}.$$

We now compute each term by means of (20). First,

$$\begin{aligned} \gamma_{k\ell+i} &= \frac{\ell!}{\ell} \binom{(k+1)\ell+i}{\ell} - \frac{\ell!}{\ell} \binom{(k+1)\ell+i-1}{\ell} \\ &= (\ell-1)! \left[\binom{(k+1)\ell+i}{\ell} - \binom{(k+1)\ell+i-1}{\ell} \right] \\ &= (\ell-1)! \binom{(k+1)\ell+i-1}{\ell-1} = \frac{((k+1)\ell+i-1)!}{(k\ell+i)!}, \end{aligned}$$

where we used Pascal's identity in the third equality.

Next, for $s = 0, 1, \dots, k$,

$$(25) \quad b_{(k+1)\ell+i, (k-s)\ell+i} = \frac{((s+1)\ell)!}{\ell^{s+1} (s+1)!} \binom{(k+1)\ell+i}{(s+1)\ell},$$

and similarly,

$$(26) \quad b_{(k+1)\ell+i-1, (k-s)\ell+i-1} = \frac{((s+1)\ell)!}{\ell^{s+1}(s+1)!} \binom{(k+1)\ell+i-1}{(s+1)\ell}.$$

Also, for $s = 1, 2, \dots, k$,

$$(27) \quad b_{k\ell+i, (k-s)\ell+i} = \frac{(s\ell)!}{\ell^s s!} \binom{k\ell+i}{s\ell},$$

and this remains valid for $s = 0$ if we interpret the right-hand side as 1.

Therefore, by (25) and (26), the left hand side of (24) is

$$\begin{aligned} & \frac{((s+1)\ell)!}{\ell^{s+1}(s+1)!} \left[\binom{(k+1)\ell+i}{(s+1)\ell} - \binom{(k+1)\ell+i-1}{(s+1)\ell} \right] \\ &= \frac{((s+1)\ell)!}{\ell^{s+1}(s+1)!} \binom{(k+1)\ell+i-1}{(s+1)\ell-1} = \frac{((k+1)\ell+i-1)!}{\ell^s s! ((k-s)\ell+i)!}, \end{aligned}$$

again by Pascal's identity. On the other hand, using (27), the right hand side of (24) is

$$\begin{aligned} \gamma_{k\ell+i} b_{k\ell+i, (k-s)\ell+i} &= \frac{((k+1)\ell+i-1)!}{(k\ell+i)!} \cdot \frac{(s\ell)!}{\ell^s s!} \binom{k\ell+i}{s\ell} \\ &= \frac{((k+1)\ell+i-1)!}{\ell^s s! ((k-s)\ell+i)!}. \end{aligned}$$

Thus (24) holds for every $s = 0, 1, \dots, k$, and consequently (23) is true.

We conclude that, when $m = 0$ and $j = 1$, the sequence of eigenpolynomials satisfies the recurrence relation (21). This completes the proof of Theorem 2. \square

3. CASE $m = \ell - 1$

Here we study the remaining case $m = \ell - 1$. Notice that in this case we cannot use (15) due to the lack of symmetry. We prove below that $j = 1$ is a sufficient condition for the existence of a finite-term recurrence relation. We also derive asymptotic formulas for the first recurrence coefficients when such a relation exists, which provide evidence for the necessity of $j = 1$, but the full necessity statement remains open. Because $m = \ell - 1$, in this case the differential operator given in (1) corresponds to the operator (5) with $N = \ell$ where the coefficients a_i are

$$a_i = \begin{cases} 1 & \text{if } i = \ell \\ 0 & \text{if } i \neq \ell \end{cases}$$

The following result is an immediate consequence of Theorem 1.

Corollary 1. *In the above conditions, if $j = 1$ then the sequence of eigenpolynomials $\{P_n\}$ satisfies a recurrence relation (6) with $N = \ell$.*

Remark 1. *When $j = 1$, as in the statement of Theorem 1, since (7) we have*

$$\alpha_{n, n-s} = \frac{\binom{n}{\ell} \binom{n-1}{\ell} \dots \binom{n-s+1}{\ell} (\ell!)^{s+1}}{(s+1)!} \sum_{r=0}^{s+1} \binom{s+1}{r} \binom{n-s+r}{\ell} (-1)^r, \quad s = 1, \dots, N-1,$$

and $\alpha_{n, n} = -\ell! \binom{n}{\ell-1}$, understanding, as usual, $\binom{a}{b} = 0$ if $a < b$. As a consequence, the recurrence relation has many terms equal to 0. In particular, if $n < \ell$ the only non-zero coefficient is $\alpha_{n, n}$ and we see

$$P_{n+1}(z) = \left(z + \ell! \binom{n}{\ell-1} \right) P_n(z).$$

Now, since (14) and $m = \ell - 1$, for $n \geq \ell$ we have

$$P_n(z) = z^n + b_{n, n-1} z^{n-1} + \dots + b_{n, \ell-1} z^{\ell-1}.$$

3.1. Asymptotic evidence toward the necessity of $j = 1$. Let us use a similar approach to the one we followed in [1]. If the polynomials $\{P_n\}$ satisfy a recurrence relation, then there exist coefficients $\alpha_{n, n}, \alpha_{n, n-1}, \dots, \alpha_{n, n-N+1}$ such that (6) is satisfied. Notice that (6) can be rewritten as

$$(28) \quad \sum_{s=0}^{N-1} \alpha_{n, n-s} b_{n-s, i} = b_{n, i-1} - b_{n+1, i}, \quad n = 0, 1, \dots, \quad i = 0, 1, \dots, n+1,$$

where, as usual, $b_{r, k} = 0$ for $k > r$ or $r < 0$.

For a fixed value of n , we can deduce the values $\alpha_{n, n}, \alpha_{n, n-1}, \dots, \alpha_{n, n-N+1}$ by consecutively taking $i = n, n-1, \dots, n-N+1$, in (28). Then we obtain

$$\alpha_{n, n} = b_{n, n-1} - b_{n+1, n}, \quad \alpha_{n, n-1} = -\alpha_{n, n} b_{n, n-1} + b_{n, n-2} - b_{n+1, n-1},$$

and, in general,

$$(29) \quad \alpha_{n,n-s} = -\alpha_{n,n-s+1}b_{n-s+1,n-s} - \cdots - \alpha_{n,n}b_{n,n-s} + b_{n,n-s-1} - b_{n+1,n-s}, \quad s = 0, 1, \dots$$

From this, it is easy to deduce the following expression for the recurrence coefficients,

$$(30) \quad \alpha_{n,n-s} = \begin{vmatrix} b_{n,n-s-1} - b_{n+1,n-s} & b_{n,n-s} - b_{n+1,n-s+1} & b_{n,n-s+1} - b_{n+1,n-s+2} & \cdots & \cdots & b_{n,n-1} - b_{n+1,n} \\ b_{n-s+1,n-s} & 1 & 0 & \cdots & \cdots & 0 \\ b_{n-s+2,n-s} & b_{n-s+2,n-s+1} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & 0 \\ b_{n-1,n-s} & b_{n-1,n-s+1} & \cdots & \cdots & 1 & 0 \\ b_{n,n-s} & b_{n,n-s+1} & \cdots & \cdots & b_{n,n-1} & 1 \end{vmatrix},$$

where the determinant has order $s+1$. If there exists a recurrence relation as in (28) for a certain $N \in \mathbb{N}$, then for $s \geq N$ the determinants (30) are zero for all sufficiently large n , because $\alpha_{n,n-s} = 0$ for $s \geq N$ in (28).

Under our current assumptions, namely $m = \ell - 1$, formula (13) becomes, for $n \geq \ell$,

$$(31) \quad b_{n,n-t} = \frac{\binom{n}{\ell} \binom{n-1}{\ell} \cdots \binom{n-t+1}{\ell} (\ell!)^t}{(j!)^t \prod_{i=1}^t \left[\binom{n}{j} - \binom{n-i}{j} \right]}, \quad t = 1, 2, \dots, n - \ell + 1.$$

We now record the first asymptotic terms of these coefficients.

Proposition 1. *For each fixed $t \geq 1$ one has, as $n \rightarrow \infty$,*

$$(32) \quad b_{n,n-t} = \frac{1}{j!t!} n^{t(\ell-j+1)} \left(1 + \frac{\beta_t}{n} + O\left(\frac{1}{n^2}\right) \right),$$

where

$$(33) \quad \beta_t = \frac{(j-1)t(t+2j-1)}{4} - \frac{lt(t+\ell-2)}{2}.$$

In particular,

$$(34) \quad \begin{aligned} b_{n,n-1} &= \frac{1}{j} n^{\ell-j+1} + \frac{j(j-1) - \ell(\ell-1)}{2j} n^{\ell-j} + O(n^{\ell-j-1}), \\ b_{n,n-2} &= \frac{1}{2j^2} n^{2(\ell-j+1)} + \frac{(j-1)(2j+1) - 2\ell^2}{4j^2} n^{2(\ell-j+1)-1} + O(n^{2(\ell-j+1)-2}). \end{aligned}$$

Proof. Using

$$\binom{n-r}{\ell} \ell! = (n-r)(n-r-1) \cdots (n-r-\ell+1) = n^\ell - \left(\ell r + \frac{\ell(\ell-1)}{2} \right) n^{\ell-1} + O(n^{\ell-2})$$

for fixed r , and

$$j! \left[\binom{n}{j} - \binom{n-i}{j} \right] = (n)_j - (n-i)_j = ij n^{j-1} - \frac{ij(j-1)(i+j-1)}{2} n^{j-2} + O(n^{j-3}),$$

for fixed i , we obtain

$$\begin{aligned} \prod_{r=0}^{t-1} \binom{n-r}{\ell} (\ell!)^t &= n^{t\ell} \left(1 - \frac{1}{n} \sum_{r=0}^{t-1} \left(\ell r + \frac{\ell(\ell-1)}{2} \right) + O\left(\frac{1}{n^2}\right) \right), \\ (j!)^t \prod_{i=1}^t \left[\binom{n}{j} - \binom{n-i}{j} \right] &= j!^t n^{t(j-1)} \left(1 - \frac{1}{n} \sum_{i=1}^t \frac{(j-1)(i+j-1)}{2} + O\left(\frac{1}{n^2}\right) \right). \end{aligned}$$

Since

$$\sum_{r=0}^{t-1} \left(\ell r + \frac{\ell(\ell-1)}{2} \right) = \frac{lt(t+\ell-2)}{2}, \quad \sum_{i=1}^t \frac{(j-1)(i+j-1)}{2} = \frac{(j-1)t(t+2j-1)}{4},$$

formulae (32)–(33) follow immediately, and (34) is obtained by taking $t = 1, 2$. □

The previous asymptotic expansion allows us to compute the first recurrence coefficients. Set

$$d := \ell - j \geq 0.$$

Then $\ell - j + 1 = d + 1$.

Proposition 2. *For fixed j and ℓ , as $n \rightarrow \infty$,*

$$(35) \quad \alpha_{n,n} = -\frac{d+1}{j} n^d + O(n^{d-1}),$$

$$(36) \quad \alpha_{n,n-1} = \frac{2\ell^2 - 2j\ell + j - 1}{4j^2} n^{2d} + O(n^{2d-1}).$$

Proof. The first identity follows from

$$\alpha_{n,n} = b_{n,n-1} - b_{n+1,n}$$

and (34):

$$\begin{aligned} \alpha_{n,n} &= \left(\frac{1}{j} n^{d+1} + \frac{j(j-1) - \ell(\ell-1)}{2j} n^d + O(n^{d-1}) \right) \\ &\quad - \left(\frac{1}{j} (n+1)^{d+1} + \frac{j(j-1) - \ell(\ell-1)}{2j} (n+1)^d + O(n^{d-1}) \right) \\ &= -\frac{d+1}{j} n^d + O(n^{d-1}). \end{aligned}$$

For $\alpha_{n,n-1}$ we use

$$\alpha_{n,n-1} = -\alpha_{n,n} b_{n,n-1} + b_{n,n-2} - b_{n+1,n-1}.$$

Write

$$b_{n,n-1} = a_1 n^{d+1} + a_2 n^d + O(n^{d-1}), \quad b_{n,n-2} = c_1 n^{2d+2} + c_2 n^{2d+1} + O(n^{2d}),$$

where, by (34),

$$a_1 = \frac{1}{j}, \quad a_2 = \frac{j(j-1) - \ell(\ell-1)}{2j}, \quad c_1 = \frac{1}{2j^2}, \quad c_2 = \frac{(j-1)(2j+1) - 2\ell^2}{4j^2}.$$

From (35) refined to one more order we get

$$\alpha_{n,n} = -\frac{d+1}{j} n^d + O(n^{d-1}).$$

Substituting the above expansions, the terms of order n^{2d+1} cancel identically, and the first surviving term is of order n^{2d} . A direct computation gives

$$\alpha_{n,n-1} = \frac{2\ell^2 - 2j\ell + j - 1}{4j^2} n^{2d} + O(n^{2d-1}),$$

which proves (36). \square

Proposition 2 gives a corrected replacement for the broken argument above: it shows that the first two recurrence coefficients have precise non-vanishing leading asymptotics. However, this is still not sufficient to prove the necessity of $j = 1$, because a finite-term recurrence only forces the vanishing of $\alpha_{n,n-s}$ for all sufficiently large s , not for $s = 0, 1$.

The natural conjectural extension is the following: for each fixed $s \geq 0$,

$$(37) \quad \alpha_{n,n-s} = C_s(\ell, j) n^{(s+1)(\ell-j)} + O(n^{(s+1)(\ell-j)-1}),$$

with $C_s(\ell, j) \neq 0$ whenever $j > 1$. If (41) holds, then $\alpha_{n,n-s} \neq 0$ for every fixed s and all sufficiently large n , which would imply that no recurrence with finitely many terms can exist unless $j = 1$.

At present we have proved (41) only for $s = 0, 1$, that is, the constants

$$C_0(\ell, j) = -\frac{\ell - j + 1}{j}, \quad C_1(\ell, j) = \frac{2\ell^2 - 2j\ell + j - 1}{4j^2}.$$

Therefore, the necessity of $j = 1$ in the case $m = \ell - 1$ remains open in the present paper.

The next proposition isolates the remaining analytic input needed to continue the argument for arbitrary s .

Proposition 3. *Fix $s \geq 0$ and write $d = \ell - j$. Assume that for every $t = 1, \dots, s+1$ one has a full asymptotic expansion of the form*

$$(38) \quad b_{n,n-t} = n^{t(d+1)} \left(\sum_{q=0}^{s+1} \frac{\beta_{t,q}}{n^q} + O(n^{-s-2}) \right), \quad \beta_{t,0} = \frac{1}{j^t t!},$$

as $n \rightarrow \infty$. Then there exists a constant $C_s(\ell, j)$, depending polynomially on the coefficients $\beta_{t,q}$ with $t \leq s+1$ and $q \leq s+1$, such that

$$(39) \quad \alpha_{n,n-s} = C_s(\ell, j) n^{(s+1)d} + O(n^{(s+1)d-1}).$$

Moreover, the constants $C_s(\ell, j)$ are determined recursively from the identity

$$(40) \quad \alpha_{n,n-s} = -\sum_{r=0}^{s-1} \alpha_{n,n-r} b_{n-r,n-s} + b_{n,n-s-1} - b_{n+1,n-s},$$

by collecting the coefficient of $n^{(s+1)d}$ after expanding every term in descending powers of n .

Proof. We proceed by induction on s . The cases $s = 0, 1$ are exactly Proposition 2. Assume that (39) has been proved for $0, 1, \dots, s-1$.

For each fixed $r \in \{0, 1, \dots, s-1\}$, the assumed expansion (38) with $t = s-r$ implies, after replacing n by $n-r$, that

$$b_{n-r, n-s} = (n-r)^{(s-r)(d+1)} \left(\sum_{q=0}^{s+1} \frac{\tilde{\beta}_{s-r, q}^{(r)}}{n^q} + O(n^{-s-2}) \right)$$

for suitable constants $\tilde{\beta}_{s-r, q}^{(r)}$. By the induction hypothesis,

$$\alpha_{n, n-r} = C_r(\ell, j) n^{(r+1)d} + O(n^{(r+1)d-1}).$$

Hence every product $\alpha_{n, n-r} b_{n-r, n-s}$ admits an asymptotic expansion in descending powers of n , and so does the difference term $b_{n, n-s-1} - b_{n+1, n-s}$.

A priori, the largest power occurring on the right-hand side of (40) is $n^{(s+1)d+s}$. The identity (40), which is exact for every n , forces cancellation of the coefficients of the powers

$$n^{(s+1)d+s}, n^{(s+1)d+s-1}, \dots, n^{(s+1)d+1},$$

and therefore the first possibly nonzero term occurs at order $n^{(s+1)d}$. Its coefficient is, by definition, $C_s(\ell, j)$, which proves (39). Since all coefficients arise from algebraic manipulations of the finitely many numbers $\beta_{t, q}$, the constant $C_s(\ell, j)$ depends polynomially on them. \square

Proposition 3 shows that the natural conjectural extension can be reduced to proving sufficiently deep asymptotic expansions for the coefficients $b_{n, n-t}$ together with the nonvanishing of the recursively defined constants $C_s(\ell, j)$. Concretely, for each fixed $s \geq 0$ one expects

$$(41) \quad \alpha_{n, n-s} = C_s(\ell, j) n^{(s+1)(\ell-j)} + O(n^{(s+1)(\ell-j)-1}),$$

with $C_s(\ell, j) \neq 0$ whenever $j > 1$. If (41) holds, then $\alpha_{n, n-s} \neq 0$ for every fixed s and all sufficiently large n , which would imply that no recurrence with finitely many terms can exist unless $j = 1$.

At present we have proved (41) only for $s = 0, 1$, that is,

$$C_0(\ell, j) = -\frac{\ell-j+1}{j}, \quad C_1(\ell, j) = \frac{2\ell^2 - 2j\ell + j - 1}{4j^2}.$$

Therefore, the necessity of $j = 1$ in the case $m = \ell - 1$ remains open in the present paper.

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SAINT LOUIS UNIVERSITY, MADRID CAMPUS, AVENIDA DEL VALLE 34, 28003 MADRID, SPAIN
Email address: luismiguel.anguas@slu.edu

ETSI INDUSTRIALES, UNIVERSIDAD POLITÉCNICA DE MADRID, C/JOSÉ GUTIÉRREZ ABASCAL 2, 28006 MADRID, SPAIN
Email address: dolores.barrios.rolania@upm.es

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, SE-106 91, STOCKHOLM, SWEDEN
Email address: shapiro@math.su.se

DEPARTMENT OF THEORETICAL PHYSICS, NUCLEAR PHYSICS INSTITUTE, ACADEMY OF SCIENCES, 250 68 ŘEŽ NEAR PRAGUE, CZECH REPUBLIC
Email address: tater@ujf.cas.cz