PÓLYA'S SHIRE THEOREM FOR ALGEBRAIC FUNCTIONS

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Abstract. Below we extend Pólya's shire theorem describing the asymptotic root distribution of iterated derivatives of a given rational function to the case of iterated derivatives of arbitrary algebraic functions.

To György Pólya[1] with admiration

1. Introduction

The mathematical legacy of G. Pólya stretches over a variety of mathematical areas. Even more impressive is the influence of his (joint with other famous authors) textbooks as well his books popularising and demystifying the mathematical profession and explaining its connection with the real life and other branches of science.

Below we dwell over one topic to which G. Pólya devoted a number of papers over the years. Namely, he studied what happens with the zeros of a function under its consecutive differentiation, see [7] – [13]. The first of these papers was published when G. Pólya was 35 and last when he was 89 years old.

The result which we mainly concentrate on is the so-called Pólya's shire theorem. It is proven 101 years ago in [7] and its (somewhat frivolous) translation into English reads as follows.

Theorem A. Let f be a meromorphic function in the plane, with at least one pole, and think of each pole A as the capital city of a shire, defined as the set of points in $\mathbb{C}$ closer to A than to any other pole. Then the final set is precisely the union of the boundaries of all the shires.

In Theorem A by a *final set* he calls the limit set of the zero loci of the sequence $\{\frac{d^n}{dx^n} f\}$. Pólya also described the situation as follows: "the poles repel the zeros, which then have no choice but to hover in between".

Pólya's "union of the boundaries of all the shires" is the 1-skeleton of the Voronoi diagram of the set of poles of the meromorphic function under consideration, see e.g. [18].

Certain generalizations of Theorem A have been found over the years, see e.g. [14], [15]. Recently in [2] and [3] the authors found the asymptotic root-counting measure of the asymptotic root distribution of iterative derivatives for rational functions and certain class of meromorphic functions.

The goal of this paper is to extend his study of the asymptotic root distributions of consecutive derivatives to the case of algebraic functions. Note that papers [14],

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[1] György's father Jakab Pollák changed his Jewish lastname to a Hungarian sounding Pólya (and also baptized the whole family) in order to improve his career chances in Hungarian academia, but whether it contributed to his success in getting an appointment as a Privatdozent at the University of Budapest, one cannot say for sure. In any case, he received such a post shortly before he died in his early fifties when György was only ten years old.

[15] contain results in this direction, but for a rather restrictive class of algebraic functions.

To formulate the main result of this paper we need to introduce some notation.

Notation 1. Consider a plane algebraic curve $\Gamma \subset \mathbb{C}_x \times \mathbb{C}_y$ with coordinates (x, y) given as the zero locus of a bivariate polynomial $\Phi(x, y) = 0$. Take the projectivization $\widetilde{\Gamma} \subset \mathbb{C}P^1_x \times \mathbb{C}P^1_y$ of Γ in $\mathbb{C}P^1_x \times \mathbb{C}P^1_y$ with homogeneous coordinates $(X_0 : X_1), (Y_0 : Y_1)$. Denote the bidegree of $\widetilde{\Gamma}$ as (k, ℓ). (To avoid trivialities we assume that is both k and ℓ are positive.) The case of rational functions considered in Theorem A corresponds to the situation $\ell = 1,\ k \geq 2$.

Notation 2. In the above notation, consider the projection $\pi_x : \widetilde{\Gamma} \to \mathbb{C}P^1_x$. Under the assumption that Γ is reduced (which we always assume valid below) the preimage $\pi_x^{-1}(\hat{x}) \subset \widetilde{\Gamma}$ of a generic point $\hat{x} \in \mathbb{C}P^1$ consists of ℓ distinct points. By the *branching locus* $\mathcal{B}(\Gamma) \subset \mathbb{C}P^1_x$ we mean the set of points $\hat{x} \in \mathbb{C}P^1_x$ for which $\pi_x^{-1}(\hat{x}) \subset \mathbb{C}P^1_y$ is a positive divisor of degree ℓ with less than ℓ distinct points, i.e. some point comes with a multiplicity exceeding 1. Points of multiplicity more than 1 in the preimage $\pi_x^{-1}(\hat{x})$ of a branching point $\hat{x}$ will be called the *critical points* of (the meromorphic function) $\pi_x : \widetilde{\Gamma} \to \mathbb{C}P^1_x$. We call a branching point $\hat{x} \in \mathcal{B}_a(\Gamma)$ *essential* if for some affine critical point $\hat{y} \in \pi_x^{-1}(\hat{x})|_{\mathbb{C}_y}$ at least one of the local branches of Γ near $\hat{y}$ has a critical point at $\hat{y}$ as well, i.e. its projection on a small neighborhood of $\hat{x}$ is not a local biholomorphism. (In other words, this branch needs fractional powers for its presentation as a Puiseux series at $\hat{x}$ or, alternatively, $\hat{x}$ is a branch point of the lift $n\pi_x : N\widetilde{\Gamma} \to \mathbb{C}P^1_x$ where $N\widetilde{\Gamma}$ is the normalization of $\widetilde{\Gamma}$.) Let $\mathcal{B}_a^{ess}(\Gamma) \subset \mathcal{B}_a(\Gamma)$ denote the set of all essential affine branching points.

Finally, the *affine branching locus* $\mathcal{B}_a(\Gamma)$ is not just the restriction of $\mathcal{B}(\Gamma)$ to $\mathbb{C}_x$, but is its subset for which we additionally require that for $\hat{x} \in \mathcal{B}_a(\Gamma)$ at least one critical point of the divisor $\pi_x^{-1}(\hat{x})$ is affine, i.e. lies in $\mathbb{C}_y$ and not at $\infty \in \mathbb{C}P^1_y$.

By the *locus of poles* $\mathcal{P}(\Gamma) \subset \mathbb{C}P^1_x$ we mean the set of points $\hat{x} \in \mathbb{C}P^1_x$ for which $\pi_x^{-1}(\hat{x}) \subset \mathbb{C}P^1_y$ contains $\infty \in \mathbb{C}P^1_y$ where $\mathbb{C}P^1_y = \mathbb{C}_y \cup \infty$. The *affine locus of poles* $\mathcal{P}_a(\Gamma) \subset \mathcal{P}(\Gamma)$ is the restriction of $\mathcal{P}(\Gamma)$ to $\mathbb{C}_x$.

Notation 3. Given Γ as above, define the algebraic curve $\Gamma' \subset \mathbb{C}_x \times \mathbb{C}_y$ obtained by taking the derivatives of all branches of Γ considered of (local) algebraic function $y(x)$ w.r.t the variable x. Denote by $\widetilde{\Gamma}' \subset \mathbb{C}P^1_x \times \mathbb{C}P^1_y$ its projectivization. (It is a known fact that derivative of an algebraic function is algebraic. FIND A REFERENCE.)

Similarly, define the sequence of algebraic curves $\Gamma^{(n)} \subset \mathbb{C}_x \times \mathbb{C}_y$ and $\widetilde{\Gamma^{(n)}} \subset \mathbb{C}P^1_x \times \mathbb{C}P^1_y$ consecutively defining $\Gamma^{(n)}$ as the derivative of all branches of $\Gamma^{(n-1)}$ w.r.t the variable x. (Similarly to the above, $\widetilde{\Gamma^{(n)}}$ is, by definition, the projectivization of $\Gamma^{(n)}$.)

Remark 1. $\widetilde{\Gamma^{(n)}}$ can be alternatively defined by taking the branches of the n-th derivative of the original plane curve Γ and taking their projectivization in $\mathbb{C}P^1_x \times \mathbb{C}P^1_y$.

Definition 1. We define the *zero locus* $\widetilde{\mathcal{Z}}^{(n)}(\Gamma)$ *of the n-th derivative* of the algebraic function given by the algebraic curve Γ as the intersection locus of $\widetilde{\Gamma}^{(n)}$ with $(\mathbb{C}P^1_x, 0) \subset \mathbb{C}P^1_x \times \mathbb{C}P^1_y$. The *affine zero locus* $\mathcal{Z}_a^{(n)}(\Gamma)$ is the restriction of $\widetilde{\mathcal{Z}}^{(n)}(\Gamma)$ to $\mathbb{C}_x \subset \mathbb{C}P^1_x$. The *final set* $\mathcal{F}(\Gamma) \subset \mathbb{C}P^1_x$ is the limiting set of the (supports) of the sequence $\{\widetilde{\mathcal{Z}}^{(n)}(\Gamma)\}$ and the *affine final set* $\mathcal{F}_a(\Gamma) \subset \mathbb{C}_x$ is the limiting set of the (supports) of the sequence $\{\widetilde{\mathcal{Z}}_a^{(n)}(\Gamma)\} \subset \mathbb{C}_x$.

Now we are finally ready to formulate the main result of the paper.

Theorem 1. *For any reduced algebraic curve $\Gamma \subset \mathbb{C}_x \times \mathbb{C}_y$, the final set $\mathcal{F}(\Gamma) \subset \mathbb{C}_x$ coincides with the 1-skeleton of the Voronoi diagram of the finite set of points $\Theta(\Gamma) \subset \mathbb{C}_x$ where $\Theta(\Gamma) = \mathcal{P}_a(\Gamma) \cup \mathcal{B}_a^{ess}(\Gamma)$.*

Let us now describe the asymptotic density of roots of the derivatives $\Gamma^{(n)}$. Define a plane measure with support on the line $L_{ij} : |z - z_i| = |z - z_j|$ by

$$\mu_{ij} := \frac{\pi \cdot \mathrm{card}\Theta(\Gamma)}{2} \frac{|z_i - z_j|}{|(z - z_i)(z - z_j)|} ds,$$

where s is Euclidean length measure in the complex plane, and z_i, z_j are distinct points in $\Theta(\Gamma)$, i.e. poles or branching points of Γ. Restricting the measure to the segment of L_{ij} belonging to the Voronoi diagram of $\Theta(\Gamma)$, and summing over all lines gives a measure μ_Γ, supported on the Voronoi diagram. This will in fact be a probability measure canonically associated with the diagram.

Theorem 2. *For any reduced algebraic curve $\Gamma \subset \mathbb{C}_x \times \mathbb{C}_y$, the asymptotic root-counting measure of $\Gamma^{(n)}$ when $n \to \infty$ coincides with the above μ_Γ.*

Our final asymptotic result extends the previous result to the case of the sequence of currents of $\Gamma^{(n)}$. Namely, given an algebraic curve Y in $\mathbb{C}^2$ associated to it $\partial\bar{\partial}$-current $\mathcal{C}Y$ given by ... Given a generic curve Γ as above consider the sequence $\{\mathcal{C}\Gamma^{(n)}\}_{n=0}^{\infty}$ of currents.

Theorem 3. *The limiting current $\mathcal{C}_\Gamma = \lim_{n\to\infty} \mathcal{C}\Gamma^{(n)}$ is given as follows. Its support is the cylinder over the Voronoi diagram and its density equals the above density times the positive function $\Theta(q)$ given by ...*

Besides these asymptotic results in the next section we present a number of algebraic geometric statements about the sequence of algebraic curves $\{\widetilde{\Gamma}^{(n)}\}$.

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2. Preliminaries on derivatives of algebraic functions

2.1. Numerical invariants of $\widetilde{\Gamma}^{(n)}$ and related curves.

Lemma 1. *In the above notation, the following facts hold.*
(i) *$\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Gamma') \subseteq \mathcal{P}(\Gamma'')$ and $\mathcal{B}_a(\Gamma) \supseteq \mathcal{B}_a(\Gamma') \supseteq \mathcal{B}(\Gamma'')$. Finally, $\Theta(\Gamma) = \Theta(\Gamma') = \Theta(\Gamma'') = \dots$.*
(ii) *For sufficiently large n, $\mathcal{B}_a(\Gamma)^{(n)} = \emptyset$ and $\Theta(\Gamma^{(n)}) = \mathcal{P}(\Gamma^{(n)})$.*

Proof. To settle the first assertion observe that the first derivative of any usual or algebraic pole $f(x) = \sum_{s=m}^{\infty} a_s(x - x_0)^{s/q}$, m/q being a rational negative number decreases is order by 1 and thus has itself a pole. For a similar reason, an essential branching point must have at least one branch of the form $f(x) = \sum_{s=m}^{\infty} a_s(x - x_0)^{s/q}$, with p/q rational and positive $(q > 1)$. After a finite number of derivations its order becomes negative, i.e. it becomes an algebraic pole. For fewer derivations the essential branching point is still such for its respective derivatives. Finally, let us observe that no essential branching point can occur from nowhere since integration gives non-trivial Puiseux terms. At the same time non-essential branching points, i.e. intersections of smooth branches can appear and disappear. □

Proposition 1. (i) *For any (reduced) $\widetilde{\Gamma}$ of bidegree (k, ℓ) and any positive integer n, the curves $\widetilde{\Gamma}$ and $\widetilde{\Gamma}^{(n)}$ are birationally equivalent. In particular, for $\widetilde{\Gamma}$ generic, the genus of any $\widetilde{\Gamma}^{(n)}$ equals $(k - 1)(\ell - 1)$.*

(ii) *For any $\widetilde{\Gamma}$ of bidegree (k,ℓ) and any positive integer n, the monodromy group of $\widetilde{\Gamma^{(n)}}$ coincides with the monodromy group of $\widetilde{\Gamma}$.*
(iii) *For a generic $\widetilde{\Gamma}$ of bidegree (k,ℓ) and any positive integer n, the bidegree of $\widetilde{\Gamma^{(n)}}$ equals $((4n-2)k\ell-(n+1)\ell-3(n-1)k,\ell)$. In particular, the bidegree of $\widetilde{\Gamma}'$ equals $(2k\ell-2\ell,\ell)$ and the bidegree of $\widetilde{\Gamma''}$ equals $(6k\ell-3k-3\ell,\ell)$.*
(iv) *If a reduced Γ does not contain a polynomial irreducible component then for any positive integer n, $\widetilde{\mathcal{Z}}^{(n)}$ is a positive divisor of degree $(4n-2)k\ell-(n+1)\ell-3(n-1)k$.*

Proof of Proposition 1. Item (i) trivially follows from the fact the taking the derivative of order n is a local biholomorphism of $\widetilde{\Gamma}$ and $\widetilde{\Gamma}^{(n)}$ except for finitely many points.

To prove item (ii), let us start with the first derivative. The (affine) set of critical points on Γ for the meromorphic function $\pi_x:\Gamma\to\mathbb{C}_x$ is given by a system of two algebraic equations

$$\begin{cases} P(x,y)=0 \\ \frac{d}{dx}\left(P(x,y(x))\right)=-\frac{\partial P}{\partial x}/\frac{\partial P}{\partial y}=0 \end{cases} \quad\leftrightarrow\quad \begin{cases} P(x,y)=0 \\ \frac{\partial P}{\partial x}(x,y)=0. \end{cases} \tag{2.1}$$

The bidegree of $\frac{\partial P}{\partial x}(x,y)$ equals $(k-1,\ell)$. System (2.1) is a complete intersection. An analog of Bezout's formula for the number of interesection points of two curves of bidegrees (a,b) and (c,d) is $ad+bc$, see [6]. The number of solutions of (2.1) counting multiplicities in $\mathbb{C}P^1_x\times\mathbb{C}P^1_y$ equals $k\ell+(k-1)\ell=2k\ell-\ell$. This number equals the number of branching points of $\widetilde{\Gamma}$ counting multiplicitites, i.e. the degree of $\widetilde{\Gamma}'$ w.r.t the x-coordinate. The degree of the y-coordinate does not change under differentiation.

Similarly, for the derivative of order n, the (affine) set of critical points on $\Gamma^{(n-1)}$ for the meromorphic function $\pi_x:\Gamma^{(n-1)}\to\mathbb{C}_x$ is given by a system of two algebraic equations

$$\begin{cases} P(x,y)=0 \\ \frac{d^n}{dx^n}\left(P(x,y(x))\right)=0, \end{cases} \tag{2.2}$$

where $\frac{d^n}{dx^n}\left(P(x,y(x))\right)$ is the implicit derivative of the algebraic function $y(x)$ of order n. In general, it has a rather complicated explicit expression which for $n=2$ is given by

$$\frac{d^2y}{dx^2}=-\frac{P''_{xx}(P'_y)^2-2P'_xP'_yP''_{yx}+P''_{yy}(P'_x)^2}{(P'_y)^3}. \tag{2.3}$$

Applied to a polynomial $P(x,y)$ of bidegree (k,ℓ) the numerator of the right-hand part of (2.3) produces a polynomial of bidegree $(3k-2,3\ell-2)$. By the same formula as above the number of solutions to (2.3) in $\mathbb{CP}^1_x\times\mathbb{C}P^1_y$ equals $k(3\ell-2)+\ell(3k-2)=6k\ell-2k-2\ell$. (Notice that each differential monomial in the numerator of (2.3) applied to $P(x,y)$ gives the same bidegree of the result.)

For $n>2$, we proceed by induction. We assume that

$$\frac{d^{n-1}y}{dx^{n-1}}=\frac{differential\,polynomial}{(P'_y)^{2n-1}}$$

and prove that the same shape holds for $\frac{d^ny}{dx^n}$ with the index shift $n-1\mapsto n$. (Base of induction is presented above.) Indeed,

$$\frac{d^ny}{dx^n}=\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=\frac{\frac{d}{dx}(enum)\cdot P_y^{(2n-1)}-(enum)\cdot\frac{d}{dx}(P_y^{(2n-1)})}{P_y^{(2n-1)}}$$

We use the relation

$$\frac{d}{dx}\left(\Phi(x,y(x))\right)=\frac{\partial}{\partial x}(\Phi(x,y(x)))+\frac{\partial}{\partial y}(\Phi(x,y(x))\cdot\frac{dy}{dx},$$

where $\frac{dy}{dx}=-\frac{P'_x}{P'_y}$.

Claim. The numerator of $\frac{d^n y}{dx^n}$ applied to a curve $\widetilde{\Gamma}$ of bidegree (k,ℓ) produces a curve of bidegree $((2n-1)k-n,(2n-1)\ell-2(n-1))$. For example, for $n=1$, we get $(k-1,\ell)$ and for $n=2$ we get $(3k-2,3\ell-2)$.

Using this claim we get the following answer for the bidegree of $\widetilde{\Gamma}^{(n)}$ in $\mathbb{C}P^1\times\mathbb{C}P^1$. It equals

$$(((2n-1)k-n)\ell+((2n-1)\ell-2(n-1))k=2(2n-1)k\ell-n\ell-2(n-1)k.$$

□

Proposition 2. *For a generic curve $\widetilde{\Gamma}$ of bidegree (k,ℓ) and a positive integer n, the degree of the affine divisor $\widetilde{\mathcal{Z}}_a^{(n)}(\Gamma)$ equals $(4n-2)k\ell-(n+1)\ell-3(n-1)k$.*

In particular, $\deg\widetilde{\mathcal{Z}}'_a(\Gamma)=2k\ell-2\ell$; $\deg\widetilde{\mathcal{Z}}''_a(\Gamma)=6k\ell-3k-3\ell$ for $\ell>1$ and $2k-2$ for $\ell=1$; $\deg\widetilde{\mathcal{Z}}_a^{(3)}(\Gamma)=10k\ell-4k-6\ell$.

Proof. Again we try to use induction on n starting with $n=1,2$. Observing that

$$\deg\widetilde{\mathcal{Z}}_a^{(n)}(\Gamma)=\deg\widetilde{\mathcal{Z}}^{(n)}(\Gamma)-\deg_\infty\widetilde{\mathcal{Z}}^{(n)}(\Gamma)$$

we need to calculate the last term which is the multiplicity of $\widetilde{\mathcal{Z}}^{(n)}(\Gamma)$ at $\infty\in\mathbb{C}P^1_z$. For generic curve $\widetilde{\Gamma}$ of bidegree (k,ℓ), the total (conjectural) number of affine solutions is $2(2n-1)k\ell-(n+1)\ell-3(n-1)k$ which means that the multiplicity at ∞ should be equal $\ell+(n-1)k$. Let us prove it ....

□

2.2. **Singularities of $\widetilde{\Gamma}^{(n)}$.** To calculate numerical invariants of $\widetilde{\Gamma}^{(n)}$ we need the information about its singularities. Fortunately, if the original $\widetilde{\Gamma}$ is a generic curve of bidegree (k,ℓ) we can explicitly describe all the singularities of $\widetilde{\Gamma}^{(n)}$ except for intersections of smooth branches which locally biholomorphically project onto $\mathbb{C}P^1_x$.

Theorem 4. *Assume that $\widetilde{\Gamma}$ is a generic curve of bidegree (k,ℓ) in $\mathbb{C}P^1_x\times\mathbb{C}P^1_y$. Then the following facts hold:*
I. *Projection $\pi_x:\widetilde{\Gamma}\to\mathbb{C}P^1_x$ has $2k\ell-2k$ simple branch points all located in $\mathbb{C}_x\subset\mathbb{C}P^1_x$.*
II. *$\widetilde{\Gamma}$ has k simple poles, i.e. intersections with the fiber $(\mathbb{C}P^1_x,\infty)$. All these poles lie in $(\mathbb{C}_x,\infty)$.*
III. *$\widetilde{\Gamma}$ has genus $(k-1)(\ell-1)$.*
IV. *For any positive integer n, the curve $\widetilde{\Gamma}^{(n)}\subset\mathbb{C}P^1_x\times\mathbb{C}P^1_y$ has the following singularities located in $\mathbb{C}_x\times\mathbb{C}P^1_y$ except possibly for intersections of smooth branches:*
a) k $n+1$-tuple poles at the same location as the simple poles of $\widetilde{\Gamma}$;
b) $2k\ell-2k$ algebraic poles of order $n-\frac{1}{2}$ at the positions of the branch points of $\widetilde{\Gamma}$.
V. *The generic curve $\widetilde{\Gamma}^{(n)}\subset\mathbb{C}P^1_x\times\mathbb{C}P^1_y$ has the following singularities located in the fiber $(\infty,\mathbb{C}P^1_y)$. There are ℓ algebraic poles ...*

3. Proof of Theorem 1

Let U be the complex plane cut by some parallell rays originating in each point of $\Theta(\Gamma)$ and having the same direction $v\in\mathbb{C}^*$. (Note that we may change this direction.) At a small enough neighbourhood $D(d,\epsilon)\subset\mathbb{C}P^1$ of a point $d\in\Theta(\Gamma)$ we have two possibilities. There are 3 cases: either a smooth branch or a pole or a critical point/algebraic pole. Either $g(z)$ can be extended to a holomorphic function in $D(d, \epsilon) \setminus p$, and d is *irrelevant*(for h), or $g(z)$ has monodromy. In the latter case there is locally a Puiseaux series expansion

$$h(z) = \sum_{c \geq a} (z - d)^{\frac{c}{k}}, \tag{3.1}$$

for some integers $k > 1$ and a, where $(z - d)^{\frac{1}{k}}$ is defined using a branch of the logarithm $\log(z - d)$, defined in U. We denote the points of B where we have monodromy by B_h. We may assume (by changing the direction of the cuts) that the cuts are made so that $g(U)$ contains all poles of f in $\overline{g(U)}$. Together with the set of poles $P_h \subset \mathbb{C}P^1$ of h, B_h are the points that define the Voronoi diagram that is relevant for our problem.

We can extend the Puiseaux series expansion (3.1) to larger open sets in the following way. Let $t(z) := (z - d)^{\frac{1}{k}}, z \in U$ so that $s(t(z)) := t^k + d = z$ in U. Then $t : U \mapsto t(U)$ is a biholomorphic map with inverse s.

Lemma 2. *Suppose that $V \subset U$ is a simply-connected open set, that $h(z)$ is holomorphic in V and $d \in \bar{U}$. Then there is $r_d(z)$ that is holomorphic in $D(0, \epsilon) \cup t(V)$(for small enough ϵ) and satisfies $r_d(0) \neq 0$, such that*

$$h(z) = (z - d)^{\frac{c}{k}} r_d((z - d)^{\frac{1}{k}}), \tag{3.2}$$

in $(D(d, \epsilon_1) \cap U) \cup V$. Here $c = c(p) \leq 0$ and $k = k(p) > 0$ are integers. If d is instead a pole of $h(z)$ there is similarily a function r_d such that (3.2) *holds with $c < 0$ and $k = 1$.*

Note that we may assume that $c < 0$, for all the points in the (relevant) discriminant, by substituting a high enough derivative for f, since we study the asymptotics of the sequence of derivatives. Let $\alpha := 1/k$.

Proof. Clearly, by (3.1)

$$h(z)((z - d)^{\frac{-c}{k}} = r_d((z - d)^{\frac{1}{k}}), \tag{3.3}$$

in a neighbourhood of d, where $r_d(t)$ is holomorphic in a neighbourhood $D(0, \epsilon)$ of 0(in fact the series representation has convergence radius in 0 equal to the distance to the nearest pole or relevant discriminant point). The left hand side equals $h(s(t))t^c$ and is holomorphic in the variable t for $t \in t(V)$. Hence $r_d(t)$ may be continued to a holomorphic function in $t(V) \cup D(0, \epsilon)$. □

++++(the above is trivial, could probably be avoided, but it is crucial for the proof that r_d is holomorphic in 0 and can be extended to V)+++++

The following lemma is a local extension of the first part of Polya's proof to algebraic functions, which inter alia shows that the shires in the Voronoi diagram associated to the discriminant and the poles are asymptotically zero-free. Prather and Shaw [14] prove (in a different way) a similar result for a less extensive class of functions. Namely, [14] (+++precise reference++++) assumes that the function involved is a sum of functions with globally defined Puiseaux series. This class contains e.g.$\sqrt{z} + \sqrt{z-1}$ but not $1/(\sqrt{z} + 1)$... +++perhaps expand+++. Our proof uses ideas from both these papers; the skeleton is Polya's proof, and the added complexity is resolved by a nice application of Faa di Bruno's formula for the derivative of a composed function, asymptotics of Pochhammers symbols from [14] and finally Abel' s theorem applied to certain Bell polynomials.

Proposition 3. *Let h be an algebraic function as in Lemma 2, with pole set P_h and relevant discriminant set B_h. Let $V \subset U \subset \mathbb{C}$ be an open simply-connected* *region. For a point $z_0 \in V$ let $D := D^h_{z_0}$ be the disc of convergence of h centered at z_0. Assume that $z_0 \in V \setminus P$ satisfies that the intersection*

$$\bar{D} \cap \partial V = \{p\} \subset P \cup B_h. \tag{3.4}$$

Then,

$$\frac{1}{n} \log |\frac{f^{(n)}(z_0)}{n!}| \to M(z_0) = -\log |z_0 - p|. \tag{3.5}$$

All points in a small enough punctured disc around p will satisfy (3.4). Denote by V_p the set of points that satisfy (3.4), together with the interior point p. In this set the convergence is uniform on compact subsets, and a compact subset only contains a finite number of zeros of any $f^{(n)}(z)$, $n \in \mathbb{N}$.

Condition (3.4) implies that p is the point in $P \cup B_h$ closest to z_0, and so $Min_{a \in P \cup B_h}(\log |z_0 - a|) = \log |z_0 - p|$. Defining

$$M(z) := Max_{a \in P \cup B_h}(-\log |z - a|) = -Min_{a \in P \cup B_h}(\log |z - a|,) \tag{3.6}$$

it follows that $M(z_0) = -\log |z_0 - p|$. Note also that if V is a bounded region then only points that are closer to $P \cup B_h$ than to ∂V will satisfy (3.4).(??perhaps a figure??.)If h stems from a function f that is meromorphic on Z all points in an open Voronoi cell will satisfy (3.4). Hence we get the following result.

Corollary 1. *(See also ([14]) Let $f(z)$ be an algebraic function. In the open dense union of all open Voronoi cells $V_{max} := \cup V^0_p, p \in P \cup B_h$,*

$$\frac{1}{n} \log |\frac{f^{(n)}(z)}{n!}| \to M(z), \tag{3.7}$$

uniformly on compact subsets of V_{max}. Hence the dense subset V_{max} is asymptotically zero-free.

Proof of Proposition 3. By Lemma 2 we have

$$h(z) = (z - d)^{\frac{c}{k}} r_d((z - d)^{\frac{1}{k}}), \tag{3.8}$$

where $r_d(t)$ is holomorphic at $t = 0$ and $r_d(0) \neq 0$. In the following we assume that $d = 0$. Let $h_1(z) := z^{c\alpha}$(where without loss of generality we assume that $c < 0$), $h_2(z) := r_d(z)$, and $h_3(z) := z^\alpha$, with $\alpha = 1/k$, so that

$$h(z) = h_1(z)h_2(h_3(z)). \tag{3.9}$$

As a consequence of the last 5 formulas from the appendix (FIX THIS), we get:

$$\begin{gathered}
\frac{(z^{c\alpha} r_d(z^\alpha))^{(n)}}{n!} \frac{z^n}{z^{c\alpha}} = \\
r_d(z^\alpha) \frac{\langle c\alpha \rangle}{n!} + \sum_{l=1}^{n} \frac{r_d^{(l)}(z^\alpha)(z^\alpha)^l}{l!} (-1)^l \left((-1)^l \sum_{i=l}^{n} \frac{\langle c\alpha \rangle_{n-i}}{(n-i)!} \frac{l!}{i!} B_{i,l} \right) \\
\Longleftrightarrow \\
\frac{(z^{c\alpha} h(z^\alpha))^{(n)}}{n!} \frac{z^n}{z^{c\alpha}} \frac{n!}{\langle c\alpha \rangle_n} = \\
r_d(z^\alpha) + \sum_{l=1}^{n} \frac{r_d^{(l)}(z^\alpha)(-z^\alpha)^l}{l!} \left(\sum_{i=l}^{n} (-1)^l \frac{\langle c\alpha \rangle_{n-i}}{\langle c\alpha \rangle_n} \frac{n!}{(n-i)!} \frac{l!}{i!} B_{i,l} \right).
\end{gathered} \tag{3.10}$$

By formula (6.11) from Appendix the expression in large parenthesis in the last part of (3.10) tends to 1. COnsequently the whole expression converges to $r_d(0) \neq 0$.

Now we can apply Polya's original argument to r_d. Let $\rho(t)$ denote the radius of convergence of the Taylor series of $r_d(t)$ at a point t, and consider $\alpha(t) := |t|/\rho(t)$.

This function is continuous at every point at which $\rho(t) > 0$, and hence in $t(V_p)$, since the nearest singularity of $r_d(t)$ is either on the boundary of $t(U)$ or a pole other than 0. The disc of convergence around z_0 contains (by the previous lemma) 0, and hence $\rho(z_0^\alpha) > |z_0^\alpha|$ Denote by A a compact region in V_p, by ρ_0 the minimal value of $\rho(t(z))$ in A and by $\alpha_0 < 1$ the maximal value of $\alpha(t(z))$ in A. Choose $1 > \beta > \alpha_0$ and surround each point of A by a disc of radius $\beta\rho(z)$, covering a region A^*. each point of which is distant at least $(1-\beta)\rho_0 > 0$ from the nearest singularity. Let M be a majorant of $|r_d(t(z))|$ in A^*. By Cauchy's inequalities

$$\frac{|r_d^{(k)}(z^\alpha)|}{k!} \leq \frac{M}{(\beta\rho(z))^k} \implies \frac{|r_d^{(k)}(z^\alpha)(z^\alpha)|^k}{k!} \leq M\big(\frac{\alpha_0}{\beta}\big)^k. \tag{3.11}$$

Hence the series

$$\sum_k^\infty \frac{|r_d^{(k)}(z_0^\alpha)(z_0^\alpha)|^k}{k!}$$

converges uniformly for $z \in A$.. The disk of convergence of $h_2(t) = r_d(t)$ at $t = z_0^\alpha$ contains, by Lemma 2 , $t = 0$, and hence the limit is

$$h_2(0) = \sum_{n\leq 0} h_2^{(n)}(z_0)(-z_0)^n \neq 0. \tag{3.12}$$

By a similar argument as in the preceding paragraph, from this follows uniform convergence of the series (3.10), and then taking log(and dividing by n of both sides) proves the proposition.

□

4. Proof of Theorem 2

BLA

5. Proof of Theorem 3

In this section we describe the limit of the sequence of curves $\{\Gamma^{(n)}\}$ as a current.

CONNECT THE ENUMERATIVE RESULTS OF SECTION 2 TO THE MAIN RESULT OF SECTIONS 3-4!!!

6. Examples

Example 1. Let $\Gamma_0 := \Gamma$ be the curve defined by $y^\ell = \frac{P(x)}{Q(x)}$ for some polynomials $P(x)$ and $Q(x)$.

Lemma 3. *The curve* $\Gamma_n := \Gamma^{(n)}$ *is given by the equation*

$$y^\ell = \frac{V_n(x)^\ell}{\ell^{n\ell}P(x)^{n\ell-1}Q(x)^{n\ell+1}} \tag{6.1}$$

where $V_n(X)$ *is a polynomial determined by the recurrence relation*

- $V_0(x) = 1$,
- $V_{n+1}(x) = \ell P(x)Q(x)V_n'(x) - ((n\ell-1)P'(x)Q(x) + (n\ell+1)P(x)Q'(x))V_n(x)$.

In particular, if P and Q has degrees d_1 and d_2, respectively, then

1. $V_n(x)$ *has degree* $n(d_1 + d_2 - 1)$ *and*
2. Γ_n *has bidegree* $(\max\{n\ell(d_1+d_2) - n\ell, n\ell(d_1+d_2) + (d_2-d_1)\}, \ell)$.

Proof. The base case $n = 0$ is clear. Assuming Γ_n has the form in (6.1) and writing the denominator in (6.1) as W_n, we have

$$\ell(y^{(n)})^{\ell-1}y^{(n+1)} = \frac{V_n^{\ell-1}(\ell W_n V_n' - V_n W_n')}{W_n^2}$$

and

$$(y^{(n+1)})^\ell = \left(\frac{W_n^{(\ell-1)/\ell}}{\ell V_n^{\ell-1}} \cdot \frac{V_n^{\ell-1}(\ell W_n V_n' - V_n W_n')}{W_n^2}\right)^\ell$$
$$= \frac{(\ell W_n V_n' - V_n W_n')^\ell}{\ell^\ell W_n^{\ell+1}}.$$

Substituting back W_n and simplifying yields

$$(y^{(n+1)})^\ell = \frac{(\ell^{n\ell} P^{n\ell-2} Q^{n\ell})^\ell (\ell PQV_n' - V_n((n\ell+1)PQ' + (n\ell-1)P'Q)^\ell}{\ell^\ell (\ell^{n\ell} P^{n\ell-1} Q^{n\ell+1})^{\ell+1}}$$
$$= \frac{(\ell PQV_n' - V_n((n\ell+1)PQ' + (n\ell-1)P'Q)^\ell}{\ell^{\ell(n+1)} P^{\ell(n+1)-1} Q^{\ell(n+1)+1}}.$$

The claims about the degrees follow directly. □

Example 1. Let $\Gamma := \Gamma_0$ be the unit circle given by $y^2 + x^2 = 1 \leftrightarrow y^2 = 1 - x^2$.

Lemma 4. *The curve* $\Gamma^{(n)} := \Gamma_n$ *is given by the equation*

$$y^2 = -\frac{U_n^2(x)}{(x^2-1)^{2n-1}},$$

where the polynomial sequence $\{U_n(x)\}$ *is given by the following recurrence relation.*

$$U_1(x) = x;\ U_n(x) = (2n-3)xU_{n-1}(x) + (1-x^2)U_{n-1}'(x) \text{ for } n \geq 2.$$

In particular, $U_n(1) = (2^n - 1)!!.????$

Notice that the bidegree of this equation DOES NOT agree with the formulas given in Proposition 1 (ii). WHY???

2. Elliptic curve in the Weierstrass form. Let $\Gamma := \Gamma_0$ be the special cubic in the Weierstrass form given by $y^2 = x^3 - 1$.

Lemma 5. *The curve* $\Gamma^{(n)} := \Gamma_n$ *is given by the equation*

$$y^2 = -\frac{V_n^2(x)}{4^n(x^3-1)^{2n-1}},$$

where the polynomial sequence $\{V_n(x)\}$ *is given by the following recurrence relation.*

???

3. Let $\Gamma_0 := \Gamma$ be the curve defined by $y^\ell = P(x)$ for some polynomial $P(x)$ of degree k.

Lemma 6. *The curve* $\Gamma_n := \Gamma^{(n)}$ *is given by the equation*

$$y^\ell = \frac{U_n^\ell(x)}{\ell^{n\ell} P(x)^{n\ell-1}}$$

where $U_n(X)$ *is a polynomial determined by the recurrence relation*

$$U_0(x) = 1; \quad U_{n+1}(x) = \ell P(x)U_n'(x) - (n\ell-1)P'(x)U_n(x).$$

In particular, $U_n(x)$ *has degree* $n(k-1)$ *and* Γ_n *has bidegree* $(\max\{k\ell n - \ell n, k\ell n - k\}, \ell)$.

7. Final remarks

1. Does the same approach work for meromorphic functions $f : Y \to \mathcal{P}^1$ where Y is a compact Riemann surface and we have chosen an affine coordinate z in $\mathbb{C}P^1$?
2. Under some simple restrictions on Γ, each $\mathcal{Z}_a^{(n)}(\Gamma)$ is a positive divisor on $\mathbb{C}_x$ and we can associate to it its *root-counting measure* $\mu_n(\Gamma)$ obtained by placing to each point in the support of $\mu_n(\Gamma)$ the point mass equal to the multiplicity of this point in the divisor divided by the total degree of the divisor. Obviously, $\mu_n(\Gamma)$ is a probability measure.

Describe the limit (if it exists) of the sequence $\{\mu_n(\Gamma)\}$ of root-counting measures in the sense of a weak convergence.
3. What is the limit of the currents given by the sequence of curves $\widetilde{\Gamma}^{(n)}$?

8. Appendix

8.1. **Faa di Bruno and Leibniz formulas.** Leibniz rule applied to (3.9) gives

$$\frac{(h_1h_2)^{(n)}}{n!} = \frac{h_1^{(n)}}{n!}h_2 + \sum_{\substack{k_1+k_2=n\\ k_2\geq 1}} \frac{h_1^{(k_1)}}{k_1!}\frac{h_2^{(k_2)}}{k_2!}. \tag{8.1}$$

and Faà di Bruno's formula(see Comtet [1, p.137]) gives

$$\frac{h_2(h_3(z))^{(k_2)}}{k_2!} = \sum_{l=1}^{k_2} \frac{h_2^{(l)}(h_3(z))}{l!}\frac{l!}{k_2!}B_{k_2,l}(h_3^{(1)}(z), h_3^{(2)}(z), ..., h_3^{(k_2-l+1)}(z)). \tag{8.2}$$

Bell polynomials are (weighted) homogeneous(loc.cit. p.133):

$$B_{k_2,l}(stx_1, st^2x_2, ..., st^{k_2-l+1}x_{k_2-l+1}) = s^lt^{k_2}B_{k_2,l}(x_1, x_2, ..., x_{k_2-l+1}).$$

Hence, with $h_3(z) = z^\alpha$ and

$$x_i = \langle\alpha\rangle_i = \alpha(\alpha-1)...(\alpha-i+1) \tag{8.3}$$

(Pochhammer descending factorial) so that

$$h_3^{(s)}(z) = x_sz^\alpha/z^s \tag{8.4}$$

and

$$B_{k_2,l}(h_3^{(1)}(z), h_3^{(2)}(z), ..., h_3^{(k_2-l+1)}(z)) = \frac{z^{l\alpha}}{z^{k_2}}B_{k_2,l}(x_1, x_2, ..., x_{k_2-l+1}) =: \frac{z^{l\alpha}}{z^{k_2}}B_{k_2,l}. \tag{8.5}$$

8.2. **Bell polynomials.** We will now prove that the sum of Bell polynomials in the last expression

$$t_n(l) := \sum_{i=l}^{n}(-1)^l\frac{\langle c\alpha\rangle_{n-i}}{\langle c\alpha\rangle_n}\frac{n!}{(n-i)!}\frac{l!}{i!}B_{i,l} \tag{8.6}$$

converges to 1 as $n \to \infty$. We need some more information on the Bell polynomials involved. Using the (formal) generating series(see [1, p.134]):

$$H_l(t) := \frac{1}{l!}\Big(\sum_{j=1}^{\infty}x_j\frac{t^j}{j!}\Big)^l = \sum_{i=l}^{\infty}B_{i,l}(x_1, x_2, ..., x_{i-l+1})\frac{t^i}{i!}. \tag{8.7}$$

For $x_i = \langle\alpha\rangle_i$ as above, Newton's binomial theorem gives that

$$H_l(t) = \frac{1}{l!}\big((1+t)^\alpha - 1\big)^l = \sum_{i=l}^{\infty}B_{i,l}\frac{t^i}{i!}, \tag{8.8}$$

where $B_{i,l} := B_{i,l}(x_1, x_2, ..., x_{i-l+1})$. For $\mathrm{Re}\, t > -1$ $H_l(t)$ is a holomorphic function. Now already Abel studied the convergence of $H_1(t)$ on the boundary of the disk of convergence $|t| < 1$. First note that $\langle\alpha\rangle_i = (-1)^i(\alpha)_i := (-1)^i(-\alpha(-\alpha+1)...(-\alpha+i-1)$(Pochhammers ascending factorial), so that $(-1)^{i+1}\langle\alpha\rangle_i > 0$ if $\alpha = 1/k < 1$. Hence by the homogeneity $B_{i,l} = (-1)^{i+l}B_{i,l}(\alpha, (-\alpha)_2, ..., (-\alpha)_{n-i+1}$. From the definition, the Bell polynomials of a positive sequence are positive, and hence

$$(-1)^{i+l}B_{i,l} > 0. \tag{8.9}$$

It is well- known (and also proved by e.g. Raabe's convergence criterion) that the binomial series for $(1+t)^\alpha$ converges at $t = 1$, and a fortiori, at $t = -1$. By (8.7) the same remains true for $H_l(t)$, and by letting $t \to -1$, and using Abel's theorem,

$$(-1)^l = \sum_{i\geq l}(-1)^i\frac{l!}{i!}B_{i,l}. \tag{8.10}$$

This series is absolutely convergent by (8.9), and this has the following easy consequence: Let $s_n(i)$, $n = 1, 2, 3...$ be uniformly bounded sequences such that $\lim_{n\to\infty} s_n(l) = (-1)^i$ for all n. Then

$$\lim_{n\to\infty}\sum_{n\geq l}(-1)^l s_n(i)B_{i,l} = 1. \tag{8.11}$$

It remains to examine

$$s_n(i) := \frac{\langle c\alpha\rangle_{n-i}n!}{\langle c\alpha\rangle_n(n-i)!} = \frac{(-1)^{n-i}(-c\alpha)_{n-i}n!}{(-1)^n(-c\alpha)_n(n-i)!} = (-1)^i\frac{\Gamma(n+1)}{\Gamma(-c\alpha+n)}\frac{\Gamma(-c\alpha+n-i)}{\Gamma(n-i+1)}, \tag{8.12}$$

(using that $\langle -x\rangle = (-1)^n(x)_n = (-1)^n\Gamma(x+n-1)/\Gamma(x)$, for $x > 0$ and recalling our reduction to $c < 0$). An application of Stirling's formula to the Gamma function gives that $\frac{\Gamma(a+n)}{\Gamma(b+n)} \sim n^{a-b}$, as $n \to \infty$ and hence $lim_{n\to\infty}s_n(i) = (-1)^i$. By (8.11) this proves (8.6), as desired.

8.3. **Implicit derivatives.** Let $P(x, y(x))$ be a polynomial. For a pair of nonnegative integers (r, s), we denote the (r, s)-mixed partial derivative of P by

$$P^{(r,s)} := \frac{\partial^{r+s}}{\partial_x^r\partial_y^s}P.$$

A *d-partition* of an ordered pair of non-negative integers (a, b) is a list of d (not necessarily distinct) non-zero ordered pairs of non-negative integers such that the componentwise sum is (a, b). Let $\mathcal{S}_d(a, b)$ be the set of d-partitions of (a, b).

For any d-partition $\mu = (r_1^\mu, s_1^\mu), \ldots, (r_d^\mu, s_d^\mu)$ of (a, b), we write

$$P^\mu := \prod_{i=1}^d P^{(r_i^\mu, s_i^\mu)}.$$

If P is a monomial of bidegree (k, ℓ), then

$$\text{the bidegree of } P^{(r,s)} = \begin{cases}(k-r, \ell-s) & \text{if } r \leq k \text{ and } s \leq \ell\\ (-\infty, -\infty) & \text{otherwise.}\end{cases}$$

Moreover, if $r_i^\mu \leq k$ and $s_i^\mu \leq \ell$ for all i, then the bidegree of P^μ is

$$\sum_{i=1}^d(k - r_i^\mu, \ell - s_i^\mu) = (dk - \sum_{i=1}^d r_i^\mu, d\ell - \sum_{i=1}^d s_i^\mu) = (dk - a, d\ell - b).$$

Otherwise, $P^\mu = 0$.

Lemma 7. *Let $P(x,y)$ be a polynomial of bidegree (k,ℓ). Then, for any positive integer n, the n-th implicit derivative of $y=y(x)$ subject to $P(x,y(x))=0$ is*

$$\frac{d^n y}{dx^n} = \frac{1}{(P^{(0,1)})^{2n-1}} \cdot \sum_{\mu\in\mathcal{S}_{2n-1}(n,2n-2)} c_\mu P^\mu \tag{8.13}$$

for some coefficients $c_\mu \in \mathbb{Z}$.

Proof. We proceed by induction. The base case $n=1$ follows from the implicit derivative formula

$$\frac{dy}{dx} = \frac{-P^{(1,0)}}{P^{(0,1)}}. \tag{8.14}$$

We now assume that $\frac{d^n y}{dx^n}$ is of the form (8.13) and will show that it also holds for $\frac{d^{n+1}y}{dx^{n+1}}$. Each of the terms in the summand of (8.13) corresponds to a $(2n-1)$-partition $\mu = (r_1^\mu, s_1^\mu),\dots,(r_{2n-1}^\mu, s_{2n-1}^\mu)$, so we consider

$$\begin{aligned}\frac{d}{dx}\frac{c_\mu P^\mu}{(P^{(0,1)})^{2n-1}} &= c_\mu \frac{(P^{(0,1)})^{2n-1}\frac{d}{dx}P^\mu - P^\mu(2n-1)(P^{(0,1)})^{2n-2}\frac{d}{dx}P^{(0,1)}}{(P^{(0,1)})^{4n-2}}\\ &= c_\mu \frac{P^{(0,1)}\frac{d}{dx}P^\mu - (2n-1)P^\mu\frac{d}{dx}P^{(0,1)}}{(P^{(0,1)})^{2n}}.\end{aligned}$$

Using the product rule and (8.14), we have

$$\begin{aligned}\frac{d}{dx}P^\mu &= \sum_{i=1}^{2n-1}\left(P^{(r_i^\mu+1,s_i^\mu)} - P^{(r_i^\mu,s_{i+1}^\mu)}\frac{P^{(1,0)}}{P^{(0,1)}}\right)\prod_{j\neq i}P^{(r_j^\mu,s_j^\mu)}\\ &= \frac{1}{P^{(0,1)}}\sum_{i=1}^{2n-1}\left(P^{(r_i^\mu+1,s_i^\mu)}P^{(0,1)} - P^{(r_i^\mu,s_{i+1}^\mu)}P^{(1,0)}\right)\prod_{j\neq i}P^{(r_j^\mu,s_j^\mu)}.\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dx}P^{(0,1)} &= P^{(1,1)} - P^{(0,2)}\frac{P^{(1,0)}}{P^{(0,1)}}\\ &= \frac{1}{P^{(0,1)}}\left(P^{(1,1)}P^{(0,1)} - P^{(0,2)}P^{(1,0)}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dx}\frac{c_\mu P^\mu}{(P^{(0,1)})^{2n-1}} = \frac{c_\mu}{(P^{(0,1)})^{2n+1}}&\Bigg[\sum_{i=1}^{2n-1}\left(P^{(r_i^\mu+1,s_i^\mu)}P^{(0,1)}P^{(0,1)}\right)\prod_{j\neq i}P^{(r_j^\mu,s_j^\mu)}\\ &-\sum_{i=1}^{2n-1}\left(P^{(r_i^\mu,s_i^\mu+1)}P^{(1,0)}P^{(0,1)}\right)\prod_{j\neq i}P^{(r_j^\mu,s_j^\mu)}\\ &-(2n-1)P^\mu P^{(1,1)}P^{(0,1)} - (2n-1)P^\mu P^{(0,2)}P^{(1,0)}\Bigg].\end{aligned} \tag{8.15}$$

Notice that each differential monomial appearing in the summand of the numerator of (8.15) is of the form $cP^{\mu'}$ where c is some integer and μ' is a $(2n+1)$-partition of $(n+1,2n)$. Thus, it must also be the case for $\frac{d^{n+1}y}{dx^{n+1}}$, and the induction step is proved. □

Corollary 2. *For a polynomial $P(x,y)$ with Newton polygon $\mathfrak{P}$, the numerator differential polynomial of $\frac{d^n y}{dx^n}$ in (8.13) applied to P is a polynomial whose Newton polygon is the convex hull of the integral points of $(2n-1)\cdot\mathfrak{P} - \{(n,2n-2)\}$ intersected with the positive quadrant in the plane. In particular, if P is generic of bidegree (k,ℓ), the numerator of $\frac{d^n y}{dx}$ is a polynomial of bidegree $((2n-1)k - n, (2n-1)\ell-(2n-2))$.*

References

[1] Analyse Combinatoire. By Louis Comtet. Two volumes: Pp. 192, 190.
[2] R. Bøgvad, Ch. Hägg, A refinement for rational functions of Pólya's method to construct Voronoi Diagram, Journal of Mathematical Analysis and Applications (2017), 452(1), 312-334.
[3] Ch. Hägg, The asymptotic zero-counting measure of iterated derivatives of a class of meromorphic functions, Arkiv för Matematik, Volume 57 (2019) Number 1, 107–120.
[4] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[5] V. Keo, Generalization of Pólya's Theorem for Voronoi Diagram Construction, Master Thesis, Royal University of Phnom Penh, 2021.
[6] V. Kiritchenko, Intersection theory course notes, https://users.mccme.ru/valya/Notes.pdf
[7] G. Pólya, Über die Nullstellen sukzessiver Derivierten, Math. Z. 12 (1922), no. 1, 36–60.
[8] G. Pólya, On the zeros of the derivatives of a function and its analytic character, Bulletin of the American Mathematical Society,(1943) 49(3),178-191.
[9] G. Pólya, On functions whose derivatives do not vanish in a given interval, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 216–218.
[10] G. Pólya, (with R. P. Boas, Jr.) Influence of the signs of the derivatives of a function on its analytic character, Duke Math. J. 9 (1942), 406–424.
[11] G. Pólya, (with N. Wiener) On the oscillation of the derivatives of a periodic function, Trans. Amer. Math. Soc. 52 (1942), 249–256.
[12] G. Pólya, On the zeros of the derivatives of a function and its analytic character, Bull. Amer. Math. Soc. 49 (1943), 178–191.
[13] G. Pólya, On the zeros of successive derivatives–an example, J. Analyse Math. 30 (1976), 452–455.
[14] C. L.. Prather and J. K. Shaw, Zeros of Successive Derivatives of Functions Analytic in a Neighborhood of a Single Pole, Michigan Math.J. 29 (1982), 111-119.
[15] C.L. Prather, J. K. Shaw, A Shire Theorem for Functions with Algebraic Singularities, Virginia Polytechnic and State university, 1982.
[16] M. Shobha, Some G. Pólya Germs from Complex Analysis, Resonance (2014)
[17] E. C. Titchmarsh, The theory of functions, 2nd ed., Oxford University Press, Oxford, 1939.
[18] https://en.wikipedia.org/wiki/Voronoi_diagram

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