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On mother body measures with algebraic Cauchy transform

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Abstract. Below we discuss the existence of a mother body measure for the exterior inverse problem in potential theory in the complex plane. More exactly, we study the question of representability almost everywhere (a.e.) in $\mathbb{C}$ of (a branch of) an irreducible algebraic function as the Cauchy transform of a signed measure supported on a finite number of compact semi-analytic curves and a finite number of isolated points. Firstly, we present a large class of algebraic functions for which there (conjecturally) always exists a positive measure with the above properties. This class was discovered in our earlier study of exactly solvable linear differential operators. Secondly, we investigate in detail the representability problem in the case when the Cauchy transform satisfies a quadratic equation with polynomial coefficients a.e. in $\mathbb{C}$. Several conjectures and open problems are posed.

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1. Introduction

The study of local and global properties of the Cauchy transform and the Cauchy-Stieltjes integral was initiated by A. Cauchy and T. Stieltjes in the middle of the 19th century. Large numbers of papers and several books are partially or completely devoted to this area which is closely connected with the potential theory in the complex plane and, especially, to the inverse problem and to the inverse moment problem, see, e.g., [Bel, CMR, Gar, Mur, Zal].

For the convenience of our readers, let us briefly recall some basic facts about the Cauchy transform. Let μ be a finite compactly supported complex measure on the complex plane $\mathbb{C}$. Define the *logarithmic potential* of μ as

$$\mathrm{u}_\mu(z) := \int_{\mathbb{C}} \ln|z-\xi| d\mu(\xi)$$

and the *Cauchy transform* of μ as

$$\mathcal{C}_\mu(z) := \int_{\mathbb{C}} \frac{d\mu(\xi)}{z-\xi}.$$

Standard facts about the Cauchy transform include:

- $\mathcal{C}_\mu$ is locally integrable; in particular it defines a distribution on $\mathbb{C}$ and therefore can be acted upon by $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$.
- $\mathcal{C}_\mu$ is analytic in the complement in $\mathbb{C} \cup \{\infty\}$ to the support of μ. For example, if μ is supported on the unit circle (which is the most classical case), then $\mathcal{C}_\mu$ is analytic both inside the open unit disc and outside the closed unit disc.
- the relations between μ, $\mathcal{C}_\mu$ and u_μ are as follows:
$$\mathcal{C}_\mu = \frac{\partial u_\mu}{\partial z} \quad \text{and} \quad \mu = \frac{1}{\pi}\frac{\partial \mathcal{C}_\mu}{\partial \bar{z}} = \frac{1}{\pi}\frac{\partial^2 u_\mu}{\partial z \partial \bar{z}}$$
which should be understood as equalities of distributions.
- the Laurent series of $\mathcal{C}_\mu$ around ∞ is given by
$$\mathcal{C}_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \dots,$$
where
$$m_k(\mu) = \int_{\mathbb{C}} z^k d\mu(z),\ k = 0, 1, \dots$$
are the harmonic moments of measure μ. (These moments are important conserved quantities in the theory of Hele-Shaw flow, [KMWWZ].)

For more relevant information on the Cauchy transform we will probably recommend a short and well-written treatise [Gar].

During the last decades the notion of a mother body of a solid domain or, more generally, of a positive Borel measure was discussed both in geophysics and mathematics, see, e.g., [Sjo, SSS, Gus, Zid]. It was apparently pioneered in the 1960's by a Bulgarian geophysicist D. Zidarov [Zid] and later mathematically developed by B. Gustafsson [Gus]. Although a number of interesting results about mother bodies was obtained in several special cases, [SSS, Gus, Zid] there is still no consensus about its appropriate general definition. In particular, no general existence and/or uniqueness results are known at present.

Below we use one possible definition of a mother body (measure) and study a natural exterior inverse mother body problem in the complex plane. (In what follows we will only use Borel measures.)

Main problem. Given a germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i$, $a_0 \in \mathbb{R}$ of an algebraic (or, more generally, analytic) function near ∞, is it possible to find a compactly supported signed measure whose Cauchy transform coincides with (a branch of) the analytic continuation of $f(z)$ a.e. in $\mathbb{C}$? Additionally, for which $f(z)$ it is possible to find a positive measure with the above properties?

If such a signed measure exists and its support is a finite union of compact semi-analytic curves and isolated points we will call it *a real mother body measure* of the germ $f(z)$, see Definition 1 below. If the corresponding measure is positive, then we call it *a positive mother body measure* of $f(z)$.

Remark. Observe that for some germs of analytic functions near ∞, the singularities of their analytic continuation to $\mathbb{C}$ might be very ill behaved which implies that a mother body measure of such a germ does not exist for this reason. However since we consider only algebraic functions, this phenomenon never occurs in our situation.

An obvious necessary condition for the existence of a positive mother body measure is $a_0 > 0$ since a_0 is the total mass. A germ (branch) of an analytic function $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i$ with $a_0 > 0$ will be called *positive*. If $a_0 = 1$, then $f(z)$ is called a *probability* branch. (Necessary and sufficient conditions for the existence of a probability branch of an algebraic function are given in Lemma 2 below.)

The formal definition of a mother body measure that we are using is as follows.

Definition 1. Given a germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i$, $a_0 > 0$ of an analytic function near ∞, we say that a signed measure μ_f is its *mother body* if:

(i) its support $supp(\mu_f)$ is the union of finitely many points and finitely many compact semi-analytic curves in $\mathbb{C}$;

(ii) the Cauchy transform of μ_f coincides in each connected component of the complement $\mathbb{C} \setminus supp(\mu_f)$ with some branch of the analytic continuation of $f(z)$.

(Here by a compact semi-analytic curve we mean a compact fragment of a real-analytic curve in $\mathbb{C} \simeq \mathbb{R}^2$.)

Remark. Notice that by Theorem 1 of [BBB], if the Cauchy transform of a positive measure coincides a.e. in $\mathbb{C}$ with an algebraic function $f(z)$, then the support of this measure is a finite union of semi-analytic curves and isolated points. Therefore it is a mother body measure according to the above definition. Whether the latter result extends to signed measures is unknown at present which motivates the following question.

Problem 1. Is it true that if there exists a signed measure whose Cauchy transform satisfies an irreducible algebraic equation a.e. in $\mathbb{C}$, then there exists, in general, another signed measure whose Cauchy transform satisfies a.e. in $\mathbb{C}$ the same algebraic equation and whose support is a finite union of compact semi-analytic curves and isolated points? Does there exist such a measure with a singularity on each connected component of its support?

Classically the inverse problem in potential theory deals with the question of how to restore a solid body or a (positive) measure from the information about its potential near infinity. The main efforts in this inverse problem since the foundational paper of P. S. Novikov [Nov] were concentrated around the question about the uniqueness of a (solid) body with a given exterior potential, see, e.g., recent [GS1] and [GS2] and the references therein. P. S. Novikov (whose main mathematical contributions are in the areas of mathematical logic and group theory) proved uniqueness for convex and star-shaped bodies with a common point of star-shapeness. The question about the uniqueness of contractible domains in $\mathbb{C}$ with a given sequence of holomorphic moments was later posed anew by H. S. Shapiro, see [BOS, Problem 1, p. 193] and answered negatively by M. Sakai in [Sak]. A similar non-uniqueness example for non-convex plane polygons was reported by geophysicists in [BS], see also [PS].

It turns out that the existence of a compactly supported positive measure with a given Cauchy transform $f(z)$ near ∞ imposes no conditions on a germ except for the obvious $a_0 > 0$, see Theorem 1 below.

On the other hand, the requirement that the Cauchy transform coincides with (the analytic continuation) of a given germ $f(z)$ a.e. in $\mathbb{C}$ often leads to additional restrictions on the germ $f(z)$ which are not easy to describe in terms of the defining algebraic equation, see § 4.

Below we study two classes of algebraic functions of very different origin and our results for these two cases are very different as well. For the first class, the obvious necessary condition $a_0 = 1$ seems to be sufficient for the existence of a positive mother body measure. (At present we can prove this fact only under certain additional restrictions.) For the second class, the set of admissible germs has a quite complicated structure. These results together with a number of conjectures seem to indicate that it is quite difficult, in general, to answer when a given algebraic germ $f(z)$ admits a mother body measure and if it does, then how many.

Several concluding remarks are in place here. Our interest in probability measures whose Cauchy transforms are algebraic functions a.e. in $\mathbb{C}$ was sparked by the pioneering work [BR]. Since then the class of interesting examples where such situation occurs has been substantially broadened, see [BBS, HS, STT].

Some general local results when one considers a collection of locally analytic functions instead of a global algebraic function were obtained in [BB] and later extended in [BBB].

This paper is mainly the application of several results obtained earlier by the authors and their collaborators in the context of differential operators to the above Main Problem as well as the study of its underlying algebraic aspects, see, e.g., [BR, BBS]. However we also formulate several new results, see, e.g., Theorem 12.

2. Some general facts

The first rather simple result of the present paper (which apparently is known to the specialists) is as follows.

Theorem 1. *Given an arbitrary germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i$, $a_0 > 0$ of an analytic function near ∞, there exist (families of) positive compactly supported in $\mathbb{C}$ measures whose Cauchy transform near ∞ coincides with $f(z)$.*

Proof of Theorem 1. Given a branch $f(z) = a_0/z + \sum_{i\geq 2} a_i/z^i$ of an analytic function near ∞, we first take a germ of its 'logarithmic potential', i.e., a germ $h(z)$ of harmonic function such that $h(z) = a_0 \log|z| + \dots$ satisfying the relation $\partial h/\partial z = f(z)$ in a punctured neighborhood of ∞. Here $\dots$ stands for a germ of harmonic function near ∞. For any sufficiently large positive v, the connected component γ_v of the level curve $h(z) = v$ near infinity is closed and simple. It makes one turn around ∞. To get a required positive measure whose Cauchy transform coincides with $f(z)$ near ∞, take γ_v for any v large enough and consider the complement $\Omega_v = \mathbb{C}P^1 \setminus Int(\gamma_v)$, where $Int(\gamma_v)$ is the interior of γ_v containing ∞. Consider the equilibrium measure of mass a_0 supported on Ω_v. By Frostman's theorem, this measure is in fact supported on γ_v (since γ_v has no polar subsets), its potential is constant on γ_v and it is harmonic in the complement to the support. (For definition and properties of equilibrium measures as well as Frostman's theorem[1] consult [Ran].) Thus it should coincide with $h(z)$ in $Int(\gamma_v)$, since the total mass is correctly chosen. Then its Cauchy transform coincides with $f(z)$ in $Int(\gamma_v)$. □

Example 1. If we choose $f(z) = 1/z$ as our branch at ∞, then $h(z) = \log|z|$ and γ_v is the circle $|z| = e^v$. The equilibrium measure is the uniform probability measure on this circle, and its Cauchy transform $C(z)$ vanishes inside the circle and equals $1/z$ outside. Since the constant 0 is not the analytic continuation of

[1] Otto Frostman has spent a substantial part of his professional life at the same department in Stockholm where we are currently employed.

$1/z$, the uniform measure on the circle is not a mother body measure of the germ $1/z$. However the Cauchy transform of the unit point mass placed at the origin equals $1/z$ in $\mathbb{C} \setminus 0$. Therefore the unit mass at the origin is a mother body measure for the germ $1/z$ at ∞.

We now give a necessary condition and a slightly stronger sufficient condition for an algebraic curve given by the equation

$$P(\mathcal{C}, z) = \sum_{(i,j)\in S(P)} \alpha_{i,j} \mathcal{C}^i z^j = 0 \tag{2.1}$$

to have a probability branch at ∞. Here every $\alpha_{i,j} \neq 0$ and $S(P)$ is an arbitrary finite subset of pairs of non-negative integers, i.e an arbitrary set of monomials in 2 variables. In other words, $S(P)$ is the set of all monomials appearing in P with non-vanishing coefficients. (In what follows, $\mathcal{C}$ stands for the variable corresponding to the Cauchy transform.)

The following group of results consisting of Lemmas 2 and 3 as well as Corollary 1 are very straight-forward, although we could not find them in the literature.

Lemma 2. *If the algebraic curve given by equation* (2.1) *has a probability branch at* ∞, *then*

$$\sum_i \alpha_{i,i-m(P)} = 0 \quad \textit{where} \quad m(P) := \min_{(i,j)\in S(P)} i - j. \tag{2.2}$$

In particular, there should be at least two distinct monomials in $S(P)$ *whose difference of indices* $i - j$ *equals* $m(P)$.

If equation (2.2) *is satisfied, and additionally*

$$\sum_i i\alpha_{i,i-m(P)} \neq 0, \tag{2.3}$$

then there is a unique probability branch at ∞ *satisfying equation* (2.1).

Proof. Substituting $w = 1/z$ in (2.1) we get $P(\mathcal{C}, w) = \sum_{(i,j)\in S(P)} \alpha_{i,j}\mathcal{C}^i/w^j$. Assuming that the algebraic curve given by $P(\mathcal{C}, w) = 0$ has a branch

$$\mathcal{C} = w + \sum_{l=2}^{\infty} a_l w^l$$

where a_l, $l = 2, 3, \ldots$ are undetermined coefficients, we substitute the latter expression in the above equation and get

$$\sum_{(i,j)\in S(P)} \alpha_{i,j}(w + \sum_{l=2}^{\infty} a_l w^l)^i / w^j = 0. \tag{2.4}$$

Collecting the entries of the minimal total degree which is equal to

$$m(P) := \min_{(i,j)\in S(P)} i-j,$$

we obtain an obvious necessary condition for the solvability of (2.4) given by:

$$\sum \alpha_{i,i-m(P)} = 0. \tag{2.5}$$

Let us now show that (2.2), together with (2.3), are sufficient for the solvability of (2.4) with respect to the sequence of coefficients $a_2, a_3, \dots$. Indeed, due to algebraicity of $P(\mathcal{C}, z)$, it suffices to prove the formal solvability of (2.4). Let $\mathcal{C} = \mathcal{D}w$ and rewrite the equation as

$$\tilde{P}(\mathcal{D}, w) = \sum_{(i,j)\in S(P)} \alpha_{i,j}\mathcal{D}^i w^{i-j-m(P)} = 0,$$

which is now a polynomial in w and $\mathcal{D}$. Assume that $d_r = 1+a_2w+\dots+a_{r+1}w^r$ satisfies

$$\tilde{P}(d_r, w) \equiv 0 \bmod z^{r+1}. \tag{2.6}$$

The fact that this equation holds for $d_0 = 1$ is exactly the relation (2.5) which gives the basis of the inductive construction of $d_1, d_2, \dots$, as follows. Letting $\tilde{P}_1'(\mathcal{C}, w)$ be the partial derivative of $\tilde{P}(\mathcal{C}, w)$ with respect to the first variable, we have the following relation for the undetermined coefficient a_{r+2}:

$$\begin{aligned}\tilde{P}(d_r + a_{r+2}w^{r+1}, w) &\equiv \tilde{P}(d_r, w) + \tilde{P}_1'(d_r, w)a_{r+2}w^{r+1}\\ &\equiv \tilde{P}(d_r, w) + \tilde{P}_1'(1,0)a_{r+2}w^{r+1} \bmod w^{r+2}.\end{aligned}$$

Since we have assumed that $\tilde{P}_1'(1,0) = \sum_i i\alpha_{i,i-m(P)}$ is non-zero, and that, by the induction assumption, $\tilde{P}(d_r, w) \equiv bw^{r+1} \bmod w^{r+2}$, $b \in \mathbb{C}$, we can solve the latter equation for a_{r+2}. Thus, by induction, we have proven that there is a formal series solution of (2.4). Therefore conditions (2.2) and (2.3) are sufficient for the existence of a probability branch (which is also unique in this case). □

Remark. Note that for an irreducible algebraic curve defined by (2.1) the second condition (2.3) in Lemma 2 says that $z = \infty$ is not its branch point. It is clearly possible, though cumbersome, to give necessary and sufficient conditions for the existence of a probability branch in terms of algebraic relations between the coefficients of the equation. An example of an equation that does not satisfy both conditions in Lemma 2, but still has a probability branch as a solution, is $P(\mathcal{C}, z) = (\mathcal{C}z - 1)^2$.

For the polynomials whose Newton polygons (i.e., the convex hulls of the set of monomials) are shown in Fig. 1, the necessary condition of Lemma 2 says

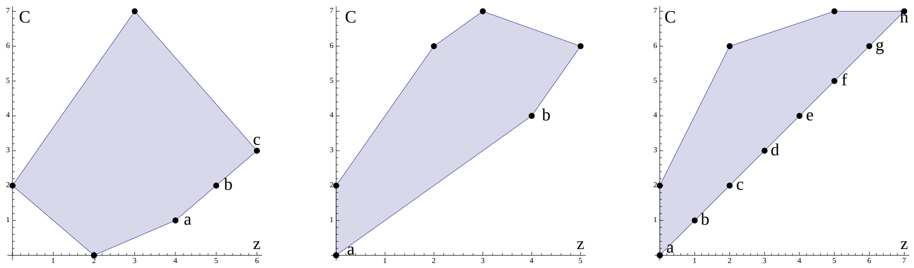


Figure 1
Three examples of Newton polygons

that the sum of the coefficients of the monomials marked by the letters a, b, … should vanish.

Since we are working with irreducible algebraic curves (obtained as the analytic continuation of given branches at ∞) we will need the following statement. Let S be an arbitrary set of monomials in 2 variables. Denote by $\mathcal{N}_S$ the *Newton polygon* of S, i.e., the convex hull of S in the plane of exponents of monomials. Denote by Pol_S the linear span of all monomials in S with complex coefficients.

Lemma 3. *A generic polynomial in Pol_S is irreducible if and only if:*

(i) *$\mathcal{N}_S$ is two-dimensional, i.e., not all monomials in S lie on the same affine line;*

(ii) *S contains a vertex on both coordinate axes, i.e., pure powers of both variables.*

Notice that (ii) is satisfied for both coordinate axis if S contains the origin (i.e., polynomials in Pol_S might have a non-vanishing constant term).

Proof. Observe that the property that a generic polynomial from Pol_S is irreducible is inherited. In other words, if S contains a proper subset S' with the same property, then it automatically holds for S. The necessity of both conditions (i) and (ii) is obvious. If (i) is violated, then any polynomial in Pol_S can be represented as a polynomial in one variable after an appropriate change of variables. If (ii) is violated, then any polynomial in Pol_S is divisible by a (power of a) variable.

To prove sufficiency, we have to consider several cases. If S contains $\mathcal{C}^i$ and z^j where both i and j are positive, then already a generic curve of the form $\alpha\mathcal{C}^i + \beta\mathcal{C}^j z^k + \gamma z^l = 0$ is irreducible unless $\mathcal{C}^i$, $\mathcal{C}^j z^k$ and z^l lie on the same line. If for some positive i, S contains $\mathcal{C}^i$ and 1 and no other pure powers of z (or, similarly, z^k and 1 for some positive k and no other pure powers of $\mathcal{C}$), then S should contain another monomial $\mathcal{C}^l z^k$ with l and k positive. But then a generic curve of the form $A + B\mathcal{C}^i + C\mathcal{C}^l z^k = 0$ is irreducible. Finally, if the only monomial on the union of the axes is 1, but there exists two monomials $\mathcal{C}^{i_1} z^{j_1}$ and $\mathcal{C}^{i_2} z^{j_2}$ with $i_1/j_1 \neq i_2/j_2$, then a generic curve of the form $A + B\mathcal{C}^{i_1} z^{j_1} + C\mathcal{C}^{i_2} z^{j_2} = 0$ is irreducible. □

Corollary 1. *An irreducible polynomial $P(\mathcal{C}, z)$ having a probability branch has a non-negative $M(P) := \max_{(i,j)\in S(P)}(i - j)$ as well as a non-positive $m(P) := \min_{(i,j)\in S(P)}(i - j)$. If $m(P) = 0$, then the set $S(P)$ of monomials of $P(\mathcal{C}, z)$ must contain the origin, i.e., must have a non-vanishing constant term.*

If we denote by $\mathcal{N}_P = \mathcal{N}_{S(P)}$ the Newton polygon of $P(\mathcal{C}, z)$, (i.e., the convex hull of its monomials), then the geometric interpretation of the latter corollary is that $\mathcal{N}_P$ should (non-strictly) intersect the bisector of the first quadrant in the plane $(z, \mathcal{C})$.

The case $m(P) = 0$ will be of special interest to us, see § 3. We say that a polynomial $P(\mathcal{C}, z)$ (and the algebraic function given by $P(\mathcal{C}, z) = 0$) is *balanced* if it satisfies the condition $m(P) = 0$. For polynomials with $m(P) < 0$ having a positive branch, the problem of existence of a mother body measure seems to be dramatically more complicated than for $m(P) = 0$, see § 4.

Notice that a rather simple situation occurs, when $\mathcal{C}$ is a rational function, i.e., $\mathcal{C}$ satisfies a linear equation.

Lemma 4. *A (germ at ∞ of a) rational function $\mathcal{C}(z) = \frac{z^n + \dots}{z^{n+1} + \dots}$ with coprime numerator and denominator admits a mother body measure if and only if it has all simple poles with real residues. If all residues are positive the corresponding mother body measure is positive.*

Proof. Recall the classical relation

$$\mu = \frac{1}{\pi} \frac{\partial \mathcal{C}_\mu(z)}{\partial \bar{z}}$$

between a measure μ and its Cauchy transform $\mathcal{C}_\mu$,where the derivative is taken in the sense of distributions. By assumption $\mathcal{C}_\mu(z)$ coincides almost everywhere with a given rational function. Therefore, by the above formula, μ is the measure supported at the poles of the rational function. Moreover the measure concentrated at each pole coincides (up to a factor π) with the residue of $\mathcal{C}_\mu(z)$ at this pole. Since by assumption μ has to be a real measure, this implies that all poles of the rational function should be simple and with real residues. □

The above statement implies that the set of rational function of degree n admitting a mother body measure has dimension equal to half of the dimension of the space of all rational functions of degree n. The case when $\mathcal{C}_\mu$ satisfies a quadratic equation is considered in some detail in § 4.

3. Balanced algebraic functions

In the case of balanced algebraic functions, i.e., for bivariate polynomials (2.1) with $m(P) := \min_{(i,j)\in S(P)}(i-j) = 0$ and satisfying the necessary condition (2.2), our main conjecture is as follows.

Conjecture 1. *An arbitrary balanced irreducible polynomial $P(\mathcal{C}, z)$ with a positive branch admits a positive mother body measure.*

Appropriate Newton polygons are shown on the central and the right-most pictures in Fig. 1. The next result essentially proven in [BBS] strongly supports the above conjecture. Further supporting statements can be found in [Ber].

Theorem 5. *An arbitrary balanced irreducible polynomial $P(\mathcal{C}, z)$ with a unique probability branch which additionally satisfies the following requirements:*

(i) *$S(P)$ contains a diagonal monomial $\mathcal{C}^n z^n$ which is lexicographically bigger than any other monomial in $S(P)$;*

(ii) *the probability branch is the only positive branch of $P(\mathcal{C}, z)$;*

admits a probability mother body measure.

By 'lexicographically bigger' we mean that the pair (n, n) is coordinate-wise not smaller than any other pair of indices in $S(P)$, see the right-most picture in Fig. 1. Condition (i) is the only essential restriction in Theorem 5 compared to Conjecture 1 since condition (ii) is generically satisfied. Observe also that an irreducible balanced polynomial $P(\mathcal{C}, z)$ must necessarily have a non-vanishing constant term, see Lemma 3. Balanced polynomials satisfying assumptions of Theorem 5 are called *excellent balanced* polynomials.

An interesting detail about the latter theorem is that its proof has hardly anything to do with potential theory. Our proof of Theorem 5 is based on certain properties of eigenpolynomials of the so-called exactly solvable differential operators, see, e.g., [BBS]. We will construct the required mother body measure as the weak limit of a sequence of root-counting measures of these eigenpolynomials.

Definition 2. A linear ordinary differential operator

$$\mathfrak{d} = \sum_{i=1}^{k} Q_i(z)\frac{d^i}{dz^i}, \tag{3.1}$$

where each $Q_i(z)$ is a polynomial of degree at most i and there exists at least one value i_0 such that $\deg Q_{i_0}(z) = i_0$ is called *exactly solvable*. An exactly solvable operator is called *non-degenerate* if $\deg Q_k(z) = k$. The symbol $T_\mathfrak{d}$ of the operator (3.1) is, by definition, the bivariate polynomial

$$T_\mathfrak{d}(\mathcal{C}, z) = \sum_{i=1}^{k} Q_i(z)\mathcal{C}^i. \tag{3.2}$$

Observe that $\mathfrak{d}$ is exactly solvable if and only if $T_\mathfrak{d}$ is balanced. (Notice that $T_\mathfrak{d}$, by definition, has no constant term.)

Given an exactly solvable $\mathfrak{d} = \sum_{i=1}^{k} Q_i(z)\frac{d^i}{dz^i}$, consider the following *homogenized spectral problem*:

$$\begin{aligned} Q_k(z)p^{(k)} &+ \lambda Q_{k-1}(z)p^{(k-1)} \\ &+ \lambda^2 Q_{k-2}(z)p^{(k-2)} + \cdots + \lambda^{k-1}Q_1(z)p' = \lambda^k p, \end{aligned} \tag{3.3}$$

where λ is called the *homogenized spectral parameter*. Given a positive integer n, we want to determine all values λ_n of the spectral parameter such that equation (3.3) has a polynomial solution $p_n(z)$ of degree n.

Using notation $Q_i(z) = a_{i,i}z^i + a_{i,i-1}z^{i-1} + \cdots + a_{i,0}$, $i = 1, \ldots, k$, one can easily check that the eigenvalues λ_n satisfy the equation:

$$\begin{aligned} a_{k,k}n(n-1)\ldots(n-k+1) &+ \lambda_n a_{k-1,k-1}n(n-1)\ldots(n-k+2) \\ &+ \cdots + \lambda_n^{k-1}na_{1,1} = \lambda_n^k. \end{aligned} \tag{3.4}$$

(If $\mathfrak{d}$ is non-degenerate, i.e., $a_{k,k} \neq 0$, then there are typically k distinct values of λ_n for n large enough.)

Introducing the normalized eigenvalues $\widetilde{\lambda}_n = \lambda_n/n$, we get that $\widetilde{\lambda}_n$ satisfies the equation:

$$\begin{aligned} a_{k,k}\frac{n(n-1)\ldots(n-k+1)}{n^k} &+ \widetilde{\lambda}_n a_{k-1,k-1}\frac{n(n-1)\ldots(n-k+2)}{n^{k-1}} \\ &+ \cdots + \widetilde{\lambda}_n^{k-1}a_{1,1} = \widetilde{\lambda}_n^k. \end{aligned}$$

If $\widetilde{\lambda} = \lim_{n\to\infty} \widetilde{\lambda}_n$ exists, then it should satisfy the relation:

$$a_{k,k} + a_{k-1,k-1}\widetilde{\lambda} + \cdots + a_{1,1}\widetilde{\lambda}^{k-1} = \widetilde{\lambda}^k. \tag{3.5}$$

If $\mathfrak{d}$ is a degenerate exactly solvable operator, then let j_0 be the maximal i such that $\deg Q_i(z) = i$. By definition, $a_{j,j}$ vanish for all $j > j_0$. Thus, the first $k - j_0$ terms in (3.4) vanish as well, implying that (3.4) has j_0 non-vanishing and $(k - j_0)$ vanishing roots.

Lemma 6. *Given an exactly solvable operator $\mathfrak{d}$ as above,*

(i) *For the sequence of vanishing 'eigenvalues' $\lambda_n = 0$, there exists a finite upper bound of the degree of a non-vanishing eigenpolynomial $\{p_n(z)\}$.*

(ii) *For any sequence $\{\lambda_n\}$ of eigenvalues such that $\lim_{n\to\infty} \frac{\lambda_n}{n} = \widetilde{\lambda}_l$, where $\widetilde{\lambda}_l$ is some non-vanishing simple root of (3.5) the sequence of their eigenpolynomials is well-defined for all $n > N_0$, i.e., for all sufficiently large n.*

Proof. Item (i) is obvious since when $\lambda = 0$, (3.3) reduces to $Q_k(z)p^{(k)} = 0$ implying that $p^{(k)} = 0$. But this is impossible for any polynomial of degree exceeding $k-1$.

To prove (ii), notice that (3.4) defines $\widetilde{\lambda}$ as an algebraic function of the (complex) variable n.. Then (3.5) defines the value of $\widetilde{\lambda}$ at $n = \infty$. By assumption there is a unique branch with the value $\widetilde{\lambda}_l$. Additionally the corresponding branch of the algebraic function will intersect other branches in at most finite number of points. Hence for n very large, we can identify $\widetilde{\lambda}_l(n)$ as belonging to this particular branch. □

The next result is central in our consideration.

Theorem 7 (see Theorem 2, [BBS]). *For any non-degenerate exactly solvable operator $\mathfrak{d}$, such that* (3.5) *has no double roots, there exists N_0 such that the roots of all eigenpolynomials of the homogenized problem* (3.3) *whose degree exceeds N_0 are bounded, i.e., there exists a disk centered at the origin containing all of them at once.*

Unfortunately the existing proof of the latter theorem is too long and technical to be reproduced in the present paper. The next local result is a keystone in the proof of Theorem 5, comp. Proposition 3 of [BBS].

Theorem 8. *Given an exactly solvable $\mathfrak{d} = \sum_{i=1}^{k} Q_i(z)\frac{d^i}{dz^i}$, consider a sequence $\{p_n(z)\}, \deg p_n(z) = n$ of the eigenpolynomials of* (3.3) *such that the sequence $\{\lambda_n\}$ of their eigenvalues satisfies the condition $\lim_{n\to\infty} \frac{\lambda_n}{n} = \widetilde{\lambda}_l$, where $\widetilde{\lambda}_l$ is some non-vanishing root of* (3.5). *Let $L_n(z) = \frac{p_n'(z)}{\lambda_n p_n(z)}$ be the normalized logarithmic derivative of $p_n(z)$. If the sequence $\{L_n(z)\}$ converges to a function $L(z)$ in some open domain $\Omega \subset \mathbb{C}$, and the derivatives of $L_n(z)$ up to order k are uniformly bounded in Ω, then in the domain Ω the function $L(z)$ satisfies the symbol equation:*

$$Q_k(z)L^k(z) + Q_{k-1}(z)L^{k-1}(z) + \cdots + Q_1(z)L(z) = 1. \tag{3.6}$$

Proof. Note that each $L_n(z) = \frac{p_n'(z)}{\lambda_n p_n(z)}$ is well-defined and analytic in any open domain Ω free from the zeros of $p_n(z)$. Choosing such a domain Ω and an appropriate branch of the logarithm such that $\log p_n(z)$ is defined in Ω, consider a primitive function $M(z) = \lambda_n^{-1} \log p_n(z)$. The latter function is also well-defined and analytic in Ω.

Straight-forward calculations give: $e^{\lambda_n M(z)} = p_n(z)$, $p_n'(z) = p_n(z)\lambda_n L_n(z)$, and $p_n''(z) = p_n(z)(\lambda_n^2 L_n^2(z) + \lambda_n L_n'(z))$. More generally,

$$\frac{d^i}{dz^i}\big(p_n(z)\big) = p_n(z)\left(\lambda_n^i L_n^i(z) + \lambda_n^{i-1} F_i\big(L_n(z), L_n'(z), \dots, L_n^{(i-1)}(z)\big)\right),$$

where the second term

$$\lambda_n^{i-1} F_i(L_n, L_n', \dots, L_n^{(i-1)}) \tag{3.7}$$

is a polynomial in λ_n of degree $i-1$. The equation $\mathfrak{d}(p_n) + \lambda_n p_n = 0$ gives us:

$$p_n(z)\left(\sum_{i=0}^{k} Q_i(z)\lambda_n^{k-i}\left(\lambda_n^i L_n^i(z) + \lambda_n^{i-1} F_i\big(L_n(z), L_n'(z), \dots, L_n^{(i-1)}(z)\big)\right)\right) = 0$$

or, equivalently,

$$\lambda_n^k \sum_{i=0}^{k} Q_i(z)\Big(L_n^i(z) + \lambda_n^{-1} F_i\big(L_n(z), L_n'(z), \dots, L_n^{(i-1)}(z)\big)\Big) = 0. \tag{3.8}$$

Letting n tend to ∞ and using the boundedness assumption for the first $k-1$ derivatives, we get the required equation (3.6). □

Sketch of Proof of Theorem 5. Take an excellent balanced irreducible polynomial $P(\mathcal{C}, z) = \sum_{i=1}^{k} Q_i(z)\mathcal{C}^i$ having a probability branch at ∞. Since $P(\mathcal{C}, z)$ has a non-vanishing constant term, we can, without loss of generality, assume that it is equal to -1. Consider its corresponding differential operator $\mathfrak{d}_P(z) = \sum_{i=1}^{k} Q_i(z)\frac{d^i}{dz^i}$, i.e., the operator whose symbol $T_{\mathfrak{d}}$ equals $P(\mathcal{C}, z) + 1$. Notice that since $P(\mathcal{C}, z)$ has a probability branch at ∞, then there exists a root of (3.5) which is equal to 1. Therefore, there exists a sequence $\{\lambda_n\}$ of eigenvalues of $\mathfrak{d}_P(z)$ satisfying the condition $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$.

To settle Theorem 5, notice that by Theorem 7 for any $\mathfrak{d}_P(z)$ as above, the union of all roots of all its eigenpolynomials of all sufficiently large degrees is bounded. Choose a sequence $\{\lambda_n\}$ of eigenvalues of $\mathfrak{d}_P(z)$ satisfying the condition $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$. (Such subsequences exist for any excellent $P(\mathcal{C}, z)$.) Consider the corresponding sequence $\{p_n(z)\}$ of eigenpolynomials of $\mathfrak{d}_P(z)$, and the sequence $\{\mu_n\}$ of the root-counting measures of these eigenpolynomials, together with the sequence $\{\mathcal{C}_n(z)\}$ of their Cauchy transforms. (Observe that the

Cauchy transform of a polynomial P of degree n equals $\frac{P'}{nP}$.) By Theorem 7, all zeros of all these $p_n(z)$ are contained in some fixed disk. Therefore the supports of all μ_n are bounded implying that there exists a weakly converging subsequences $\{\mu_{i_n}\}$ of the sequence $\{\mu_n\}$. Choosing a further appropriate subsequence, we can guarantee that the corresponding subsequence $\{\mathcal{C}_{i_n}(z)\}$ converges almost everywhere in $\mathbb{C}$. (The rigorous argument for the latter claim is rather technical, see details in [BBS].) Taking a further subsequence, if necessary, we get that the subsequences $\{\mathcal{C}'_{i_n}(z)\}, \{\mathcal{C}''_{i_n}(z)\}, \dots, \{\mathcal{C}^{(k)}_{i_n}(z)\}$ will be bounded a.e. on any compact subset of $\mathbb{C}$. Finally, notice that if the limit $\lim_{n\to\infty} \mathcal{C}_{i_n}(z) = \mathcal{C}(z)$ exists in some domain Ω, then also $\lim_{n\to\infty} L_{i_n}(z)$ exists in Ω and equals $\mathcal{C}(z)$. Here $L_n(z) = \frac{p_n'(z)}{\lambda_n p_n(z)}$ and the sequence $\{\lambda_n\}$ is chosen as above. Since it satisfies the condition $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$, we have settled the existence of a probability measure whose Cauchy transform coincides a.e. in $\mathbb{C}$ with a branch of the algebraic function given by the equation $\sum_{i=1}^{k} Q_i(z)\mathcal{C}^i = 1$. The latter equation coincides with $P(\mathcal{C}, z) = 0$. □

4. Cauchy transforms satisfying quadratic equations and quadratic differentials

In this section we discuss which quadratic equations of the form:

$$P(z)\mathcal{C}^2 + Q(z)\mathcal{C} + R(z) = 0, \tag{4.1}$$

with $\deg P = n+2$, $\deg Q \le n+1$, $\deg R \le n$ admit mother body measure(s).

For the subclass of (4.1) with $Q(z)$ identically vanishing, such results were in large obtained in [STT]. Very close statements were independently and simultaneously obtained in [MFR1] and [MFR2]. To go further, we need to recall some basic facts about quadratic differentials.

4.1. Basics on quadratic differentials. The following material can be easily found in the classical sources [Str] and [Jen], see also [Sol1, Sol2].

A (meromorphic) quadratic differential Ψ on a compact orientable Riemann surface Y without boundary is a (meromorphic) section of the tensor square $(T^*_{\mathbb{C}}Y)^{\otimes 2}$ of the holomorphic cotangent bundle $T^*_{\mathbb{C}}Y$. The zeros and the poles of Ψ constitute the set of *critical points* of Ψ denoted by $Sing_\Psi$. Zeros and poles are called *finite critical points*. (Non-critical points of Ψ are usually called *regular*.)

If Ψ is locally represented in two intersecting charts by $h(z)dz^2$ and by $\tilde{h}(\tilde{z})d\tilde{z}^2$ resp. with a transition function $\tilde{z}(z)$, then $h(z) = \tilde{h}(\tilde{z})\,(d\tilde{z}/dz)^2$. Any quadratic differential induces a canonical metric on its Riemann surface Y, whose length element in local coordinates is given by

$$|dw| = |h(z)|^{\frac{1}{2}}|dz|.$$

The above canonical metric $|dw| = |h(z)|^{\frac{1}{2}}|dz|$ on Y is closely related to two distinguished line fields given by the condition that $h(z)dz^2 > 0$ and $h(z)dz^2 < 0$ resp. The integral curves of the first field are called *horizontal trajectories* of Ψ, while the integral curves of the second field are called *vertical trajectories* of Ψ. In what follows we will mostly use horizontal trajectories of quadratic differentials and reserve the term *trajectories* for the horizontal ones.

Here we only consider rational quadratic differentials, i.e., $Y = \mathbb{CP}^1$. Any such quadratic differential Ψ will be globally given in $\mathbb{C}$ by $\phi(z)dz^2$, where $\phi(z)$ is a complex-valued rational function.

Trajectories of Ψ can be naturally parameterized by their arclength. In fact, in a neighborhood of a regular point z_0 on $\mathbb{C}$, one can introduce a local coordinate called a *canonical parameter* and given by

$$w(z) := \int_{z_0}^{z} \sqrt{\phi(\xi)}d\xi.$$

One can easily check that $dw^2 = \phi(z)dz^2$ implying that horizontal trajectories in the z-plane correspond to horizontal straight lines in the w-plane, i.e., they are defined by the condition $\operatorname{Im} w = const$.

A trajectory of a meromorphic quadratic differential Ψ given on a compact Y without boundary is called *critical* if there exists a finite critical point of Ψ belonging to its closure. For a given quadratic differential Ψ on a compact surface Y, denote by $K_\Psi \subset Y$ the union of all its critical trajectories and critical points. In general, K_Ψ can be very complicated. In particular, it can be dense in some subdomains of Y.

We say that a critical trajectory is *finite* if it approaches finite critical points in both directions, i.e., its both endpoints are finite critical points. We denote by $DK_\Psi \subseteq K_\Psi$ (the closure of) the set of finite critical trajectories of (4.2). (One can easily show that DK_Ψ is an imbedded (multi)graph in Y. Here by a *multigraph* on a surface we mean a graph with possibly multiple edges and loops.) Finally, denote by $DK_\Psi^0 \subseteq DK_\Psi$ the subgraph of DK_Ψ consisting of (the closure of) the set of finite critical trajectories whose both ends are zeros of Ψ.

A non-critical trajectory $\gamma_{z_0}(t)$ of a meromorphic Ψ is called *closed* if $\exists\ T > 0$ such that $\gamma_{z_0}(t+T) = \gamma_{z_0}(t)$ for all $t \in \mathbb{R}$. The least such T is called the *period* of γ_{z_0}. A quadratic differential Ψ on a compact Riemann surface Y without boundary is called *Strebel* if the set of its closed trajectories covers Y up to a set of Lebesgue measure zero.

4.2. General results on Cauchy transforms satisfying quadratic equations. In this subsection we relate the question for which triples of polynomials (P, Q, R) equation (4.1) admits a real mother body measure to a certain problem about rational quadratic differentials. We start with the following necessary condition.

Proposition 9. *Assume that equation* (4.1) *admits a real mother body measure* μ. *Then the following two conditions hold:*

(i) *any connected curve in the support of* μ *coincides with a horizontal trajectory of the quadratic differential*

$$\Theta = -\frac{D^2(z)}{P^2(z)}dz^2 = \frac{4P(z)R(z) - Q^2(z)}{P^2(z)}dz^2. \tag{4.2}$$

(ii) *the support of* μ *should include all branch points of* (4.1).

Remark. Observe that if $P(z)$ and $Q(z)$ are coprime, the set of all branch points coincides with the set of all zeros of $D(z)$. In particular, requirement (ii) of Proposition 9 implies that the set DK^0_Θ for the differential Θ should contain all zeros of $D(z)$.

Proof. The fact that every curve in $\operatorname{supp}(\mu)$ should coincide with some horizontal trajectory of (4.2) is well-known and follows from the Plemelj–Sokhotsky's formula, see, e.g., [Pri]. It is based on the local observation that if a real measure $\mu = \frac{1}{\pi}\frac{\partial \mathcal{C}}{\partial \bar{z}}$ is supported on a smooth curve γ, then the tangent to γ at any point $z_0 \in \gamma$ should be perpendicular to $\overline{\mathcal{C}_1(z_0)} - \overline{\mathcal{C}_2(z_0)}$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the one-sided limits of $\mathcal{C}$ when $z \to z_0$, see e.g. [BR]. (Here $\bar{\ }$ stands for the usual complex conjugation.) Solutions of (4.1) are given by:

$$\mathcal{C}_{1,2} = \frac{-Q(z) \pm \sqrt{Q^2(z) - 4P(z)R(z)}}{2P(z)},$$

their difference being

$$\mathcal{C}_1 - \mathcal{C}_2 = \frac{\sqrt{Q^2(z) - 4P(z)R(z)}}{P(z)}.$$

Being orthogonal to $\overline{\mathcal{C}_1(z_0)} - \overline{\mathcal{C}_2(z_0)}$, the tangent line to the support of the real mother body measure μ satisfying (4.1) at its (arbitrary) smooth point z_0, is given by the condition $\frac{4P(z_0)R(z_0) - Q^2(z_0)}{P^2(z_0)}dz^2 > 0$. The latter condition is exactly the one defining the horizontal trajectory of Θ at z_0.

Finally the observation that $\operatorname{supp} \mu$ should contain all branch points of (4.1) follows immediately from the fact that $\mathcal{C}_\mu$ is a well-defined function in $\mathbb{C} \setminus \operatorname{supp} \mu$. □

In many special cases statements similar to Proposition 9 can be found in the literature, see, e.g., recent [AMM] and references therein.

Proposition 9 allows us, under mild nondegeneracy assumptions, to formulate necessary and sufficient conditions for the existence of a mother body measure for (4.1) which however are difficult to verify. Namely, let $\Gamma \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ with coordinates $(\mathcal{C}, z)$ be the algebraic curve given by (the projectivization of) equation (4.1). Γ has bidegree $(2, n+2)$ and is hyperelliptic. Let $\pi_z : \Gamma \to \mathbb{C}$ be the projection of Γ along the $\mathcal{C}$-coordinate onto the z-plane $\mathbb{CP}^1$. From (4.1) we observe that π_z induces a branched double covering of $\mathbb{CP}^1$ by Γ. If $P(z)$ and $Q(z)$ are coprime and if $\deg D(z) = 2n+2$, the set of all branch points of $\pi_z : \Gamma \to \mathbb{CP}^1$ coincides with the set of all zeros of $D(z)$. (If $\deg D(z) < 2n+2$, then ∞ is also a branch point of π_z of multiplicity $2n+2-\deg D(z)$.) We need the following lemma.

Lemma 10. *If $P(z)$ and $Q(z)$ are coprime, then at each pole of* (4.1)*, i.e., at each zero of $P(z)$, only one of two branches of Γ goes to ∞. Additionally the residue of this branch at this zero equals that of $-\frac{Q(z)}{P(z)}$.*

Proof. Indeed if $P(z)$ and $Q(z)$ are coprime, then no zero z_0 of $P(z)$ can be a branch point of (4.1) since $D(z_0) \neq 0$. Therefore only one of two branches of Γ goes to ∞ at z_0. More exactly, the branch $\mathcal{C}_1 = \frac{-Q(z)+\sqrt{Q^2(z)-4P(z)R(z)}}{2P(z)}$ attains a finite value at z_0 while the branch $\mathcal{C}_2 = \frac{-Q(z)-\sqrt{Q^2(z)-4P(z)R(z)}}{2P(z)}$ goes to ∞. (Here we use the agreement that $\lim_{z\to z_0} \sqrt{Q^2 - 4P(z)R(z)} = Q(z_0)$.) Now consider the residue of the branch $\mathcal{C}_2$ at z_0. Since residues depend continuously on the coefficients $(P(z), Q(z), R(z))$, it suffices to consider only the case when z_0 is a simple zero of $P(z)$. Further if z_0 is a simple zero of $P(z)$, then

$$Res(\mathcal{C}_2, z_0) = \frac{-2Q(z_0)}{2P'(z_0)} = Res\left(-\frac{Q(z)}{P(z)}, z_0\right).$$

$\square$

By Proposition 9 (besides the obvious condition that (4.1) has a real branch near ∞ with the asymptotics $\frac{\alpha}{z}$ for some $\alpha \in \mathbb{R}$) the necessary condition for (4.1) to admit a mother body measure is that the set DK_Θ^0 for the differential (4.2) contains all branch points of (4.1), i.e., all zeros of $D(z)$. Consider $\Gamma_{cut} = \Gamma \setminus \pi_z^{-1}(DK_\Theta^0)$. Since DK_Θ^0 contains all branch points of π_z, Γ_{cut} consists of some number of open sheets each projecting diffeomorphically on its image in $\mathbb{CP}^1 \setminus DK_\Theta^0$. (The number of sheets in Γ_{cut} equals to twice the number of connected components in $\mathbb{C} \setminus DK_\Theta^0$.) Observe that since we have chosen a real branch of (4.1) at infinity with the asymptotics $\frac{\alpha}{z}$, we have a marked point $p_{br} \in \Gamma$ over ∞. If we additionally assume that $\deg D(z) = 2n+2$, then ∞ is not a branch point of π_z and therefore $p_{br} \in \Gamma_{cut}$.

Lemma 11. *If* $\deg D(z) = 2n+2$*, then any choice of a spanning* (*multi*)*subgraph* $G \subset DK^0_\Theta$ *with no isolated vertices induces the unique choice of the section* S_G *of* Γ *over* $\mathbb{CP}^1 \setminus G$ *which:*

a) *contains* p_{br}*;*

b) *is discontinuous at any point of* G*;*

c) *is projected by* π_z *diffeomorphically onto* $\mathbb{CP}^1 \setminus G$.

Here by a spanning subgraph we mean a subgraph containing all the vertices of the ambient graph. By a section of Γ over $\mathbb{CP}^1 \setminus G$ we mean a choice of one of two possible values of Γ at each point in $\mathbb{CP}^1 \setminus G$.

Proof. Obvious. □

Observe that the section S_G might attain the value ∞ at some points, i.e., contain some poles of (4.1). Denote the set of poles of S_G by $Poles_G$. Now we can formulate our necessary and sufficient conditions.

Theorem 12. *Assume that the following conditions are valid:*

(i) *equation* (4.1) *has a real branch near* ∞ *with the asymptotic behavior* $\frac{\alpha}{z}$, *for some* $\alpha \in \mathbb{R}$, *comp. Lemma* 2*;*

(ii) $P(z)$ *and* $Q(z)$ *are coprime, and the discriminant* $D(z) = Q^2(z) - 4P(z)R(z)$ *of equation* (4.1) *has degree* $2n+2$*;*

(iii) *the set* DK^0_Θ *for quadratic differential* Θ *given by* (4.2) *contains all zeros of* $D(z)$*;*

(iv) Θ *has no closed horizontal trajectories.*

Then (4.1) *admits a real mother body measure if and only if there exists a spanning* (*multi*)*subgraph* $G \subseteq DK^0_\Theta$ *with no isolated vertices, such that all poles in* $Poles_g$ *are simple and all their residues are real, see notation above.*

Proof. Indeed assume that (4.1) satisfying (ii) admits a real mother body measure μ. Assumption (i) is obviously necessary for the existence of a real mother body measure. The necessity of assumption (iii) follows from Proposition 9 if (ii) is satisfied. The support of μ consists of a finite number of curves and possibly a finite number of isolated points. Since each curve in the support of μ is a trajectory of Θ and Θ has no closed trajectories, then the whole support of μ consists of finite critical trajectories of Θ connecting its zeros, i.e., the support belongs to DK^0_Θ. Moreover the support of μ should contain sufficiently many finite critical trajectories of Θ such that they include all the branch points of (4.1). By (ii) these are exactly all zeros of $D(z)$. Therefore the union of finite critical trajectories of Θ belonging to the support of μ is a spanning (multi)graph of DK^0_Θ without isolated vertices. The isolated points in the support of μ are necessarily the poles of (4.1). Observe that the Cauchy transform of any (complex-valued) measure can only have simple poles (as opposed to the Cauchy transform of a more general distribution). Since μ is real the residue of its Cauchy transform at each pole must be real as well. Therefore the existence of a real mother body under the assumptions (i)–(iv) implies the existence of a spanning (multi)graph G with the above properties. The converse is also immediate. □

Remark. Observe that if (i) is valid, then assumptions (ii) and (iv) are generically satisfied. Notice however that (iv) is violated in the special case when $Q(z)$ is absent considered in Subsection 4.3. Additionally, if (iv) is satisfied, then the number of possible mother body measures is finite. On the other hand, it is the assumption (iii) which imposes severe additional restrictions on admissible triples $(P(z), Q(z), R(z))$. At the moment the authors have no information about possible cardinalities of the sets $Poles_G$ introduced above. Thus it is difficult to estimate the number of conditions required for (4.1) to admit a mother body measure. Theorem 12 however leads to the following sufficient condition for the existence of a real mother body measure for (4.1).

Corollary 2. *If, additionally to assumptions* (i)–(iii) *of Theorem* 12*, one assumes that all roots of $P(z)$ are simple and all residues of $\frac{Q(z)}{P(z)}$ are real, then* (4.1) *admits a real mother body measure.*

Proof. Indeed if all roots of $P(z)$ are simple and all residues of $\frac{Q(z)}{P(z)}$ are real, then all poles of (4.1) are simple with real residues. In this case for any choice of a spanning (multi)subgraph G of DK^0_Θ, there exists a real mother body measure whose support coincides with G plus possibly some poles of (4.1). Observe that if all roots of $P(z)$ are simple and all residues of $\frac{Q(z)}{P(z)}$ are real, one can omit assumption (iv). In case when Θ has no closed trajectories, then all possible real mother body measures are in a bijective correspondence with all spanning (multi)subgraphs of DK^0_Θ without isolated vertices. In the opposite case such measures are in a bijective correspondence with the unions of a spanning (multi)subgraph of DK^0_Θ and an arbitrary (possibly empty) finite collection of closed trajectories. □

Although we at present do not have rigorous results about the structure of the set of general equations (4.1) admitting a real mother body measure, we think that generalizing our previous methods and statements from [BBS, HS, STT], one would be able to settle the following conjecture.

Conjecture 2. *Fix any monic polynomial $P(z)$ of degree $n+2$ and an arbitrary polynomial $Q(z)$ of degree at most $n+1$. Let $\Omega_{P,Q}$ denote the set of all polynomials $R(z)$ of degree at most n such that* (4.1) *admits a probability measure, i.e., a positive mother body measure of mass* 1*. Then $\Omega_{P,Q}$ is a real semi-analytic variety of* (*real*) *dimension n.*

4.3. Case $Q = 0$ and Strebel differentials. In this subsection we present in more detail the situation when the middle term in (4.1) is vanishing, i.e., $Q(z) = 0$. In this case one can obtain complete information about the number of possible signed mother body measures and also a criterion of the existence of a positive mother body measure. This material is mainly borrowed from [STT] and is included here for the sake of completeness of presentation.

It is known that for a meromorphic Strebel differential Ψ given on a compact Riemann surface Y without boundary the set K_Ψ has several nice properties. In particular, it is a finite embedded multigraph on Y whose edges are finite critical trajectories. In other words, for a Strebel quadratic differential Ψ, one gets $K_\Psi = DK_\Psi$.

Our next result relates a Strebel differential Ψ on $\mathbb{CP}^1$ with a double pole at ∞ to real-valued measures supported on K_Ψ.

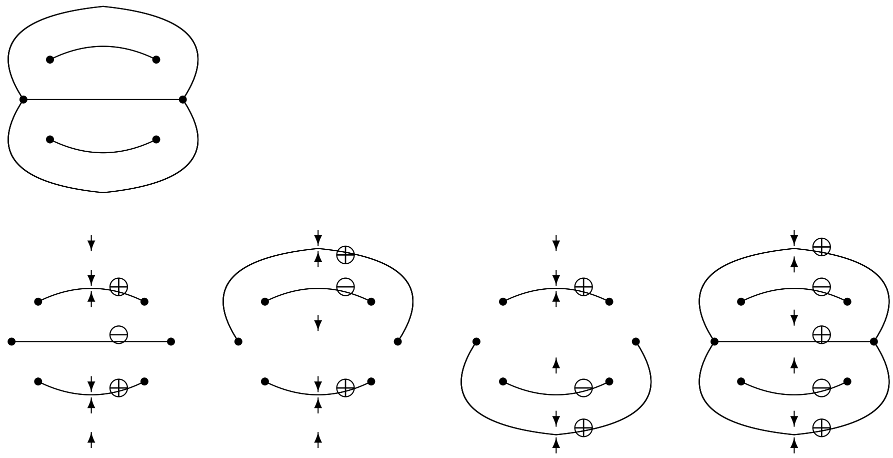
Theorem 13. i *Given two coprime polynomials $P(z)$ and $R(z)$ of degrees $n+2$ and n respectively where $P(z)$ is monic and $R(z)$ has a negative leading coefficient, the algebraic function given by the equation*

$$P(z)\mathcal{C}^2 + R(z) = 0 \tag{4.3}$$

admits a mother body measure μ_C if and only if the quadratic differential $\Psi = R(z)dz^2/P(z)$ is Strebel.

i *Such mother body measures are, in general, non-unique. If we additionally require that the support of each such measure consists only of finite critical trajectories, i.e., is a spanning subgraph of $K_\Psi = DK_\Psi$, then for any Ψ as above, there exists exactly 2^{d-1} real measures where d is the total number of connected components in $\mathbb{CP}^1 \setminus K_\Psi$. These measures are in $1-1$-correspondence with 2^{d-1} possible choices of the branches of $\sqrt{-R(z)/P(z)}$ in the union of $(d-1)$ bounded components of $\mathbb{CP}^1 \setminus K_\Psi$.*

Concerning possible positive measures, we can formulate an exact criterion of the existence of a positive measure for a rational quadratic differential



(a) 1st measure (b) 2nd measure (c) 3rd measure (d) 4th measure

FIGURE 2
The set $K_\Psi = DK_\Psi$ of a Strebel differential Ψ and 4 related real measures

$\Psi = R(z)dz^2/P(z)$ in terms of rather simple topological properties of K_Ψ. To do this, we need a few definitions. Notice that K_Ψ is a planar multigraph with the following properties. The vertices of K_Ψ are the affine critical points of Ψ (i.e., excluding ∞) and its edges are critical trajectories connecting these critical points. Each (open) connected component of $\mathbb{C} \setminus K_\Psi$ is homeomorphic to an (open) annulus. K_Ψ might have isolated vertices which are the affine double poles of Ψ. Vertices of K_Ψ having valency 1 (i.e., hanging vertices) are exactly the simple poles of Ψ. Vertices different from the isolated and hanging vertices are the zeros of Ψ. The number of edges adjacent to a given vertex minus 2 is equal to the order of the zero of Ψ at this point. Finally, the sum of the multiplicities of all poles (including the one at ∞) minus the sum of the multiplicities of all zeros equals 4.

By a *simple cycle* in a planar multigraph K_Ψ we mean any closed non-self-intersecting curve formed by the edges of K_Ψ. (Obviously, any simple cycle bounds an open domain homeomorphic to a disk which we call the *interior of the cycle*.)

Proposition 14. *A Strebel differential* $\Psi = R(z)dz^2/P(z)$ *admits a positive mother body measure if and only if no edge of* K_Ψ *is attached to a simple cycle from inside. In other words, for any simple cycle in* K_Ψ *and any edge not in the cycle, but adjacent to some vertex in the cycle, this edge does not belong to its interior. The support of the positive measure coincides with the forest obtained from* K_Ψ *after the removal of all its simple cycles.*

Remark. Notice that under the assumptions of Proposition 14, all simple cycles of K_Ψ are pairwise non-intersecting and, therefore, their removal is well-defined in an unambiguous way.

In particular, the compact set shown on the right picture of Fig. 3 admits no positive measure since it contains an edge cutting a simple cycle (the outer boundary) in two smaller cycles. The left picture on Fig. 3 has no such edges and, therefore, admits a positive measure (whose support consists of the four horizontal edges of K_Ψ).

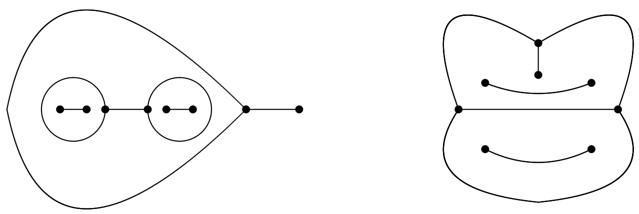


Figure 3
Two examples of K_Ψ admitting and not admitting a positive measure

Let us finish the paper with the following important observation.

Proposition 15. *For any monic* $P(z)$ *of degree* $n+2$, *the set of all polynomials* $R(z)$ *of degree* n *and with leading coefficient* -1 *such that the differential* $\frac{R(z)dz^2}{P(z)}$ *is Strebel is dense in the space of all polynomials of degree* n *with leading coefficient* -1. *In fact, this set is the countable union of real semi-analytic varieties of positive codimension.*

This circumstance illustrates the difficulty of the general problem to determine for which algebraic equations a real mother body measure exists.

5. Final remarks

1. The natural question about which algebraic functions of degree exceeding 2 whose Newton polygon intersects the diagonal in the $(\mathcal{C}, z)$-plane nontrivially admit a real mother body measure is hard to answer. Some steps in this direction can be found in [HS]. This topic is apparently closely related to the (non-existing) notion of Strebel differential of higher order which we hope to develop in the future. In any case, it is clear that no results similar to Theorem 5 can be true and one needs to impose highly non-trivial additional conditions on such functions to ensure the existence of a probability measure.

2. Problem. Given a finite set S of monomials satisfying the assumptions of Lemma 2 and Lemma 3, consider the linear space Pol_S of all polynomials $P(\mathcal{C}, z)$ whose Newton polygon is contained in S. What is the (Hausdorff) dimension of the subset $MPol_S \subseteq Pol_S$ of polynomials admitting a mother body measure?

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References

[AMM] M. J. Atia, A. Martínez-Finkelshtein, P. Martínez-González and F. Thabet, Quadratic differentials and asymptotics of Laguerre polynomials with varying complex parameters. *J. Math. Anal. Appl.* **416** (2014), 52–80. Zbl 1295.30015 MR 3182748

[Bar] Y. Baryshnikov, On stokes sets. New developments in singularity theory. *NATO Science Series* **21** (2001), 65–86. Zbl 1065.58028 MR 1849304

[Bel] S. R. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mapping*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992. MR 1228442

[BOS] O. B. Bekken, B. K. Oksendal and A. Stray (Editors), *Spaces of Analytic Functions*, Lecture Notes in Math., vol. 512, Springer-Verlag, Berlin, 1976. Zbl 0315.00005

[Ber] T. Bergkvist, On asymptotics of polynomial eigenfunctions for exactly solvable differential operators. *J. Approx. Theory* **149** (2007), 151–187. Zbl 1155.34045 MR 2374603

[BR] T. Bergkvist and H. Rullgård, On polynomial eigenfunctions for a class of differential operators. *Math. Res. Lett.* **9** (2002), 153–171. Zbl 1155.34045 MR 1909635

[BB] J. Borcea and R. Bøgvad, Piecewise harmonic subharmonic functions and positive Cauchy transforms. *Pacific J. Math.* **240** (2009), 231–265. Zbl 1163.31001 MR 2485464

[BBB] J.-E. Björk, J. Borcea and R. Bøgvad, Subharmonic configurations and algebraic Cauchy transforms of probability measures. *Notions of Positivity and the Geometry of Polynomials*, Trends in Mathematics 2011, pp. 39–62. Zbl 1253.31001 MR 3051160

[BBS] J. Borcea, R. Bøgvad and B. Shapiro, Homogenized spectral pencils for exactly solvable operators: Asymptotics of polynomial eigenfunctions, *Publ. RIMS* **45** (2009), 525–568. Corrigendum: "Homogenized spectral pencils for exactly solvable operators: asymptotics of polynomial eigenfunctions", *Publ. RIMS* **85** (2012), 229–233. Zbl 1235.30003 MR 2888043

[BS] M. A. Brodsky and V. N. Strakhov, On the uniqueness of the inverse logarithmic potential problem. *SIAM Journal on Applied Mathematics* **46** (1986), 324–344. Zbl 0604.31004 MR 0833480

[CMR] J. A. Cima, A. L. Matheson and W. T. Ross, *The Cauchy Transform*. Mathematical Surveys and Monographs, 125. American Mathematical Society, Providence, RI, 2006. Zbl 1096.30046 MR 2215991

[GS1] S. J. Gardiner and T. Sjödin, Convexity and the exterior inverse problem of potential theory. *Proceedings of the AMS* **136** (2008), 1699–1703. Zbl 1138.31003 MR 2373599

[GS2] S. J. Gardiner and T. Sjödin, Partial balayage and the exterior inverse problem of potential theory. *Potential Theory and Stochastics in Albac*, 111–123, Bucharest, Theta (2009). Zbl 1199.31009 MR 2681841

[Gar] J. B. Garnett, *Analytic Capacity and Measure*. LNM 297, Springer-Verlag, 1972. Zbl 1196.28007 MR 0454006

[Gus] B. Gustafsson, On mother bodies of convex polyhedra. *SIAM J Math. Anal.* **29** (1998), 1106–1117. Zbl 0913.31002 MR 1613731

[Hoe] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis.* Reprint of the second (1990) edition. Classics in Mathematics. Springer-Verlag, Berlin, 2003. Zbl 1028.35001 MR 1996773

[HS] T. Holst and B. Shapiro, On higher Heine–Stieltjes polynomials. *Isr. J. Math.* **183** (2011), 321–347. Zbl 1227.34087 MR 2811162

[Jen] J. A.Jenkins, *Univalent Functions and Conformal Mapping*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 18. Reihe: Moderne Funktionentheorie Springer-Verlag, Berlin-Göttingen-Heidelberg, (1958). Zbl 0521.30012 MR 0096806

[KMWWZ] I. Krichever, M. Mineev-Weinstein, P. Wiegmann and A. Zabrodin, Laplacian growth and Whitham equations of soliton theory. *Phys. D* **198** (2004), 1–28. Zbl 1078.76025 MR 2096577

[MFR1] A. Martínez-Finkelshtein and E. A. Rakhmanov, Critical measures, quadratic differentials, and weak limits of zeros of Stieltjes polynomials. Commun. *Math. Phys.* **302** (2011), 53–111. Zbl 1226.30005 MR 2770010

[MFR2] A. Martínez-Finkelshtein and E. A. Rakhmanov, On asymptotic behavior of Heine-Stieltjes and Van Vleck polynomials. *Contemp. Math.* **507** (2010), 209–232. Zbl 1207.30058 MR 2647571

[Mur] T. Murai, *A Real Variable Method for the Cauchy Transform, and Analytic Capacity.* Lecture Notes in Mathematics, 1307. Springer-Verlag, Berlin, 1988. Zbl 0645.30016 MR 0944308

[Nov] P. S. Novikoff, Sur le probléme inverse du potentiel, *C. R.* (*Dokl.*) *Acad. Sci. URSS* (*N.S.*) **18** (1938), 165–168. Zbl 0018.30901

[PS] D. Pasechnik and B. Shapiro, On polygonal measures with vanishing harmonic moments. *Journal d'Analyse mathematique* **123** (2014), 281–301. Zbl 1298.28010 MR 3233582

[Pri] I. Pritsker, How to find a measure from its potential. CMFT **8** (2008), 597–614. Zbl 1160.31004 MR 2419497

[Ran] T. Ransford, *Potential Theory in the Complex Plane*, LMS Student Texts 28, 1995. Zbl 0828.31001 MR 1334766

[ST] E. B. Saff and V. Totik, *Logarithmic Potentials With External Fields*. Appendix B by Thomas Bloom. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 316. Springer-Verlag, Berlin, 1997. Zbl 0881.31001 MR 1485778

[Sak] M. Sakai, A moment problem on Jordan domains. *Proc. Amer. Math. Soc.* **70** (1978), 35–38. Zbl 0394.30027 MR 0470216

[SSS] T. V. Savina, B. Yu. Sternin and V. E. Shatalov, On a minimal element for a family of bodies producing the same external gravitational field. *Appl. Anal.* **84** (2005), 649–668. Zbl 1115.86002 MR 2152681

[Sha1] H. S. Shapiro, *The Schwarz Function and its Generalization to Higher Dimensions*. Univ. of Arkansas Lect. Notes Math. Vol. 9, Wiley N.Y., 1992. Zbl 0784.30036 MR 1160990

[Sha2] B. Shapiro, Algebro-geometric aspects of Heine–Stieltjes theory. *J. London Math. Soc.* **83** (2011), 36–56. Zbl 1210.34127 MR 2763943

[STT] B. Shapiro, K. Takemura and M. Tater, On spectral polynomials of the Heun equation. II. *Comm. Math. Phys.* **311** (2012), 277–300. Zbl 1266.34139 MR 2902190

[Sol1] A. Yu. Solynin, Modules and extremal metric problems. *Algebra i Analiz* **11** (1999), 3–86; English transl. *St. Petersburg Math. J.* **11** (2000), 1–65. Zbl 0935.30019

[Sol2] A. Yu. Solynin, *Quadratic Differentials and Weighted Graphs on Compact Surfaces*. Analysis and mathematical physics, 473–505, Trends Math., Birkhäuser, Basel, 2009. Zbl 1297.30068 MR 2724628

[Sjo] T. Sjödin, Mother bodies of algebraic domians in the complex plane. *Complex Variables and Elliptic Equations* **51** (2006), 357–369. Zbl 1101.31003 MR 2218726

[Str] K. Strebel, *Quadratic Differentials*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 5, Springer-Verlag, Berlin, 1984. Zbl 0547.30001 MR 0743423

[Zal] L. Zalcmann, *Analytic Capacity and Rational Approximation*. Lecture Notes in Mathematics, No. 50, Springer-Verlag, Berlin-New York, 1968. MR 0227434

[Zid] D. Zidarov, *Inverse Gravimetric Problem in Geoprospecting and Geodesy*. Amsterdam: Elsevier, 1990. Zbl 0718.31001

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