ALGEBRO-GEOMETRIC ASPECTS OF HEINE-STIELTJES THEORY

BORIS SHAPIRO

Dedicated to Heinrich Eduard Heine and his 140 years old riddle

Abstract. The goal of the paper is to develop a Heine-Stieltjes theory for univariate linear differential operators of higher order. Namely, for a given linear ordinary differential operator $\mathfrak{d}(z) = \sum_{i=1}^{k} Q_i(z)\frac{d^i}{dz^i}$ with polynomial coefficients set $r = max_{i=1,\dots,k}(\deg Q_i(z) - i)$. Following the classical approach of E. Heine and T. Stieltjes we study the multiparameter spectral problem of finding all polynomials $V(z)$ of degree at most r such that for a given positive integer n the equation:

$$\mathfrak{d}(z)S(z) + V(z)S(z) = 0$$

has a polynomial solution $S(z)$ of degree n. We show that under some mild non-degeneracy assumptions there exist exactly $\binom{n+r}{n}$ such polynomials $V(z)$ whose corresponding eigenpolynomials $S(z)$ are of degree n.

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1. Introduction and main results

The algebraic form of the classical Lamé equation (see [49], ch. 23) was introduced by Gabriel Lamé in 1830's in connection with the separation of variables in the Laplace equation in $\mathbb{R}^l$ with respect to elliptic coordinates. It has the form:

$$Q(z)\frac{d^2S}{dz^2} + \frac{1}{2}Q'(z)\frac{dS}{dz} + V(z)S = 0, \tag{1.1}$$

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where $Q_l(z)$ is a real polynomial of degree l with all real and distinct roots, and $V(z)$ is a polynomial of degree at most $l-2$ whose choice depends on what type of solution to (1.1) we are looking for. In the second half of the 19th century, several well-known mathematicians including M. Bôcher, E. Heine, F. Klein and T. Stieltjes studied the number and different properties of the so-called Lamé polynomials (sometimes also referred as Lamé **solutions**) of a given degree and certain kinds. Such solutions to (1.1) exist for some choices of $V(z)$ and are characterized by the property that their logarithmic derivative is a rational function. For a given $Q(z)$ of degree $l \geq 2$ with simple roots there exist 2^l different kinds of Lamé polynomials depending on whether this solution is smooth or has a square root singularity at each given root of $Q(z)$ (for details see [34] and [49]). An excellent modern study of these questions can be found in [22]. In what follows, we will concentrate on the usual polynomial solutions of (1.1).

A *generalized Lamé equation* (see [49]) is the second order differential equation given by

$$Q_2(z)\frac{d^2S}{dz^2} + Q_1(z)\frac{dS}{dz} + V(z)S = 0, \tag{1.2}$$

where $Q_2(z)$ is a complex polynomial of degree l and $Q_1(z)$ is a complex polynomial of degree at most $l-1$. The special case $l = 3$ is widely known as the Heun equation.

The following fundamental result announced in [19], and provided there with not quite satisfactory proof, was undoubtedly the starting point of the classical Heine-Stieltjes theory.

Theorem 1 (Heine)**.** *If the coefficients of $Q_1(z)$ and $Q_2(z)$ are algebraically independent, then for any integer $n > 0$ there exists exactly $\binom{n+l-2}{n}$ polynomials $V(z)$ of degree exactly $(l-2)$ such that the equation* (1.2) *has a unique* (*up to a constant factor*) *polynomial solution S of degree exactly n.*

Remark 1. Notice that throughout this paper, we count polynomials $V(z)$ as such and polynomials $S(z)$ up to a constant factor.

Later on, T. Stieltjes and his followers considered a special case of (1.2), which is important in physics and is closely related to the original equation (1.1). Namely, they studied the equation of the form:

$$\prod_{i=1}^{l}(z-\alpha_i)\frac{d^2S}{dz^2} + \sum_{j=1}^{l}\beta_j\prod_{j\neq i}(z-\alpha_i)\frac{dS}{dz} + V(z)S = 0, \tag{1.3}$$

with real $\alpha_1, \alpha_2, \ldots, \alpha_l$ such that $\alpha_1 < \alpha_2 < \ldots < \alpha_l$ and real positive $\beta_1, \ldots, \beta_l$. In particular, they proved the following important result:

Theorem 2 (Stieltjes-Van Vleck-Bôcher [43], [47], [8] and [44])**.** *Under the assumptions of* (1.3)*, for any integer $n > 0$:*

(1) *there exist exactly $\binom{n+l-2}{n}$ distinct polynomials V of degree $(l-2)$ such that the equation* (1.3) *has a polynomial solution S of degree exactly n;*
(2) *each root of each V and S is real and simple, and belongs to the interval (α_1, α_l);*
(3) *none of the roots of S can coincide with some of the α_i's. Moreover, $\binom{n+l-2}{n}$ polynomials S are in 1-1-correspondence with $\binom{n+l-2}{n}$ possible ways to distribute n roots of S into the $(l-1)$ open intervals (α_1, α_2), (α_2, α_3),..., (α_{l-1}, α_l).*

The polynomials V and the corresponding polynomial solutions S of the equation (1.2) (or, equivalently, of (1.3)) are called *Van Vleck* and *Stieltjes* polynomials, respectively.

The case when α_i's and/or β_j's are not real is substantially less studied, see [28] and [29]. One nice result in this case is as follows (see [33]):

Theorem 3 (Polya)**.** *If in the notation of* (1.3)*, all α_i's are arbitrary complex and all β_j's are real positive, then all roots of each V and each S belong to the convex hull $Conv_{Q_2}$ of the root set $\{\alpha_1, \dots, \alpha_l\}$ of $Q_2(z)$.*

Remark 2. The situation when all the numbers β_j are negative (for example, when $Q_1(z) = -Q_2'(z)$) or have different signs seems to differ drastically from the latter case, see e.g. [46] and [15]. Further interesting results on the distribution of the zeros of Van Vleck and Stieltjes polynomials under weaker assumptions on α_i's and β_j's were obtained in [23], [24], [1], [2], [50].

In the present article, we extend the above three fundamental results on generalized Lamé equations of the second order to the case of higher orders and/or complex coefficients. Namely, consider an arbitrary linear ordinary differential operator

$$\mathfrak{d}(z) = \sum_{i=1}^{k} Q_i(z) \frac{d^i}{dz^i}, \tag{1.4}$$

with polynomial coefficients. The number $r = max_{i=1,\dots,k}(\deg Q_i(z) - i)$ will be called the *Fuchs index* of $\mathfrak{d}(z)$. The operator $\mathfrak{d}(z)$ is called a *higher Lamé operator* if its Fuchs index r is non-negative. If $r = 0$, then $\mathfrak{d}(z)$ is usually called *exactly solvable* in the physics literature, see [45]. This case is also of special interest in connection with the classical Bochner-Krall problem in the theory of orthogonal polynomials.

The operator $\mathfrak{d}(z)$ is called *non-degenerate* if $\deg Q_k(z) = k+r$. Notice that non-degeneracy of $\mathfrak{d}(z)$ is equivalent to the requirement that $\mathfrak{d}(z)$ has either a regular or a regular singular point at ∞.

Given a higher Lamé operator $\mathfrak{d}(z)$, consider the multiparameter spectral problem as follows.

Problem. For each positive integer n find all polynomials $V(z)$ of degree at most r such that the equation

$$\mathfrak{d}(z)S(z) + V(z)S(z) = 0 \tag{1.5}$$

has a polynomial solution $S(z)$ of degree n.

Following the classical terminology, we call (1.5) a *higher Heine-Stieltjes spectral problem*, $V(z)$ is called a *higher Van Vleck polynomial*, and the corresponding polynomial $S(z)$ is called a *higher Stieltjes polynomial*. Below, we will often skip mentioning "higher".

Remark 3. Obviously, any differential operator (1.4) has either a non-negative or a negative Fuchs index. In the latter case, it can be easily transformed into the operator with a non-negative Fuchs index by the change of variable $y = \frac{1}{z}$. Notice also that the condition of non-degeneracy is generically satisfied. In what follows, we will always assume that the leading coefficient of $\mathfrak{d}(z)$ is a monic polynomial.

1.1. **Generalizations of Heine's Theorem, degeneracies and non-resonance condition.** We start with a number of generalizations of Heine's Theorem 1. Following Heine's original proof, one can obtain the following straightforward generalization.

Theorem 4. *For any non-degenerate higher Lamé operator $\mathfrak{d}(z)$ with algebraically independent coefficients of its polynomial coefficients $Q_i(z)$, $i = 1, \dots, k$ and for any $n \geq 0$ there exist exactly $\binom{n+r}{r}$ distinct Van Vleck polynomials $V(z)$ whose*

corresponding Stieltjes polynomials $S(z)$ are unique (*up to a constant factor*) *and of degree n.*

Our next result, obtained by a linear-algebraic interpretation of (1.5), has no genericity assumptions and is crucial in the problem concerning existence of solutions to (1.5) (compare [29], Problem 1).

Theorem 5. *For any non-degenerate operator $\mathfrak{d}(z)$ with a Fuchs index $r \geq 0$ and any positive integer n, the total number of Van Vleck polynomials $V(z)$* (*counted with natural multiplicities*) *having a Stieltjes polynomial $S(z)$ of degree at most n equals $\binom{n+r+1}{r+1}$.*

Remark 4. Note that in Theorem 5, we do not require that there is a unique (up to constant factor) Stieltjes polynomial corresponding to a given Van Vleck polynomial. For the exact (rather lengthy) description of the notion of the *natural multiplicity* of a Van Vleck polynomial consult Definition 2 in § 2.

On degeneracies. Notice that Theorem 4 claims that a generic operator $\mathfrak{d}(z)$ has for any positive n exactly $\binom{n+r}{r}$ distinct Van Vleck polynomials each of which has a unique Stieltjes polynomial and this polynomial is of degree exactly n. If we drop the genericity assumptions on $\mathfrak{d}(z)$, then the question about possible degeneracies occurring in Problems (1.2) and (1.5) is quite delicate. Then it may happen that Van Vleck polynomials attain nontrivial multiplicities or the corresponding Stieltjes polynomials form linear spaces of dimension bigger than one, or Stieltjes polynomials of certain degrees do not exist at all. In particular, for any polynomial $Q(z)$ of degree l no choice of a polynomial $V(z)$ of degree at most $l-2$ will supply the equation

$$Q(z)\frac{d^2S}{dz^2} - Q'(z)\frac{dS}{dz} + V(z)S = 0$$

with a polynomial solution S of degree $l+1$. (This follows from the Proposition 5 and Lemma 4 of [15].) The fact that (1.2) can admit families (i.e. linear spaces of dimension at least 2) of polynomial solutions S corresponding to one and the same V was already mentioned by Heine in his original proof. More exact information is available nowadays. For example, a result of Varchenko-Scherbak gives necessary and sufficient condition for a Fuchsian second order equation to have 2 independent polynomial solutions, see [37] and [15]. Finally, high multiplicity of Van Vleck polynomials occur, for example, in the case $Q_2(z) = z^l$, $Q_1(z) = 0$. Then one can easily show that for all $n \geq 2$, there exists just one and only polynomial $V_n(z) = -n(n-1)z^{l-2}$ solving the above problem; its corresponding Stieltjes polynomial equals $S_n(z) = z^n$. The multiplicity of the latter Van Vleck polynomial $V_n(z)$ is $\binom{n+l-2}{n}$.

To formulate necessary and sufficient conditions under which the conclusion of Heine's theorem holds for all positive integers n is apparently an impossible task. Heine himself mentions that for the validity of his result for a given fixed positive integer n, one has to avoid a certain discriminantal hypersurface (similar to the usual discriminant of univariate polynomials) but this hypersurface is difficult to obtain explicitly.

Below, we formulate a simple sufficient condition which allows us to avoid many of the above degeneracies and guarantees the existence of Stieltjes polynomials of a given degree. Namely, consider an arbitrary non-degenerate operator $\mathfrak{d}(z)$ of the form (1.4) with the Fuchs index r. Denote by $A_k, A_{k-1}, ..., A_1$ the coefficients at the highest possible degrees $k+r, k+r-1, ..., r+1$ in the polynomials $Q_k(z), Q_{k-1}(z), ..., Q_1(z)$ resp. (Notice that any subset of A_j's can vanish but $A_k \neq 0$ due to the non-degeneracy of $\mathfrak{d}(z)$.) In what follows, we will often use the

notation

$$(j)_i = j(j-1)(j-2)...(j-i+1),$$

where j is a non-negative and i is a positive integer. In case $j = i$ one has $(j)_i = j!$ and in case $j < i$ one gets $(j)_i = 0$. For any non-negative n, we call the expression:

$$\mathbb{L}_n = (n)_k A_k + (n)_{k-1} A_{k-1} + ... + (n)_1 A_1 \tag{1.6}$$

the nth diagonal coefficient.

Proposition 1. *Let n be a positive integer. If in the above notation, the n***th nonresonance condition** *holds, that is,*

$$\mathbb{L}_n \neq \mathbb{L}_j, \ j = 0, 1, ..., n-1, \tag{1.7}$$

then there exist Van Vleck polynomials which possess Stieltjes polynomials of degree exactly n and no other Stieltjes polynomials of degree smaller than n. In this case, the total number of such Van Vleck polynomials (*counted with natural multiplicities*) *equals* $\binom{n+r}{r}$.

Remark 5. The above nonresonance condition is quite natural. It says that if the equation (1.5) has a polynomial solution of degree n, then it has no polynomial solutions of smaller degrees. Another way to express this fact is that if the indicial equation of (1.5) at ∞ has $-n$ as its root, then it has no roots among non-positive integers $0, -1, -2, ..., 1-n$, see e.g. [34], ch. V.

Explicit formula (1.6) for $\mathbb{L}_n$ immediately shows that Theorem 5 and Proposition 1 are valid for any non-degenerate $\mathfrak{d}(z)$ and all sufficiently large n.

Corollary 1. *For any non-degenerate higher Lamé operator $\mathfrak{d}(z)$ and all sufficiently large n the nth nonresonance condition holds. In particular, for any problem* (1.5) *there exist finitely many* (*up to a scalar multiple*) *Stieltjes polynomials of any sufficiently large degree.*

Remark 6. Notice that for an arbitrary non-degenerate operator $\mathfrak{d}(z)$ and a given integer n it is difficult to find explicitly all Van Vleck polynomials which possess a Stieltjes polynomial of degree at most n. By this we mean that in order to do this one has, in general, to solve an overdetermined system of algebraic equations in the coefficients of V, since the set of Van Vleck polynomials under consideration is not a complete intersection. (In other words, this system of determinantal equations contains many more equations than variables.) However, one consequence of Heine's way to prove his Theorem 1 is as follows. For a given non-degenerate operator $\mathfrak{d}(z)$ with Fuchs index r and a positive integer n denote by $\mathfrak{V}_n \subset Pol_r$ the set of all its Van Vleck polynomials possessing a Stieltjes polynomial of degree exactly n.

Theorem 6. *If in the above notation the nth nonresonance condition* (1.7) *holds then $\mathfrak{V}_n$ is a complete intersection and the corresponding system of equations can be obtained explicitly in each concrete case.*

Explicit example of the defining system of r algebraic equations in r variables can be found in § 2, see Example 1. In purely linear algebraic setting, this result and further information about relevant discriminants can be found in [41].

1.2. **Generalizations of Stieltjes's Theorem.** We continue with a conceptually new generalization of Theorem 2. It was conjectured by the present author after extensive computer experiments and was later proved by P. Bränden in [9]. Here we only quote a somewhat simplified version of his result. For further details consult [9].

Definition 1. A differential operator $\mathfrak{d}(z)=\sum_{i=m}^{k} Q_i(z)\frac{d^i}{dz^i}$, $1\le m\le k$, where all $Q_i(z)$'s are polynomials with real coefficients is called a *strict hyperbolicity preserver* if for any real polynomial $P(z)$ with all real and simple roots the image $\mathfrak{d}(P(z))$ either vanishes identically or is a polynomial with only real and simple roots.

Theorem 7. *For any strict hyperbolicity preserving non-degenerate Lamé operator $\mathfrak{d}(z)$ with the Fuchs index r as above and any integer $n\ge m$, we have:*

1. *there exist exactly $\binom{n+r}{n}$ distinct polynomials $V(z)$ of degree exactly r such that the equation (1.5) has a polynomial solution $S(z)$ of degree exactly n;*
2. *all roots of each such $V(z)$ and $S(z)$ are real, simple and coprime;*
3. *$\binom{n+r}{n}$ polynomials $S(z)$ are in 1-1-correspondence with $\binom{n+r}{n}$ possible arrangements of r real roots of polynomials $V(z)$ and n real roots of the corresponding polynomials $S(z)$.*

Using Theorem 5, one immediately sees that the latter result describes the set of all possible pairs (V,S) with $m\le n=\deg S$ for any hyperbolicity preserver $\mathfrak{d}(z)$.

Remark 7. The interested reader can check that the sum of the first two terms in (1.3) is indeed a strict hyperbolicity preserver. It looks very tempting and important to find an analog of the electrostatic interpretation of the roots of classical Heine-Stieltjes and classical Van Vleck polynomials (alias 'Bethe ansatz') in the case of higher Heine-Stieltjes and Van Vleck polynomials (compare [30]).

Remark 8. Notice that the converse to the above theorem is false. Namely, one can show that the exactly solvable operator $\mathfrak{d}(z)(f)=f'+z(z+1)f''$ has all hyperbolic eigenpolynomials but is not a hyperbolicity preserver.

In the present paper we use Polya's original argument to prove the following statement (which also follows from a more general Theorem 7). Consider a differential equation

$$\prod_{i=1}^{l}(z-\alpha_i)\frac{d^kS}{dz^k}+\sum_{j=1}^{l}\beta_j\prod_{j\ne i}(z-\alpha_i)\frac{d^{k-1}S}{dz^{k-1}}+V(z)S=0, \tag{1.8}$$

where $2\le k\le l$, $\alpha_1<\alpha_2<\ldots<\alpha_l$ and $\beta_1,\ldots,\beta_l$ are positive.

Corollary 2. *Under the assumptions concerning (1.8) and for any $n\ge k-1$:*

1. *there exist exactly $\binom{n+l-k}{n}$ polynomials $V(z)$ of degree $(l-k)$ such that the equation (1.8) has a polynomial solution $S(z)$ of degree exactly n;*
2. *all roots of each $V(z)$ and $S(z)$ are real, simple, coprime and belong to the interval (α_1,α_l);*
3. *$\binom{n+l-k}{n}$ polynomials $S(z)$ are in 1-1-correspondence with $\binom{n+l-k}{n}$ possible arrangements of $(l-k)$ real roots of a polynomial $V(z)$ and n real roots of the corresponding polynomial $S(z)$ on the interval (α_1,α_l).*

It seems that Theorem 7 and Corollary 2 might contradict to the classical Theorem 2 in case of (1.3). However, the following two statements proven by G. Shah explain this mystery (see Theorem 3 of [38] and Theorem 3 of [40]).

Proposition 2. *Under the assumptions of Theorem 2 the roots of any Van Vleck polynomial $V(z)$ and its corresponding $S(z)$ are coprime.*

Moreover, the following result is valid:

Proposition 3. *If $v_1<v_2<\ldots<v_r$, $r=l-2$ denote the roots of some Van Vleck polynomial $V(z)$ in the classical situation (1.3), then for each $i=2,\ldots,l-1$ the interval (v_i,α_{i+1}) contains no roots of the corresponding $S(z)$. Therefore, for each polynomial $S(z)$ the distribution of its n roots into $(l-1)$ intervals*

$(\alpha_1, \alpha_2), \dots, (\alpha_{l-1}, \alpha_l)$ *coincides with the distribution of these roots defined by the roots of its Van Vleck polynomial* $V(z)$.

Remark 9. Note that we do not claim $v_i < \alpha_{i+1}$, i.e. the endpoints of the interval (v_i, α_{i+1}) may be placed in the wrong order or may coincide.

1.3. **Generalizations of Polya's Theorem.** We start with a simple-minded extension of Polya's Theorem 3, [33].

Theorem 8. *If the zeros* $\alpha_1, \dots, \alpha_l$ *in (1.8) are arbitrary complex numbers and the constants* $\beta_1, \dots, \beta_l$ *are real and non-negative, then all the roots of* V*'s and* S*'s lie in the (closed) convex hull* $Conv_{Q_k}$ *of the roots* $(\alpha_1, \dots, \alpha_l)$ *of the polynomial* $Q_k(z)$.

The next Theorem 9 is far more general. It shows that Theorem 8 is *asymptotically* true for any non-degenerate Lamé operator.

Theorem 9. *For any non-degenerate higher Lamé operator* $\mathfrak{d}(z)$ *and any* $\epsilon > 0$ *there exists a positive integer* N_ϵ *such that the zeros of all Van Vleck polynomials* $V(z)$ *possessing a Heine-Stieltjes polynomial* $S(z)$ *of degree* $n \geq N_\epsilon$ *and well as all zeros of these Stieltjes polynomials belong to* $Conv^\epsilon_{Q_k}$. *Here* $Conv_{Q_k}$ *is the convex hull of all zeros of the leading coefficient* Q_k *and* $Conv^\epsilon_{Q_k}$ *is its* ϵ*-neighborhood in the usual Euclidean distance on* $\mathbb{C}$.

The latter theorem is closely related to the next somewhat simpler localization result having independent interest.

Proposition 4. *For any non-degenerate higher Lamé operator* $\mathfrak{d}(z)$ *there exists a positive integer* N_0 *and a positive number* R_0 *such that all zeros of all Van Vleck polynomials* $V(z)$ *possessing a Stieltjes polynomial* $S(z)$ *of degree* $n \geq N_0$ *as well as all zeros of these Stieltjes polynomials lie in the disk* $|z| \leq R_0$.

Remark 10. The question for which operators $\mathfrak{d}(z)$ the roots of all its Van Vleck and Stieltjes polynomials lie exactly (and not just asymptotically) in the convex hull of its leading coefficient seems to be very hard even in the classical case of the equation (1.2). Notice that the roots of absolutely all Van Vleck polynomials (and not just those whose Stieltjes polynomials are of sufficiently large degree) of any non-degenerate higher Lamé operator $\mathfrak{d}(z)$ lie in some disk. But this is no longer true for Stieltjes polynomials. If the set of all Stieltjes polynomials is discrete (up to a scalar multiple) then their roots are bounded. But as soon as some Van Vleck polynomial admits at least a 2-dimensional linear space of Stieltjes polynomials then these roots become unbounded for obvious reasons. However for sufficiently large n no Van Vleck polynomial admits such families and the localization result holds, see Corollary 1.

Remark 11. Similar and stronger localization results with explicit constants and degree bounds were independently obtained by late J. Borcea (private communication).

Let us now show a typical behavior of the zeros of Van Vleck polynomials and the corresponding Stieltjes polynomials obtained in numerical experiments. Below we consider an example of the operator $\mathfrak{d}(z) = Q(z)\frac{d^3}{dz^3}$ with $Q(z) = (z^2+1)(z-3I-2)(z+2I-3)$. For $n = 24$, we calculate all 25 pairs (V, S) with $\deg S = 24$ (notice that V in this case is linear). The asymptotic behavior of the union of zeros of all Van Vleck polynomials whose Stieltjes polynomials have a given degree n when $n \to \infty$ as well as the asymptotics of the zeros of subsequences of Stieltjes polynomials of increasing degrees whose corresponding (monic) Van Vleck polynomials have a limit seems to be an extremely rich and interesting topic, see first steps in [42], [20].

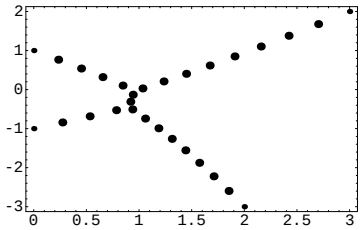

FIGURE 1. The union of the roots of 25 different linear Van Vleck polynomials whose Stieltjes polynomials are of degree 24. (Four average size dots are the zeros of $Q(z) = (z^2+1)(z-3I-2)(z+2I-3)$ and large dots are the zeros of 25 different $V(z)$.)

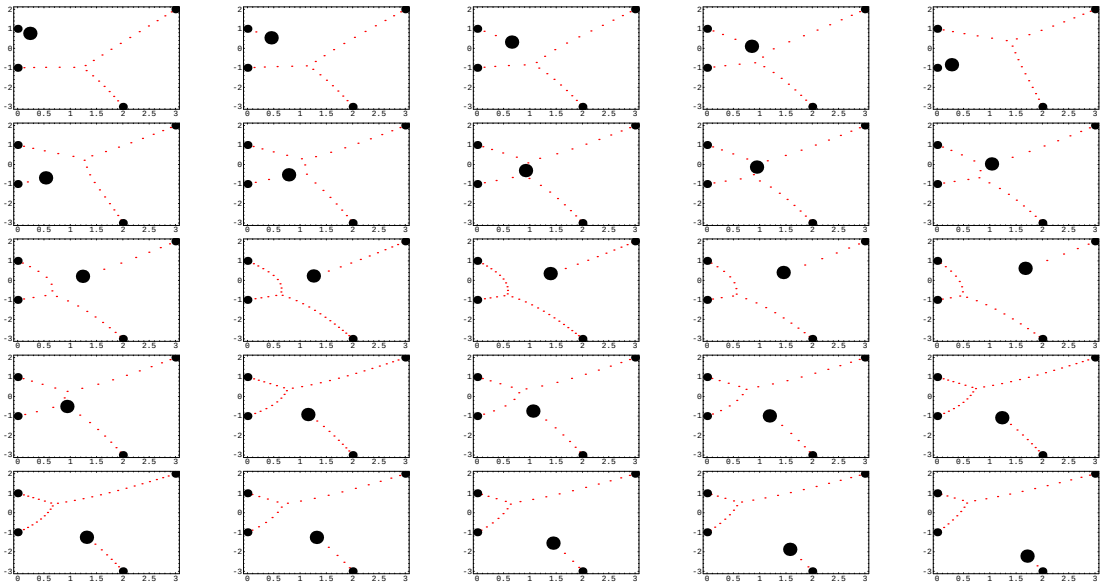

FIGURE 2. Zeros of 25 different Stieltjes polynomials of degree 24 for the above $\mathfrak{d}(z)$. (The small dots on each picture are the 24 zeros of $S(z)$; 4 average size dots are the zeros of $Q(z)$ and the single large dot is the (only) zero of the corresponding $V(z)$.)

1.4. **Comments on Heine's result and history around it.** As Heine himself mentions in [19], the requirement of algebraic independence of the coefficients of $Q_1(z)$ and $Q_2(z)$ is too strong and restrictive for his purposes. However, he fails to give any other explicit condition guaranteeing the same result (see Theorem 1). Heine's original motivation for the consideration (1.2) comes from the classical Lamé equation (1.1) in which case the coefficient of $Q_1(z)$ and $Q_2(z)$ are in fact algebraically dependent, namely, $Q_1(z) = Q_2'(z)/2$. Also in order to prove that the upper bound $\binom{n+l-2}{n}$ is actually achieved for algebraically independent $Q_1(z)$ and $Q_2(z)$, Heine uses an inductive argument, where he forces the coefficient of $Q_1(z)$ and $Q_2(z)$ to become algebraically dependent in a special way. Theorem 2 is another

clear indication that the algebraic independence is apparently an inappropriate condition for the goal.

Interpretation of Heine's text written in a rather cumbersome 19th century German, seems to create difficulties for mathematicians already in 1870-ties and up to now, see e.g. [29]. The main classical sources, namely, [43] from 1885, [33] from 1912 and [44] from 1939 are not too clear about what Heine actually proved and under which assumptions on $Q_1(z)$ and $Q_2(z)$ one can guarantee that for a given positive n the number of possible polynomial pairs (V, S) such that the corresponding S has degree exactly n is finite and bounded by $\binom{n+l-2}{n}$. Being aware of a gap in his proof, Heine seems to cover it by a reference to a letter of his friend Leopold Kronecker, who has (under unknown conditions) shown that for a given degree n the eliminant of the system of algebraic equations defining the coefficients of the polynomial $V(z)$ does not vanish identically. This statement is equivalent to the finiteness of the number of these polynomials. Heine mentions also a short note of Kronecker on this topic presented in the January issue of Monatsbericht der Berliner Akademie from 1864. Unfortunately, the track ends here. All one can find in this issue is a phrase about a meeting of the section of physics and mathematics of the Prussian Academy of Science held on November 14, 1864 "Herr Kronecker las über die verschieden Faktoren des Discriminante von Eliminantions-Gleichungen", i.e. "Herr Kronecker gave a lecture on different factors of the discriminant of the elimination equation." In an attempt to overbridge this gap, we undertook the task on translating and decoding Heine's arguments. The result of these efforts can be found in the preprint version of the present paper available at the address: www.math.su.se/˜shapiro.

1.5. **Some literature.** Let us finally mention a few relatively recent references on (generalized) Lamé equation. Being an object of substantial physical and mathematical importance, it gives, in particular, an example of an equation whose monodromy group can be analyzed in details, see [7], [21]-[22]. It is also closely related to the so-called quasi-exact solvability and integrable models, see [16]. Theory of multiparameter spectral problems originating from Heine-Stieltjes pioneering studies was developped in sixtees, see e.g. [48] and references therein. Recently the interest in Heine-Stieltjes polynomials has been stimulated by an unexpected extension of the Bethe ansatz in representation theory, see [35], [31], [36], [37] and a further series of articles by A. Varchenko and his coauthors, e.g. [32]. Starting with [27] a substantial progress has been made in the understanding of the asymptotics of the root distributions for these polynomials when either $l \to \infty$ (thermodynamic asymptotics) or $n \to \infty$ (semi-classical asymptotics), see [11], and [10]. Asymptotic root distribution for the eigenpolynomials of non-degenerate exactly solvable operators was studied in [26] and [6]. Interesting preliminary results of a similar flavor in the case of degenerate exactly solvable operators were very recently obtained by T. Bergkvist, [5].

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2. Proof of generalized Heine's theorems

We start with Theorem 4 (see Introduction). To settle it, we need a detailed description of the action of a non-degenerate operator $\mathfrak{d}(z)$ on the linear space Pol_n of all univariate polynomials of degree at most n.

Proof. Substituting $V(z) = v_r z^r + v_{r-1} z^{r-1} + \ldots + v_0$ and $S(z) = s_n z^n + s_{n-1} z^{n-1} + \ldots + s_0$ in (1.5), we get the following system of $(n+r+1)$ equations of a band shape (i.e. only a fixed and independent of n number of diagonals is non-vanishing in this system):

$$\begin{cases} 0 = s_n(v_r + L_{n,n+r}); \\ 0 = s_n(v_{r-1} + L_{n,n+r-1}) + s_{n-1}(v_r + L_{n-1,n+r-1}); \\ 0 = s_n(v_{r-2} + L_{n,n+r-2}) + s_{n-1}(v_{r-1} + L_{n-1,n+r-2}) + s_{n-2}(v_r + L_{n-2,n+r-2}); \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ 0 = s_n(v_0 + L_{n,n}) + s_{n-1}(v_1 + L_{n-1,n}) + ... + s_{n-r}(v_r + L_{n-r,n}); \\ 0 = s_n L_{n,n-1} + s_{n-1}(v_0 + L_{n-1,n-1}) + ... + s_{n-r-1}(v_r + L_{n-r-1,n-1}); \\ 0 = s_n L_{n,n-2} + s_1 L_{1,r+1} + s_2(v_r + L_{2,r}) + ... + s_{n-r-2}(v_r + L_{n-r-2,n-2}); \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ 0 = s_n L_{n,r} + s_{n-1} L_{n-1,r} + s_{n-2} L_{n-2,r} + ... + s_0(v_r + L_{0,r}); \\ 0 = s_n L_{n,r-1} + s_{n-1} L_{n-1,r-1} + s_{n-2} L_{n-2,r-1} + ... + s_0(v_{r-1} + L_{0,1}); \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ 0 = s_n L_{n,1} + s_{n-1} L_{n-1,1} + ... + s_1(v_0 + L_{1,1}) + s_0(v_1 + L_{0,1}); \\ 0 = s_n L_{n,0} + s_{n-1} L_{n-1,0} + ... + s_1 L_{0,1} + s_0(v_0 + L_{0,0}). \end{cases} \tag{2.1}$$

Here $L_{p,q}$ is a polynomial which expresses the coefficient containing s_p at the power z^q in $\sum_{i=1}^k Q_i(z) S^{(i)}$. Obviously, it is linear in the coefficients of $Q_k(z), ..., Q_1(z)$ and is explicitly given by the relation

$$L_{p,q} = \sum_{r=1}^{k} (p)_r A_{r,q-p+r},$$

where $A_{r,q-p+r}$ is the coefficient at z^{q-p+r} in $Q_r(z)$. In the notation used in the definition (1.6), we have $L_{m,m+r} = \mathbb{L}_m, m = 0, ..., n$. We use the convention that $L_{p,q}$ vanishes outside the admissible range of indices and, therefore, many of the above coefficients $L_{p,q}$ are in fact equal to 0. (In the system (2.1), we assumed that $n \geq r$ for simplicity.) Notice that all equations in (2.1) depend linearly on the variables $v_r, ..., v_0$ and $s_n, ..., s_0$ as well as on the coefficients of polynomials $Q_i(z)$, $i = 1, \ldots, k$. Note additionally that (2.1) is lower-triangular with respect to the coefficients $s_n, ..., s_0$, which allows us to perform the following important elimination. Let us enumerate the equations of (2.1) from 0 to $n+r$ assigning the number j to the equation describing the vanishing of the coefficient at the power z^{n+r-j}. Then if $\mathbb{L}_n = L_{n,n+r} \neq 0$, one has that the 0th equation has a solution $s_n = 1$ and $v_r = -L_{n,n+r} \neq 0$. The next n equations are triangular with respect to the coefficients $s_n, ..., s_0$, i.e. jth equation in this group contains only the variables $s_n, s_{n-1}, \ldots, s_{n-j}$ (among all s_j's) along with other types of variables. Thus under the assumption that all the diagonal terms $v_r + L_{n-i,n+r-i} = \mathbb{L}_{n-i} - \mathbb{L}_n$, $i = 0, 1, ..., n$ are nonvanishing, we can express all s_{n-i}, $i = 0, 1, ..., n$ consecutively as rational functions of the remaining variables. We get the reduced system of r rational equations containing only $(v_{r-1}, \ldots, v_0)$ as unknowns. Notice that in view

of $v_r = -L_{n,n+r} \neq 0$, the non-vanishing of the diagonal entries $v_r + L_{n-i,n+r-i},\ i = 0, 1, ..., n$ coincides exactly with the nonresonance condition (1.7).

Cleaning the common denominators, we get a reduced system of polynomial equations. We show now that this polynomial system is quasi-homogeneous in the variables v_j with the quasi-homogeneous weights $w(v_j) = r - j$. Thus, using the weighted-homogeneous version of the Bezout theorem (see e.g [14]), we get that the total number of solutions of this system (counted with multiplicities) equals $\binom{n+r}{r}$ in the corresponding weighted projective space. For this we need to verify that the system under consideration defines a complete intersection, i.e. has only isolated solutions. To check the quasi-homogenuity, note that the standard action of $\mathbb{C}^*$ on the set of roots of the polynomial $V(z)$ by simultaneous multiplication assigns the weight $r - j$ to its coefficient v_j. These weights are still valid in the reduced system with the variables $s_n, ..., s_0$ eliminated. Finally, we have to show that if the coefficients of $Q_k(z), ..., Q_1(z)$ are algebraically independent then the eliminated system has exactly $\binom{n+r}{r}$ simple solutions. Indeed, consider the linear space EQ of all systems of r quasi-homogeneous equations in the variables $(v_r, ..., v_0)$ with the weights $w(v_j) = r - j$ and where the ith equation is weighted-homogeneous of degree $n + i$. We equip this space with the standard monomial basis.

To accomplish the proof of Theorem 4, we need two additional standard facts.

Lemma 1. *The discriminant $Discr \subset EQ$ (i.e. the set of the coefficients of monomials in the equations for which the system has at least one solution of multiplicity greater than 1) is given by an algebraic equation with rational coefficients in the standard monomial basis of EQ.*

Proof. See [18], ch. 13. □

Consider some linear parameter space Λ with a chosen basis. Assume that there is a rational map: $\Phi : \Lambda \to EQ$ where each coordinate in the standard monomial basis of EQ is given by a rational function with rational coefficients w.r.t. the chosen basis in Λ. The next statement is obvious.

Lemma 2. *In the above notation the pullback of $\Phi^{-1}(Discr)$ in Λ either a) coincides with the whole Λ or b) is given in the chosen basis by an algebraic equation with rational coefficients.*

It remains to show that there are some values of the coefficients of the polynomials $Q_k(z), ..., Q_1(z)$ for which (1.5) has exactly $\binom{n+r}{r}$ distinct solutions. Here we are not able to follow the nice inductive argument of [19]. Heine's original proof does not generalize straightforwardly to higher order equations. Instead, we can, for example, invoke Theorem 7 whose proof is completely independent of the present arguments. It claims, in particular, that for any strict hyperbolicity preserver of the form $\mathfrak{d}(z) = \sum_{i=m}^{k} Q_i(z)\frac{d^i}{dz^i}$ and any $n \geq m$, there exist exactly $\binom{n+r}{r}$ pairs (V, S). One can additionally choose such a hyperbolicity preserver with $m = 1$ and, in this way get a necessary example of an operator with given k and r such that for any $n \geq 1$ it has exactly the maximal number of pairs (V, S). □

To settle Theorem 5 (see Introduction), let us first reinterpret Problem (1.5) in purely linear algebraic terms.

2.1. **On eigenvalues for rectangular matrices.** We start with the following natural question.

Problem. Given an $(l + 1)$-tuple of $(m_1 \times m_2)$-matrices $A, B_1, ..., B_l$ where $m_1 \leq m_2$, describe the set of all values of parameters $\lambda_1, ...\lambda_l$ for which the rank of the linear combination $A + \lambda_1 B_1 + ... + \lambda_l B_l$ is less than m_1. In other words, find such parameters for which the linear system $v * (A + \lambda_1 B_1 + ... + \lambda_l B_l) = 0$ has a

nontrivial (left) solution $v \neq 0$. This left solution will be called an *eigenvector of A w.r.t. the linear span of $B_1, ..., B_l$.*

Let $\mathcal{M}_{m_1,m_2}$ denote the linear space of all $(m_1 \times m_2)$-matrices with complex entries. Below, we will consider l-tuples of $(m_1 \times m_2)$-matrices $B_1, ..., B_l$, which are linearly independent in $\mathcal{M}_{m_1,m_2}$ and denote their linear span by $\mathcal{L} = \mathcal{L}(B_1, ..., B_l)$. Given a matrix pencil $\mathcal{P} = A + \mathcal{L}$ where $A \in \mathcal{M}_{m_1,m_2}$ denote by $\mathcal{E}_{\mathcal{P}} \subset \mathcal{P}$ its *eigenvalue locus*, i.e. the set of matrices in $\mathcal{P}$ whose rank is less than the maximal one. Denote by $\mathcal{M}^1 \subset \mathcal{M}_{m_1,m_2}$ the set of all $(m_1 \times m_2)$ matrices with positive corank, i.e whose rank is less than m_1. Its co-dimension equals $m_2 - m_1 + 1$ and its degree as an algebraic variety equals $\binom{m_2}{m_1-1}$, see [13], Prop. 2.15. Consider the natural left-right action of the group $GL_{m_1} \times GL_{m_2}$ on $\mathcal{M}_{m_1,m_2}$, where GL_{m_1} (resp. GL_{m_2}) acts on $(m_1 \times m_2)$-matrices by the left (resp. right) multiplication. This action on $\mathcal{M}_{m_1,m_2}$ has finitely many orbits, each orbit being the set of all matrices of a given (co)rank, see e.g. [4], ch.1 §2. Notice that due to the well-known formula of the product of coranks, the codimension of the set of matrices of rank $\leq r$ equals $(m_1 - r)(m_2 - r)$. Obviously, for any pencil $\mathcal{P}$, one has that the eigenvalue locus coincides with $\mathcal{E}_{\mathcal{P}} = \mathcal{M}^1 \cap \mathcal{P}$. Thus for a generic pencil $\mathcal{P}$ of dimension l the eigenvalue locus $\mathcal{E}_{\mathcal{P}}$ is a subvariety of $\mathcal{P}$ of codimension $m_2 - m_1 + 1$ if $l \geq m_2 - m_1 + 1$ and it is empty otherwise. The most interesting situation for applications occurs when $l = m_2 - m_1 + 1$ in which case $\mathcal{E}_{\mathcal{P}}$ is generically a finite set. From now on, let us assume that $l = m_2 - m_1 + 1$. Denoting as above by $\mathcal{L}$ the linear span of $B_1, ..., B_l$, we say that $\mathcal{L}$ is *transversal to* $\mathcal{M}^1$ if the intersection $\mathcal{L} \cap \mathcal{M}^1$ is finite and *non-transversal to* $\mathcal{M}^1$ otherwise. Notice that due to homogeneity of $\mathcal{M}^1$ any $(m_2 - m_1 + 1)$-dimensional linear subspace $\mathcal{L}$ transversal to it intersects $\mathcal{M}^1$ only at 0 and that the multiplicity of this intersection at 0 equals $\binom{m_2}{m_1-1}$.

We start with the following obvious statement, which will later imply Theorem 5.

Lemma 3. *If $(m_2 - m_1 + 1)$-dimensional linear space $\mathcal{L}$ is tranversal to $\mathcal{M}^1$, then for any matrix $A \in \mathcal{M}_{m_1,m_2}$ the eigenvalue locus $\mathcal{E}_{\mathcal{P}}$ of the pencil $\mathcal{P} = A + \mathcal{L}$ consists of exactly $\binom{m_2}{m_1-1}$ points counted with multiplicitites.*

Remark 12. Notice that since $\mathcal{M}^1 \subset \mathcal{M}(m_1, m_2)$ is an incomplete intersection, then in order to explicitly determine the eigenvalue locus of a given matrix A w.r.t. some $(m_2 - m_1 + 1)$-dimensional linear subspace $\mathcal{L} \subset \mathcal{M}(m_1, m_2)$, one has to solve an overdetermined system of $\binom{m_2}{m_1}$ equations describing the vanishing of all maximal minors of a $(m_1 \times m_2)$-matrix depending on parameters. Fortunately, for the multi-parameter spectral problem (1.5), we encounter only "triangular" rectangular matrices (i.e. with the left-lower corner vanishing) for which the determination of the eigenvalue locus often reduces to a complete intersection, see proofs of Theorem 4 and Theorem 6.

Let us explain how Lemma 3 implies Theorem 5. Namely, given a non-degenerate operator $\mathfrak{d}(z)$ in order to find all its Van Vleck polynomials having (at least one) Stieltjes polynomial of degree at most n, we need to study the action of $\mathfrak{d}(z)$ on the linear space Pol_n of all univariate polynomials of degree at most n. If $\mathfrak{d}(z)$ has the Fuchs index r, then $\mathfrak{d}(z)$ maps Pol_n to Pol_{n+r}. Using the standard monomial basis $1, z, z^2, ..., z^l$ in Pol_l, we get that if $n \geq k = \mathrm{ord}(\mathfrak{d}(z))$, then the action of $\mathfrak{d}(z)$ in this basis is represented by a "triangular" band $(n+1) \times (n+r+1)$-matrix $A_{\mathfrak{d}(z),n}$ with at most $r+k$ non-vanishing diagonals. Here "triangular" means that all entries $a_{i,j}$ of $A_{\mathfrak{d}(z),n}$ with $i < j$ vanish. Denote by $I_s,\ s = 0, ..., r$ the $(n+1) \times (n+r+1)$-matrix whose entries are given by $a_{i,j} = 0$ if $i - j \neq s$ and 1 otherwise. Denote by $\mathfrak{L}$ the linear span of $I_0, ..., I_r$ and notice that $\mathfrak{L}$ is transversal to $\mathcal{M}^1 \subset M_{n+1,n+r+1}$, since any matrix belonging to the pencil $\mathfrak{L}$ and different from 0 has full rank.

Notice that adding to $\mathfrak{d}(z)$ an arbitrary polynomial $V(x) = v_r z^r + v_{r-1} z^{r-1} + ... + v_0$ of degree at most r corresponds, on the matrix level, to adding of the linear combination $v_r I_0 + v_{r-1} I_1 + ... + v_0 I_r$ to the initial matrix $A_{\mathfrak{d}(z),n}$. The existence of a non-trivial Stieltjes polynomial of degree at most n corresponds to the fact that the matrix $A_{\mathfrak{d}(z),n} + v_r I_0 + v_{r-1} I_1 + ... + v_0 I_r$ has a non-trivial (left) kernel. Thus, for a given non-degenerate operator $\mathfrak{d}(z)$ the problem of finding all Van Vleck polynomials whose Stieltjes polynomials are of degree at most n is exactly equivalent to the determination of all the eigenvalues of its matrix $A_{\mathfrak{d}(z),n}$ w.r.t. the linear space $\mathfrak{L}$ in the above-mentioned sense. Lemma 3 has a simple analog for "triangular" rectangular matrices which is equivalent to Theorem 5.

Namely, denote by $TM(m_1, m_2) \subset \mathcal{M}(m_1, m_2), \quad m_1 \leq m_2$, the set of all "triangular" $m_1 \times m_2$-matrices, i.e. with $a_{i,j} = 0$ for $i < j$. Let $\mathfrak{L} \subset TM(m_1, m_2)$ be the linear subspace spanned by all $(m_2 - m_1 + 1)$ possible unit matrices $I_1, ..., I_{m_2 - m_1 + 1} \in TM(m_1, m_2)$. Finally, denote by $TM^1 \subset TM(m_1, m_2)$ the set of all "triangular" matrices with positive corank.

Lemma 4. *For any matrix $A \in TM(m_1, m_2)$, the eigenvalue locus $\mathcal{E}_{\mathcal{P}}$ of the pencil $\mathcal{P} = A + \mathfrak{L}$ consists of exactly $\binom{m_2}{m_1 - 1}$ points counted with multiplicitites.*

Proof. The same as above. □

The latter Lemma settles Theorem 5. Let us now prove Proposition 1 (see Introduction).

Proof. In the above notation, consider the pencil $\mathcal{A}_n(v_r, v_{r-1}, ..., v_0) = A_{\mathfrak{d}(z),n} + v_r I_0 + v_{r-1} I_1 + ... + v_0 I_r$ of $(n+1) \times (n+r+1)$-matrices. One has

$$\mathcal{A}_n(v_r, v_{r-1}, ..., v_0) = \begin{pmatrix} \mathbb{L}_n + v_r & * & * & * & * & \cdots \\ 0 & \mathbb{L}_{n-1} + v_r & * & * & * & \cdots \\ 0 & 0 & \mathbb{L}_{n-2} + v_r & * & * & \cdots \\ 0 & 0 & 0 & \mathbb{L}_{n-3} + v_r & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where $*$ stands for possibly non-vanishing entries. The obvious necessary condition for such a matrix to have a positive corank it that one of the elements on the shown above main diagonal vanishes, i.e. there exists $i = 0, ..., n$ such that $\mathbb{L}_i + v_r = 0$ or, equivalently, $v_r = -\mathbb{L}_i$. Set $v_r = -\mathbb{L}_n$ thus making the entry in the left-upper corner equal to 0. Recall that the nth nonresonance condition requires that $\mathbb{L}_n \neq \mathbb{L}_j, j = 0, ..., n-1$. Therefore, the subtraction of $\mathbb{L}_n$ along the main diagonal will keep all other diagonal entries except for the left-upper corner non-vanishing. The r-dimensional pencil $\mathcal{A}_n(-\mathbb{L}_n, v_{r-1}, ..., v_0) = A_{\mathfrak{d}(z),n} - \mathbb{L}_n I_0 + v_{r-1} I_1 + ... + v_0 I_r$ has its 1-st column of the above matrix presentation vanishing and its main diagonal being the same for all possible values of $v_{r-1}, ..., v_0$. Therefore, by Lemma 3, there exist exactly $\binom{n+r}{r}$ eigenvalues $v_{r-1}, v_{r-2}, ..., v_0$ counted with multiplicities such that $\mathcal{A}_n(-\mathbb{L}_n, v_{r-1}, ..., v_0)$ has a positive corank. Finally, notice that since for any matrix from the pencil $\mathcal{A}_n(-\mathbb{L}_n, v_{r-1}, ..., v_0)$ its entries along the main diagonal, except for the left-upper corner, are non-vanishing, its corank can be at most 1. Moreover, when the corank of such a matrix is 1, then the occurring non-trivial linear combination of the rows, which vanishes, must necessarily include the first row, since the second, the third etc rows are linearly independent for the above reason. The coefficients of this linear dependence of rows are exactly the coefficients of the corresponding Stieltjes polynomial. The fact that the first row must be in the linear dependence means that the leading coefficient of the corresponding Stieltjes polynomial must be non-vanishing, i.e. this Stieltjes polynomial is of degree exactly n. Proposition 1 is settled. □

To prove Theorem 6 (see Introduction), we need to take a more careful look at the proof of Theorem 4. Namely, consider again the system (2.1) determining the set of all pairs (V, S), where V is a Van Vleck polynomial and S is the corresponding Stieltjes polynomial of degree at most n. As in the proof of Theorem 4, we solve the 0th equation in (2.1) by taking $s_n = 1$ and $v_r = -L_{n,n+r} = -\mathbb{L}_n$. (By Proposition 1 finding a solution of (2.1) with $s_n = 1$ and $v_r = -L_{n,n+r} = -\mathbb{L}_n$ leads to a pair (V, S) such that V is of degree exactly equal to r and S is of degree exactly equal to n.) Then we express consecutively the variables $s_{n-1}, s_{n-2}, ..., s_0$ from the next n equations of (2.1). The crucial circumstance here is that while doing this we only divide by the differences of the form $\mathbb{L}_n - \mathbb{L}_j,\ j = n-1, n-2, ..., 0$, which are non-vanishing due to the validity of our nonresonance condition. Substituting the obtained expressions for $s_j,\ j = 0, ..., n$ in the remaining r equations in (2.1), we get the required eliminated system of algebraic equations in the variables $v_{r-1}, ..., v_0$, which determines a complete intersection. □

To illustrate the above procedure take a concrete example.

Example 1. Consider the action of some operator $\mathfrak{d}(z)$ with the Fuchs index $r = 2$ on the space Pol_1. It maps Pol_1 to Pol_3 and, say, is represented in the monomial bases of Pol_1 and Pol_3, by the matrix

$$\begin{pmatrix} \mathbb{L}_1 & L_{1,2} & L_{1,1} & L_{1,0} \\ 0 & \mathbb{L}_0 & L_{0,1} & L_{0,0} \end{pmatrix}.$$

(Here we used the notation from the proof of Theorem 4.) Since $r = 2$, we need to add to $\mathfrak{d}(z)$ a quadratic Van Vleck polynomial $V(z) = v_2 z^2 + v_1 z + v_0$ with the undetermined coefficients v_2, v_1, v_0, which modifies the above matrix as follows:

$$\begin{pmatrix} \mathbb{L}_1 + v_2 & L_{1,2} + v_1 & L_{1,1} + v_0 & L_{1,0} \\ 0 & \mathbb{L}_0 + v_2 & L_{0,1} + v_1 & L_{0,0} + v_0 \end{pmatrix}.$$

The operator $\mathfrak{d}(z) + V(z)$ has a linear Stieltjes polynomial $S(z) = s_1 z + s_0$ if and only if the vector (s_1, s_0) is the left kernel of the latter matrix, which leads to the system:

$$\begin{cases} 0 = s_1(\mathbb{L}_1 + v_2); \\ 0 = s_1(L_{1,2} + v_1) + s_0(\mathbb{L}_0 + v_2); \\ 0 = s_1(L_{1,1} + v_0) + s_0(L_{0,1} + v_1); \\ 0 = s_1 L_{1,0} + s_0(L_{0,0} + v_0). \end{cases}$$

Setting $s_1 = 1$ and $v_2 = -\mathbb{L}_1$ as was explained earlier, we get $s_0 = \frac{L_{1,2}+v_1}{\mathbb{L}_1 - \mathbb{L}_0}$ from the 2-nd equation. Substituting the obtained variables in the remaining two equations, we get the system of two equations:

$$\begin{cases} (\mathbb{L}_1 - \mathbb{L}_0)(v_0 + L_{1,1}) + (v_1 + L_{1,1})(v_1 + L_{1,2}) = 0; \\ (v_0 + L_{0,0})(v_1 + L_{1,2}) + (\mathbb{L}_1 - \mathbb{L}_0)L_{1,0} = 0. \end{cases}$$

This system determines three (not necessarily distinct) pairs (v_1, v_0), which together with $v_2 = -\mathbb{L}_1$ gives us three (not necessarily distinct) quadratic Van Vleck polynomials, whose Stieltjes polynomials are of degree exactly 1.

Now we finally describe the notion of natural multiplicity of a given Van Vleck polynomial of an operator $\mathfrak{d}(z)$ used in the Introduction. Let $\widetilde{V}(z) = \tilde{v}_r z^r + \tilde{v}_{r-1} z^{r-1} + ... + \tilde{v}_0$ be some fixed Van Vleck polynomial of the Heine-Stieltjes problem (1.5), i.e. there exists a (not necessarily unique) polynomial solution $\widetilde{S}(z)$ of the equation of (1.5) with the chosen $V(z) = \widetilde{V}(z)$. (Below we use notation from the proof of Proposition 1.)

Definition 2. Given a positive integer n let us define the nth multiplicity $\sharp_n(\widetilde{V})$ of $\widetilde{V}(z)$ as the usual local algebraic multiplicity of the intersection of the $(r+1)$-dimensional matrix pencil $\mathcal{A}_n(v_r, v_{r-1}, ..., v_0) = A_{\mathfrak{d}(z),n} + v_r I_0 + v_{r-1} I_1 + ... + v_0 I_r$ consisting of "triangular" $(n+1)\times(n+r+1)$-matrices with the set $TM^1 \subset TM(n+1, n+r+1)$ of positive corank matrices at the matrix $A_{\mathfrak{d}(z),n}+\tilde{v}_r I_0+\tilde{v}_{r-1} I_1+...+\tilde{v}_0 I_r$.

Here (as above) $A_{\mathfrak{d}(z),n}$ denotes the matrix of the action of $\mathfrak{d}(z)$ on the space Pol_n taken w.r.t the standard monomial basis and $A_{\mathfrak{d}(z),n} + \tilde{v}_r I_0 + \tilde{v}_{r-1} I_1 + ... + \tilde{v}_0 I_r$ is, therefore, the matrix of action of the operator $\mathfrak{d}(z) + \widetilde{V}(z)$ on Pol_n. We set $\sharp_n(\widetilde{V}) = 0$ in case, when $A_{\mathfrak{d}(z),n} + \tilde{v}_r I_0 + \tilde{v}_{r-1} I_1 + ... + \tilde{v}_0 I_r$ does not belong to $TM^1 \subset TM(n+1, n+r+1)$, i.e. the operator $\mathfrak{d}(z) + \widetilde{V}(z)$ does not annihilate any polynomial of degree at most n.

Remark 13. The natural multiplicity of Van Vleck polynomials in Theorems 4 and 5, while counting those with Stieltjes polynomials of degree at most n, is exactly the nth multiplicity from Definition 2.

Obviously, for any given Van Vleck polynomial $\widetilde{V}(z)$, the sequence $\{\sharp_n(\widetilde{V})\}$, $n = 0, 1, ...$ is a non-decreasing sequence of non-negative integers. Moreover the following stabilization result holds.

Lemma 5. *For any non-degenerate operator $\mathfrak{d}(z)$, the sequence $\{\sharp_n(\widetilde{V})\}$ of multiplicities of any its Van Vleck polynomial $\widetilde{V}(z)$ stabilizes, i.e there exists $n_{\widetilde{V}}$ such that for all $n > n_{\widetilde{V}}$ one has $\sharp_n(\widetilde{V}) = \sharp_{n_{\widetilde{V}}}(\widetilde{V})$.*

Proof. Indeed, as was mentioned in e.g. the proof of Proposition 1, the leading coefficient $\tilde{v}_r$ of $\widetilde{V}(z)$ must necessarily coincide with $-\mathbb{L}_m$ for some non-negative m. The sequence $\{|\mathbb{L}_j|\}$ is strictly increasing starting from some j_0, see (1.6). Moreover, by Proposition 1, if $\mathbb{L}_n \neq L_j, j = 0, 1, ..., n-1$, then the total multiplicity of all Van Vleck polynomials whose leading term equals $-\mathbb{L}_n$ is equal to $\binom{n+r}{r}$. Therefore, if we take the index value j_0 such $|\mathbb{L}_j| > |\mathbb{L}_m|$ for all $j \geq j_0$, then the multiplicities $\sharp_j(\widetilde{V})$ can not change for $j \geq j_0$, since the total multiplicity increase is obtained on Van Vleck polynomials with a different leading coefficient when j grows. □

3. Proof of generalized Pólya's theorems

Let us now prove Theorem 8 following straightforwardly the recipe of [33], which in turn is closely related to the proof of the classical Gauss-Lukas theorem.

Proof. Let $(z_1, \ldots, z_n)$ denote the set of all roots of a Stieltjes polynomial $S(z)$ of some degree n satisfying the equation (1.8) with α_i's being arbitrary complex and β_j's being real positive. Then for each z_i one has

$$\frac{S^{(k)}(z_i)}{S^{(k-1)}(z_i)} + \sum_{j=1}^{l} \frac{\beta_j}{z_i - \alpha_j} = 0.$$

This equation has the form

$$\sum_{s=1}^{p} \frac{m_s}{z_i - \xi_s} + \sum_{j=1}^{l} \frac{\beta_j}{z_i - \alpha_j} = 0, \tag{3.1}$$

where $(\xi_1, \ldots, \xi_p)$ is the set of all roots of $S^{(k-1)}(z)$ with $p = n - k + 1$ and $(m_1, ..., m_p)$ is the set of multiplicities of the roots $(\xi_1, \ldots, \xi_p)$. Notice that by the standard Gauss-Lukas theorem, all $(\xi_1, \ldots, \xi_p)$ lie in the convex hull of the set of roots $(z_1, \ldots, z_n)$. Assume now that the convex hull of $(z_1, \ldots, z_n)$ is not contained in the convex hull of $(\alpha_1, \ldots, \alpha_l)$. Then there exists some root z_i and an affine line

$L \subset \mathbb{C}$ separating z_i from the rest of z_j's together with all α_j's and ξ_m's. But then the equation (3.1) can not hold, since all the vectors $z_i - \xi_s$ and $z_i - \alpha_j$ lie in the same half-plane. □

Remark 14. The above argument works for the roots of Van Vleck polynomials $V(z)$ as well and extends to the case $\beta_j \geq 0$.

To settle a much more delicate Theorem 9, we will prove a number of localization results having an independent interest.

3.1. **Root localization for Van Vleck and Stieltjes polynomials.**

Definition 3. Given a finite (complex-valued) measure μ supported on $\mathbb{C}$, we call by its *total mass* the integral $\int_{\mathbb{C}} d\mu(\zeta)$. The Cauchy transform $\mathcal{C}_\mu(z)$ of μ is standardly defined as

$$\mathcal{C}_\mu(z) = \int_{\mathbb{C}} \frac{d\mu(\zeta)}{z-\zeta}. \tag{3.2}$$

Obviously, $\mathcal{C}_\mu(z)$ is analytic outside the support of μ and $\mu = \frac{1}{\pi}\frac{\mathcal{C}_\mu(z)}{\partial \bar{z}}$ understood in the distributional sense. Detailed information about Cauchy transforms can be found in [17].

Definition 4. Given a (monic) polynomial $P(z)$ of some degree m, we associate with $P(z)$ its *root-counting measure* $\mu_P(z) = \frac{1}{m}\sum_j \delta(z - z_j)$, where $\{z_1, ..., z_m\}$ stands for the set of all roots of $P(z)$ with repetitions and $\delta(z - z_j)$ is the usual Dirac delta-function supported at z_j.

Directly from the definition of $\mu_P(z)$, one has that for any given polynomial $P(z)$ of degree m its Cauchy transform is given by $\mathcal{C}_{\mu_P}(z) = \frac{P'(z)}{mP(z)}$.

We start with a rather simple estimate of the absolute value of the Cauchy transform of a probability measure, (comp. Lemma 2 in [5]).

Lemma 6. *Let μ be a probability measure supported in a disk D_0 of radius R_0 and centered at z_0. Then for any z outside D_0, one has:*

$$\frac{1}{|z - z_0 - R_0|} \geq |\mathcal{C}_\mu(z)| \geq \frac{1}{2|z - z_0|}. \tag{3.3}$$

Proof. The l.h.s. of the above inequality is quite obvious. By (3.2), one has that $|\mathcal{C}_\mu(z)|$ will be maximal if one places the whole unit mass of μ at the point which has the least distance to z in the admissible support. In our case, such a point p is the intersection of the boundary circle of D_0 with the segment (z, z_0). Its distance to z equals $|z - z_0 - R_0|$, which gives the required inequality. To settle the r.h.s. let us assume for simplicity that $z_0 = 0$. Translation invariance of our considerations is obvious. Let us use (3.2) and change the integration variable as follows:

$$\frac{1}{z-\zeta} = \frac{1}{z}\cdot\frac{1}{1-\zeta/z} = \frac{1}{z}\cdot\frac{1}{1-\theta}$$

where $\theta = \frac{\zeta}{z}$. Since z lies outside D_0 and ζ lies inside D_0, one has $|\theta| < 1$ which implies for $w = \frac{1}{1-\theta}$ that one has $\mathrm{Re}(w) \geq \frac{1}{2}$. Indeed,

$$|w-1| = \frac{|\theta|}{|1-\theta|} = |\theta||w| \leq |w| \Leftrightarrow |w-1| \leq |w| \Leftrightarrow \mathrm{Re}(w) \geq \frac{1}{2}.$$

Therefore,

$$|\mathcal{C}_\mu(z)| = \left|\int_{\mathbb{C}} \frac{d\mu(\zeta)}{z-\zeta}\right| = \frac{1}{|z|}\left|\int_{\mathbb{C}} \frac{d\mu(\zeta)}{1-\theta}\right| = \frac{1}{|z|}\left|\int_{\mathbb{C}} w d\mu(\zeta)\right| \geq \frac{1}{|z|}\left|\int_{\mathbb{C}} \mathrm{Re}(w) d\mu(\zeta)\right| \geq \frac{1}{2|z|}.$$

□

Using Lemma 6, we now settle Proposition 4 (see Introduction).

Proof. Take a pair $(V(z), S(z))$ where $V(z)$ is some Van Vleck polynomial and $S(z)$ is its corresponding Stieltjes polynomial of degree n. Let ξ be the root of either $V(z)$ or $S(z)$ which has the maximal modulus among all roots of the chosen $V(z)$ and $S(z)$. We want to show that there exists a radius $R > 0$ such that $|\xi| \leq R$ for any ξ as above and as soon as n is large enough. Substituting $V(z), S(z), \xi$ in (1.5) and using (1.4), we get the relation:

$$Q_k(\xi)S^{(k)}(\xi) + Q_{k-1}(\xi)S^{(k-1)}(\xi) + ... + Q_1(\xi)S'(\xi) = 0,$$

dividing which by its first term, we obtain:

$$1 + \sum_{j=1}^{k-1} \frac{Q_j(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)} = 0. \tag{3.4}$$

Notice that the rational function $b_i(z) := \frac{S^{(i+1)}(z)}{(n-i)S^{(i)}(z)}$ is the Cauchy transform of the polynomial $S^{(i)}(z)$. Easy calculation shows that

$$S^{(i)}(z) = \frac{S^{(k)}(z)}{(n-k+1)...(n-i)\prod_{j=i}^{k-1} b_j(z)} \Leftrightarrow \frac{S^{(i)}(z)}{S^{(k)}(z)} = \frac{(n-k)!}{(n-i)!\prod_{j=i}^{k-1} b_j(z)}.$$

Notice additionally, that by the usual Gauss-Lucas theorem all roots of any $S^{(i)}(z)$ lie within the convex hull of the set of roots of $S(z)$. In particular, all these roots lie within the disk of radius $|\xi|$. Therefore, using Lemma 6, we get

$$\left|\frac{Q_i(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| \leq \frac{|Q_i(\xi)|}{|Q_k(\xi)|}\frac{(n-k)!}{(n-i)!}2^{k-i}|\xi|^{k-i}. \tag{3.5}$$

Notice that since $Q_k(z)$ is a monic polynomial of degree $k+r$ (recall that r is the Fuchs index of the operator $\mathfrak{d}(z)$), then one can choose a radius R such that for any z with $|z| > R$ one has $|Q_k(z)| \geq \frac{|z|^{k+r}}{2}$. Now since for any $i = 1, ..., k-1$ one has $\deg Q_i(z) \leq i + r$, we can choose a positive constant K such that $|Q_i(z)| \leq K|z|^{i+r}$ for all $i = 1, ..., k-1$ and $|z| > R$. We want to show that ξ can not be too large for a sufficiently large n. Using our previous assumptions and assuming additionally that $|\xi| > R$, we get

$$\left|\frac{Q_i(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| \leq \frac{|Q_i(\xi)|}{|Q_k(\xi)|}\frac{(n-k)!}{(n-i)!}2^{k-i}|\xi|^{k-i} \leq \frac{K \cdot 2^{k-i+1}}{(n-i)...(n-k+1)}.$$

Now we can finally choose N_0 large enough such that for all $n \geq N_0$, all $i = 1, ..., k-1$ and any $|\xi| > R$, one has that

$$\left|\frac{Q_i(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| \leq \frac{K \cdot 2^{k-i+1}}{(n-i)...(n-k+1)} < \frac{1}{k-1}.$$

But then obviously the relation (3.4) can not hold for all $n \geq N_0$ and any $|\xi| > R$, since

$$\left|\sum_{j=1}^{k-1} \frac{Q_j(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| \leq \sum_{j=1}^{k-1} \left|\frac{Q_j(\xi)S^{(i)}(\xi)}{Q_k(\xi)S^{(k)}(\xi)}\right| < \sum_{i=1}^{k-1} \frac{1}{k-1} < 1.$$

□

Now we will strengthen the arguments in the proof of Proposition 4 in order to settle Theorem 9. Denote by $\mathcal{R}_{Q_k}$ the maximal distance between the origin and $Conv_{Q_k}$. The following statement holds.

Lemma 7. *For any non-degenerate higher Lamé operator $\mathfrak{d}(z)$ and a given number $\delta > 0$ there exists a positive integer N_δ such that the roots of all Van Vleck polynomials $V(z)$ possessing a Stieltjes polynomial $S(z)$ of degree $\geq N_\delta$ as well as the roots of these Stieltjes polynomials lie in the disk $|z| \leq \mathcal{R}_{Q_k} + \delta$.*

Proof. Notice that once δ is fixed the quotient $\frac{|Q_i(z)|}{|Q_k(z)|}$ is bounded from above for each $i = 1, ..., k-1$ if we assume that $|z| \geq \mathcal{R}_{Q_k} + \delta$. Indeed, all the roots of $Q_k(z)$ lie within the disk of radius $\mathcal{R}_{Q_k}$ centered at the origin and each $Q_i(z)$ has a smaller degree than $Q_k(z)$. Consider now again the estimate (3.5). Since we now know that ξ lies in some bounded domain for all possible polynomials $V(z)$ and $S(z)$ of sufficiently high degree and that the quotient $\frac{|Q_i(z)|}{|Q_k(z)|}$ is bounded from above outside the disk of radius $\mathcal{R}_{Q_k} + \delta$, we get that the right-hand side of (3.5) goes to 0 when $n \to \infty$ under the assumption that ξ stays outside the latter disk. Looking again at (3.4), we see that by the latter argument it can not hold for $|\xi| \geq \mathcal{R}_{Q_k} + \delta$ when $n \to \infty$. This contradiction proves the lemma. □

To finish the proof of Theorem 9, notice that the choice of the origin is in our hands, i.e we can make an arbitrary affine shift of the independent variable z and use the same arguments. Since the convex hull $Conv_{Q_k}$ is the intersection of all disks centered at different points and containing $Conv_{Q_k}$, we can for any chosen $\epsilon > 0$ find the intersection $\mathcal{K}$ of finitely many disks in $\mathbb{C}$ such that $\mathcal{K}$ contains $Conv_{Q_k}$ but is contained in $Conv^{\epsilon}_{Q_k}$. (One can choose one such disk for each edge of the boundary of $Conv_{Q_k}$ putting its center sufficiently far away on the line perpendicular to the edge and passing through its middle point.) Then since $\mathcal{K}$ is the intersection of finitely many disks, we can applying Lemma 7 find such N_ϵ that all roots of all $V(z)$ and $S(z)$ for all $n \geq N_\epsilon$ lie in $\mathcal{K}$. □

4. Final Remarks

Let us formulate a number of relevant questions and conjectures.

Problem 1. Is it possible to describe when a linear ordinary differential equation with polynomial coefficients admits at least 2 linearly independent polynomial solutions?

The prototype result of Varchenko-Scherbak gives a satisfactory answer for equations of the second order. An answer to the latter question will allow to detect the appearance of multi-dimensional families of Stieltjes polynomials.

Problem 2. Under the nonresonance assumption (1.7) is it possible to obtain explicitly the discriminantal surface which shows when a Van Vleck polynomial attains a non-trivial multiplicity.

This question addresses the problem of explicit determination of the discriminantal surface mentioned in Heine's proof. Some discussion of this problem can be found in [41].

Problem 3. Explain how the number of Van Vlecks polynomials having Stieltjes polynomials for a certain given degree n can drop below $\binom{n+r}{r}$?

The next question addresses the issue of location of the roots of Van Vleck and Stieltjes polynomials.

Problem 4. Under which assumptions on $\mathfrak{d}(z)$ the roots of any its Van Vleck and Stieltjes polynomials lie in the convex hull of its leading coefficient $Q_k(z)$?

The basic examples are provided by Stieltjes's and Polya's theorems.

Finally,

Problem 5. Is it possible to extend the results of this paper to the case of degenerate higher Lamé operators?

T. Bergkvist [5] has obtained a number of interesting results and conjectures in the case of degenerate exactly solvable operators. Motivated by her results, we formulate the following conjecture.

Conjecture 1. *For any degenerate Lamé operator and any positive integer N_0 the union of all the roots to polynomials V and S taken over $\deg S \geq N_0$ is always unbounded. Therefore, this property is a key distinction between non-degenerate and degenerate Lamé operators.*

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Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
E-mail address: shapiro@math.su.se