EIGENPOLYNOMIALS OF DEGENERATE EXACTLY SOLVABLE OPERATORS: ALGEBRAIC CURVES, ZERO ASYMPTOTICS, AND RECURRENCES

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Dedicated to our friend, late Professor Jan-Erik Björk

Abstract. We study the monic eigenpolynomials of the degenerate exactly solvable operators

$$T_{m,\ell,j} = z^m D^\ell + z^j D^j, \qquad 0 \le m < \ell > j \ge 1.$$

Their scaled zero distributions are governed by the algebraic equation $\alpha z^m \mathcal{C}^\ell + z^j \mathcal{C}^j = 1$. We determine the branching data and genus of this curve, classify the rational cases, and construct explicitly the corresponding probability measure: its support is a star with $\ell - m$ equal legs, under $z \mapsto z^{\ell-m}$ it becomes the Raney distribution $\mathcal{R}_{\ell/j,1/j}$, and it factorises as $\mu_{\ell-1,\ell,j} = \mathrm{Beta}(\frac{1}{j}, 1-\frac{1}{j}) \boxtimes \pi^{\boxtimes(\ell-j)}$.

At finite degree we prove the ray-root conjecture for all $j \le 2$: the zeros of $p_n^{(m,\ell,j)}$ then lie on $\ell - m$ equally spaced rays. Together with a weighted Gershgorin bound in the eigenpolynomial basis this settles both of Bergkvist's conjectures for every $T_{m,\ell,1}$ with $(\ell - m) \mid \ell$, and settles the weak convergence of the scaled zero-counting measures for all $j \le 2$. We also give, for every $j = 1$, a closed formula for all multiplication coefficients, including the infinite-bandwidth cases. For arbitrary j we give explicit telescoping weights and reduce the matching upper root bound to a uniform estimate on normalized multiplication coefficients. We also formulate the sign and decay pattern suggested by exact computations and, for $j = 2$, identify the triangular system with a Gould–Hsu inversion problem.

For a general degenerate exactly solvable operator we reduce Bergkvist's conjectural Cauchy-transform equation to a superelliptic curve, identify its nonzero branching points, prove moment asymptotics, and obtain an explicit unconditional lower bound for the root growth.

1. Introduction

A linear differential operator

$$T = \sum_{i=1}^{\ell} Q_i(z) D^i \tag{1}$$

is called *exactly solvable* if $\deg Q_i \le i$ for every i, with equality for at least one index; it is *non-degenerate* when $\deg Q_\ell = \ell$ and *degenerate* otherwise. For all sufficiently large n an exactly solvable operator has a unique monic eigenpolynomial p_n^T of degree n.

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In the non-degenerate case the zeros of p_n^T stay in a fixed compact set and their limiting distribution is classical [BR]. In the degenerate case the zeros escape to infinity, and their growth rate and limiting distribution were described conjecturally by Bergkvist [Be, BeBj]; these conjectures are recalled in §2. A parallel algebraic question, going back to the Bochner–Krall problem [BRSh, KL] and posed in the present generality by Horozov–Shapiro–Tater [HST], asks when multiplication by z has finite lower bandwidth in the eigenpolynomial basis. For the family studied here that question was answered completely by Anguas–Barrios Rolanía–Shapiro–Tater [ABST]: finite lower bandwidth occurs precisely for $j = 1$ and $(\ell - m) \mid \ell$, and the associated difference operator then has order ℓ.

We consider the three-parameter family

$$T_{m,\ell,j} = z^m D^\ell + z^j D^j, \qquad 0 \le m < \ell > j \ge 1. \tag{2}$$

These operators are a basic model for degenerate exactly solvable operators and already display the main difficulties of [Be, BeBj]; at the same time their eigenpolynomials have explicit coefficients and strong symmetries, which allows substantially sharper results than are available in general.

Main results. For an arbitrary degenerate exactly solvable operator, §2 turns Bergkvist's conjectural Cauchy-transform equation into a superelliptic equation $\zeta^j = P_T(v)$ (Proposition 1), identifies its nonzero branching points as the critical values of P_T (Corollary 2), proves moment asymptotics (Theorem 3), and deduces an explicit unconditional lower bound $\liminf r_n^T/n^{d_T} \ge L_T > 0$ (Theorem 4), where L_T is the reciprocal radius of convergence of the principal branch. This makes the lower half of Bergkvist's conjecture effective in complete generality.

For the family (2) our main finite-degree result is Theorem 14: *for $j \le 2$ all zeros of $p_n^{(m,\ell,j)}$ lie on the $\ell - m$ rays* $\arg z = (2t+1)\pi/(\ell - m)$. The case $j = 2$ is due to Alexandersson [Al]; the case $j = 1$, which is the case where all the remaining results of the paper live, is new. Both cases follow from one hypergeometric representation (Lemma 13) combined with the Schur–Szegő composition. Together with the weighted Gershgorin bound of Theorem 51 this yields Bergkvist's conjecture in full for every $T_{m,\ell,1}$ with $(\ell - m) \mid \ell$ (Corollary 52), and the weak convergence of the scaled zero-counting measures for all $j \le 2$ (Theorem 39).

Sections 4–6 study the algebraic equation $\alpha z^m \mathcal{C}^\ell + z^j \mathcal{C}^j = 1$: we compute its discriminant and genus, classify the rational cases (Theorem 23), construct the associated probability measure (Theorem 28), and reduce the general m to the Laguerre-like case by $z \mapsto z^{\ell-m}$. The latter is the Raney distribution $\mathcal{R}_{\ell/j,1/j}$, whence the free multiplicative factorisation of Theorem 34. Related Fuss–Catalan limiting zero distributions also arise for Jacobi–Piñeiro multiple orthogonal polynomials [KLS]. Finally, §8 derives all multiplication coefficients for $j = 1$ (Theorem 44) and uses the resulting Hessenberg matrix to obtain sharp root bounds.

Throughout we write

$$c := \ell - m, \qquad d := \frac{\ell - j}{\ell - m}, \qquad g := \gcd(j, \ell), \qquad K_T := \frac{\ell^{\frac{\ell}{j(\ell-m)}}}{j^{\frac{1}{\ell-m}} (\ell - j)^{\frac{\ell-j}{j(\ell-m)}}}.$$

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2. Bergkvist's conjectures for arbitrary degenerate exactly solvable operators

Let r_n^T denote the largest modulus of a zero of p_n^T; Bergkvist proved in [Be] that $r_n^T \to \infty$. For a degenerate operator (1) let j_T be the largest index with $\deg Q_{j_T} = j_T$; then $j_T < \ell$. Set

$$d_T := \max_{i>j_T,\ \deg Q_i<i} \frac{i-j_T}{i-\deg Q_i}, \qquad A_T := \left\{ i : \ \frac{i-j_T}{i-\deg Q_i} = d_T \right\},$$

and normalise T so that Q_{j_T} is monic.

Conjecture A. *For any degenerate operator T there exists a positive number K_T such that*

$$\lim_{n\to\infty} \frac{r_n^T}{n^{d_T}} = K_T \qquad \text{(Claim a)}, \tag{3}$$

and the zero-counting measures of the scaled eigenpolynomials $p_n^T(n^{d_T} z)$ converge weakly to a compactly supported probability measure μ_T (Claim b).

Corollary A (Follows if Conjecture A and some additional statements are valid)**.** *The Cauchy transform $\mathcal{C} = \mathcal{C}_{\mu_T}$ satisfies almost everywhere*

$$z^{\,j_T} \mathcal{C}^{\,j_T} + \sum_{i\in A_T} \alpha_i \, z^{\deg Q_i} \mathcal{C}^{\,i} = 1, \tag{4}$$

where α_i is the leading coefficient of Q_i.

Conjecture A was settled in several special cases in [Be]. For an arbitrary degenerate T a qualitative estimate $\liminf r_n^T/n^{d_T} \geq c_0 > 0$ was announced in [BeBj]; Theorem 4 below makes such a bound explicit. Every instance in which Conjecture A is settled also supports the main conjecture of §3 of [BoSh].

2.1. **A universal substitution.** Put

$$N_i := i - \deg Q_i = \frac{i-j_T}{d_T} \ \ (i \in A_T), \qquad N_T := \gcd_{i\in A_T} N_i, \tag{5}$$

$$P_T(v) := v^{\,j_T} + \sum_{i\in A_T} \alpha_i v^{\,i} \in \mathbb{C}[v]. \tag{6}$$

Proposition 1. *Set $\zeta := z^{-1/d_T}$ and $v := \zeta\, z\mathcal{C}$. Then* (4) *becomes*

$$\zeta^{\,j_T} = P_T(v). \tag{7}$$

Proof. With $u = z\mathcal{C}$ we have $z^{\deg Q_i}\mathcal{C}^{\,i} = z^{\deg Q_i - i} u^i = z^{-N_i} u^i = \zeta^{\,i-j_T} u^i$, so (4) reads $u^{j_T} + \sum_i \alpha_i u^i \zeta^{\,i-j_T} = 1$. Substituting $u = v/\zeta$ turns each term into $\zeta^{-j_T} v^{\,i}$; multiplying by $\zeta^{\,j_T}$ gives (7). □

Thus every equation (4) is, after the change of variable $z = \zeta^{-d_T}$ (a local homeomorphism off $0, \infty$), a cyclic j_T-sheeted cover of the v-line branched over the roots of P_T.

Corollary 2 (Branching points in the general case)**.** *The branching points of the projection of* (4) *to* $\mathbb{C}^\star_z$ *are exactly*

$$z_* = P_T(v_*)^{-d_T/j_T} \qquad \text{(all branches of the root)}, \tag{8}$$

where v_* *runs over the nonzero critical points of* P_T *with* $P_T(v_*) \neq 0$.

Proof. Off $z = 0, \infty$ the branching of v over z is that of v over ζ. Differentiating (7) gives $j_T \zeta^{\,j_T - 1} d\zeta = P_T'(v)\, dv$, so the critical points are the nonzero zeros of P_T', with critical value $\zeta_* = P_T(v_*)^{1/j_T}$, i.e. $z_* = \zeta_*^{-d_T}$. □

For the family (2) one has $P_T(v) = v^j + \alpha v^\ell$, and (8) reproduces Lemma 18 below; in particular $|z_*| = K_T(\alpha)$.

2.2. **Moments and the general lower bound.**

Theorem 3. *Let* T *be an arbitrary degenerate exactly solvable operator normalised as above, and let* $s_k(n)$ *be the* k*-th power sum of the zeros of* p_n^T*. Then for every fixed* $k \geq 0$

$$\lim_{n\to\infty} \frac{s_k(n)}{n^{1+d_T k}} = M_k, \tag{9}$$

where $\mathfrak{M}(Z) = \sum_{k\geq 0} M_k Z^{-k-1}$ *is the unique formal solution of* (4) *with* $\mathfrak{M} = Z^{-1} + O(Z^{-2})$*. Moreover* $M_k = 0$ *unless* $N_T \mid k$.

Proof. Put $L := p_n'/p_n = \sum_{k\geq 0} s_k z^{-k-1}$ (convergent for $|z| > r_n^T$) and $y_i := p_n^{(i)}/p_n = \sum_{\pi\vdash[i]} \prod_{B\in\pi} L^{(|B|-1)}$, the complete Bell polynomials in the derivatives of L. The eigenvalue is $\lambda_n = \sum_i q_{i,i}(n)_i$ with $q_{i,i} = 0$ for $i > j := j_T$ and $q_{j,j} = 1$, so $\lambda_n/n^j \to 1$. Rescale by $z = n^d Z$, $d := d_T$, and put

$$M(Z) := n^{d-1} L(n^d Z) = \sum_{k\geq 0} m_k^{(n)} Z^{-k-1}, \qquad m_k^{(n)} = \frac{s_k}{n^{1+dk}},$$

so that $L^{(t)}(z) = n^{1-d(t+1)} M^{(t)}(Z)$ and $y_i = \sum_{\pi\vdash[i]} n^{|\pi|-di} \prod_{B\in\pi} M^{(|B|-1)}$. Dividing the spectral equation $\sum_i Q_i(z) y_i = \lambda_n$ by n^j, the term $q_{i,a} z^a y_i$ acquires the factor $n^{\,da+|\pi|-di-j}$, and

$$da + |\pi| - di - j \leq (i-j) - d(i-a) \leq 0$$

for every i and every $a \leq \deg Q_i$: for $i \leq j$ because $i - j \leq 0 \leq d(i-a)$, and for $i > j$ because $d \geq \frac{i-j}{i-\deg Q_i}$ by the definition of d_T. Equality forces $|\pi| = i$ and either $i = j$, $a = j$, or $i > j$, $a = \deg Q_i$ and $i \in A_T$. Hence

$$Z^j M^{\,j} + \sum_{i\in A_T} \alpha_i Z^{\deg Q_i} M^{\,i} = \frac{\lambda_n}{n^j} + O(n^{-\delta_0}) \cdot (\cdots) \tag{10}$$

for some $\delta_0 > 0$, the omitted terms having coefficients bounded in n.

We show by induction on k that $m_k^{(n)}$ is bounded and converges to M_k. A product $\prod_{B\in\pi} M^{(|B|-1)}$ with $\pi \vdash [i]$ starts with Z^{-i}, and its coefficient at Z^{-N} is a polynomial in the m_b with $b \leq N - i$. Hence in (10) the coefficient of Z^{-k} involves, for $i > j$, only m_b with $b \leq k + \deg Q_i - i \leq k - 1$; the terms with $i < j$ carry a negative power of n; and the coefficient of m_k coming from $Z^j M^j$ equals $j \neq 0$. Thus

$$\big(j + O(n^{-\delta_0})\big) m_k^{(n)} + F_k\big(m_1^{(n)}, \ldots, m_{k-1}^{(n)}\big) + O(n^{-\delta_0}) G_k\big(m_1^{(n)}, \ldots, m_{k-1}^{(n)}\big) = 0$$

with universal polynomials F_k, G_k, which gives the claim and, in the limit, the recursion determining $\mathfrak{M}$.

Finally, substituting $u = Z\mathfrak{M}$ and $x = Z^{-N_T}$ in (4) gives

$$u^j = 1 - \sum_{i \in A_T} \alpha_i \, u^{\,i} x^{\,n_i}, \qquad n_i := N_i/N_T \in \mathbb{Z}_{>0}, \tag{11}$$

which has a unique solution $u \in \mathbb{C}[[x]]$ with $u(0) = 1$; hence $M_k = 0$ unless $N_T \mid k$. □

Theorem 4. *With the notation above put*

$$L_T := \limsup_{k\to\infty} |M_k|^{1/k}. \tag{12}$$

Then $0 < L_T < \infty$ *and, unconditionally,*

$$\liminf_{n\to\infty} \frac{r_n^T}{n^{d_T}} \;\geq\; L_T. \tag{13}$$

Moreover L_T is the largest modulus of a finite singularity of the branch $\mathfrak{M}$; such a singularity is either a pole of that branch or a branching point from Corollary 2. In particular, if Conjecture A and Corollary A hold for T, then their constant satisfies $K_T = L_T$.

Proof. Since $|s_k(n)| \leq n(r_n^T)^k$, Theorem 3 gives $\liminf_n r_n^T/n^{d_T} \geq |M_k|^{1/k}$ for every k with $M_k \neq 0$; take the lim sup over k. The series u solving (11) is analytic near $x = 0$ by the implicit function theorem (the u-derivative of $u^j - 1 + \sum \alpha_i u^i x^{n_i}$ equals $j \neq 0$ at $(1, 0)$), so $L_T < \infty$. If $L_T = 0$, then u is entire and algebraic, hence a polynomial in x of some degree $D \geq 0$; comparing degrees in (11), the left side has degree jD while on the right the degrees $n_i + iD$ are strictly increasing in i, so the maximum is attained by the single term $i^* = \max A_T$ and the right side has degree $n_{i^*} + i^*D > jD$, a contradiction. Finally $\limsup_q |M_{N_T q}|^{1/q} = \rho^{-1}$, where ρ is the radius of convergence in x, and $L_T = \rho^{-1/N_T}$; as $|x| = |z|^{-N_T}$, the singularity of u nearest $x = 0$ is the singularity of $\mathfrak{M}$ of largest modulus in the z-plane. Singularities of an algebraic function are branching points or poles. □

Remark 5 (Not every critical value need be visible). Only the critical values reachable by continuation of the principal branch occur in (12). For $T = 2z^3D^5 + 3z^2D^3 + zD$ one has $j_T = 1$, $d_T = 2$, $A_T = \{3, 5\}$, $N_T = 1$ and $P_T(v) = v + 3v^3 + 2v^5$; the two critical-value moduli are $18.5590\ldots$ and $84.1910\ldots$, while the moments $M_1, \ldots, M_5 = -3,\ 25,\ -276,\ 3485,\ -47619$ give, by the ratio test, $L_T = 18.559\ldots$. For the family (2) all critical values have the same modulus, so there $L_T = K_T$.

Remark 6 (A general symmetry). By Theorem 3 all limiting power sums vanish except those of order divisible by N_T. This suggests that in general the support of μ_T is invariant under $z \mapsto e^{2\pi i/N_T} z$; for (2) one has $N_T = \ell - m$, in accordance with Corollary 29.

2.3. **The case of** (2)**.** For $T = T_{m,\ell,j}$ one has $j_T = j$, $d_T = \frac{\ell - j}{\ell - m}$, $A_T = \{\ell\}$, $N_T = \ell - m$, and (4) becomes

$$z^m \mathcal{C}^\ell + z^j \mathcal{C}^j = 1. \tag{14}$$

We shall study the slightly more general equation

$$\alpha z^m \mathcal{C}^\ell + z^j \mathcal{C}^j = 1, \qquad \alpha \in \mathbb{C}^\star. \tag{15}$$

Remark 7 (Normalisation of α). Substituting $z \mapsto \lambda z$, $\mathcal{C} \mapsto \lambda^{-1}\mathcal{C}$ multiplies α by $\lambda^{\ell-m}$. Hence all equations (15) with a given $|\alpha|$ are equivalent up to a rotation of the z-plane, and any $\alpha \neq 0$ can be normalised to $\alpha = -1$, which is the convenient choice in §5. The operator itself corresponds to $\alpha = 1$.

3. Eigenpolynomials of $T_{m,\ell,j}$ and the ray-root theorem

Proposition 8. *For any $0 \le m < \ell > j \ge 1$ and any $n \ge 0$ the monic eigenpolynomial of $T_{m,\ell,j}$ equals*

$$p_n(z) = \sum_{r \ge 0} \gamma_r^{(n,m,\ell,j)}\, z^{\,n-(\ell-m)r}, \tag{16}$$

where $\gamma_0 = 1$, the sum runs over $0 \le r \le \lfloor \frac{n-m}{\ell-m} \rfloor$ and

$$\gamma_r^{(n,m,\ell,j)} = \prod_{s=1}^{r} \frac{\big(n-(s-1)(\ell-m)\big)_\ell}{(n)_j - \big(n-s(\ell-m)\big)_j} > 0, \tag{17}$$

$(N)_i := N(N-1)\cdots(N-i+1)$ denoting the falling factorial.

Proof. The eigenvalue of T on a monic polynomial of degree n is $\lambda_n = (n)_j$. Writing $p_n = \sum_r \gamma_r z^{n-cr}$ with $c = \ell - m$ and comparing the coefficients of z^{n-cr} in $z^m p_n^{(\ell)} + z^j p_n^{(j)} = (n)_j p_n$ gives

$$\gamma_{r-1}\big(n-(r-1)c\big)_\ell + \gamma_r\big(n-rc\big)_j = (n)_j \gamma_r,$$

which is (17); the exponents $n - cr$ are exactly those congruent to n modulo c, and positivity is clear since $(n)_j - (n-sc)_j > 0$. □

Formula (17) shows that $\{p_n\}$ splits into $c = \ell - m$ subsequences: writing $n = kc + i$ with $0 \le i < c$,

$$p_{kc+i}(z) = z^{\,i}\, Q_k^{[i]}(z^{\,c}), \tag{18}$$

where $Q_k^{[i]}$ is monic of degree k with nonnegative coefficients. We shall also use the relation, immediate from (17),

$$\gamma_r^{(n,m+1,\ell+1,j)} = \gamma_r^{(n,m,\ell,j)} \prod_{s=1}^{r} \big(n-\ell-(s-1)c\big), \tag{19}$$

in which the factor on the right does not depend on j.

3.1. The ray-root theorem.

Conjecture 9 (Ray-root conjecture)**.** *For all $0 \le m < \ell > j \ge 1$ and all k, i the polynomial $Q_k^{[i]}(\tau)$ is real-rooted with all roots in $(-\infty, 0]$. Equivalently, all zeros of $p_n^{(m,\ell,j)}$ lie on the union of the $\ell - m$ rays $\arg z = \frac{(2t+1)\pi}{\ell-m}$, $t = 0, \dots, \ell-m-1$.*

This is a natural finite-n strengthening of Claim b): the conjectured rays are exactly the rays supporting the limiting measure of §5. We prove it for all $j \le 2$ (Theorem 14) and for $c = \ell - m = 1$ (Theorem 12); it remains open for $j \ge 3$ and $c \ge 2$.

Both proofs use the Schur–Szegő composition of two polynomials of degree N,

$$P \star Q(x) := \sum_{r=0}^{N} \frac{a_r b_r}{\binom{N}{r}} x^r, \qquad P = \sum a_r x^r, \quad Q = \sum b_r x^r,$$

and the following classical facts; see [RS, Sz2, KS].

Proposition 10 (Theorem 5.5.5 and Corollary 5.5.10 of [RS])**.** *If P is real-rooted and Q is real-rooted with all roots of the same sign, then $P \star Q$ is real-rooted; moreover all roots of $P \star Q$ lie in $[-M, -\varkappa]$, where M and $\varkappa$ are the maximal and the minimal pairwise product of roots of P and of Q.*

Theorem 11 (Generalized Malo–Schur–Szegő composition theorem, Theorem 2.5 of [CC])**.** *Let $A(z) = \sum_{r=0}^{M} a_r z^r$ and $B(z) = \sum_{r=0}^{N} b_r z^r$ with $a_M b_N \neq 0$, and let $C(z) = \sum_{r=0}^{\min(M,N)} r!\, a_r b_r z^r$. If all zeros of A lie in a sector S_α ($\alpha \leq \pi$) and all zeros of B lie in a sector S_β ($\beta \leq \pi$), then all zeros of C lie in $-S_\alpha S_\beta$. In particular, if A and B have only non-positive zeros, so does C.*

Theorem 12. *Let $m = \ell - 1$, i.e. $c = 1$. Then for every $j < \ell$ and every n the eigenpolynomial $p_n^{(\ell-1,\ell,j)} = Q_n^{[0]}$ is real-rooted with all roots non-positive.*

Proof. Fix n and induct on ℓ. By (19) with $c = 1$ the ratio $\gamma_r^{(n,\ell,\ell+1,j)} / \gamma_r^{(n,\ell-1,\ell,j)} = r!\binom{n-\ell}{r}$ does not depend on j, so

$$p_n^{(\ell,\ell+1,j)} = p_n^{(\ell-1,\ell,j)} \star Q_n^{(\ell)}, \qquad Q_n^{(\ell)}(z) = \sum_{r=0}^{n-\ell} r! \binom{n-\ell}{r} \binom{n}{r} z^{\,n-r}.$$

The reversal $U_n^{(M)}(z) = \sum_{r=0}^{M} r!\binom{M}{r}\binom{n}{r} z^r$ with $M = n - \ell$ is obtained from $(1+x)^n$ and $(1+x)^M$ by the operation of Theorem 11, hence has only negative zeros; therefore so does $Q_n^{(\ell)}$, and Proposition 10 propagates real-rootedness from ℓ to $\ell + 1$. For the base of the induction one may take $p_n^{(j-1,j,j)}$, whose coefficients are given by the same formulas and whose zeros lie in $(-1, 0)$ and are simple by Corollary 3 and Theorem 3 of [MS]. (For $j = 1$ the sequence $\{p_n^{(\ell-1,\ell,1)}\}$ is a d-orthogonal polynomial sequence, see e.g. [BchD, Ho], and its real-rootedness also follows from known results.) □

For $c \geq 2$ this argument breaks down, as explained in Remark 15 below. The next lemma provides a different route, which settles all $j \leq 2$.

Fix (m, ℓ, j), write $n = kc + i$ as in (18), and put

$$\nu_i := \max\left(0, \left\lceil \frac{m-i}{c} \right\rceil\right), \qquad K := k - \nu_i, \qquad s := i + c\,\nu_i. \tag{20}$$

Since $\gamma_r = 0$ for $r > \lfloor \frac{n-m}{c} \rfloor$, the polynomial $Q_k^{[i]}$ has the forced factor τ^{ν_i}:

$$Q_k^{[i]}(\tau) = \tau^{\nu_i} R_k^{[i]}(\tau), \qquad \deg R_k^{[i]} = K, \quad R_k^{[i]}(0) \neq 0, \tag{21}$$

and by the minimality of ν_i

$$n = Kc + s, \qquad m \leq s \leq m + c - 1 = \ell - 1. \tag{22}$$

Finally put

$$B_t := \frac{s + c - t}{c} \quad (0 \leq t \leq \ell - 1), \qquad \mathcal{B} := \{B_t : \ t \neq s\}. \tag{23}$$

Then $B_s = 1$, and every B_t is positive because $t \leq \ell - 1 = m + c - 1 \leq s + c - 1$ by (22).

Lemma 13 (Hypergeometric form)**.** *Let $K \geq 0$ and let ρ_r denote the coefficient of τ^r in $R_k^{[i]}(\tau)/R_k^{[i]}(0)$. Then*

$$\frac{\rho_{r+1}}{\rho_r} = \frac{(n)_j - \big(n - c(K-r)\big)_j}{c^{\,\ell}\prod_{t=0}^{\ell-1}(r+B_t)}. \tag{24}$$

Consequently

$$\frac{R_k^{[i]}(\tau)}{R_k^{[i]}(0)} = {}_1F_{\ell-1}\Big(-K;\ \mathcal{B};\ -\frac{\tau}{c^{\,\ell-1}}\Big) = \sum_{r=0}^{K}\binom{K}{r}\frac{1}{\prod_{v\in\mathcal{B}}(v)^{\overline{r}}}\Big(\frac{\tau}{c^{\,\ell-1}}\Big)^r \qquad (j=1), \tag{25}$$

and, with $A := \frac{2s-1}{c}$,

$$\frac{R_k^{[i]}(\tau)}{R_k^{[i]}(0)} = {}_2F_{\ell-1}\left(\begin{matrix}-K,\ K+A\\ \mathcal{B}\end{matrix};\ -\frac{\tau}{c^{\,\ell-2}}\right) \qquad (j=2), \tag{26}$$

where $(a)^{\overline{r}} := a(a+1)\cdots(a+r-1)$.

Proof. By (21) the coefficient of τ^r in $R_k^{[i]}$ equals $\gamma_{K-r}(n)$, so $\rho_{r+1}/\rho_r = \gamma_{K-r-1}(n)/\gamma_{K-r}(n)$, which by (17) is the left-hand side of (24) divided by $\big(n-c(K-r-1)\big)_\ell$. Now $n-c(K-r-1) = c(r+1)+s$ by (22), hence

$$\big(n-c(K-r-1)\big)_\ell = \prod_{t=0}^{\ell-1}\big(c(r+1)+s-t\big) = c^{\,\ell}\prod_{t=0}^{\ell-1}(r+B_t),$$

which gives (24). For $j=1$ the numerator equals $c(K-r)$, and since $B_s = 1$ contributes the factor $r+1$,

$$\frac{\rho_{r+1}}{\rho_r} = \frac{-(r-K)}{c^{\,\ell-1}(r+1)\prod_{v\in\mathcal{B}}(r+v)},$$

which is exactly the coefficient ratio of the right-hand side of (25). For $j=2$ one has $(n)_2 - (n-cq)_2 = cq\,(2n-cq-1)$, so the numerator equals $c(K-r)\big(c(K+r)+2s-1\big)$, and (26) follows in the same way. □

Theorem 14 (Ray-root theorem for $j \leq 2$)**.** *Let $j \in \{1,2\}$. Then for all $0 \leq m < \ell$, all $0 \leq i < c = \ell - m$ and all $k \geq 0$, every zero of $Q_k^{[i]}$ is real and non-positive. Equivalently, all zeros of $p_n^{(m,\ell,j)}$ lie on the $\ell - m$ rays*

$$\arg z = \frac{(2t+1)\pi}{\ell-m}, \qquad 0 \leq t < \ell - m.$$

Proof. If $k < \nu_i$ then $kc + i < m$, so $p_n(z) = z^n$ and there is nothing to prove; assume $k \geq \nu_i$ and keep the notation (20)–(23). It suffices to show that all zeros of $R_k^{[i]}$ are negative, the factor τ^{ν_i} only adding zeros at the origin. For every $v > 0$ put

$$H_{K,v}(\tau) := \sum_{r=0}^{K}\binom{K}{r}\frac{\tau^r}{(v)^{\overline{r}}} = {}_1F_1(-K; v; -\tau) = \frac{K!}{(v)^{\overline{K}}}\,L_K^{(v-1)}(-\tau), \tag{27}$$

a polynomial of degree exactly K all of whose zeros are negative, because the zeros of the Laguerre polynomial $L_K^{(v-1)}$ are positive for $v > 0$. Since $\star$ divides the

product of the coefficients of τ^r by $\binom{K}{r}$, for $v_1, \dots, v_p > 0$ one has

$$H_{K,v_1} \star \cdots \star H_{K,v_p}(\tau) = \sum_{r=0}^{K} \binom{K}{r} \frac{\tau^r}{\prod_{q=1}^{p} (v_q)^{\overline{r}}}. \tag{28}$$

Case $j = 1$. By (25) and (28) applied to the $\ell - 1$ positive parameters $\mathcal{B}$,

$$\frac{R_k^{[i]}\big(c^{\,\ell-1}\tau\big)}{R_k^{[i]}(0)} = \underset{v\in\mathcal{B}}{\star} H_{K,v}(\tau).$$

All the factors are real-rooted with all zeros negative, so Proposition 10, applied $\ell - 2$ times, shows that the left-hand side is real-rooted with all zeros negative; rescaling τ by $c^{\,\ell-1} > 0$ finishes this case.

Case $j = 2$ (Alexandersson [Al]). Single out one parameter $u \in \mathcal{B}$: take $u = B_{\ell-1} = \frac{s-m+1}{c}$ if $s \le \ell - 2$, and $u = B_{\ell-2} = \frac{c+1}{c}$ if $s = \ell - 1$. Put $\alpha = u - 1 > -1$ and $\beta = A - u$; then

$$\beta = \frac{s+m-2}{c} > -1 \ \ (s \le \ell - 2), \qquad \beta = \frac{2m+c-4}{c} > -1 \ \ (s = \ell - 1),$$

the strict inequalities following from $s \ge m$ and $m + c = \ell > 2$. Hence

$$F_K(\tau) := {}_2F_1\left(\begin{matrix} -K,\ K+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; -\frac{\tau}{c^{\,\ell-2}}\right) = \frac{K!}{(\alpha+1)^{\overline{K}}} P_K^{(\alpha,\beta)}\left(1 + \frac{2\tau}{c^{\,\ell-2}}\right)$$

has all its zeros in $(-c^{\,\ell-2}, 0)$. Composing F_K successively with $H_{K,v}$ for all $v \in \mathcal{B}\setminus\{u\}$ multiplies the coefficient of τ^r by $1/(v)^{\overline{r}}$ and therefore produces exactly the right-hand side of (26); Proposition 10 again gives the assertion.

Finally, z^c is a negative real number exactly on the c displayed rays, so the statement about p_n follows from (18). □

Remark 15. For $j \ge 3$ the numerator of (24) no longer splits into real linear factors: for $j = 3$ one has $(n)_3 - (n - cs)_3 = cs\big(c^2s^2 - 3c(n-1)s + 3n^2 - 6n + 2\big)$, whose quadratic factor has discriminant $-c^4(3n^2 - 6n - 1) < 0$ for $n \ge 2$. Thus $R_k^{[i]}$ is a ${}_jF_{\ell-1}$ with non-real numerator parameters, and no classical real-rooted building block is available. The route used in Theorem 12 also fails for $c \ge 2$: by Proposition 16 below it would require real-rootedness of a truncated binomial series $\sum_{r\le k}\binom{\beta}{r}x^r$ with non-integer $\beta > 0$, which is false in general.

3.2. **Two structural identities in the case** $j = 1$**.** The case $j = 1$ is exceptional in several further respects, the reason being that the denominators in (17) degenerate: $(n)_1 - (n - sc)_1 = sc$, so that

$$\gamma_r^{(n,m,\ell,1)} = \frac{\Pi_r(n)}{r!\,c^{\,r}}, \qquad \Pi_r(n) := \prod_{s=0}^{r-1} \big(n - sc\big)_\ell. \tag{29}$$

It is convenient to encode p_n by the reversed generating polynomial

$$f_n(w) := \sum_{r\ge 0} \gamma_r^{(n,m,\ell,1)} w^{\,r}, \qquad \text{so that} \qquad p_n(z) = z^{\,n} f_n\big(z^{-c}\big). \tag{30}$$

Proposition 16. *Let* $j = 1$ *and* $\beta_i := \frac{n-i}{c}$, $i = 0, \dots, \ell - 1$. *Then*

$$f_n(w) = \sum_{r\ge 0} \frac{x^r}{r!} \prod_{i=0}^{\ell-1} (\beta_i)_r = \sum_{r\ge 0} (r!)^{\ell-1} \prod_{i=0}^{\ell-1} \binom{\beta_i}{r} x^r, \qquad x := c^{\,\ell-1} w,$$

f_n is annihilated by $c\,\theta - w\prod_{i=0}^{\ell-1}(n-i-c\theta)$ with $\theta = w\frac{d}{dw}$, and the series terminates at $r = k = \lfloor n/c \rfloor$.

Proof. By (29), $\Pi_r(n) = \prod_{s<r}\prod_i (n - i - sc) = c^{\ell r}\prod_i (\beta_i)_r$, whence the two displayed formulas. Since $\gamma_r/\gamma_{r-1} = \frac{(n-(r-1)c)_\ell}{rc}$, multiplying by $rc\,w^r$ and summing gives $c\,\theta f_n = w\prod_i (n - i - c\theta) f_n$. Finally, if $n = kc + i_0$ with $0 \le i_0 < c$, then $\beta_{i_0} = k \in \mathbb{Z}_{\ge 0}$, so $(\beta_{i_0})_r = 0$ for $r > k$. □

Proposition 17 (Lowering relation)**.** *Let $j = 1$. Then for every $n \ge 0$*

$$(n+c)_\ell\, p_n(z) = \Big((n+c) - z\frac{d}{dz}\Big) p_{n+c}(z), \tag{31}$$

equivalently $f_n = \frac{c}{(n+c)_\ell} f'_{n+c}$.

Proof. By (29), $\Pi_r(n) = (n)_\ell \Pi_{r-1}(n - c)$, hence $\gamma_r^{(n)} = \frac{(n)_\ell}{rc}\gamma_{r-1}^{(n-c)}$ and $f'_n = \frac{(n)_\ell}{c} f_{n-c}$. Putting $w = z^{-c}$ and using $\theta_w = -\frac{1}{c}\theta_z$ gives $p_n(z) = -\frac{z^{n+c}}{(n+c)_\ell}\theta_z\big[z^{-(n+c)} p_{n+c}(z)\big]$, which is (31). □

4. The curve $\alpha z^m \mathcal{C}^\ell + z^j \mathcal{C}^j = 1$

Throughout this section $\alpha \in \mathbb{C}^\star$, $0 \le m < \ell > j \ge 1$ and

$$P(z,\mathcal{C}) := \alpha z^m \mathcal{C}^\ell + z^j \mathcal{C}^j - 1, \qquad \Gamma_\alpha := \{P = 0\} \subset \mathbb{C}^2,$$

with $\pi : \Gamma_\alpha \to \mathbb{C}_z$ the projection, a degree ℓ covering away from $z = 0$.

4.1. Branching points and discriminant.

Lemma 18. *The branching points of π different from the origin are the solutions of*

$$z^{\frac{j(\ell-m)}{\ell-j}} = \frac{\ell}{\ell-j}\left(\frac{-j}{\alpha\ell}\right)^{-\frac{j}{\ell-j}}. \tag{32}$$

All of them have absolute value

$$K_T(\alpha) = \frac{|\alpha|^{1/(\ell-m)}\,\ell^{\ell/(j(\ell-m))}}{j^{1/(\ell-m)}(\ell-j)^{(\ell-j)/(j(\ell-m))}}$$

(for $|\alpha| = 1$ this is the constant K_T of Conjecture A).

Proof. The branching points are given by $P = \partial_{\mathcal{C}} P = 0$. Neither $z = 0$ nor $\mathcal{C} = 0$ satisfies $P = 0$, so we may cancel $z^j\mathcal{C}^{j-1}$ in the second equation and obtain $\mathcal{C}^{\ell-j} = A z^{\,j-m}$ with $A := \frac{-j}{\alpha\ell}$. Substituting this into $P = 0$, both terms acquire the same power $z^{\frac{j(\ell-m)}{\ell-j}}$ with total coefficient $A^{\frac{j}{\ell-j}}(\alpha A + 1) = A^{\frac{j}{\ell-j}}\frac{\ell-j}{\ell}$, which gives (32). Taking absolute values and using $\frac{\ell-j}{j(\ell-m)} + \frac{1}{\ell-m} = \frac{\ell}{j(\ell-m)}$ gives the stated modulus. □

Proposition 19. *Put $g = \gcd(j,\ell)$, $A = \frac{-j}{\alpha\ell}$ and $\kappa := \left(\frac{\ell-j}{\ell}\right)^{\frac{\ell-j}{g}} A^{\frac{j}{g}}$. Then*

$$\operatorname{Disc}_{\mathcal{C}}\big(P(z,\mathcal{C})\big) = (-1)^{\frac{(\ell-1)(\ell+2)}{2}}\alpha^{\ell-1}\ell^\ell\, z^{\,m(\ell-1)}\left(1 - \kappa z^{\frac{j(\ell-m)}{g}}\right)^g. \tag{33}$$

Proof. Write $Q := \partial_{\mathcal{C}} P = \mathcal{C}^{j-1}\big(\alpha\ell z^m \mathcal{C}^{\ell-j} + j z^j\big)$, of degree $\ell-1$ in $\mathcal{C}$ with leading coefficient $\alpha\ell z^m$; its roots are $\mathcal{C}=0$ with multiplicity $j-1$ and the $\ell-j$ numbers β_t with $\beta_t^{\ell-j} = A z^{j-m}$. Then

$$\operatorname{Res}_{\mathcal{C}}(P,Q) = (\alpha\ell z^m)^{\ell} P(0)^{j-1} \prod_t P(\beta_t), \qquad P(0) = -1,$$

and, as in the previous proof, $P(\beta_t) = \frac{\ell-j}{\ell} z^j \beta_t^j - 1 = \lambda \beta_t^j - 1$ with $\lambda = \frac{\ell-j}{\ell} z^j$. Since $\gcd(j, \ell-j) = g$, the numbers β_t^j run g times over the $\frac{\ell-j}{g}$ roots of $X^{\frac{\ell-j}{g}} = (A z^{j-m})^{\frac{j}{g}}$, so

$$\prod_t P(\beta_t) = (-1)^{\ell-j}\Big[1 - \lambda^{\frac{\ell-j}{g}} (A z^{j-m})^{\frac{j}{g}}\Big]^g = (-1)^{\ell-j}\big[1 - \kappa z^{\frac{j(\ell-m)}{g}}\big]^g.$$

Now use $\operatorname{Disc}_{\mathcal{C}}(P) = \frac{(-1)^{\ell(\ell-1)/2}}{\alpha z^m} \operatorname{Res}_{\mathcal{C}}(P, \partial_{\mathcal{C}} P)$. □

Lemma 20. *With multiplicities understood as multiplicities of roots of* $\operatorname{Disc}_{\mathcal{C}}(P)$*:* (i) *the branching points are the solutions of* (32) *together with the origin when* $m > 0$*;* (ii) *their total multiplicity is* $j\ell + m(\ell - j - 1)$*;* (iii) *each nonzero branching point has multiplicity* g*, there are* $\frac{j(\ell-m)}{g}$ *of them and they are the vertices of a regular* $\frac{j(\ell-m)}{g}$*-gon inscribed in* $|z| = K_T(\alpha)$*;* (iv) *the multiplicity of the origin is* $m(\ell-1)$*. Moreover over each nonzero branching point there lie exactly* g *simple critical points of* π*.*

Proof. Everything except the last claim is read off from (33), the degree count being $m(\ell-1) + g \cdot \frac{j(\ell-m)}{g} = j\ell + m(\ell-j-1)$. For the last claim, by the proof of Proposition 19 the critical points of π over $z_0 \neq 0$ are those β_t with $\frac{\ell-j}{\ell} z_0^j \beta_t^j = 1$; since the $\ell-j$ numbers β_t^j take $\frac{\ell-j}{g}$ distinct values, each attained g times, exactly g of them qualify, and the total multiplicity g forces each to be simple. □

4.2. **Rational parametrisation and genus.**

Proposition 21. *Set* $u := z\mathcal{C}$*. Then on* $\Gamma_\alpha \cap \{z \neq 0\}$

$$z^{\,\ell-m} = R(u), \qquad R(u) := \frac{\alpha u^\ell}{1-u^j}, \qquad \mathcal{C} = \frac{u}{z}. \tag{34}$$

Proof. Multiply (15) by $z^{\ell-m}$ and use $z^j \mathcal{C}^j = u^j$, $z^m \mathcal{C}^\ell = u^\ell z^{m-\ell}$. □

The change of variables $(z, \mathcal{C}) \mapsto (z, u)$ is birational, so Γ_α is birationally the curve

$$X_\alpha : \quad z^{\,c} = R(u), \qquad c = \ell - m, \tag{35}$$

a cyclic c-sheeted cover of the u-line branched over the zeros and poles of R. Its deck group $\mathbb{Z}_c$ acts by

$$(z, \mathcal{C}) \mapsto (\varepsilon z, \varepsilon^{-1}\mathcal{C}), \qquad \varepsilon^{\,c} = 1, \tag{36}$$

which is the source of the $(\ell-m)$-fold symmetry of all pictures below. For $m = \ell - 1$ the curve is rational, parametrised by $z = R(u)$.

Theorem 22. *For all* $0 \le m < \ell > j \ge 1$ *and all* $\alpha \in \mathbb{C}^\star$ *the curve* Γ_α *is irreducible and the genus of its normalisation equals*

$$\mathfrak{g} = 1 + \frac{j(\ell-m-1) - \gcd(m,\ell) - \gcd(\ell-j, \ell-m)}{2}. \tag{37}$$

Proof. The divisor of R on $\mathbb{CP}^1$ is $\ell\cdot[0]-\sum_{\omega^j=1}[\omega]-(\ell-j)\cdot[\infty]$. All poles being simple, R is not a p-th power for any prime p, so the cyclic cover (35) is irreducible. Over a point where R has a zero or pole of order e there are $\gcd(e,c)$ preimages of ramification index $c/\gcd(e,c)$, so the total ramification of $X_\alpha\to\mathbb{CP}^1_u$ equals

$$\mathcal{R}=\big(c-\gcd(\ell,c)\big)+j(c-1)+\big(c-\gcd(\ell-j,c)\big),$$

the three groups coming from $u=0$, the j simple poles, and $u=\infty$. Since $\gcd(\ell,c)=\gcd(m,\ell)$, Riemann–Hurwitz $2\mathfrak{g}-2=-2c+\mathcal{R}$ gives (37). □

Theorem 23 (Classification of the rational cases)**.** *Γ_α is rational if and only if one of the following holds:* (a) *$c=\ell-m=1$;* (b) *$j=1$ and $c\mid\ell$ or $c\mid(\ell-1)$;* (c) *$j=2$, $c=2$ and ℓ is even.*

Proof. By (37), $\mathfrak{g}=0$ reads

$$j(c-1)+2=\gcd(\ell,c)+\gcd(\ell-j,c). \tag{38}$$

For $c=1$ this is automatic. Let $c\ge 2$; since the right-hand side is at most $2c$, (38) forces $j\le 2$. If $j=2$, both gcds must equal c, so $c\mid\ell$ and $c\mid(\ell-2)$, whence $c=2$ and ℓ even; the converse is clear. If $j=1$, put $a=\gcd(\ell,c)$, $b=\gcd(\ell-1,c)$, so that $a+b=c+1$. These divisors of c are coprime, hence $ab\mid c$ and $a+b=c+1\ge ab+1$, i.e. $(a-1)(b-1)\le 0$. Thus $a=1$ or $b=1$, i.e. $b=c$ or $a=c$, i.e. $c\mid(\ell-1)$ or $c\mid\ell$; the converse again follows from (38). □

Remark 24. For $j=1$, rationality of Γ_α is strictly weaker than the existence of a finite recurrence (Theorem 45): $c\mid\ell$ gives both, whereas $c\mid(\ell-1)$, $c\nmid\ell$ gives a rational curve and an infinite recurrence. The smallest such example is $(m,\ell,j)=(3,5,1)$. For $(m,\ell,j)=(1,4,2)$ formula (37) gives $\mathfrak{g}=2$.

Lemma 25. *Over $z=0$ the ℓ Puiseux branches of Γ_α are $\mathcal{C}\sim c_0 z^{-m/\ell}$ with $\alpha c_0^{\ell}=1$, forming $\gcd(m,\ell)$ cycles of length $\ell/\gcd(m,\ell)$. Over $z=\infty$ there are j unramified branches $\mathcal{C}\sim c_1/z$ with $c_1^{j}=1$, and $\ell-j$ branches $\mathcal{C}\sim c_2 z^{-\frac{m-j}{\ell-j}}$ with $\alpha c_2^{\ell-j}=-1$, forming $\gcd(\ell-j,c)$ cycles of length $\frac{\ell-j}{\gcd(\ell-j,c)}$. The resulting ramification data*

$$\big(\ell-\gcd(m,\ell)\big)+\big((\ell-j)-\gcd(\ell-j,c)\big)+jc$$

reproduce (37) *through Riemann–Hurwitz.*

Proof. Substituting $\mathcal{C}=cz^{-\rho}$ into $P=0$, the three monomials have orders $m-\ell\rho$, $j-j\rho$, 0. Near $z=0$ the dominant balance must involve the constant term and, since $m<\ell$, the only admissible one is $\rho=m/\ell$, $\alpha c^{\ell}=1$. Near $z=\infty$ the two balances are $\rho=1$, $c^{j}=1$ and $\rho=\frac{m-j}{\ell-j}$, $\alpha c^{\ell-j}=-1$. The cycle lengths are the denominators of the exponents in lowest terms, and $\gcd(m-j,\ell-j)=\gcd(\ell-j,c)$; the first group at infinity is unramified because in (34) $z\to\infty$ with $u\to\omega$, $\omega^j=1$, gives $u-\omega=O(z^{-c})$. The finite nonzero branching points contribute jc by Lemma 20. □

5. The measure attached to $\alpha z^m\mathcal{C}^\ell+z^j\mathcal{C}^j=1$

Recall that the Cauchy transform of a compactly supported measure μ is $\mathcal{C}_\mu(z)=\int\frac{d\mu(\zeta)}{z-\zeta}\in L^1_{loc}(\mathbb{C})$, with $\partial_{\bar z}\mathcal{C}_\mu=\pi\mu$ in $\mathcal{D}'(\mathbb{C})$. By Remark 7 we assume $\alpha=-1$ until further notice, so that (15) reads

$$-z^m\mathcal{C}^\ell+z^j\mathcal{C}^j=1, \tag{39}$$

equivalently, by (34),

$$z^{\,c} = R(u) := \frac{u^\ell}{u^j - 1}, \qquad u = z\mathcal{C}, \quad c = \ell - m. \tag{40}$$

Set

$$z_{cr} := \frac{\ell^{\ell/j}}{j(\ell-j)^{(\ell-j)/j}}, \quad K := z_{cr}^{1/c} = K_T, \quad \varepsilon := e^{2\pi i/c}, \quad \Sigma := \bigcup_{t=0}^{c-1} \big[0, \varepsilon^{\,t} K\big]. \tag{41}$$

5.1. **The one-leg (Laguerre-like) case.**

Lemma 26. *Let $\ell > j \geq 1$, $R(u) = \frac{u^\ell}{u^j-1}$, and for $\theta \in (0, \pi/\ell)$ put $\rho(\theta) := \Big(\frac{\sin \ell\theta}{\sin(\ell-j)\theta}\Big)^{1/j} > 0$, extended by $\rho(-\theta) = \rho(\theta)$, $\rho(0) = u_0 := \Big(\frac{\ell}{\ell-j}\Big)^{1/j}$ and $\rho(\pm\pi/\ell) = 0$. Then:*

(a) *$\mathcal{J} := \{\rho(\theta)e^{i\theta} : |\theta| \leq \pi/\ell\}$ is a Jordan curve whose interior U contains $u = 1$;*

(b) *for $\theta \in (0, \pi/\ell)$,*

$$R\big(\rho(\theta)e^{i\theta}\big) = t(\theta) := \frac{(\sin \ell\theta)^{\ell/j}}{\sin(j\theta)\big(\sin(\ell-j)\theta\big)^{(\ell-j)/j}} \in (0, z_{cr}), \tag{42}$$

and t is a decreasing homeomorphism of $(0, \pi/\ell)$ onto $(0, z_{cr})$; in particular $R(\mathcal{J}) = [0, z_{cr}]$;

(c) *R maps $U \cap R^{-1}(W)$ biholomorphically onto $W := \widehat{\mathbb{C}} \setminus [0, z_{cr}]$.*

Proof. (a) On $(0, \pi/\ell)$ both $\sin \ell\theta$ and $\sin(\ell-j)\theta$ are positive, so ρ is positive and continuous, with the stated limits at the endpoints. Hence $\mathcal{J}$ is the polar graph of a positive continuous function on an interval of length $2\pi/\ell \leq \pi$, closed up at $u = 0$; it is therefore a Jordan curve bounding $U = \{re^{i\varphi} : |\varphi| < \pi/\ell,\ 0 \leq r < \rho(\varphi)\}$, and $u_0 > 1$ gives $1 \in U$.

(b) Write $S_a := \sin(a\theta)$ and $u = \rho e^{i\theta}$. From $\rho^j = S_\ell / S_{\ell-j}$ and the identities $\sin \ell\theta \cos j\theta = \frac{1}{2}[\sin(\ell+j)\theta + S_{\ell-j}]$, $\frac{1}{2}[\sin(\ell+j)\theta - S_{\ell-j}] = \cos \ell\theta\, S_j$ we get

$$\mathrm{Im}(u^j - 1) = \frac{S_\ell S_j}{S_{\ell-j}}, \qquad \mathrm{Re}(u^j - 1) = \frac{\cos(\ell\theta) S_j}{S_{\ell-j}}, \qquad \text{i.e.} \qquad u^j - 1 = \frac{S_j}{S_{\ell-j}} e^{i\ell\theta}.$$

Hence $R(u) = \rho^\ell \frac{S_{\ell-j}}{S_j} = t(\theta) > 0$, which is (42), with $t \to z_{cr}$ as $\theta \to 0^+$ and $t \to 0$ as $\theta \to (\pi/\ell)^-$. Monotonicity: $t'(\theta) = R'(\gamma)\gamma'$ with $\gamma(\theta) = \rho e^{i\theta}$, $\gamma' \neq 0$, and the critical points of R in $\mathbb{C}^\star$ are the j numbers $u_0\omega$, $\omega^j = 1$, whose arguments $\frac{2\pi k}{j}$ avoid $(0, \pi/\ell)$ because $2\ell > j$.

(c) Since $R(\partial U) \subset [0, z_{cr}]$, the winding number of $R \circ \partial U$ around any $w \notin [0, z_{cr}]$ vanishes, so by the argument principle the number of solutions of $R = w$ in U equals the number of poles of R in U, which is one ($u = 1$; the other j-th roots of unity have argument outside $(-\pi/\ell, \pi/\ell)$). The same holds for $w = \infty$. □

Theorem 27. *Let $\ell > j \geq 1$ and let ν be the measure on $[0, z_{cr}]$ with density*

$$\varrho_\nu\big(t(\theta)\big) = \frac{\sin\theta\, \sin(j\theta)\big(\sin(\ell-j)\theta\big)^{\frac{\ell-j-1}{j}}}{\pi\big(\sin \ell\theta\big)^{\frac{\ell-1}{j}}}, \qquad 0 < \theta < \frac{\pi}{\ell}, \tag{43}$$

$t(\theta)$ as in (42). Then ν is a probability measure, and $\mathcal{C}_\nu$ is the unique function holomorphic on $\widehat{\mathbb{C}} \setminus [0, z_{cr}]$ with $\mathcal{C}_\nu(z) = \frac{1}{z} + O(z^{-2})$ at infinity satisfying $-z^{\ell-1}\mathcal{C}_\nu^\ell +$

$z^j\mathcal{C}_\nu^j = 1$. *Moreover* $\varrho_\nu(t) \sim C_0 t^{-1+\frac{1}{\ell}}$ *as* $t \to 0^+$ *and* $\varrho_\nu(t) \sim C_1\sqrt{z_{cr}-t}$ *as* $t \to z_{cr}^-$, *with* $C_0, C_1 > 0$.

Proof. Let $u : W \to U \cap R^{-1}(W)$ be the inverse of the biholomorphism of Lemma 26(c) and put $\mathcal{C}(z) := u(z)/z$. The relation $u^\ell = z(u^j - 1)$ divided by z^ℓ is exactly $-z^{\ell-1}\mathcal{C}^\ell + z^j\mathcal{C}^j = 1$, and $u(\infty) = 1$ gives $\mathcal{C} = \frac{1}{z} + O(z^{-2})$. Both R and U are invariant under $u \mapsto \bar{u}$, so u commutes with conjugation; since $u(z) = 1 + \frac{1}{jz} + O(z^{-2})$, we get $u(\mathbb{H}^+) \subset \mathbb{H}^-$ and therefore $u_+(t) = \rho(\theta)e^{-i\theta}$ for $t = t(\theta)$. Hence

$$-\frac{1}{\pi}\operatorname{Im}\mathcal{C}_+(t) = \frac{\rho(\theta)\sin\theta}{\pi t(\theta)} = \varrho_\nu(t) > 0.$$

Endpoint asymptotics: as $\theta \to (\pi/\ell)^-$ one has $S_\ell \to 0$ linearly, so $t \asymp S_\ell^{\ell/j}$ and $\varrho_\nu \asymp t^{-\frac{\ell-1}{\ell}}$, which is integrable; as $\theta \to 0^+$, t extends to an even real-analytic function, so $z_{cr} - t \asymp \theta^2$ while $\varrho_\nu \asymp \theta$.

Both $\mathcal{C}$ and $\mathcal{C}_\nu$ are holomorphic on W, vanish at ∞ and, by Sokhotski–Plemelj, have the same jump across $(0, z_{cr})$; hence $\mathcal{C} - \mathcal{C}_\nu$ extends holomorphically across it. Near z_{cr} both are bounded, and near $z = 0$ we have $u(z) = O(|z|^{1/\ell})$ by (40), so $\mathcal{C}(z) = o(|z|^{-1})$, and likewise $\mathcal{C}_\nu(z) = o(|z|^{-1})$ because $\varrho_\nu \in L^1$. By Liouville $\mathcal{C} \equiv \mathcal{C}_\nu$, and $\int_0^{z_{cr}} \varrho_\nu = \lim_{z\to\infty} z\mathcal{C}(z) = 1$. Uniqueness holds because any such solution is a branch of the algebraic function (40), and the normalisation singles out the branch with $u(\infty) = 1$ among the ℓ branches of Lemma 25. □

5.2. **The general case.**

Theorem 28. *For any* $\alpha \in \mathbb{C}^\star$ *and any* $0 \le m < \ell > j \ge 1$ *there is a probability measure* μ *whose Cauchy transform satisfies* (15) *a.e. For* $\alpha = -1$ *it is*

$$\mu = \frac{1}{c}\sum_{t=0}^{c-1}(\sigma_t)_*\nu, \qquad \sigma_t(s) := \varepsilon^{\,t}s^{1/c}, \tag{44}$$

with ν *as in Theorem 27. Its support is the star* Σ *of* (41), *i.e.* $\ell - m$ *segments joining the origin to every* $\frac{j}{g}$*-th branching point of* (15), *see Figures 1–2; its density with respect to arclength on the leg* $\{s\varepsilon^t : 0 < s < K\}$ *equals*

$$\varrho_\mu(s) = s^{\,c-1}\varrho_\nu(s^{\,c}). \tag{45}$$

Finally μ *is the unique probability measure supported on the union of the* c *rays* $\{s\varepsilon^t : s \ge 0\}$ *whose Cauchy transform satisfies* (15) *a.e.; for general* α *the measure is the push-forward of* (44) *under* $z \mapsto \lambda z$ *with* $\lambda^{\ell-m} = -\alpha$.

Proof. Let $\alpha = -1$. The measure (44) is a probability measure on Σ, and (45) follows by the change of variables $s \mapsto s^{1/c}$. For $|z| > K$, the identity $\frac{1}{c}\sum_t \frac{1}{z-\varepsilon^t w} = \frac{z^{c-1}}{z^c - w^c}$ gives

$$\mathcal{C}_\mu(z) = z^{\,c-1}\int_0^{z_{cr}} \frac{d\nu(s)}{z^{\,c} - s} = z^{\,c-1}\mathcal{C}_\nu\big(z^{\,c}\big),$$

and both sides are holomorphic on $\mathbb{C}\setminus\Sigma$, since $z \mapsto z^c$ maps $\mathbb{C}\setminus\Sigma$ into $\mathbb{C}\setminus[0, z_{cr}]$. Writing $u = z\mathcal{C}_\mu(z) = z^c\mathcal{C}_\nu(z^c)$ and using Theorem 27 in the form $u_\nu^j = 1 + w^{-1}u_\nu^\ell$ with $w = z^c$, we obtain $u^j = 1 + z^{-c}u^\ell$, which is (39).

The support equals Σ because $\varrho_\nu > 0$ on $(0, z_{cr})$. For $\alpha = -1$ one has $A = \frac{j}{\ell} > 0$, hence $\kappa > 0$, and by Lemma 20(iii) the nonzero branching points are $K \exp\left(\frac{2\pi i \tau g}{jc}\right)$; the endpoints $\varepsilon^t K$ of the legs correspond to $\tau = \frac{tj}{g}$, i.e. to every $\frac{j}{g}$-th of them.

Uniqueness: let $\tilde\mu$ be supported on the union of the c rays, hence on $\Sigma_M = \bigcup_t [0, \varepsilon^t M]$ for some $M \geq K$, with $\mathcal{C}_{\tilde\mu}$ satisfying (39) a.e. On the simply connected domain $\widehat{\mathbb{C}} \setminus \Sigma_M$ both $\mathcal{C}_{\tilde\mu}$ and $\mathcal{C}_\mu$ are holomorphic and equal $\frac{1}{z} + O(z^{-2})$ at infinity; by the uniqueness statement in Theorem 27 they agree near ∞, hence everywhere there. As Σ_M has zero area, $\mathcal{C}_{\tilde\mu} = \mathcal{C}_\mu$ a.e., and applying $\partial_{\bar z}$ gives $\tilde\mu = \mu$. □

Corollary 29. *The support of the measure satisfying* (15) *consists of $\ell - m$ legs of equal length K_T at angles $\frac{2\pi}{\ell - m}$, and all its moments vanish except those of order divisible by $\ell - m$.*

Remark 30 (Uniqueness without a support restriction). Let $\tilde\mu$ be an arbitrary compactly supported probability measure with $\mathcal{C}_{\tilde\mu}$ satisfying (15) a.e. Away from the discriminant locus $\mathcal{C}_{\tilde\mu}$ coincides a.e. with a locally finite patchwork $\sum_i \chi_{A_i} \mathcal{C}_i$ of the ℓ local branches, so $\pi\tilde\mu = \sum_i \mathcal{C}_i \partial_{\bar z} \chi_{A_i}$; since $\tilde\mu \geq 0$, each A_i has locally finite perimeter and $\tilde\mu$ lives on a rectifiable set of zero area. On a C^1-arc along which $\mathcal{C}_{\tilde\mu}$ jumps from $\mathcal{C}_1$ to $\mathcal{C}_2$, positivity forces $\operatorname{Re}[(\mathcal{C}_1 - \mathcal{C}_2)dz] = 0$, i.e. $\operatorname{supp}\tilde\mu$ is a union of trajectories of the corresponding quadratic differential, terminating only at the branching points of Lemma 20; moreover $\tilde\mu$ has the same moments as μ by Lemma 31. We expect that a classification of the admissible trajectory configurations, in the spirit of [BoSh], shows that Σ is the only one, but we do not pursue this here.

6. Moments and densities: Raney and Fuss–Catalan distributions

Throughout this section $\alpha = -1$, $c = \ell - m$, and ν, μ are as in Theorems 27 and 28. Formula (44) reduces the three-parameter family to the Laguerre-like one: writing $\mu_{m,\ell,j}$ for the measure of Theorem 28,

$$(z \mapsto z^{\ell-m})_* \, \mu_{m,\ell,j} = \mu_{\ell-1,\ell,j} = \nu, \qquad \mathcal{C}_{\mu_{m,\ell,j}}(z) = z^{\,c-1} \mathcal{C}_\nu\big(z^{\,c}\big), \tag{46}$$

so that $\mu_{m,\ell,j}$ depends on m only through $\ell - m$.

Lemma 31. *Let $\alpha = -1$ and let $\mathfrak{M}(z) = \sum_{s \geq 0} M_s z^{-s-1}$ be the unique formal solution of* (15) *with $\mathfrak{M} = z^{-1} + O(z^{-2})$. Then $M_s = 0$ unless $(\ell - m) \mid s$ and*

$$M_{(\ell-m)q} = \mathcal{R}_q := \frac{1}{q\ell + 1} \binom{\frac{q\ell+1}{j}}{q}, \qquad q \geq 0. \tag{47}$$

For $\alpha = 1$ one has instead $M_{(\ell-m)q} = (-1)^q \mathcal{R}_q$. Consequently these are the moments of any *compactly supported probability measure whose Cauchy transform satisfies* (15) *a.e.*

Proof. In terms of $u := z\mathfrak{M} = \sum_s M_s z^{-s}$ and $x := z^{-(\ell-m)}$ equation (15) with $\alpha = -1$ reads $u^{\,j} = 1 + x u^{\,\ell}$, which has a unique solution in $\mathbb{C}[[x]]$ with constant term 1 and involves only integral powers of x; hence $M_s = 0$ unless $(\ell - m) \mid s$. Put $y = u^j$, so that $y = 1 + x y^p$ with $p = \ell/j$ and $u = y^{1/j}$. Writing $y = 1 + w$, i.e. $w = x(1+w)^p$, the Lagrange–Bürmann formula gives $[x^q](1+w)^r = \frac{r}{pq+r}\binom{pq+r}{q}$ for $q \geq 1$; with $r = 1/j$ this is $\mathcal{R}_q$. Passing from $\alpha = -1$ to $\alpha = 1$ replaces x by $-x$ (Remark 7). Finally, if $\tilde\mu$ is a compactly supported probability

measure whose Cauchy transform satisfies (15) a.e., then near $z=\infty$ the function $\mathcal{C}_{\tilde{\mu}}$ is holomorphic, equals $\sum_s \left(\int \zeta^s d\tilde{\mu}\right) z^{-s-1}$ and solves (15), hence coincides with $\mathfrak{M}$. □

Theorem 32. *The measure $\nu=\mu_{\ell-1,\ell,j}$ is the Raney distribution $\mathcal{R}_{\ell/j,1/j}$, i.e. its q-th moment is $\mathcal{R}_q$. In particular, for $j=1$ the moments are the Fuss–Catalan numbers $\frac{1}{(\ell-1)q+1}\binom{\ell q}{q}$ and $\nu=\pi^{\boxtimes(\ell-1)}$, the $(\ell-1)$-st free multiplicative power of the Marchenko–Pastur law π of rate 1; and $\mathcal{R}_q^{1/q}\to z_{cr}$ as $q\to\infty$.*

Proof. Everything except the last statement is contained in Lemma 31 and Theorem 27, the identification of the Fuss–Catalan case being classical [Mlo, NS, PZ]. The last statement holds for any probability measure on $[0,z_{cr}]$ whose density is positive near z_{cr}. □

Remark 33. Positivity of the Raney distributions $\mathcal{R}_{p,r}$ for $p\geq 1$, $0<r\leq p$ is known [MPZ], and our parameters $p=\ell/j$, $r=1/j$ satisfy these restrictions; Theorem 27 gives a self-contained proof together with the explicit density. Fuss–Catalan laws also occur in the asymptotic zero distribution of Jacobi–Piñeiro multiple orthogonal polynomials; see [KLS] for strong uniform asymptotics obtained by Riemann–Hilbert steepest descent.

6.1. **A free-probabilistic factorisation.** For $0<a<1$ let $\beta_a:=\mathrm{Beta}(a,1-a)$, i.e. $d\beta_a(t)=\frac{\sin\pi a}{\pi}t^{a-1}(1-t)^{-a}dt$ on $(0,1)$, with the convention $\beta_1=\delta_1$.

Theorem 34. *Let $\nu=\mu_{\ell-1,\ell,j}$. Then*

(i) $S_\nu(y)=\dfrac{(1+y)^j-1}{y(1+y)^{\ell-1}}$;

(ii) *$\beta_{1/j}$ is the Raney distribution $\mathcal{R}_{1,1/j}$ and* $S_{\beta_{1/j}}(y)=\dfrac{(1+y)^j-1}{y(1+y)^{j-1}}$;

(iii) *consequently*

$$\mu_{\ell-1,\ell,j}=\beta_{1/j}\boxtimes\pi^{\boxtimes(\ell-j)}, \tag{48}$$

and $\mu_{\ell,\ell+1,j}=\mu_{\ell-1,\ell,j}\boxtimes\pi$ for fixed j;

(iv) *for general m, $\mu_{m,\ell,j}$ is the law of $\omega Y^{1/c}$, where $Y\sim\nu$ and ω is uniform on the c-th roots of unity.*

Proof. Let $\psi(x)=\sum_{q\geq1}\mathcal{R}_q x^q$ and $u=1+\psi$. By Lemma 31, $u^j=1+xu^\ell$; solving for x in terms of $y=\psi$ gives $\chi(y)=\frac{(1+y)^j-1}{(1+y)^\ell}$ and $S_\nu(y)=\frac{1+y}{y}\chi(y)$, which is (i). For $a=1/j$ one has $\int_0^1 t^q d\beta_a=\frac{\Gamma(q+a)}{\Gamma(a)\Gamma(q+1)}=[x^q](1-x)^{-1/j}$, which is the Raney sequence $\mathcal{R}_{1,1/j}$, i.e. the case $\ell=j$ of the above computation; this gives (ii). Since $S_\pi(y)=(1+y)^{-1}$, multiplicativity of the S-transform yields $S_{\beta_{1/j}}S_\pi^{\ell-j}=S_\nu$, which is (iii). Part (iv) is (46) rewritten probabilistically. □

Remark 35 (Schur–Szegő versus free multiplicative convolution). For $j=1$, (48) is the Fuss–Catalan identity $\nu=\pi^{\boxtimes(\ell-1)}$; for $j=2$, $\beta_{1/2}$ is the arcsine law on $[0,1]$, i.e. the law of $|(1+U)/2|^2$ for a Haar unitary U, so $\mu_{\ell-1,\ell,2}$ has the standard random-matrix product model with one arcsine and $\ell-2$ Marchenko–Pastur factors. Moreover the identity $\mu_{\ell,\ell+1,j}=\mu_{\ell-1,\ell,j}\boxtimes\pi$ is the asymptotic counterpart of the finite-n relation $p_n^{(\ell,\ell+1,j)}=p_n^{(\ell-1,\ell,j)}\star Q_n^{(\ell)}$ used in the proof of Theorem 12: the j-independence of the Schur–Szegő factor $Q_n^{(\ell)}$ becomes the j-independence of the

Marchenko–Pastur factor. It would be interesting to know the scope of this $\star \leftrightarrow \boxtimes$ correspondence.

Example 36. (i) For $\ell = 2$, $j = 1$, $m = 1$ one has $z_{cr} = 4$, $t(\theta) = 4\cos^2\theta$ and $\varrho_\nu(t) = \frac{1}{2\pi}\sqrt{\frac{4-t}{t}}$: the Marchenko–Pastur law. Correspondingly $p_n^{(1,2,1)}$ is, up to $z \mapsto -z$, the monic Laguerre polynomial $L_n^{(-1)}$.

(ii) For $\ell = 2$, $j = 1$, $m = 0$, i.e. for the Hermite operator $D^2 + zD$, we have $c = 2$ and, by (45), $\varrho_\mu(x) = |x|\varrho_\nu(x^2) = \frac{1}{2\pi}\sqrt{4 - x^2}$: Wigner's semicircle law. Here $p_n(z)$ is proportional to $H_n\Big(\frac{iz}{\sqrt{2}}\Big)$, so for $\alpha = 1$ the semicircle appears on the imaginary axis, in accordance with Remark 7.

(iii) For $j = 1$ and arbitrary ℓ the Laguerre-like density becomes the classical parametrisation of the Fuss–Catalan density on $\Big[0, \frac{\ell^\ell}{(\ell-1)^{\ell-1}}\Big]$, cf. [HM, Neu, PZ].

Remark 37 (Hard and soft edges). By Theorem 27 and (45), the density of μ on a leg behaves like $s^{c/\ell - 1}$ near the origin and like $\sqrt{K_T - s}$ near the tip. In particular it is unbounded at the origin exactly when $m > 0$, which matches Lemma 20(i): the origin is a branching point precisely for $m > 0$.

7. Bergkvist's conjectures for $T_{m,\ell,j}$

We return to the operator itself, i.e. to $\alpha = 1$, denote by $z_1, \dots, z_n$ the zeros of $p_n = p_n^{(m,\ell,j)}$ and by $s_k = s_k(n) = \sum_i z_i^k$ their power sums. Recall $c = \ell - m$, $d = \frac{\ell - j}{\ell - m}$, $r_n^T = \max_i |z_i|$.

Theorem 38. *For $T_{m,\ell,j}$ the following hold.*

(a) *For every fixed $k \geq 0$*

$$\lim_{n\to\infty} \frac{s_k(n)}{n^{1+dk}} = M_k, \qquad M_k = \begin{cases} (-1)^q \mathcal{R}_q, & k = cq, \\ 0, & c \nmid k, \end{cases} \tag{49}$$

with $\mathcal{R}_q$ the Raney numbers (47).

(b) *Unconditionally,* $\liminf\limits_{n\to\infty} \dfrac{r_n^T}{n^d} \geq K_T$.

(c) *If Conjecture 9 holds for (m, ℓ, j) — in particular for every $j \leq 2$ by Theorem 14 and for every $c = 1$ by Theorem 12 — then for all $q, n \geq 1$*

$$\left(\frac{|s_{cq}(n)|}{n}\right)^{\frac{1}{cq}} \leq r_n^T \leq \big|s_{cq}(n)\big|^{\frac{1}{cq}}. \tag{50}$$

(d) *Under the hypothesis of* (c)*, Claim a) holds as soon as the two limits $n \to \infty$ and $q \to \infty$ may be interchanged in the left inequality of* (50)*, e.g. as soon as $|s_{cq}(n)|/n \leq C\mathcal{R}_q n^{dcq}$ uniformly for $q \leq \log^2 n$.*

Proof. (a) is the case $A_T = \{\ell\}$ of Theorem 3, the identification of M_k following from Lemma 31 applied with $\alpha = 1$: writing $u = Z\mathfrak{M}$ and $x = Z^{-c}$, the equation $Z^m\mathfrak{M}^\ell + Z^j\mathfrak{M}^j = 1$ becomes $u^j = 1 - xu^\ell$.

(b) By the triangle inequality $|s_{cq}| \leq n(r_n^T)^{cq}$, so by (a)

$$\liminf_{n\to\infty} \frac{r_n^T}{n^d} \;\geq\; \lim_{n\to\infty} \left(\frac{|s_{cq}|}{n\, n^{dcq}}\right)^{\frac{1}{cq}} = \mathcal{R}_q^{\frac{1}{cq}},$$

and $\mathcal{R}_q^{1/q} \to z_{cr} = K_T^c$ by Theorem 32.

(c) Under Conjecture 9 every z_i^c is a non-positive real, so all the numbers z_i^{cq} are real of the same sign $(-1)^q$; hence $|s_{cq}| = \sum_i |z_i|^{cq}$ exactly, which gives both inequalities.

(d) The right inequality in (50) gives $\frac{r_n^T}{n^d} \le n^{\frac{1}{cq}} \left|m_{cq}^{(n)}\right|^{\frac{1}{cq}}$ with $m_k^{(n)} = s_k/n^{1+dk}$. If $|m_{cq}^{(n)}| \le C\mathcal{R}_q$ uniformly for $q \le \log^2 n$, the choice $q = \lfloor \log^2 n \rfloor$ makes $n^{1/(cq)} \to 1$ and $(C\mathcal{R}_q)^{1/(cq)} \to K_T$; combined with (b) this is Claim a). □

Part (b) sharpens the qualitative lower bound of [BeBj]: it produces the constant predicted by Conjecture A, which is exactly the modulus of the nonzero branching points of (14), cf. Lemma 18. The only missing ingredient in Claim a) is thus the absence of *outlying* zeros. Classical estimates are of no help here: already for $(m, \ell, j) = (1, 2, 1)$ the Cauchy bound β_n, the positive root of $\beta^n = \sum_{r \ge 1} \gamma_r \beta^{n-r}$, satisfies $\beta_n/n \to \infty$. Section 8 removes this obstacle for $j = 1$ by working in the eigenpolynomial basis instead.

Theorem 39 (Tightness without an a priori root bound)**.** *Assume Conjecture 9 for (m, ℓ, j). Then*

(i) *for every $A > K_T$, $\lim_{n\to\infty} \frac{1}{n} \#\{i : \ |z_i| \ge An^d\} = 0$;*

(ii) *the zero-counting measures μ_n of the scaled eigenpolynomials $p_n(n^d z)$ are tight and converge weakly to the measure μ_T of Theorem 28 with $\alpha = 1$.*

Thus Claim b) holds with no assumption $r_n^T = O(n^d)$. In particular, by Theorems 14 and 12, Claim b) holds unconditionally whenever $j \le 2$, and also whenever $m = \ell - 1$.

Proof. (i) By Conjecture 9, $\left|\frac{s_{cq}(n)}{n\, n^{dcq}}\right| = \frac{1}{n} \sum_i \left(\frac{|z_i|}{n^d}\right)^{cq}$, so Chebyshev's inequality and Theorem 38(a) give, for each fixed q,

$$\limsup_{n\to\infty} \frac{1}{n} \#\{i : |z_i| \ge An^d\} \le \frac{\mathcal{R}_q}{A^{cq}} = \left(\frac{\mathcal{R}_q^{1/q}}{A^{\,c}}\right)^q ;$$

since $\mathcal{R}_q^{1/q} \to K_T^c < A^{\,c}$, letting $q \to \infty$ proves (i).

(ii) Part (i) gives tightness. Let μ_∞ be a weak limit point; by (i) it is supported in $\{|Z| \le K_T\}$ and, by Conjecture 9, on the c rays. By (18) the nonzero zeros of p_n occur in full $\mathbb{Z}_c$-orbits, and the possible atom at the origin has mass at most $(c-1)/n$, so μ_∞ is $\mathbb{Z}_c$-invariant. Push μ_n forward by $Z \mapsto Z^{\,c}$: the resulting measures ν_n live on $(-\infty, 0]$ and have moments $\int t^q d\nu_n = \frac{s_{cq}(n)}{n\, n^{dcq}} \to (-1)^q \mathcal{R}_q$. By (i) and Hölder's inequality (using convergence of one higher moment) every fixed moment passes to any weak limit, which is therefore a measure on $[-z_{cr}, 0]$ with moments $(-1)^q \mathcal{R}_q$; compactly supported measures on $\mathbb{R}$ being moment-determinate, ν_n converges to the push-forward of μ_T. Finally a $\mathbb{Z}_c$-invariant probability measure is determined by this push-forward: for bounded continuous f one has $\int f\, d\mu = \int g\, d\big[(Z \mapsto Z^c)_* \mu\big]$ with $g(w) = \frac{1}{c} \sum_t f(\varepsilon^t w^{1/c})$, a bounded continuous function of w. Hence $\mu_\infty = \mu_T$ for every limit point. □

Remark 40. Theorem 39(i) says that outlying zeros are always statistically negligible; what remains for Claim a) is the exclusion of $o(n)$ exceptional zeros, possibly even of a single one. The sign information supplied by the ray-root theorem is essential here: without it the power sums can be small because of cancellation.

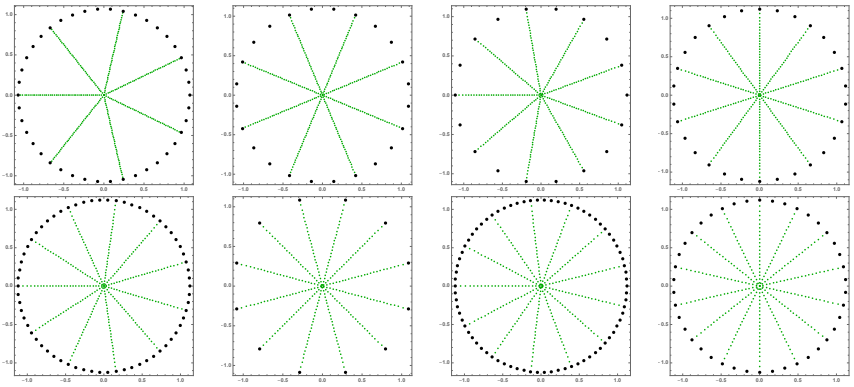

FIGURE 1. Roots of $p_n(z)$ divided by $n^{\frac{\ell-j}{\ell-m}}$ for $n = 300$, $j = 6$, $m = 0$ and $\ell = 7, 8, \dots, 14$ shown in green. Black dots are the branching points (32).

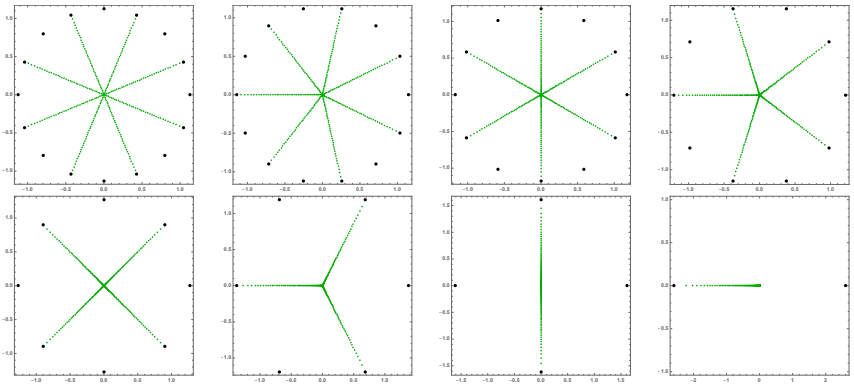

FIGURE 2. Roots of $p_n(z)$ divided by $n^{\frac{\ell-j}{\ell-m}}$ for $n = 300$, $j = 6$, $\ell = 9$ and $m = 1, 2, \dots, 8$.

8. Multiplication recurrences and sharp root bounds

Definition 41. The sequence $\{p_n\}$ *satisfies a linear recurrence relation of finite order* (equivalently, has finite lower bandwidth) if there are an integer $N \geq 1$ independent of n and numbers $a_{n,i}$ with $zp_n(z) = p_{n+1}(z) + \sum_{i=0}^{N-1} a_{n,n-i} p_{n-i}(z)$ for all large n. The smallest such N is the *optimal lower bandwidth*. Some coefficients inside the window may vanish identically; following [ABST] we do not identify the bandwidth with the number of nonzero terms.

This is a special case of the generalized algebraic Bochner–Krall problem of [HST], and for the present family it was solved completely in [ABST]: finite lower bandwidth occurs if and only if $j = 1$ and $c = \ell - m$ divides ℓ, in which case the optimal bandwidth is ℓ. Below we give a short independent derivation which, in addition, computes *all* multiplication coefficients for every $j = 1$, including the infinite-bandwidth cases, and then uses them to bound the zeros.

Throughout, $\gamma_r(N) := \gamma_r^{(N,m,\ell,j)}$ and

$$B_s(N) := (N)_j - (N - sc)_j, \qquad \gamma_r(N) = \prod_{s=1}^{r} \frac{(N-(s-1)c)_\ell}{B_s(N)}.$$

All quantities below are rational functions of n; two rational functions agreeing for infinitely many integers agree identically, so we treat n as a variable.

Lemma 42. *There are unique rational functions $\delta_r = \delta_r(n)$, $r \geq 1$, with*

$$p_{n+1}(z) - z\,p_n(z) = \sum_{r\geq 1} \delta_r\, p_{n+1-cr}(z), \tag{51}$$

determined by the triangular system

$$\gamma_r(n+1) - \gamma_r(n) = \sum_{q=1}^{r} \delta_q\, \gamma_{r-q}(n+1-cq), \qquad r \geq 1. \tag{52}$$

Moreover $\{p_n\}$ has finite lower bandwidth if and only if $\delta_r \equiv 0$ for all $r > r_0$ for some r_0, and then the optimal bandwidth equals $c\,r_0$ with r_0 the largest index with $\delta_{r_0} \not\equiv 0$.

Proof. All monomials in $p_{n+1} - zp_n$ have exponents $\equiv n+1 \pmod c$ and degree at most $n+1-c$, while the p_{n+1-cr} are monic of pairwise distinct degrees $\equiv n+1 \pmod c$; hence they form a triangular basis, which gives existence and uniqueness, and comparing coefficients of z^{n+1-cr} gives (52). Conversely, in any finite relation $zp_n = p_{n+1} + \sum_i a_{n,n-i} p_{n-i}$ the coefficients are unique, and comparing exponents modulo c shows $a_{n,n-i} = 0$ unless $c \mid (i+1)$; so the relation is (51) truncated. □

Lemma 43. *Put $D(r) := \frac{\gamma_r(n)}{\gamma_r(n+1)}$, $\lambda_q := \frac{\delta_q}{\gamma_q(n+1)}$,*

$$\Psi_q(r) := \prod_{s=q+1}^{r} \frac{U(s)}{V_q(s)}, \qquad U(s) := B_s(n+1), \quad V_q(s) := (n+1-cq)_j - (n+1-cs)_j.$$

Then $\Psi_q(q) = 1$ and, for every $r \geq 1$,

$$1 - D(r) = \sum_{q\geq 1} \lambda_q \Psi_q(r). \tag{53}$$

Moreover, for fixed generic n and $r \to \infty$, $D(r) = C r^{\,e_D}(1 + o(1))$ and $\Psi_q(r) = C_q r^{\,e_q}(1+o(1))$ with $C, C_q \neq 0$,

$$e_D = \frac{\ell}{c}\ (j=1), \quad e_D = \frac{\ell - j}{c}\ (j \geq 2); \qquad e_q = q\ (j=1), \quad e_q = 0\ (j \geq 2).$$

For $j = 1$ one has $\Psi_q(r) = \binom{r}{q}$.

Proof. Dividing (52) by $\gamma_r(n+1)$, it suffices to check $\frac{\gamma_{r-q}(n+1-cq)}{\gamma_r(n+1)} = \frac{\Psi_q(r)}{\gamma_q(n+1)}$; this follows by shifting $s \mapsto s - q$, since $B_{s-q}(n+1-cq) = V_q(s)$ and all numerators with $s > q$ cancel.

For the asymptotics recall that $\prod_{s\leq r} \psi(s) = C r^e (1 + o(1))$ with $C \neq 0$ whenever $\psi(s) = 1 + \frac{e}{s} + O(s^{-2})$ is finite and nonzero on $\mathbb{N}$. Both U and V_q are polynomials in s of degree j with the same leading coefficient, so e_q equals the difference of the sums of their roots. For $j = 1$, $U(s) = cs$ and $V_q(s) = c(s-q)$, giving $\Psi_q(r) = \prod_{s=q+1}^{r} \frac{s}{s-q} = \binom{r}{q}$ and $e_q = q$. For $j \geq 2$ the sum of the roots in τ of $B_\tau(N) =$

$(N)_j - (N - c\tau)_j$ equals $\frac{jN - \binom{j}{2}}{c}$, whence $\sum \mathrm{roots}(V_q) = jq + \frac{j(n+1-cq) - \binom{j}{2}}{c} = \sum \mathrm{roots}(U)$ and $e_q = 0$. Similarly $D(r) = \prod_{s \le r} \phi(s)$ with

$$\phi(s) = \frac{(n-(s-1)c)_\ell}{(n+1-(s-1)c)_\ell} \cdot \frac{U(s)}{B_s(n)} = \frac{s-(1+A)}{s-(1+B)} \cdot \frac{U(s)}{B_s(n)},$$

where $A = \frac{n-\ell+1}{c}$ and $B = \frac{n+1}{c}$; and the same root count gives $e_D = \frac{\ell}{c} + \Delta_j$ with $\Delta_j = 0$ for $j = 1$ and $\Delta_j = -\frac{j}{c}$ for $j \ge 2$. □

8.1. **All multiplication coefficients for $j = 1$, and the classification.** For real x and $\sigma \ge 0$ we use the generalized falling factorial and binomial coefficient

$$(x)_\sigma := \frac{\Gamma(x+1)}{\Gamma(x+1-\sigma)}, \qquad \binom{\sigma}{r} := \frac{\sigma(\sigma-1)\cdots(\sigma-r+1)}{r!}.$$

Theorem 44. *Let $j = 1$, $c = \ell - m$, $\sigma = \ell/c$, and let $\delta_r(n)$ be as in Lemma 42. Then for every $r \ge 1$*

$$\boxed{\delta_r(n) = (-1)^{r+1} \binom{\sigma}{r} \prod_{t=0}^{r-1} (n - tc)_{\ell-1}} \tag{54}$$

equivalently, with $\theta = w\frac{d}{dw}$ and $\beta_0 = \frac{n+1}{c}$,

$$f_n = \mathcal{D}_\sigma(\theta) f_{n+1}, \qquad \mathcal{D}_\sigma(\theta) = \frac{(\beta_0 - \theta)_\sigma}{(\beta_0)_\sigma}. \tag{55}$$

Proof. For $j = 1$ one has $U(s) = B_s(n) = cs$, so the function D of Lemma 43 satisfies $D(0) = 1$ and $\frac{D(r+1)}{D(r)} = \frac{r - \beta_0 + \sigma}{r - \beta_0}$. The Gamma-function identity $\frac{(x-1)_\sigma}{(x)_\sigma} = \frac{x-\sigma}{x}$ shows that $r \mapsto \frac{(\beta_0 - r)_\sigma}{(\beta_0)_\sigma}$ satisfies the same recursion and the same initial condition, so, with no integrality assumption on σ,

$$D(r) = \frac{(\beta_0 - r)_\sigma}{(\beta_0)_\sigma}. \tag{56}$$

By Lemma 43, $1 - D(r) = \sum_{q \ge 1} \lambda_q \binom{r}{q}$, and for each integer $r \ge 0$ this is a *finite* triangular system, because $\binom{r}{q} = 0$ for $q > r$; its unique solution is the Newton forward-difference expansion $\lambda_q = -\Delta^q D(0)$. Applying

$$\Delta\big[(\beta - \theta)_\sigma\big] = (\beta - \theta)_\sigma \left(\frac{\beta - \theta - \sigma}{\beta - \theta} - 1 \right) = -\sigma(\beta - \theta - 1)_{\sigma - 1}$$

repeatedly gives $\Delta^q\big[(\beta_0 - \theta)_\sigma\big] = (-1)^q (\sigma)_q (\beta_0 - \theta - q)_{\sigma - q}$, hence

$$\lambda_q = (-1)^{q+1} (\sigma)_q \frac{(\beta_0 - q)_{\sigma - q}}{(\beta_0)_\sigma}.$$

Now $\delta_q = \lambda_q \gamma_q(n+1)$ with $\gamma_q(n+1) = \frac{\Pi_q(n+1)}{q!\, c^q}$ by (29); splitting off the first factor of each falling factorial,

$$\frac{\Pi_q(n+1)}{c^q} = \prod_{t=0}^{q-1} \frac{n+1-tc}{c} \cdot \prod_{t=0}^{q-1} (n - tc)_{\ell-1} = (\beta_0)_q \prod_{t=0}^{q-1} (n - tc)_{\ell-1},$$

and $(\beta_0)_q (\beta_0 - q)_{\sigma - q} = (\beta_0)_\sigma$, which gives (54). Formula (55) is the same expansion written in the diagonal operator θ; indeed by (31) $f_{n+1-cr} = \frac{c^r}{\Pi_r(n+1)} f^{(r)}_{n+1}$ and $w^r \frac{d^r}{dw^r} = (\theta)_r$. □

Theorem 45 (Finite-recurrence classification, [ABST])**.** *The monic eigenpolynomials of $T_{m,\ell,j}$ have finite lower bandwidth if and only if $j = 1$ and $c = \ell - m$ divides ℓ. In that case $\sigma = \ell/c \in \mathbb{N}$,*

$$p_{n+1} = z\,p_n + \sum_{r=1}^{\sigma} \delta_r\, p_{n+1-cr}, \qquad \delta_r = (-1)^{r+1}\binom{\sigma}{r}\prod_{t=0}^{r-1}(n-tc)_{\ell-1}, \tag{57}$$

and the optimal lower bandwidth equals $c\sigma = \ell$, i.e. the associated difference operator has order ℓ. The right-hand side has $\sigma + 1$ nonzero eigenpolynomials, not $\ell + 1$.

Proof. Let $j \geq 2$ and suppose $\delta_q \equiv 0$ for $q > r_0$. Then by (53) the finite sum $\sum_{q\leq r_0} \lambda_q \Psi_q(r)$ converges as $r \to \infty$ to $\sum_{q\leq r_0} \lambda_q C_q$, whereas $|D(r)| \sim |C| r^{(\ell-j)/c} \to \infty$ by Lemma 43 — a contradiction.

Let $j = 1$. By (54) the coefficients δ_r vanish for large r if and only if $\binom{\sigma}{r}$ does, i.e. if and only if $\sigma = \ell/c \in \mathbb{N}$; and then $\delta_\sigma = (-1)^{\sigma+1}\prod_{t<\sigma}(n-tc)_{\ell-1} \neq 0$ for large n, so the extreme index really occurs and the optimal bandwidth is $c\sigma = \ell$ by Lemma 42. □

Corollary 46. *Let $j = 1$. The normalized coefficients $\widehat{\delta}_r := \delta_r(n)/\prod_{t<r}(n - tc)_{\ell-1} = (-1)^{r+1}\binom{\sigma}{r}$ do not depend on n. If $\sigma \notin \mathbb{N}$ then*

$$|\widehat{\delta}_r| = \frac{|\sin \pi\sigma|\,\Gamma(\sigma+1)}{\pi}\, r^{-\sigma-1}\big(1 + O(r^{-1})\big), \qquad r \to \infty, \tag{58}$$

and all δ_r with $r > \sigma$ have the sign of $\sin(\pi\sigma)$.

Proof. Use the reflection formula $\binom{\sigma}{r} = (-1)^{r+1}\frac{\sin(\pi\sigma)\Gamma(\sigma+1)}{\pi}\frac{\Gamma(r-\sigma)}{\Gamma(r+1)}$ and the standard Gamma-ratio asymptotic. □

Thus for every $j = 1$ the normalized multiplication coefficients are universal and n-independent; the whole n-dependence is carried by the product produced by iterating the lowering relation (31), and the obstruction to a finite recurrence is the nonintegrality of the single rational number $\ell/(\ell - m)$. When $\sigma \notin \mathbb{N}$ the recurrence is "almost finite": its normalized coefficients decay like $r^{-\sigma-1}$.

Example 47. (i) For $(m, \ell, j) = (2, 4, 1)$ one has $c = 2$, $\sigma = 2$ and $p_{n+1} = zp_n + 2(n)_3 p_{n-1} - (n)_3(n-2)_3 p_{n-3}$; admissible m other than 0 and $\ell - 1$ exist exactly when ℓ is composite.

(ii) For $(m, \ell, j) = (1, 3, 1)$ one has $c = 2$, $\sigma = 3/2$ and

$$p_{n+1} = zp_n + \tfrac{3}{2}(n)_2 p_{n-1} - \tfrac{3}{8}(n)_2(n-2)_2 p_{n-3} - \tfrac{1}{16}(n)_2(n-2)_2(n-4)_2 p_{n-5} - \cdots,$$

the smallest instance with no finite recurrence; the normalized coefficients decay as $r^{-5/2}$.

Remark 48. The structural interpretation proved in [ABST] is relevant to the question posed in [HST]: if $j = 1$ and $c \mid \ell$, then

$$T_{m,\ell,1} = z\partial_z + q'(G)G, \qquad G = p(z\partial_z)\partial_z, \quad p(t) = \prod_{r=1}^{\sigma-1}\big(t - (rc-1)\big), \quad q(t) = \frac{t^{\,c}}{c},$$

so every admissible operator of our family has exactly the Type (2) form of Conjecture 1.10 of *loc. cit.*, and the order equality predicted by Conjecture 1.9 holds.

Note also the trap emphasized in [ABST]: for $c \geq 2$ the sequence $\{p_n\}$ is $(c-1)$-symmetric in the sense of Douak–Maroni [DM], but symmetry alone does not force a recurrence with a single lower term, and (57) indeed has σ lower diagonals. Finally, Theorem 45 refines Example 7.3 of [Ho2] with $\beta = -1$; see also [AB].

8.2. **A Hessenberg matrix and sharp root bounds for $j = 1$.** Formula (54) makes it possible to bound the zeros in the eigenpolynomial basis rather than through the coefficients of p_n in the monomial basis; this removes the extra factor that makes Cauchy–Fujiwara bounds ineffective.

Lemma 49 (Multiplication matrix)**.** *For every j, the zeros of p_n, with multiplicities, are the eigenvalues of the $n \times n$ matrix $H^{(n)} = (H_{i,t})_{0 \leq i,t < n}$ with*

$$H_{t+1,t} = 1, \qquad H_{t+1-cr,t} = -\delta_r(t) \quad (r \geq 1), \tag{59}$$

for those r for which $t + 1 - cr \geq 0$, all other entries being zero. If $j = 1$, then more precisely $\delta_r(t) \neq 0$ implies $t + 1 - cr \geq m$.

Proof. Since $p_0, \dots, p_{n-1}$ form a basis of $\mathbb{C}[z]/(p_n)$, the recurrence $z p_t = p_{t+1} - \sum_r \delta_r(t) p_{t+1-cr}$ shows that (59) represents multiplication by z in this quotient. Its spectrum is therefore the multiset of zeros of p_n.

For the last assertion, in the range $cr \leq t + 1$ we have $t - uc \geq c - 1 \geq 0$ for $u \leq r - 1$, and $(N)_{\ell-1} = 0$ for $0 \leq N \leq \ell - 2$; so nonvanishing of the product in (54) forces $t - (r-1)c \geq \ell - 1$, i.e. $t + 1 - cr \geq \ell - c = m$. Hence the $j = 1$ recurrence never uses an index below m. □

For $s \geq m$ define

$$\eta_s := c^{-d} \, \frac{\Gamma\!\left(\frac{s-\ell+1}{c} + 1\right)}{\Gamma\!\left(\frac{s}{c} + 1\right)}, \qquad d = \frac{\ell - 1}{c}. \tag{60}$$

Lemma 50 (Telescoping weights)**.** *The numbers* (60) *are positive, $1/\eta_s$ is increasing, and*

$$\frac{\eta_s}{\eta_{s-c}} = \frac{s - \ell + 1}{s}, \qquad \frac{1}{\eta_s} = s^{\,d}\big(1 + O(s^{-1})\big), \tag{61}$$

$$\prod_{s=b-c+1}^{b} \eta_s = \frac{1}{(b)_{\ell-1}} \quad (b \geq \ell - 1), \qquad \prod_{u=0}^{r-1} (t - uc)_{\ell-1} \prod_{s=t+1-cr}^{t-1} \eta_s = \frac{1}{\eta_t} \tag{62}$$

whenever $\delta_r(t) \neq 0$.

Proof. The first identity is the functional equation of Γ, and $1/\eta_s = c^{\,d}\Gamma(x+1)/\Gamma(x+1-d)$ with $x = s/c$ is increasing and asymptotic to $s^{\,d}$. For the first identity in (62) put $R(b) := (b)_{\ell-1} \prod_{s=b-c+1}^{b} \eta_s$; using $\frac{(b+1)_{\ell-1}}{(b)_{\ell-1}} = \frac{b+1}{b-\ell+2}$ and (61) one gets $R(b+1) = R(b)$, while the asymptotics give $R(b) \to 1$; hence $R \equiv 1$. The second identity follows by induction on r: for $r = 1$ it is the first one with $b = t$, and passing from r to $r + 1$ multiplies the left-hand side by $(t - rc)_{\ell-1} \prod_{s=t+1-c(r+1)}^{t-cr} \eta_s = 1$, again by the first identity with $b = t - rc$. By Lemma 49 all indices occurring lie in the allowed range. □

Theorem 51 (Weighted Gershgorin bound)**.** *Let $j = 1$, $c = \ell - m$, $\sigma = \ell/c$, $d = \frac{\ell-1}{c}$, and put*

$$F_\sigma(\Lambda) := \sum_{r \geq 0} \left| \binom{\sigma}{r} \right| \Lambda^r, \qquad \widehat{K} := \inf_{0 < \Lambda < 1} \frac{F_\sigma(\Lambda)}{\Lambda^{1/c}}. \tag{63}$$

Then every zero of $p_n^{(m,\ell,1)}$ with $n \geq \ell$ satisfies

$$|z| \leq \frac{\widehat{K}}{\eta_{n-1}} = \widehat{K}\, c^{\,d}\, \frac{\Gamma\big(\frac{n-1}{c}+1\big)}{\Gamma\big(\frac{n-\ell}{c}+1\big)} = \widehat{K}\, n^{\,d}\big(1+O(n^{-1})\big). \tag{64}$$

In particular $r_n^T = O(n^d)$ for every $T_{m,\ell,1}$. If $c \mid \ell$ then $F_\sigma(\Lambda) = (1+\Lambda)^\sigma$ and

$$\widehat{K} = \min_{\Lambda>0} \frac{(1+\Lambda)^{\ell/c}}{\Lambda^{1/c}} = \frac{\ell^{\ell/c}}{(\ell-1)^{(\ell-1)/c}} = K_T, \tag{65}$$

the minimum being attained at $\Lambda = \frac{1}{\ell-1}$.

Proof. Fix $0 < \Lambda < 1$, put $\mu = \Lambda^{1/c}$ and define positive weights

$$d_t := \prod_{s<t} \rho_s, \qquad \rho_s := \begin{cases} 1, & s < m, \\ \mu\, \eta_s, & s \geq m, \end{cases} \qquad D := \operatorname{diag}(d_0, \ldots, d_{n-1}).$$

By Lemma 49 the spectral radius of $H^{(n)}$ is at most $\|D^{-1}H^{(n)}D\|_1$, i.e. at most the largest weighted column sum. In column t, the superdiagonal entry contributes $\frac{d_t}{d_{t+1}} = \frac{1}{\mu\eta_t}$, while for each r with $\delta_r(t) \neq 0$, Theorem 44 and (62) give

$$|\delta_r(t)| \frac{d_t}{d_{t+1-cr}} = \left|\binom{\sigma}{r}\right| \mu^{\,cr-1} \prod_{u<r} (t-uc)_{\ell-1} \prod_{s=t+1-cr}^{t-1} \eta_s = \left|\binom{\sigma}{r}\right| \frac{\Lambda^r}{\mu\,\eta_t}$$

(all weights used have index $\geq m$ by Lemma 49). Hence the column sum is at most $\frac{F_\sigma(\Lambda)}{\Lambda^{1/c}\eta_t} \leq \frac{F_\sigma(\Lambda)}{\Lambda^{1/c}\eta_{n-1}}$, since $1/\eta_t$ increases; the columns $t \leq \ell-2$, for which all $\delta_r(t)$ vanish, satisfy the same bound because $F_\sigma \geq 1$. Taking the infimum over Λ gives (64). Finally, for $\sigma \in \mathbb{N}$ the sum (63) is $(1+\Lambda)^\sigma$ and $\frac{d}{d\Lambda}\log\frac{(1+\Lambda)^\sigma}{\Lambda^{1/c}} = 0$ gives $\sigma c\Lambda = 1+\Lambda$, i.e. $\Lambda = \frac{1}{\ell-1}$, with value (65). □

Corollary 52. *Let $j = 1$ and $(\ell - m) \mid \ell$. Then Bergkvist's Conjecture A holds in full for $T_{m,\ell,1}$: both*

$$\lim_{n\to\infty} \frac{r_n^T}{n^{(\ell-1)/(\ell-m)}} = K_T \qquad \text{(Claim a)}$$

and the weak convergence $\mu_n \to \mu_T$ (Claim b) hold, and $\mathcal{C}_{\mu_T}$ satisfies (14), i.e. Corollary A is valid as well. In particular, for $T = T_{\ell-1,\ell,1}$ one has the explicit bound

$$r_n^T \leq \frac{\ell^\ell}{(\ell-1)^{\ell-1}}\,(n-1)(n-2)\cdots(n-\ell+1).$$

Proof. Claim a) follows by combining (65) with Theorem 38(b), and Claim b) from Theorem 39 together with the ray-root theorem for $j = 1$ (Theorem 14); the equation for $\mathcal{C}_{\mu_T}$ is Theorem 28. For $c = 1$ one has $\eta_s = 1/(s)_{\ell-1}$ and $\sigma = \ell$, and (64) becomes the displayed inequality. □

Remark 53 (The nonintegral case). When $c \nmid \ell$ the only loss in (64) comes from taking absolute values of the one-signed tail of Corollary 46. Since $F_\sigma(\Lambda) \geq (1+\Lambda)^\sigma$ we have $\widehat{K} \geq K_T$, but numerically the loss is very small:

(m,ℓ)	$(1,3)$	$(1,4)$	$(2,5)$	$(1,5)$	$(2,6)$	$(3,7)$
$\widehat{K}/K_T$	1.00825	1.00247	1.00132	1.00092	1.00076	1.00060

Closing this gap requires exploiting cancellation rather than a pure column-sum norm. Note that, by Theorems 14 and 39, Claim b) is in any case a theorem for all $j = 1$.

8.3. **A sharpened recurrence-coefficient reduction for $j \geq 2$.** The weights required by the preceding argument can be written explicitly for every j. More importantly, no prescribed decay rate is needed: the free parameter in the weighted Gershgorin estimate absorbs any uniform geometric growth of the normalized multiplication coefficients.

Put

$$\Theta(b) := \frac{\Gamma(b+\ell-j+1)}{\Gamma(b+1)} = (b+1)(b+2)\cdots(b+\ell-j) \tag{66}$$

and, for $s \geq 0$,

$$\eta_s^{[j]} := c^{-d}\,\frac{\Gamma(\frac{s}{c}+1)}{\Gamma(\frac{s+\ell-j}{c}+1)}. \tag{67}$$

Lemma 54 (Telescoping weights for every j)**.** *The weights* (67) *are positive,* $1/\eta_s^{[j]}$ *is increasing, and*

$$\frac{1}{\eta_s^{[j]}} = c^d \frac{\Gamma(\frac{s+\ell-j}{c}+1)}{\Gamma(\frac{s}{c}+1)} = s^d(1+O(s^{-1})). \tag{68}$$

Moreover,

$$\prod_{s=b-c+1}^{b} \eta_s^{[j]} = \frac{1}{\Theta(b)} \qquad (b \geq c-1), \tag{69}$$

$$\prod_{u=0}^{r-1} \Theta(t-uc) \prod_{s=t+1-cr}^{t-1} \eta_s^{[j]} = \frac{1}{\eta_t^{[j]}} \qquad (r \geq 1,\ t+1-cr \geq 0). \tag{70}$$

Proof. Positivity, monotonicity, and (68) follow from the standard Gamma-ratio properties. For (69), let $R(b) = \Theta(b)\prod_{s=b-c+1}^{b} \eta_s^{[j]}$. Then

$$\frac{R(b+1)}{R(b)} = \frac{b+\ell-j+1}{b+1}\,\frac{\eta_{b+1}^{[j]}}{\eta_{b+1-c}^{[j]}} = 1.$$

Thus R is constant, and the Gamma-ratio asymptotics show that its limit is 1. Formula (70) follows by induction on r: its $r = 1$ case is (69), and the step is obtained by multiplying by $\Theta(t-rc)\prod_{s=t+1-c(r+1)}^{t-cr} \eta_s^{[j]} = 1$. □

For an admissible pair (n, r), namely $r \geq 1$ and $n+1-cr \geq 0$, define

$$\widehat{\delta}_r(n) := \frac{\delta_r(n)}{\prod_{t=0}^{r-1}\Theta(n-tc)}, \qquad S_\Lambda(n) := \sum_{\substack{r\geq 1 \\ n+1-cr\geq 0}} |\widehat{\delta}_r(n)|\Lambda^r \quad (\Lambda > 0). \tag{71}$$

Theorem 55 (Sharpened Gershgorin reduction)**.** *For every n and every $\Lambda > 0$,*

$$r_n^T \leq \frac{1+\max_{0\leq t<n} S_\Lambda(t)}{\Lambda^{1/c}}\,\frac{1}{\eta_{n-1}^{[j]}}. \tag{72}$$

Consequently $r_n^T = O(n^d)$ whenever $S_\Lambda(n)$ is bounded uniformly in n for one value of $\Lambda > 0$. In particular, any uniform geometric bound $|\widehat{\delta}_r(n)| \le CM^r$ suffices (choose $0 < \Lambda < M^{-1}$); neither alternating signs nor a closed formula is required.

Proof. Set $\mu = \Lambda^{1/c}$, $d_t = \prod_{0\le s<t} \mu\eta_s^{[j]}$, and $D = \mathrm{diag}(d_0, \dots, d_{n-1})$. Lemma 49 and the column-sum norm give $r_n^T \le \|D^{-1}H^{(n)}D\|_1$. The superdiagonal contribution in column t is at most $1/(\mu\eta_t^{[j]})$. For $i = t + 1 - cr \ge 0$, equations (71) and (70) give

$$|\delta_r(t)|\frac{d_t}{d_i} = |\widehat{\delta}_r(t)|\,\frac{\Lambda^r}{\mu\eta_t^{[j]}}.$$

Hence the t-th column sum is at most $(1 + S_\Lambda(t))/(\Lambda^{1/c}\eta_t^{[j]})$. Now use the monotonicity in Lemma 54 and (68). □

For $j = 1$ this is consistent with Theorem 51: apart from the bounded change caused by using the rising factorial (66) instead of the falling factorial in (54), the series $1 + S_\Lambda$ is $F_\sigma(\Lambda)$. The usefulness of Theorem 55 for $j \ge 2$ is that it isolates the only missing estimate in a directly computable quantity.

Exact rational computations from the triangular system (52) suggest a particularly simple answer. In all tested families with $j = 2, 3, 4, 5$ (up to $r = 45$ and $n = 480$), the nonzero admissible coefficients have sign $(-1)^{r+1}$. The normalized magnitudes decay geometrically, and the partial sums $S_1(n)$ appear to converge to finite limits as $n \to \infty$. For example, at $n = 900$ the ratios $|\widehat{\delta}_{25}/\widehat{\delta}_{24}|$ are 0.2496, 0.2485, 0.4655, 0.5828, and 0.6550 for $(m, \ell, j) = (2, 3, 2), (2, 4, 2), (3, 4, 3), (4, 5, 4), (5, 6, 5)$, respectively.

Conjecture 56 (Normalized multiplication coefficients)**.** *For every $0 \le m < \ell > j \ge 2$ the nonzero admissible coefficients satisfy $\operatorname{sign}\delta_r(n) = (-1)^{r+1}$. Moreover, there exist constants $C, M > 0$, depending only on (m, ℓ, j), such that $|\widehat{\delta}_r(n)| \le CM^r$ for all admissible (n, r). More strongly, $S_1(n)$ has a finite limit as $n \to \infty$.*

The geometric part alone, by Theorem 55, rules out outlying zeros on scales larger than n^d. The stronger S_1 assertion is consistent with the observed limiting ratios, which are below 1 but appear to depend on (m, ℓ, j).

8.4. **A Gamma-function and Gould–Hsu route for $j = 2$.** When $j = 2$, all entries in Lemma 43 become explicit Gamma ratios. Put

$$a := \frac{2n+1}{c}, \qquad a' := \frac{2n-1}{c}, \qquad A := \frac{n-\ell+1}{c}, \qquad B := \frac{n+1}{c}.$$

Proposition 57. *For $j = 2$ one has*

$$\frac{U(s)}{V_q(s)} = \frac{s(a-s)}{(s-q)(a-q-s)}, \tag{73}$$

$$\Psi_q(r) = \binom{r}{q}\frac{\Gamma(a-q)\Gamma(a-q-r)}{\Gamma(a-r)\Gamma(a-2q)}, \tag{74}$$

$$D(r) = \prod_{s=1}^{r}\frac{s-(1+A)}{s-(1+B)}\,\frac{a-s}{a'-s}. \tag{75}$$

Consequently, if $x_q := \lambda_q \Gamma(a-q)/\Gamma(a-2q)$, *then* (53) *becomes*

$$1 - D(r) = \sum_{q\geq 1} x_q \frac{\binom{r}{q}}{(a-r-1)^{(q)}} = \sum_{q\geq 1} x_q \frac{(-r)^{(q)}}{q!(r+1-a)^{(q)}}, \tag{76}$$

where $(x)^{(q)} = x(x+1)\cdots(x+q-1)$.

Proof. Here $U(s) = sc(2n+1-sc) = c^2 s(a-s)$ and $V_q(s) = c^2(s-q)(a-q-s)$, proving (73). Taking the product from $s = q+1$ to r gives (74). Formula (75) follows from the formula for $\phi(s)$ in the proof of Lemma 43, since $U(s)/B_s(n) = (a-s)/(a'-s)$. Finally, $\Gamma(a-q-r)/\Gamma(a-r) = 1/(a-r-1)^{(q)}$, and the second expression in (76) is the corresponding elementary rising-factorial identity. □

The kernel in (76) is precisely of Gould–Hsu (Carlitz) inversion type: a binomial coefficient divided by a product of q factors linear in r, while its left-hand side is the explicit four-Gamma ratio (75). Inverting this system should give x_q, hence λ_q and δ_q, as a terminating hypergeometric sum. A usable bound for that sum would prove the $j = 2$ case of Conjecture 56 and, through Theorem 55, the desired root upper bound.

9. Open problems

Problem 58. Prove Conjecture 9 in the remaining range $j \geq 3$, $\ell - m \geq 2$. By Lemma 13 this amounts to the real-rootedness of an explicit ${}_jF_{\ell-1}$ whose numerator parameters are non-real (Remark 15); already the case $(m, \ell, j) = (0, 5, 3)$ is open.

Problem 59. Complete Bergkvist's Claim a) outside the range of Corollary 52. For $j = 1$ and $(\ell - m) \nmid \ell$ only the constant is missing (Remark 53); for $j \geq 2$ the geometric estimate in Conjecture 56, by Theorem 55, would give the correct order of the largest zero. Identifying the sharp constant remains a separate problem.

Problem 60. Prove Conjecture 56. For $j = 2$, carry out the Gould–Hsu inversion in (76) and obtain a closed formula or a uniform bound for the multiplication coefficients δ_r in (51).

Problem 61. Prove that the measure of Theorem 28 is the unique compactly supported probability measure whose Cauchy transform satisfies (15) a.e., without a support restriction; cf. Remark 30.

Problem 62. Characterize the exactly solvable operators all of whose eigenpolynomials are real-rooted. This is related to, but more general than, hyperbolicity preservation; see [BB].

Problem 63. Is there a direct finite-n explanation of the correspondence between Schur–Szegő composition and free multiplicative convolution in Remark 35?

Problem 64. For a general degenerate exactly solvable operator, characterize the critical values of P_T visible from the principal branch of (7), and decide whether the support of μ_T is a union of N_T congruent arms as suggested by Remark 6.

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