STUDYING EIGENPOLYNOMIALS OF DEGENERATE EXACTLY SOLVABLE OPERATORS. A PRIMER

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Dedicate to our friend, late Professor Jan-Erik Björk

Abstract. In this paper we discuss a number of properties of eigenpolynomials to the degenerate exactly solvable differential operators given by

\[ T_{m,\ell,j} = z^m \frac{d^\ell}{dz^\ell} + z^j \frac{d^j}{dz^j}, \]

where \(0 \leq m < \ell > j \geq 1\) are integers. The obtained sequences of eigenpolynomials generalize several important families of multiorthogonal polynomials and, at the same time, have some new features. This study is a part of an ongoing project aimed to settle a general conjecture posed in [Be] and is also related to the main conjecture of [HST].

1. Introduction

We start with some basic notions.

Definition 1. A linear univariate differential operator

\[ T = \sum_{j=1}^\ell Q_j(z) \frac{d^j}{dz^j} \]

with polynomial coefficients is called exactly solvable if

(i) \( \deg Q_j \leq j \) for all \( j = 1, \ldots, \ell \);

(ii) \( \exists j_0 \) such that \( \deg Q_{j_0} = j_0 \).

An exactly solvable operator (1) is called non-degenerate if \( \deg Q_{\ell} = \ell \) and degenerate otherwise.

One can easily show that any exactly solvable operator \( T \) has a polynomial solution of the spectral equation \( T(p) = \lambda p \) in each degree \( n \) which we call an eigenpolynomial. Moreover for \( n \) large enough this eigenpolynomial is unique up to a scalar factor. In what follows, we denote by \( \{p_n^T\}_{n=0}^\infty \) the sequence of monic eigenpolynomials of the exactly solvable operator \( T \). By the above, for any exactly solvable \( T \), its entries are uniquely defined for all sufficiently large \( n \).

The class of exactly solvable differential operators is the main object of study in the classical Bochner-Krall problem asking for which exactly solvable \( T \), the sequence \( \{p_n^T\}_{n=0}^\infty \) consists of orthogonal polynomials. In [HST] we generalized this problem by asking for which \( T \), the sequence \( \{p_n^T\}_{n=0}^\infty \) satisfies a linear recurrence relation of finite order, i.e. consists of multi-orthogonal polynomials. Another

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natural problem related to exactly solvable $T$ and their sequences $\{p_n^T\}_{n=0}^{\infty}$ is to study the root asymptotics of the latter eigenpolynomials which was addressed in a number of earlier papers including [Be, BR, MS] to mention a few. For non-degenerate operators satisfactory results about the root asymptotics were obtained in [BR]. On the other hand, for degenerate operators the detailed conjectures about root asymptotics and some initial results can be found in [Be, BeBj], but these conjectures are still widely open. The present paper is an additional step in the direction motivated by [Be] and [HST]. At the same time, sequences of eigenpolynomials studied below possess a number of additional nice features not available in the general case which make their study more interesting.

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2. Basic properties of eigenpolynomials for $T = z^m \frac{d^\ell}{dz^\ell} + z^j \frac{d^j}{dz^j}$

Let as above $p_n(z) := p_n^{(m,\ell,j)}(z)$ denote the sequence of eigenpolynomials to $T := T_{m,\ell,j} = z^m \frac{d^\ell}{dz^\ell} + z^j \frac{d^j}{dz^j}$. Observe that the $n$-th eigenvalue for $T := T_{m,\ell,j}$ corresponding to $p_n(z)$ is given by

$$\lambda_n = (n)_j,$$

where $(a)_b$ with $b$ being a positive integer denotes the falling factorial $(a)_b := a(a-1)\ldots(a-b+1)$.

Proposition 1. In the above notation, for any $0 \leq m < \ell > j \geq 1$,

$$p_n^{(m,\ell,j)}(z) = z^n + \sum_{r=1}^{[n/\ell-m]} \gamma_r z^{n-(\ell-m)r},$$

where the coefficients $\gamma_r$, $r = 1, \ldots, [n/(\ell-m)]$ is given by

$$\gamma_r := \gamma_r^{(m,\ell,j)} = \frac{\prod_{s=1}^{r-1}((n)_j - (n-s(\ell-m)))_\ell}{\prod_{s=1}^{r}((n)_j - (n-s(\ell-m)))_\ell}.$$  

Proof. Substituting (2) in the spectral equation $T(p_n) = \lambda_n p_n$, one gets the simple recurrent system of equations satisfied by $\gamma_r$ of the form

$$\begin{cases}
(n)_\ell = ((n)_j - (n-\ell-m))_\ell \gamma_1; \\
(n-(\ell-m))_\ell \gamma_1 = ((n)_j - (n-2(\ell-m)))_\ell \gamma_2; \\
(n-2(\ell-m))_\ell \gamma_2 = ((n)_j - (n-3(\ell-m)))_\ell \gamma_3; \\
\vdots \quad \vdots \quad \vdots \\
\end{cases}$$

solving which consecutively we get exactly expression (3).

Examples and special cases:

For $m = \ell - 1$ and $j = 1$,

$$\gamma_r^{(\ell-1,\ell,1)} = \frac{(n)_\ell (n-1)_\ell \cdots (n-r+1)_\ell}{r!} \quad ; \quad r = 1, \ldots, n-\ell+1, \quad \gamma_{n-\ell+2}^{(\ell-1,\ell,1)} = \cdots = \gamma_n^{(\ell-1,\ell,1)} = 0;$$
In particular, for \( r = 1, \ldots, n - \ell \),
\[
\gamma_r^{(n, \ell, \ell+1, 1)} = \gamma_r^{(n, \ell-1, \ell, 1)}(n - \ell)(n - \ell - 1)(n - \ell - r + 1) = \gamma_r^{(n, \ell-1, \ell, 1)} r! \left( \frac{n-\ell}{r} \right).
\]

For \( m = \ell - 1 \) and \( j = 2 \),
\[
\gamma_r^{(n, \ell-1, \ell, 2)} = \frac{(n)\ell(n-1)\ell \cdots (n-r+1)\ell}{r!(2n-2)\ell}; \quad r = 1, \ldots, n-\ell+1; \quad \gamma_{n-\ell+2}^{(n, \ell-1, \ell, 2)} = \cdots = \gamma_{n}^{(n, \ell-1, \ell, 2)} = 0.
\]

In particular, for \( r = 1, \ldots, n - \ell \),
\[
\gamma_r^{(n, \ell, \ell+1, 2)} = \gamma_r^{(n, \ell-1, \ell, 2)} r! \left( \frac{n-\ell}{r} \right),
\]
which is the same relation as for \( m = \ell - 1 \) and \( j = 1 \).

For \( m = \ell - 1 \) and any \( j < \ell \),
\[
\gamma_r^{(n, \ell-1, \ell, j)} = \frac{(n)\ell(n-1)\ell \cdots (n-r+1)\ell}{\prod_{i=1}^{r}((n)_{j} - (n-i)_{j})}; \quad r = 1, \ldots, n-\ell+1; \quad \gamma_{n-\ell+1}^{(n, \ell-1, \ell, j)} = \cdots = \gamma_{n}^{(n, \ell-1, \ell, j)} = 0;
\]

One still has the same relation, for \( r = 1, \ldots, n - \ell \),
\[
\gamma_r^{(n, \ell, \ell+1, j)} = \gamma_r^{(n, \ell-1, \ell, j)} r! \left( \frac{n-\ell}{r} \right),\]
as for \( m = \ell - 1 \) and \( j = 1 \).

For \( m = \ell - 2 \) and any \( j < \ell \),
\[
\gamma_r^{(n, \ell-2, \ell, j)} = \frac{(n)\ell(n-2)\ell \cdots (n-2r+2)\ell}{\prod_{i=1}^{r}((n)_{j} - (n-2i)_{j})}; \quad r = 1, \ldots, n-\ell+1; \quad \gamma_{n-\ell+1}^{(n, \ell-2, \ell, j)} = \cdots = \gamma_{n}^{(n, \ell-2, \ell, j)} = 0; \quad \text{CHECK!}
\]

For \( r = 1, \ldots, n - \ell \),
\[
\gamma_r^{(n, \ell-1, \ell+1, j)} = \gamma_r^{(n, \ell-1, \ell, j)}(n - \ell)(n - \ell - 2) \cdots (n - \ell - 2(r + 1)).
\]

Further, in view of formula (2), the sequence \( \{p^{(m, \ell, j)}_n(z)\}_{n=0}^{\infty} \) naturally splits into \( \ell - m \) subsequences \( \{p^{(m, \ell, j)}_k(z)\}_{k=0}^{\infty} \) where \( i = 0, \ldots, \ell - m - 1 \). Namely,
\[
p^{(k, \ell-1)}_{k}(z) = z^i Q^{[i]}_k(z^{\ell-m}),
\]
where \( Q^{[i]}_k(\tau) \) is a monic polynomial of degree \( k \) in \( \tau \). Thus, we get \( \ell - m \) polynomial sequences \( \{Q^{[i]}_k(\tau)\}_{k=0}^{\infty} \) where \( i = 0, 1, \ldots, \ell - m - 1 \) with positive coefficients.

In what follows, we will need more notions. Recall that the Schur-Szegö composition of polynomials \( P(x) = \sum_{i=0}^{n} a_i x^i \) and \( Q(x) = \sum_{i=0}^{n} b_i x^i \) is given by
\[
P \ast Q(x) := \sum_{i=0}^{n} a_i \frac{b_i}{\binom{n}{i}} x^i.
\]

**Proposition 2** (see Theorem 5.5.5 and Corollary 5.5.10 of [RS]). If \( P \) is a real-rooted polynomial and if \( Q \) is a real-rooted polynomial with all roots of the same sign, then \( P \ast Q \) is real-rooted. Moreover, all roots of \( P \ast Q \) lie in \([-M, -m]\) where \( M \) is the maximal and \( m \) is the minimal pairwise product of roots of \( P \) and \( Q \).
Theorem 3 (Generalized Malo-Schur-Szegő Composition theorem, see Theorem 2.5. of [CC]). Let $A(z) = \sum_{j=0}^{m} a_j z^j$ and $B(z) = \sum_{j=0}^{n} b_j z^j$, $a_m b_n \neq 0$, and let

$$C(z) = \sum_{j=0}^{\nu} j! a_j b_k z^j, \quad \text{where} \quad \nu = \min(m, n).$$

If $A(z)$ has all its zeros in the sector $S_\alpha$ ($\alpha \leq \pi$) and if $B(z)$ has all its zeros in the sector $S_\beta$ ($\beta \leq \pi$), then $C(z)$ has all its zeros in the sector $-S_\alpha S_\beta$. In particular, if $A(z)$ has non-positive roots and $B(z)$ has non-positive roots then $C(z)$ has non-positive roots.

Theorem 4. In the above notation, each polynomial $Q_{k, \ell}^{(n)}(\tau)$ is real-rooted.

Proof. Let us start with the simplest case $T = z^{\ell - 1} \frac{d^\ell}{dz^\ell} + z \frac{d}{dz}$. (This sequence of polynomials is a sequence of $d$-orthogonal polynomials, see e.g. [BchD, Ho] and their real-rootedness follows from other arguments as well.) Then, there is just a single family

$$p_n(z) := p_{n}^{(\ell - 1, \ell, 1)}(z) = z^n + \sum_{r=1}^{n-\ell+1} \frac{(n+1)(n-1)\cdots(n-r+1)}{r!} z^{n-r}.$$

To avoid triviality, one has to assume that $n \geq \ell$. For $n < \ell$, $p_n(z) = z^n$.

For $r = 1, \ldots, n - \ell$, one has a relation

$$\gamma_{r}^{(n, \ell, \ell+1, 1)} = \gamma_{r}^{(n, \ell-1, \ell, 1)} \frac{(n-\ell)}{r}.$$

For fixed $n \geq 2$, we will prove the real-rootedness of $p_n(\ell, z)$ by induction on $\ell = 2, 3, \ldots, n$. Note that for $\ell = 2$, $p_2^{(1,2,1)}(z)$ is the monic $n$-th Laguerre polynomial with parameter $a = -1$ which is known to have all negative roots. Using (4), we present $p_{n}^{(\ell+1, \ell+1)}(z)$ as

$$p_{n}^{(\ell+1, \ell+1)}(z) = p_{n}^{(\ell-1, \ell, 1)}(z) \ast Q_{n}^{(\ell)}(z),$$

where

$$Q_{n}^{(\ell)}(z) = \sum_{r=0}^{n-\ell} \frac{r!}{r!} \binom{n}{r} \binom{n-\ell}{r} z^{n-r}.$$

If we show that for any $n \geq \ell$, $Q_{n}^{(\ell)}(z)$ has all non-positive roots that using our base of induction and (a slightly stronger version of) Proposition 2 we obtain the required result. Inverting $Q_{n}^{(\ell)}(z)$ and setting $m = n - \ell \geq 0$ we need to check that polynomials

$$U_{n}^{(m)}(z) = \sum_{r=0}^{m} \binom{m}{r} \binom{n}{r} z^r$$

have all negative roots. Take $\Phi_n(x) = (1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} z^r$ and $\Phi_m(x) = (1 + x)^m = \sum_{r=0}^{m} \binom{m}{r} z^r$ with obviously have only negative roots. The polynomial $U_{n}^{(m)}(z)$ is obtained by the operation described in Theorem 3 from $\Phi_n(x)$ and $\Phi_m(x)$. Thus all roots of $U_{n}^{(m)}(z)$ are non-positive and since its constant term is non-vanishing they are negative.
More generally, for \( m = \ell - 1 \) and an arbitrary \( 2 \geq j < \ell \), we still have the relation

\[
p_n^{(\ell, \ell+1,j)}(z) = p_n^{(\ell-1, \ell,j)}(z) \ast Q_n^{(\ell)}(z),
\]

where \( Q_n^{(\ell)}(z) \) are exactly the same polynomials as above. We need however to check that the initial polynomial for our recurrence is real-rooted. Observe that as the initial polynomial we can take \( p_n^{(j-1, \ell,j)} \), \( n \geq j \). (One can check that the coefficients of the eigenpolynomials \( p_n(z) \) for the operator \( T = (z^{-1} + z^1) \frac{d^j}{dz^j} \) are given by the same formulas and can be used as the initial polynomials.) By Corollary 3 and Theorem 3 of [MS], all roots of \( p_n(z) \), \( n > j \) lie in the interval \((-1, 0)\) and are simple.

\[
3. \text{ Bergkvist’s conjecture for } T = z^m \frac{d^j}{dz^j} + z^j \frac{d^k}{dz^k}.
\]

Let us start with formulation of the main conjectures of paper [Be] which describe the asymptotics of the root growth for the eigenpolynomials of an arbitrary degenerate operator. Namely, for any degenerate operator \( T \), denote by \( r_n^T \) the maximal absolute value of the roots of the \( n \)-th eigenpolynomial \( p_n^T \). In [Be] T. Bergkvist has shown that \( \lim_{n \to \infty} r_n^T = +\infty \).

For a degenerate exactly solvable operator (1), denote by \( j_T \) the largest integer \( i \) for which \( \deg Q_i = i \). Since \( T \) is a degenerate exactly solvable operator, then \( j_T \) exists and is strictly smaller than \( \ell \).

**Conjecture A.** In the above notation, for any degenerate operator \( T \), there exists a positive number \( K_T > 0 \) such that

\[
\lim_{n \to \infty} \frac{r_n^T}{n^{d_T}} = K_T,
\]

where

\[
d_T := \max_{i \in [j_T+1, \ell]} \left( \frac{i - j_T}{i - \deg Q_i} \right).
\]

The geometric meaning of the above exponent \( d_T \) in terms of the Newton polygon of the operator \( T \) is shown in Fig. 1. From now on we will always use the normalisation assumption that the polynomial coefficient \( Q_{j_T}(z) \) in \( T \) is monic. (Observe that by assumptions \( \deg Q_{j_T}(z) = j_T \) and \( j_T \) is the maximal positive integer with the latter property.)

Under certain additional assumptions Conjecture A implies the following.

**Corollary A** (Follows if Conjecture A and some additional statements are valid). Given an arbitrary degenerate operator \( T \) as in (1), the Cauchy transform \( \mathcal{C} := \mathcal{C}_T(z) \) of the asymptotic root counting measure \( \mu_T \) for the sequence \( \{q_n(z)\} \) of the scaled eigenpolynomials \( q_n(z) := p_n(n^d z) \) satisfies a.e. in \( \mathbb{C} \) the algebraic equation

\[
z^{j_T} \mathcal{C}^{j_T} + \sum_{i \in A_T} \alpha_i z^{\deg Q_i} \mathcal{C}^i = 1,
\]

where \( A_T \) is the set consisting of all \( i \) for which the maximum \( d := \max_{i \in [j_T+1, \ell]} \left( \frac{i - j_T}{i - \deg Q_i} \right) \) is attained, i.e. \( A_T = \{ i : (i - j_T)/(i - \deg Q_i) = d_T \} \). Here \( \alpha_i \) is the leading coefficient of the polynomial \( Q_i(z) \). (Observe that \( A_T \neq \emptyset \).)
Figure 1. Newton polygon of a degenerate exactly solvable operator of order 7 and the corresponding \( d \).

Conjecture A was settled in a number of special cases already in the original paper [Be]. Additionally for an arbitrary degenerate operator \( T \), the fact that there exists a positive constant \( c > 0 \) such that

\[
\lim_{n \to \infty} \frac{r_n^T}{n^d} \geq c
\]

was proven in a still unpublished preprint [BeBj]. In short, both Conjecture A and its Corollary A are highly plausible and are confirmed by a large number of computer experiments, see [Be]. Observe that every instance in which Conjecture A is settled also gives a special case supporting the main Conjecture 1 in § 3 of [BoSh].

For the special case \( T = z^m \frac{d^\ell}{dz^\ell} + z^j \frac{d^j}{dz^j} \), Bergkvist’s conjecture leads to the following claims:

a) \( \lim_{n \to \infty} \frac{r_n^T}{n^d} = K_T > 0 \);

b) with the rescaling \( q_n(z) = p_n(n^{\ell-m}z) \), the Cauchy transform of the limiting root counting measure for the sequence \( \{q_n(z)\} \) satisfies almost everywhere in \( C \) the algebraic equation

\[
z^m \mathcal{C}^\ell + z^j \mathcal{C}^j = 1.
\]

The main result of this section is as follows.

**Theorem 5.** For \( T = z^m \frac{d^\ell}{dz^\ell} + z^j \frac{d^j}{dz^j} \),

\[
z_{\text{max}} \sim \frac{\ell - \sqrt{-1}}{j/((\ell-m) \cdot (\ell-j)/(j(\ell-m)))} \cdot n^{\ell-m}, \quad n \to \infty.
\]

In other words, a) is valid with \( K_T \) given by \( \frac{\ell / (j(\ell-m))}{j/((\ell-m) \cdot (\ell-j)/(j(\ell-m)))} \).

Theorem 5 together with the results about the location of roots from § 2 imply that b) is also valid. Distributions of roots for some scaled eigenpolynomials are shown in Figures 2 and 3.

**Remark 1.** The constant \( K_T \) given above coincides with the absolute value of non-vanishing branch points of the branch points when the curve given by (7) on the \( z \)-plane. All the latter branch points lie on a circle.
Lemma 6. The set of branching points different from the origin for the projection of the algebraic curve given by (7) onto the $z$-plane is given by the equation:

\[ z^{j\ell/m} = \frac{\ell}{\ell-j} \left( \frac{-j}{\ell} \right)^{\ell-j}. \]

Proof. The system of equations defining the required branching points is given by

\[
\begin{align*}
z^m C^\ell + z j C^j &= 1 \\
(\ell z^m C^{\ell+1} + j z^j C^{j+1}) &= 0.
\end{align*}
\]

Since $z = 0$ does not satisfy the first equation we can factor out $z^j C^{j+1}$ from the second equation and obtain

\[
\ell z^{m-j} C^{\ell-j} + j = 0 \quad \Leftrightarrow \quad C^{\ell-j} = -\frac{j}{\ell} z^{j-m} \quad \Leftrightarrow \quad C = \left( \frac{-j}{\ell} \right)^{\ell-j} z^{j-m}.
\]

Substituting the latter expression for $C$ in the first equation, we get

\[
z^m \left( \frac{-j}{\ell} \right)^{\ell-j} z^{j\ell/m} + z^j \left( \frac{-j}{\ell} \right)^{\ell-j} z^{j\ell/m} = 1.
\]

Both terms in the left-hand side of the latter equation have the same powers of $z$ which gives after some manipulations

\[ \ell - j \left( \frac{-j}{\ell} \right)^{\ell-j} z^{j\ell/m} = 1 \quad \Leftrightarrow \quad z^{j\ell/m} = \frac{\ell}{\ell-j} \left( \frac{-j}{\ell} \right)^{\ell-j}. \]

Taking the root or order $\frac{j\ell}{\ell-j}$ of the r.h.s. we get the same expression for $K_T$ as before. \qed

Our approach to finding the root(s) with the maximal absolute value of a given polynomial $S(z)$ is based on the following statement called the Gräffe-Lobachevskii method, see e.g., [GrL].

Lemma 7. Given a univariate polynomial $S(z) = z^n + \sum_{r=0}^{n-1} \gamma_r z^r$ whose roots (with possible repetitions) are denoted by $z_1, \ldots, z_n$, assume that $S$ has exactly one root $z_{\text{max}}$ of maximal absolute value. Then

\[ \limsup_{k \to \infty} \frac{s_k}{z_{\text{max}}^k} = 1, \]

where $s_k := \sum_{j=1}^{n} z_{j}^k$.

Notice that $s_k$ can be recursively expressed using the recurrence relation

\[ s_k = -k \gamma_k - \sum_{j=1}^{k-1} \gamma_j s_{k-j}. \]

WHAT CAN WE PUT HERE?
Figure 2. Roots of $p_n(z)$ divided by $n^{\ell-1}$ for $n = 300$, $j = 6$, $m = 0$ and $\ell = 7,8,\ldots,14$ shown in green. Black dots are the branch points given by the equation (10).

Figure 3. Roots of $p_n(z)$ divided by $n^{\ell-1}$ for $n = 300$, $j = 6$, $\ell = 9$ and $m = 1,2,\ldots,8$. 

4. WHICH SEQUENCES $\{p_n^T\}$ SATISFY FINITE RECURRENCE RELATIONS

Conjecture 1. The polynomial sequence $\{p_n(z) := p_n^{(m,\ell,j)}(z)\}$ satisfies a linear finite recurrence relation if and only if $j = 1$ and $\ell = (\ell - m)s$ for some positive integer $s$. The length of this recurrence relation equals $\ell + 1 = (\ell - m)s + 1$.

Example 1. Case 1. For $m = 0$ and $j = 1$, i.e. for $T = \frac{d^\ell}{dz^\ell} + z \frac{d}{dz}$, one has the following recurrence relation:

$$p_{n+1} = z \cdot p_n + \Gamma(\ell) \binom{n}{\ell - 1} \cdot p_{n-\ell+1}.$$ 

Case 2. For $m = \ell - 1$ and $j = 1$, i.e. for $T = z^{\ell-1} \frac{d^\ell}{dz^{(\ell-1)}} + z \frac{d}{dz}$, one has the following recurrence relation:

$$p_{n+1} = \left( z^{\ell} \frac{\Gamma(n+1)}{\Gamma(n+\ell+2)} \right) p_n - \frac{\Gamma(n+1)}{\Gamma(n-\ell+2)} \sum_{r=2}^{\ell} (-1)^r \binom{\ell}{r} \cdot \prod_{s=1}^{r-1} \frac{\Gamma(n-s+1)}{\Gamma(n-\ell-s+2)} \cdot p_{n-r+1}.$$ 

{figNice}

{figNice2}

{conj:main}
Case 3. For \( n = \ell - 2 \) and \( j = 1 \), i.e. for \( T = z^{\ell-2} \frac{d^2}{dz^2} + z \frac{d}{dz} \), one has the following recurrence relation:

\[
p_{n+1} = z \cdot p_n - \sum_{r=1}^{\lfloor n/2 \rfloor} \frac{(-1)^r T(\ell + 2r - 2)}{2^r \Gamma(r+1)} \cdot \binom{n}{\ell + 2r - 3} \prod_{s=0}^{r-1} (\ell - 2s) \cdot p_{n-2r+1}.
\]

How is this conjecture related to the main conjecture of [HST]?

5. Final remarks

Observe that at present the following natural question is still open.

**Problem 1.** Characterize the class of exactly solvable operators \( T \) such that every eigenpolynomial \( p_T^n \) has all real zeros.

Apparently Problem 1 is closely related to the notion of hyperbolicity preserving operators, see e.g. [BB]. In particular, one can easily see that any hyperbolicity preserving exactly solvable operator solves Problem 1, but the converse is not true.

If we additionally assume that the positive measure defining the bilinear functional belongs to the so-called Nevai class, see [Ne] then it is known that the limiting root-counting measure of any sequence of orthogonal polynomials in the Nevai class has the arcsine distribution on some finite interval \([a,b]\). The latter fact implies that up to a constant the leading coefficient of the corresponding positive BK-operator must be of the form \((x - a)^k(x - b)^l\), where the order \( k \) of the operator equals \( 2l \), see [BRSh]. (More details about this case can be found in [KL].)

**References**


[GrL] Gräffe ....


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