THE FULL SHADOW OF A POLYNOMIAL III. SEMIALGEBRAICITY AND REFLECTION RIGIDITY

CHRISTIAN HÄGG AND BORIS SHAPIRO

Abstract. We prove that the limiting sets of zeros of all derivatives of polynomial powers form a semialgebraic family in each degree. Their limiting pooled zero measures split into planar densities, measures on reflection axes, and explicit root atoms. This rigidity classifies all zero-area limits as segments or unions of radial segments. A Cauchy-transform conservation law identifies their holomorphic moments with those of random chords and reconstructs a degree-d polynomial from its first d moments. We also establish analytic regularity of the planar density and area stability, including convergence when root masses vanish.

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1. Introduction

Successive differentiation moves polynomial zeros within their convex hull. When one differentiates all powers of a fixed polynomial and pools the resulting zeros, a limiting set and a canonical probability measure emerge. These objects are determined by a logarithmic separation problem. We prove that their geometry is nevertheless algebraic, and that their one-dimensional parts are governed by reflection symmetry.

1.1. **Shadows and their canonical measures.** For a nonzero polynomial G, let $Z(G)$ be its set of distinct zeros and let $\operatorname{div}_0 G = \sum_{a\in Z(G)} \operatorname{ord}_a(G)\delta_a$ be its zero counting measure. Throughout the polynomial part of the paper,

$$P(z) = \prod_{j=1}^{s} (z-a_j)^{r_j}, \qquad d = \sum_{j=1}^{s} r_j, \qquad p_j = \frac{r_j}{d}, \tag{1.1}$$

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where the a_j are distinct and r_j are positive integers. Write

$$\mu_P^{\mathrm{root}} = \sum_{j=1}^{s} p_j \delta_{a_j}, \qquad \mathsf{H}_P = \operatorname{conv} Z(P), \qquad \mathfrak{c}_P = \sum_{j=1}^{s} p_j a_j$$

for the root probability measure, its convex hull, and its centroid. For a compactly supported finite measure ν, our potential and Cauchy-transform conventions are

$$U_\nu(z) = \int \log|z-\xi| \,\mathrm{d}\nu(\xi), \qquad C_\nu(z) = \int \frac{\mathrm{d}\nu(\xi)}{z-\xi}. \tag{1.2}$$

In particular,

$$\mathscr{C}_P(z) := C_{\mu_P^{\mathrm{root}}}(z) = \frac{P'(z)}{dP(z)}, \qquad \mathfrak{m}_k(P) := \int \xi^k \,\mathrm{d}\mu_P^{\mathrm{root}}(\xi).$$

Set $R_{m,n} = (P^n)^{(m)}$ for $n \geq 1$ and $0 \leq m < dn$, and define

$$\mathcal{Z}_{P,n} = \bigcup_{m=0}^{dn-1} Z(R_{m,n}), \qquad \mathcal{S}_P = \overline{\bigcup_{n\geq 1} \mathcal{Z}_{P,n}}.$$

We call $\mathcal{S}_P$ the *full shadow*. If $m/n \to \alpha \in (0,d)$, the probability measures $(dn-m)^{-1} \operatorname{div}_0 R_{m,n}$ converge to a measure $\mu_{\alpha,P}$ [HS1, Theorem 3.3]. Define

$$\mathcal{K}_{\alpha,P}^{\mathrm{fix}} = \operatorname{supp} \mu_{\alpha,P}, \qquad \rho_P = \frac{2}{d^2} \int_0^d (d-\alpha)\mu_{\alpha,P} \,\mathrm{d}\alpha, \qquad \mathcal{K}_P = \operatorname{supp} \rho_P. \tag{1.3}$$

The measure ρ_P is the *shadow measure*, and $\mathcal{K}_P$ is the *core*. Equivalently, ρ_P is the weak limit of the pooled measures

$$\bar{\nu}_{P,n} := \frac{2}{dn(dn+1)} \sum_{m=0}^{dn-1} \operatorname{div}_0 R_{m,n}. \tag{1.4}$$

The complete sets $\mathcal{Z}_{P,n}$ converge to $\mathcal{K}_P$ in Hausdorff distance. The remaining points

$$T_P := \mathcal{S}_P \setminus \mathcal{K}_P$$

are isolated zeros from finite levels and are locally finite outside $\mathcal{K}_P$; we call them *transients*. Thus the core and full shadow have the same interior and area [HS1, Theorems 5.1, 5.3, and 5.7]. The distinction matters for the full set, but not for its planar geometry.

We use $\mathrm{d}A$ for planar Lebesgue measure, $\mathcal{H}^1$ for arclength, and d_{H} for Hausdorff distance. The ratio $\alpha = m/n$ is measured against the exponent of P. The total-degree ratio is $t = m/(dn) = \alpha/d$; it will be used for general root probability measures.

1.2. **Algebraic geometry and reflection rigidity.** A set is *semialgebraic* if it is a finite Boolean combination of real polynomial equalities and inequalities. Monic degree-d coefficient space is identified with $\mathbb{R}^{2d}$. A *reflection axis* of P is a line whose reflection preserves the root multiset, including multiplicities.

Theorem 1.1 (Geometry of the core)**.** *For each degree d, the family $\{(P,z) : z \in \mathcal{K}_P\}$ is semialgebraic. There is an integer D_d such that every boundary $\partial\mathcal{K}_P$ lies in a real algebraic curve of degree at most D_d. The cores in each degree have finitely many homeomorphism types; each is homeomorphic to a finite simplicial complex and has finitely many complementary components and boundary of finite length.*

The shadow measure has a decomposition

$$\rho_P = f_P \,\mathrm{d}A + \vartheta_P \,\mathcal{H}^1 \restriction \Gamma_P + \sum_{j=1}^{s} p_j^2 \delta_{a_j}, \tag{1.5}$$

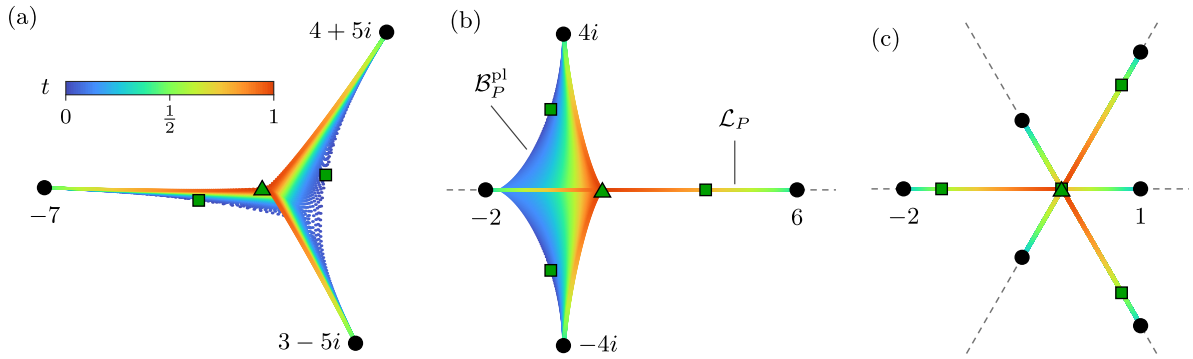


FIGURE 1. Zeros of $(P^n)^{(m)}$, $0 \le m < dn$, colored by $t = m/(dn)$. Black disks: roots; green squares: critical points; green triangle: $\mathfrak{c}_P$; dashed: reflection axes. (a) $P = (z+7)(z-3+5i)(z-4-5i)$, $n = 110$: no reflection axis, and $\mathcal{K}_P = \overline{\operatorname{int} \mathcal{K}_P}$. (b) $P = (z^2+16)(z+2)(z-6)$, $n = 120$: the planar body $\mathcal{B}_P^{\mathrm{pl}}$ and the two segments of $\mathcal{L}_P$ on the real axis. (c) $P = (z^3-1)(z^3+8)$, $n = 60$: every root lies on an axis, and every zero lies on a radial star.

where the densities are nonnegative and integrable and Γ_P is a finite union of bounded C^1 arcs of finite length. The curve measure is supported on reflection axes of P.

If $s \ge 2$, put

$$\mathcal{B}_P^{\mathrm{pl}} = \overline{\operatorname{int} \mathcal{K}_P}, \qquad \mathcal{L}_P = \overline{\mathcal{K}_P \setminus \mathcal{B}_P^{\mathrm{pl}}}.$$

Then

$$\mathcal{S}_P = \mathcal{B}_P^{\mathrm{pl}} \cup \mathcal{L}_P \cup T_P, \qquad \mathcal{B}_P^{\mathrm{pl}} = \operatorname{supp}(f_P \, \mathrm{d}A). \tag{1.6}$$

The set $\mathcal{L}_P$ is empty or a finite union of straight segments on reflection axes, and its intersection with $\mathcal{B}_P^{\mathrm{pl}}$ is finite.

In (1.5), the first two terms are the *planar part* and the *curve part*; the mass of the latter is the *curve mass*. We call $\mathcal{B}_P^{\mathrm{pl}}$ the *planar body*. If P has no reflection axis, its core equals its planar body and the only singular part of ρ_P consists of the root atoms. More generally, reflection rigidity gives a complete classification of zero-area shadows.

Theorem 1.2 (Zero-area classification)**.** *Suppose that P has at least two distinct roots. The core has zero area if and only if, after a change $P(z) \mapsto cP(az+b)$ with $ac \neq 0$, it has the form*

$$z^{r_0} H(z^q), \qquad q \ge 1, \quad r_0 \ge 0,$$

where H is a nonconstant real-rooted polynomial with $H(0) \neq 0$. In this case the full shadow equals the core and is a segment or a finite union of radial segments. In all other cases the core has nonempty interior.

A finite union of segments with a common endpoint is called a *radial star*. The signed radial segments, including those arising from negative roots of H, admit explicit sampling formulas for ρ_P. A finite coefficient test detects the relevant reflections and decides the zero-area alternative. Figure 1 illustrates the three possibilities: no reflection axis, an axis off which some roots lie, and every root on an axis.

1.3. **Conservation and reconstruction.** The analytic identity behind several of these results is

$$C'_{\rho_P}(z) = -\mathscr{C}_P(z)^2, \qquad z \in \mathbb{C} \setminus \mathcal{K}_P. \tag{1.7}$$

It holds on every complementary component, including bounded ones. A finite-degree identity for successive logarithmic derivatives first explains the exterior law; the potential characterization then gives (1.7) throughout the complement.

Let X, Y be independent with law μ_P^{root}, and let U be independent and uniform on $[0,1]$. The *chord measure*

$$\sigma_P = \mathrm{Law}((1-\mathsf{U})X + \mathsf{U}Y)$$

has the same exterior potential as ρ_P. Consequently,

$$\int z^k \,\mathrm{d}\rho_P(z) = \frac{1}{k+1}\sum_{j=0}^{k} \mathfrak{m}_j(P)\mathfrak{m}_{k-j}(P), \qquad k \geq 0. \tag{1.8}$$

For planar root configurations this identifies holomorphic moments, rather than the measures themselves.

Theorem 1.3 (Finite-moment reconstruction)**.** *The first d holomorphic moments of ρ_P determine the monic polynomial P of degree d. Explicitly, if $\mathfrak{s}_j = \int z^j \,\mathrm{d}\rho_P(z)$, then $\mathfrak{m}_0 = 1$ and*

$$\mathfrak{m}_j = \frac{1}{2}\left((j+1)\mathfrak{s}_j - \sum_{i=1}^{j-1} \mathfrak{m}_i\mathfrak{m}_{j-i}\right), \qquad 1 \leq j \leq d,$$

recover the root moments, and Newton's identities recover P. The exterior transform also determines the normalized root measure through $\mathscr{C}_P = \sqrt{-C'_{\rho_P}}$, with the branch asymptotic to z^{-1}.

Access to the whole measure gives a simpler degree-free reconstruction: its atoms have masses p_j^2, so their positive square roots recover μ_P^{root}. Thus two monic polynomials have the same shadow measure precisely when they are powers of a common monic polynomial whose root multiplicities have greatest common divisor one. Such a polynomial will be called *primitive*. The multiplicity scale can change the transient set while leaving the core and measure unchanged.

1.4. **Earlier work and the argument.** Bøgvad, Hägg, and Shapiro [BHS] studied the same descendants at fixed differentiation ratio and proved limiting zero-distribution results when P and P' have simple zeros. Their shadow-boundary conjecture [BHS, Conjecture 7.2], in degree at least three under these simplicity conditions and with roots in convex position but not forming a regular polygon, is expressed through critical values of the rational map $w \mapsto w - \alpha P(w)/P'(w)$. Our boundary theorem includes these critical values together with a second alternative, arising from pairs of inverse images with equal values of both $|P|$ and $|P'|$. It proves algebraic containment and semialgebraicity of the entire coefficient family; the collision-only boundary description and concavity in that conjecture are separate questions.

Martínez-Finkelshtein and Rakhmanov [MFR] established exterior Cauchy-transform limits for iterated differentiation of general polynomial arrays and related the evolution to a Burgers equation and fractional free convolution. Here the averaging over all derivative orders yields an exact finite-degree identity and a quantitative exterior estimate. We also establish conservation on bounded components of the complement, which is essential for the geometric continuation argument.

We begin with the potential and contour framework and the required continuity results. Conservation and moment reconstruction come next. Local changes between the harmonic branches of the potential then give the measure decomposition and the boundary alternatives. The conservation law forces consistent inclusion along the edges of a finite algebraic graph; coefficient continuity upgrades this to semialgebraicity of the family. Reflection rigidity and the explicit segment and radial models yield the classification. The remaining sections establish

regularity of the planar density and extend the arguments to finite atomic laws, including vanishing masses and area convergence. Algebraic selection at rational ratios and the potential-theoretic effect of transients are treated in the appendices.

2. Potentials, selected saddles, and support incidence

2.1. **The logarithmic separation problem.** The identities associated with (1.2) are $2\partial_z U_\nu = C_\nu$, $\Delta U_\nu = 2\pi\nu$, and $\partial_{\bar z} C_\nu = \pi\nu$, in the sense of distributions. Here $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar z} = (\partial_x + i\partial_y)/2$. We write $C_{\alpha,P} := C_{\mu_{\alpha,P}}$ and introduce the rational functions

$$\begin{aligned} &\mathsf{Q}_P(w) = \prod_{j=1}^{s}(w-a_j), \qquad \mathsf{N}_P(w) = \sum_{j=1}^{s} r_j \prod_{k\neq j}(w-a_k), \\ &\Lambda_P(w) = \frac{P'(w)}{P(w)} = \frac{\mathsf{N}_P(w)}{\mathsf{Q}_P(w)}, \qquad h_P(w) = \frac{\mathsf{Q}_P(w)}{\mathsf{N}_P(w)}, \qquad F_{P,\alpha}(w) = w - \alpha h_P(w). \end{aligned} \tag{2.1}$$

The polynomials $\mathsf{Q}_P, \mathsf{N}_P$ are coprime. In particular, $h_P = P/P'$ is understood as a reduced rational function.

For $z \notin Z(P)$ and $0 < \alpha < d$, consider the real landscape

$$L_{P,z,\alpha}(w) = \log|P(w)| - \alpha \log|w - z|.$$

Its value tends to $+\infty$ at the marked ends z and infinity and to $-\infty$ at the roots. For a real level ℓ, adjoin the marked ends to the superlevel set:

$$E_\ell(P,z,\alpha) = \{w \in \mathbb{C} \setminus (Z(P)\cup\{z\}) : L_{P,z,\alpha}(w) \geq \ell\} \cup \{z,\infty\} \subset \widehat{\mathbb{C}}.$$

The *barrier* is

$$\mathfrak{b}_P(z,\alpha) = \sup\{\ell : z \text{ and } \infty \text{ belong to the same connected component of } E_\ell(P,z,\alpha)\}. \tag{2.2}$$

At this level, let $\mathcal{U}_z$ and $\mathcal{U}_\infty$ be the components of the strict superlevel set $\{L_{P,z,\alpha} > \mathfrak{b}_P(z,\alpha)\} \cup \{z,\infty\}$ containing the respective marked ends.

An *ordinary saddle*, or simply a *saddle*, is a critical point of $L_{P,z,\alpha}$ away from the roots and marked ends. Equivalently, it solves

$$(w-z)\mathsf{N}_P(w) - \alpha \mathsf{Q}_P(w) = 0, \qquad F_{P,\alpha}(w) = z. \tag{2.3}$$

The reduced equation has degree s and leading coefficient $d-\alpha$. A saddle is *connecting*, or *selected*, precisely when it belongs to $\overline{\mathcal{U}_z} \cap \overline{\mathcal{U}_\infty}$; see Figure 2(a). It is *simple* when $F'_{P,\alpha}(w) \neq 0$; a solution with $F'_{P,\alpha}(w) = 0$ is a *saddle collision*. With local logarithm branches, the holomorphic phase is

$$\Phi_{P,z,\alpha}(w) = \log P(w) - \alpha \log(w-z), \qquad \operatorname{Re}\Phi_{P,z,\alpha} = L_{P,z,\alpha}.$$

The saddle order is the least $\kappa \geq 2$ for which $\partial_w^\kappa \Phi(w) \neq 0$; a simple saddle has order two.

The following contour facts specify the selection properties used below.

Proposition 2.1 (Separating contours and local selection)**.** [HS1, Lemmas 2.5, 2.6, and 2.8] *For $z \notin Z(P)$ and $0 < \alpha < d$:*

(i) *The barrier equals*

$$\inf_\gamma \max_{w\in\gamma} L_{P,z,\alpha}(w),$$

where γ ranges over Jordan curves avoiding $Z(P)$ and having winding number one about z. There is such a separating contour passing through the connecting saddles and lying strictly below the barrier elsewhere. This statement allows saddles of arbitrary finite order. At a unique simple connecting saddle the contour has a nondegenerate maximum; a path joining the two marked ends through that saddle has a strict minimum there.

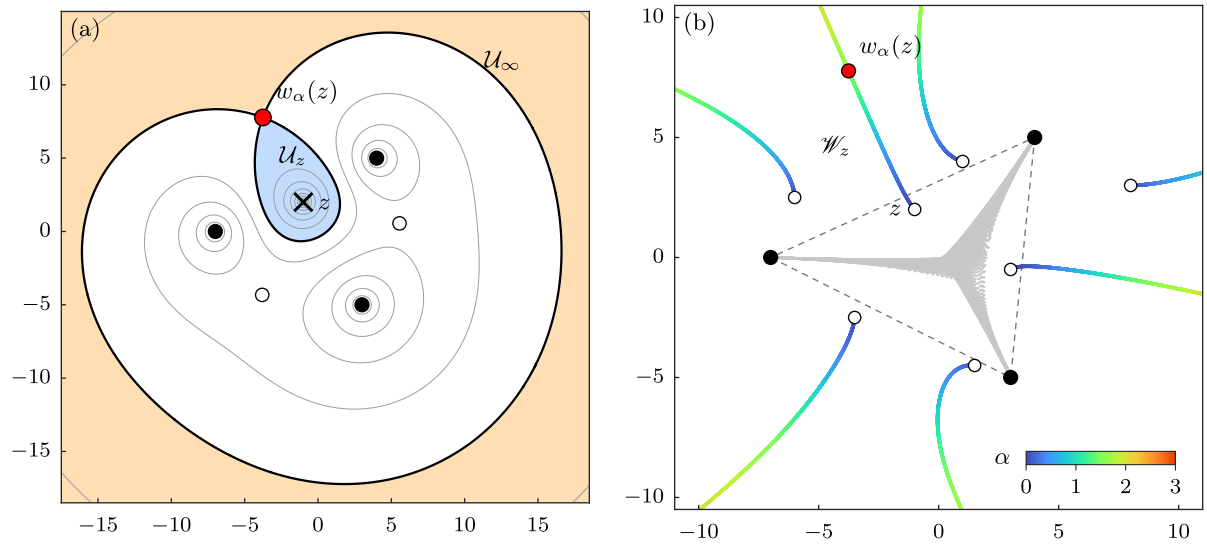


FIGURE 2. (a) The landscape $L_{P,z,\alpha}$ for the cubic of Figure 1(a), $z = -1+2i$ (cross) and $\alpha = \frac{3}{2}$. Blue and orange: the components $\mathcal{U}_z$ and $\mathcal{U}_\infty$ of $\{L_{P,z,\alpha} > \mathfrak{b}_P(z,\alpha)\}$, which meet at the connecting saddle $w_\alpha(z)$ (red); black: the level set at height $\mathfrak{b}_P(z,\alpha)$; white disks: the other two saddles. (b) Selected saddle paths $\mathscr{W}_z$ of Theorem 3.4, colored by α, from seven points $z \notin \mathcal{K}_P$ (white disks); gray: the zeros of Figure 1(a); dashed: $\partial\mathsf{H}_P$.

(ii) *If all connecting saddles at a parameter value are simple, their local analytic continuations $w_1, \dots, w_k$ satisfy*

$$\mathfrak{b}_P(z,\alpha) = \max_{1\le j\le k} L_{P,z,\alpha}(w_j(z,\alpha)) \tag{2.4}$$

in a neighborhood of that value. This holds jointly with the polynomial coefficients, even if other saddles are multiple. The corresponding contours and paths persist with uniform strict height gaps on compact pieces away from their selected saddles.

(iii) *On a root-free observation disk where $\mathfrak{b}_P(\cdot,\alpha)$ is harmonic, the connecting saddle is unique and simple. It varies holomorphically in z, and the one-saddle contour and its strict height gap persist under small changes of z, α, and the coefficients.*

2.2. **Potentials and complete support limits.** The entropy term in the normalization is

$$\mathfrak{h}_d(\alpha) = d\log d - \alpha\log\alpha - (d-\alpha)\log(d-\alpha).$$

We record the potential and support results together, since both weak limits and complete zero-set limits enter the arguments.

Proposition 2.2 (Fixed-ratio measures and selected inverses)**.** [HS1, Theorem 3.3, Propositions 3.4–3.5, and Theorem 3.9] *The measure $\mu_{\alpha,P}$ is a probability measure supported in H_P, and its potential is*

$$U_{\mu_{\alpha,P}}(z) = \frac{\mathfrak{b}_P(z,\alpha) - \mathfrak{h}_d(\alpha)}{d-\alpha} \qquad (z \notin Z(P)), \tag{2.5}$$

with the subharmonic extension at the roots. Its atomic part is

$$\sum_{j=1}^{s} \frac{(r_j-\alpha)_+}{d-\alpha}\,\delta_{a_j}, \qquad x_+ := \max\{x,0\}. \tag{2.6}$$

The measures vary weakly continuously, and their supports vary in Hausdorff distance, jointly in P and $\alpha \in (0, d)$. The complete zero sets of $R_{m,n}$ converge to these supports whenever $m/n \to \alpha$; the convergence is locally uniform in these parameters.

On $\mathbb{C} \setminus \mathcal{K}^{\mathrm{fix}}_{\alpha,P}$ the function $C_{\alpha,P}$ has no zeros, and

$$\begin{aligned} w_\alpha(z) &= z + \frac{\alpha}{(d-\alpha)C_{\alpha,P}(z)}, \\ F_{P,\alpha}(w_\alpha(z)) = z, &\qquad \Lambda_P(w_\alpha(z)) = (d-\alpha)C_{\alpha,P}(z). \end{aligned} \tag{2.7}$$

At a non-root observation point this is the unique selected simple saddle. It avoids the roots of P and the poles of h_P, is holomorphic in z, and is locally real-analytic in α.

Proposition 2.3 (Endpoints, pooling, and the full shadow)**.** [HS1, Theorems 4.5, 5.1, 5.3, 5.6, and 5.7] *As $\alpha \downarrow 0$ and $\alpha \uparrow d$, respectively,*

$$\begin{aligned} &\mu_{\alpha,P} \Longrightarrow \mu_P^{\mathrm{root}}, && \mathcal{K}^{\mathrm{fix}}_{\alpha,P} \longrightarrow Z(P) \cup Z(P'), \\ &\mu_{\alpha,P} \Longrightarrow \delta_{\mathfrak{c}_P}, && \mathcal{K}^{\mathrm{fix}}_{\alpha,P} \longrightarrow \{\mathfrak{c}_P\}, \end{aligned}$$

where the support limits are in Hausdorff distance. The pooled measures in (1.4) *converge weakly to ρ_P, and*

$$\mathcal{K}_P = \overline{\bigcup_{0<\alpha<d} \mathcal{K}^{\mathrm{fix}}_{\alpha,P}}, \qquad \mathcal{Z}_{P,n} \longrightarrow \mathcal{K}_P. \tag{2.8}$$

The latter convergence is uniform on compact fixed-degree coefficient sets, and $P \mapsto \mathcal{K}_P$ is Hausdorff continuous. The core is connected and contains $Z(P) \cup Z(P') \cup \{\mathfrak{c}_P\}$. If $s \geq 2$, it has no isolated points and

$$\mathcal{S}_P = \mathcal{K}_P \dot\cup T_P, \quad \operatorname{acc}(\mathcal{S}_P) = \mathcal{K}_P, \quad \operatorname{int} \mathcal{S}_P = \operatorname{int} \mathcal{K}_P, \quad \partial \mathcal{S}_P = \partial \mathcal{K}_P \cup T_P,$$

where $\operatorname{acc}(E)$ denotes limits of spatially distinct points of E and T_P is countable and locally finite outside $\mathcal{K}_P$. For $s = 1$, both shadows are the single root.

Integrating (2.6) gives the atomic part of the pooled measure:

$$\rho_P^{\mathrm{at}} = \sum_{j=1}^{s} p_j^2 \delta_{a_j}, \qquad \rho_P(\{a_j\}) = \frac{2}{d^2} \int_0^{r_j} (r_j - \alpha)\, \mathrm{d}\alpha. \tag{2.9}$$

The measures and cores are unchanged by replacing P by a positive integer power. They are covariant under affine changes of variables: if $\phi(z) = az + b$, $a \neq 0$, and $\widetilde{P}$ is the monic scalar multiple of $P \circ \phi$, then $\rho_{\widetilde{P}} = (\phi^{-1})_* \rho_P$ and $\mathcal{K}_{\widetilde{P}} = \phi^{-1}(\mathcal{K}_P)$; the same covariance holds for the full shadow [HS1, Proposition 5.2].

2.3. **Finite-dimensional geometry.** A set or function is *definable* here if its graph is given, with real parameters, by a first-order formula in the real exponential field $\mathbb{R}_{\exp} = (\mathbb{R}; <, +, \cdot, \exp)$. This structure is *o-minimal*: every definable subset of the real line is a finite union of intervals and points [Wil]. Definable sets admit finite C^k stratifications, compatible with any specified finite family of subsets and with the ranks of specified definable maps. A stratum is a connected C^k submanifold in such a partition of a carrier S; for each stratum C, the relative closure difference $(\overline{C} \setminus C) \cap S$ is a union of lower-dimensional strata. The dimension of a definable set is the largest stratum dimension. Projection and closure preserve definability, and closure preserves dimension [vdD].

Proposition 2.4 (Definable polynomial families)**.** [HS2, Theorem 4.1] *The barrier, fixed-ratio supports, and their fixed-degree coefficient families are definable. Their incidence sets therefore admit finite stratifications compatible with prescribed definable subsets and projection ranks.*

Proof. The cited joint definability includes every root multiplicity pattern. The changes $\alpha = dt$ and $H = dh$ recover the original ratio and barrier height for monic polynomials. For the frontier condition, start with a compatible smooth cell partition on which all prescribed map ranks are constant. Put all cell closure differences and all lower-dimensional cells into one closed definable exceptional set of smaller dimension. The connected components of the remaining top-dimensional cells retain the same chart maps. Recursively stratify the exceptional set, compatibly also with the closures of these components. This gives the relative frontier condition and retains all original constant ranks. □

A bounded definable curve is a finite union of points and C^1 arcs of finite length: subdividing into arcs on which both coordinates are monotone bounds the length by the sum of their coordinate variations. We shall also use semialgebraic triangulation, quantifier elimination, and compatible topological trivializations [BCR, Har]. Semialgebraic triviality means that a finite partition of parameter space makes the given family topologically a product over each piece, simultaneously for the specified subsets.

For an analytic saddle branch $w = w(z, \alpha)$ with real critical height $\psi(z, \alpha) = \operatorname{Re} \Phi_{P,z,\alpha}(w(z, \alpha))$, differentiation at the critical point gives

$$2\partial_z \psi = \frac{\alpha}{w - z}, \qquad \partial_\alpha \psi = -\log |w - z|. \tag{2.10}$$

Distinct saddles thus have distinct spatial gradients.

2.4. **The compact incidence space.** For fixed P, define the *support incidence space* by

$$\mathcal{I}_P := \overline{\{(z, \alpha) : 0 < \alpha < d,\ z \in \mathcal{K}^{\mathrm{fix}}_{\alpha,P}\}} \subset \mathsf{H}_P \times [0, d], \qquad \mathrm{pr}_z(z, \alpha) := z. \tag{2.11}$$

The closure includes the endpoint fibers described in Proposition 2.3. The set $\mathcal{I}_P$ is compact and definable, and $\mathrm{pr}_z(\mathcal{I}_P) = \mathcal{K}_P$. Away from roots and saddle collisions, the barrier is locally one of finitely many harmonic critical-height branches. Its Riesz measure, namely its distributional Laplacian divided by 2π, is therefore supported where at least two selected heights agree. By (2.10), distinct saddles have distinct spatial gradients, so their equal-height sets are locally smooth arcs. Consequently each fiber at a fixed interior ratio has dimension at most one, and definable dimension together with fiberwise closure gives $\dim \mathcal{I}_P \le 2$. Figure 3 shows $\mathcal{I}_P$ for a cubic.

Proposition 2.5 (Projection principle)**.** *Choose a finite C^2 stratification of $\mathcal{I}_P$ compatible with the endpoint fibers and the vertical root lines $\{a_j\} \times [0, d]$, and refine it so that $D\,\mathrm{pr}_z$ has constant rank on each stratum. Every point of $\mathcal{I}_P$ that projects to $\partial\mathcal{K}_P$ lies in a stratum on which pr_z has rank less than two. Hence the boundary is contained in the union of the images of the rank-deficient strata.*

Proof. If $D\,\mathrm{pr}_z$ has rank two at a point of a stratum, then the restriction of pr_z to that stratum maps a neighborhood of that point onto a neighborhood in the observation plane. This neighborhood lies in $\mathrm{pr}_z(\mathcal{I}_P) = \mathcal{K}_P$, so the image point is interior. □

The rank-two strata account for planar contributions. Images of the remaining strata have dimension at most one.

Lemma 2.6 (Closedness of the non-root support incidence)**.** *The set*

$$\{(z, \alpha) : z \notin Z(P),\ 0 < \alpha < d,\ z \in \mathcal{K}^{\mathrm{fix}}_{\alpha,P}\}$$

is relatively closed in $(\mathbb{C} \setminus Z(P)) \times (0, d)$.

Proof. At a point outside this set, the fixed-ratio potential is harmonic on a root-free observation disk. Proposition 2.1 gives one simple connecting saddle. By Proposition 2.1 its critical height

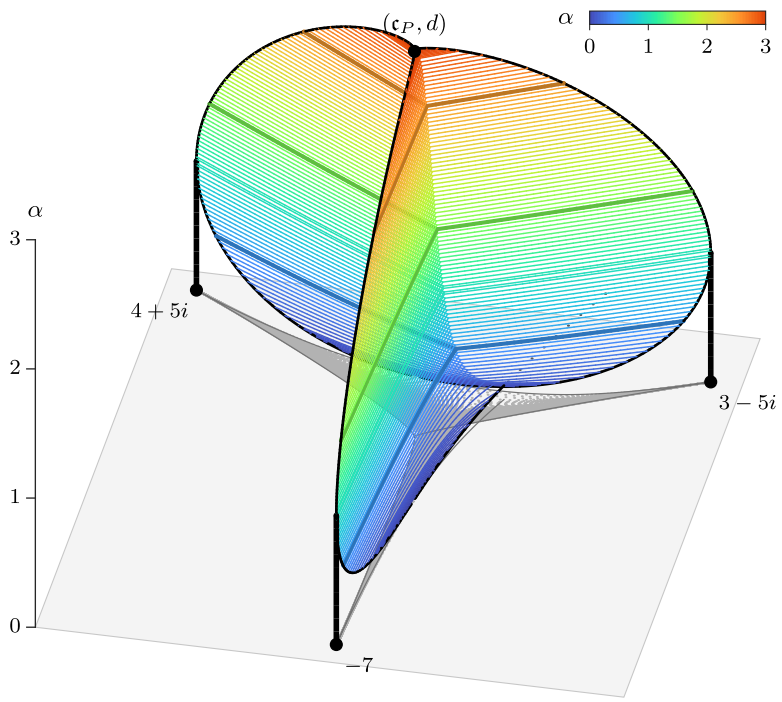


FIGURE 3. The incidence space $\mathcal{I}_P$ for the cubic of Figure 1(a): the supports $\mathcal{K}^{\mathrm{fix}}_{\alpha,P}$, traced numerically, at height α and colored by α (thicker: $\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$). Black: critical values of $F_{P,\alpha}$ that are endpoints of $\mathcal{K}^{\mathrm{fix}}_{\alpha,P}$, at height α; dotted: the other critical values; thick segments: the root lines $\{a_j\} \times [0,1]$. The floor shows the zeros of $(P^{110})^{(m)}$, approximating $\mathrm{pr}_z(\mathcal{I}_P) = \mathcal{K}_P$.

continues to equal the barrier on a joint neighborhood in (z, α). It is harmonic in z for every nearby ratio, so that neighborhood contains no support incidence. □

3. Conservation and moment reconstruction

An exact identity for finite polynomials leads to an exterior approximation by chord measures. We then use the selected inverse to prove conservation on every complementary component of the core. The resulting identities recover root moments and constrain shadow-free curves.

3.1. **An exact finite-degree identity.**

Proposition 3.1 (Telescoping logarithmic derivatives)**.** *Let G be a polynomial of degree $N \geq 1$. Its normalized root measure and root Cauchy transform are*

$$\mu_G^{\mathrm{root}} := \frac{1}{N} \operatorname{div}_0 G, \qquad \mathscr{C}_G := C_{\mu_G^{\mathrm{root}}} = \frac{1}{N} \frac{G'}{G}.$$

For $0 \leq m < N$, define the logarithmic derivative of the mth derivative by

$$\mathfrak{q}_m := \frac{G^{(m+1)}}{G^{(m)}},$$

and adopt the terminal convention $\mathfrak{q}_N := 0$. *Define the pooled derivative-zero measure and its Cauchy transform by*

$$\nu_G^{\mathrm{pool}} := \frac{2}{N(N+1)} \sum_{m=0}^{N-1} \operatorname{div}_0 G^{(m)}, \qquad C_G^{\mathrm{pool}} := C_{\nu_G^{\mathrm{pool}}}.$$

Then the following rational identity holds:

$$(C_G^{\mathrm{pool}})' = -\frac{N}{N+1}\mathscr{C}_G^2 - \frac{1}{N(N+1)} \sum_{m=0}^{N-1} \left(\frac{\mathfrak{q}_m'}{\mathfrak{q}_m}\right)^2. \tag{3.1}$$

For $G = P^n$, *the second term is* $O(n^{-1})$ *locally uniformly on every set sufficiently far from* H_P.

Proof. Logarithmic differentiation of $\mathfrak{q}_m = G^{(m+1)}/G^{(m)}$ gives the elementary identity

$$\mathfrak{q}_{m+1} = \mathfrak{q}_m + \frac{\mathfrak{q}_m'}{\mathfrak{q}_m}.$$

It is a rational identity even at multiple zeros. Squaring and subtracting gives

$$\mathfrak{q}_{m+1}^2 - \mathfrak{q}_m^2 = 2\mathfrak{q}_m' + \left(\frac{\mathfrak{q}_m'}{\mathfrak{q}_m}\right)^2.$$

Sum over $m = 0, \dots, N-1$, use $\mathfrak{q}_0 = N\mathscr{C}_G$ and $\mathfrak{q}_N = 0$, and divide by $N(N+1)$. Since $C_G^{\mathrm{pool}} = 2\sum_m \mathfrak{q}_m/(N(N+1))$, the result is (3.1).

Suppose the roots of G lie in $|z| \le R$. Gauss–Lucas gives the same confinement for every derivative. For $|z| > 3R+1$, the Cauchy transform of any probability measure supported in that disk is uniformly separated from zero after multiplication by z, while its derivative is bounded by a constant times $|z|^{-2}$. Thus $|\mathfrak{q}_m'/\mathfrak{q}_m|$ is bounded independently of m and N. There are N summands and the prefactor is $O(N^{-2})$, which proves the final assertion. □

For $G = P^n$, one has $\mathscr{C}_G = \mathscr{C}_P = P'/(dP)$ and $\nu_G^{\mathrm{pool}} = \bar{\nu}_{P,n}$. Passing to the limit in (3.1) already gives $C'_{\rho_P} = -\mathscr{C}_P^2$ near infinity and an $O(n^{-1})$ error for each fixed moment of $\bar{\nu}_{P,n}$.

3.2. **A uniform exterior estimate.** The preceding identity gives a quantitative result for every polynomial. For a polynomial G of degree N, let σ_G be its chord probability measure: draw two independent roots with law $\mu_G^{\mathrm{root}} = N^{-1}\operatorname{div}_0 G$ and then a uniform point on their segment. All transforms below are evaluated outside $\mathsf{H}_G = \operatorname{conv} Z(G)$.

Theorem 3.2 (Uniform exterior chord approximation)**.** *For every nonconstant polynomial* G *of degree* N, *every* $z \notin \mathsf{H}_G$, *and* $\delta = \operatorname{dist}(z, \mathsf{H}_G)$,

$$\left|(C_G^{\mathrm{pool}} - C_{\sigma_G})'(z)\right| \le \frac{2}{(N+1)\delta^2}, \tag{3.2}$$

$$\left|C_G^{\mathrm{pool}}(z) - C_{\sigma_G}(z)\right| \le \frac{2}{(N+1)\delta}. \tag{3.3}$$

If Γ *is a rectifiable Jordan curve surrounding* H_G, *at positive distance* δ_Γ *from it, and* f *is holomorphic on a neighborhood of its closure and interior, then*

$$\left|\int f \,\mathrm{d}\nu_G^{\mathrm{pool}} - \int f \,\mathrm{d}\sigma_G\right| \le \frac{\operatorname{length}(\Gamma)}{\pi(N+1)\delta_\Gamma} \max_\Gamma |f|. \tag{3.4}$$

Proof. We first prove the estimate on each summand in (3.1). Choose a nearest point $b \in \mathsf{H}_G$ to z and put $e = (z-b)/\delta$. Convexity gives $\operatorname{Re}(\overline{e}(z-a)) \geq \delta$ for every $a \in \mathsf{H}_G$. For any nonconstant polynomial G with roots $a_j \in \mathsf{H}_G$, listed with multiplicity, write $q = G'/G$. Then

$$|q'(z)| \leq \sum_j |z-a_j|^{-2}, \qquad \operatorname{Re}(eq(z)) = \sum_j \frac{\operatorname{Re}(\overline{e}(z-a_j))}{|z-a_j|^2} \geq \delta \sum_j |z-a_j|^{-2}.$$

Consequently $q(z) \neq 0$ and

$$|q'(z)/q(z)| \leq 1/\delta. \tag{3.5}$$

Gauss–Lucas makes this applicable to every $G = G^{(m)}$, $m < N$. The chord integral for a fixed pair of roots, differentiated in z, gives $C'_{\sigma_G} = -\mathscr{C}_G^2$. Therefore (3.1) yields

$$\bigl(C_G^{\mathrm{pool}} - C_{\sigma_G}\bigr)' = \frac{\mathscr{C}_G^2}{N+1} - \frac{1}{N(N+1)} \sum_{m=0}^{N-1} (\mathfrak{q}'_m/\mathfrak{q}_m)^2.$$

Since $|\mathscr{C}_G| \leq 1/\delta$, the derivative estimate follows.

On the exterior normal ray $z + te$, $t \geq 0$, the distance to H_G is at least $\delta + t$. Both probability Cauchy transforms tend to zero at infinity, so integration of their derivative difference along this ray gives

$$|C_G^{\mathrm{pool}}(z) - C_{\sigma_G}(z)| \leq \frac{2}{N+1} \int_0^\infty (\delta+t)^{-2}\,\mathrm{d}t = \frac{2}{(N+1)\delta}.$$

Finally the winding index n of Γ is constant and nonzero on H_G. Cauchy's integral formula and Fubini express n times the difference of the two tests as $(2\pi i)^{-1}$ times the contour integral of $f(C_G^{\mathrm{pool}} - C_{\sigma_G})$. Since $|n| \geq 1$, the factor $1/(2\pi)$ and (3.3) give (3.4). □

Corollary 3.3 (Exterior universality for arbitrary bounded root arrays)**.** *Let* $\deg G_j = N_j \to \infty$, *suppose all input roots lie in one compact convex set* K, *and suppose* $N_j^{-1} \operatorname{div}_0 G_j \Longrightarrow \eta$. *Here* η *is an arbitrary probability measure supported in* K. *If* σ_η *is the law of a uniform point on a chord between two independent samples from* η, *then*

$$C_{G_j}^{\mathrm{pool}} \longrightarrow C_{\sigma_\eta} \quad \textit{locally uniformly on } \mathbb{C} \setminus K.$$

Every weak subsequential limit of $\nu_{G_j}^{\mathrm{pool}}$ *has the same logarithmic potential as* σ_η *on* $\mathbb{C} \setminus K$ *and has moments*

$$\lim_{j\to\infty} \int z^k \,\mathrm{d}\nu_{G_j}^{\mathrm{pool}} = \frac{1}{k+1} \sum_{\ell=0}^{k} \left(\int z^\ell \,\mathrm{d}\eta\right)\left(\int z^{k-\ell}\,\mathrm{d}\eta\right) \qquad (k \geq 0). \tag{3.6}$$

For $G_j = P^j$, (3.3) *gives the explicit error* $2/[(dj+1)\operatorname{dist}(z, \mathsf{H}_P)]$ *throughout the hull complement.*

Proof. Products of weakly convergent probabilities on the compact set K converge weakly; the continuous chord map then gives $\sigma_{G_j} \Longrightarrow \sigma_\eta$. Their Cauchy transforms converge locally uniformly off K, and the finite-degree estimate proves the first assertion. Gauss–Lucas gives compactness of the pooled measures. Every weak cluster has the stated exterior transform; the difference of the logarithmic potentials has zero derivative along each exterior normal ray and tends to zero at infinity, so it vanishes. The expansion of the exterior Cauchy transform at infinity identifies all complex moments. Compactness thus gives the same limit for every subsequence of moments, and hence convergence of the whole sequence. Finally $C'_{\sigma_\eta} = -C_\eta^2$; equating coefficients at infinity gives the convolution in (3.6). □

3.3. **Conservation on complementary components.**

Theorem 3.4 (Cauchy-transform conservation law)**.** *For every $z \notin \mathcal{K}_P$,*

$$C'_{\rho_P}(z) = -\frac{1}{d^2}\left(\frac{P'(z)}{P(z)}\right)^2 = -\mathscr{C}_P(z)^2. \tag{3.7}$$

More precisely, let $w_\alpha(z)$ be the selected inverse branch from Proposition 2.2. The oriented selected saddle path $\mathscr{W}_z$ *is the path*

$$\alpha \longmapsto w_\alpha(z), \qquad 0 < \alpha < d,$$

with increasing α. Its endpoint limits are z as $\alpha \downarrow 0$ and infinity as $\alpha \uparrow d$, and

$$C_{\rho_P}(z) = \frac{1}{d^2}\int_{\mathscr{W}_z}\left(\frac{P'(w)}{P(w)}\right)^2 \mathrm{d}w. \tag{3.8}$$

The orientation in (3.8) *is that of increasing ratio.*

Proof. Fix a disk compactly contained in $\mathbb{C} \setminus \mathcal{K}_P$. Every $\mu_{\alpha,P}$ is supported away from that disk, so its Cauchy transform is holomorphic in z there and continuous in α. Proposition 2.2 and the implicit-function theorem make w_α locally real-analytic in α, because $F'_{P,\alpha}(w_\alpha) \neq 0$. The selected branch avoids the roots of P and the poles of h_P.

As $\alpha \downarrow 0$, the transforms $C_{\alpha,P}$ converge uniformly on the disk to $\mathscr{C}_P$. Every zero of $\mathscr{C}_P$ belongs to $\mathcal{K}_P$ by Proposition 2.3; hence $w_\alpha(z) \to z$. As $\alpha \uparrow d$, the same theorem gives $C_{\alpha,P}(z) \to (z - \mathfrak{c}_P)^{-1}$ uniformly on the disk, and therefore $w_\alpha(z) \to \infty$.

Recall $\Lambda_P = P'/P$. Along the selected path, the saddle relation $\alpha = (w - z)\Lambda_P(w)$ gives

$$\Lambda_P(w)\,\mathrm{d}\alpha = \big(\Lambda_P(w)^2 + (w - z)\Lambda_P(w)\Lambda'_P(w)\big)\,\mathrm{d}w = \tfrac{1}{2}\,\mathrm{d}\big((w - z)\Lambda_P(w)^2\big) + \tfrac{1}{2}\Lambda_P(w)^2\,\mathrm{d}w.$$

The boundary term vanishes at both endpoints: it contains the factor $w - z$ at the initial endpoint and is $O(w^{-1})$ at infinity. The averaging formula now yields

$$C_{\rho_P}(z) = \frac{2}{d^2}\int_0^d (d - \alpha)C_{\alpha,P}(z)\,\mathrm{d}\alpha = \frac{2}{d^2}\int_0^d \Lambda_P(w_\alpha(z))\,\mathrm{d}\alpha,$$

and the preceding differential identity gives (3.8), with $|(d - \alpha)C_{\alpha,P}(z)| \leq (d - \alpha)/\operatorname{dist}(z, \mathcal{K}_P)$.

To prove the differential identity, set

$$\widehat{C}_{\alpha,P}(z) = (d - \alpha)C_{\alpha,P}(z) = \Lambda_P(w_\alpha(z)).$$

Differentiating $w_\alpha = z + \alpha/\widehat{C}_{\alpha,P}$ and $\widehat{C}_{\alpha,P} = \Lambda_P(w_\alpha)$ gives

$$\widehat{C}_{\alpha,P}\,\partial_\alpha \widehat{C}_{\alpha,P} = \partial_z \widehat{C}_{\alpha,P}, \qquad \partial_\alpha(\widehat{C}^2_{\alpha,P}) = 2\partial_z \widehat{C}_{\alpha,P}.$$

If the observation disk is at distance $\delta > 0$ from $\mathcal{K}_P$, then $|\partial_z \widehat{C}_{\alpha,P}| \leq (d - \alpha)/\delta^2$. Differentiation may therefore be interchanged with integration in α. The endpoint limits are $\widehat{C}_{\alpha,P}(z) \to \Lambda_P(z)$ as $\alpha \downarrow 0$ and $\widehat{C}_{\alpha,P}(z) \to 0$ as $\alpha \uparrow d$. Consequently,

$$C'_{\rho_P}(z) = \frac{2}{d^2}\int_0^d \partial_z \widehat{C}_{\alpha,P}(z)\,\mathrm{d}\alpha = \frac{1}{d^2}[\widehat{C}_{\alpha,P}(z)^2]_{\alpha=0}^{\alpha=d} = -\frac{\Lambda_P(z)^2}{d^2}.$$

The physical transforms extend continuously and without zeros to $[0, d]$ at this observation, so they are bounded away from zero. The first differential identity also gives $\partial_\alpha \widehat{C}_{\alpha,P} = (\partial_z C_{\alpha,P})/C_{\alpha,P}$, which is bounded. In (3.8) the path integral means the improper parameter integral of $\Lambda_P(w_\alpha)^2 \partial_\alpha w_\alpha$. Differentiating the explicit selected inverse identifies this integrand with $\widehat{C}_{\alpha,P} - \alpha\partial_\alpha \widehat{C}_{\alpha,P}$; thus the path integral is absolutely convergent.

The argument applies to every disk outside $\mathcal{K}_P$, and hence to every complementary component. □

Figure 2(b) shows selected saddle paths for several points outside the core.

3.4. **Random chords and moments.**

Theorem 3.5 (The chord law)**.** *Let $\sigma_P = \mathrm{Law}((1-\mathsf{U})X + \mathsf{U}Y)$ be the chord measure from the introduction. Then*

$$C_{\rho_P} = C_{\sigma_P}, \qquad U_{\rho_P} = U_{\sigma_P} \qquad \text{on } \mathbb{C} \setminus \mathsf{H}_P.$$

The moment identity (1.8) *holds for every $k \geq 0$.*

Proof. For $a, b \in Z(P)$ and $z \notin \mathsf{H}_P$, the elementary identity

$$\int_0^1 \frac{\mathrm{d}t}{(z-(1-t)a-tb)^2} = \frac{1}{(z-a)(z-b)}$$

follows from the primitive $t/((z-a)(z-(1-t)a-tb))$, including $a=b$. Differentiating the chord Cauchy integral and averaging over the independent random roots gives $C'_{\sigma_P} = -\mathscr{C}_P^2$. By Theorem 3.4, C_{ρ_P} has the same derivative on $\mathbb{C} \setminus \mathsf{H}_P$, and both transforms tend to zero at infinity. On the outward normal ray from any point outside the compact convex hull, their difference has zero derivative and zero limit, so it vanishes at the initial point. Their logarithmic potentials also differ by a constant along each such ray; the common probability-mass normalization at infinity fixes this constant to zero.

For the moments, expand the random chord directly:

$$\mathbb{E}[((1-\mathsf{U})X + \mathsf{U}Y)^k] = \sum_{j=0}^{k} \binom{k}{j} \mathfrak{m}_j(P)\mathfrak{m}_{k-j}(P) \int_0^1 (1-t)^j t^{k-j}\,\mathrm{d}t = \frac{1}{k+1}\sum_{j=0}^{k} \mathfrak{m}_j(P)\mathfrak{m}_{k-j}(P).$$

Equivalently, comparing Taylor coefficients at infinity in $C'_{\sigma_P} = -\mathscr{C}_P^2$ gives the same convolution. Equality of the exterior transforms identifies these quantities with the moments of ρ_P. □

Figure 4(a) compares the potentials of ρ_P and σ_P for a cubic, and Figure 4(b) illustrates the rate in Theorem 3.2.

3.5. **What the canonical measure remembers.** Call a monic polynomial *primitive* if the greatest common divisor of its positive root multiplicities is one.

Theorem 3.6 (The fibers of the polynomial shadow measure)**.** *For nonconstant monic polynomials P, Q, the following are equivalent:*

(i) *$\rho_P = \rho_Q$;*
(ii) *their normalized root measures are equal;*
(iii) *$P = {P_*}^k$ and $Q = {P_*}^\ell$ for a common primitive monic polynomial P_* and positive integers k, ℓ.*

In particular, a shadow measure has exactly the fiber $\{P_*, {P_*}^2, {P_*}^3, \ldots\}$ for a unique primitive P_*. If the degree d is known, the first d holomorphic moments of ρ_P determine P.

More explicitly, put $\mathfrak{s}_j = \int z^j\,\mathrm{d}\rho_P(z)$, $\mathfrak{m}_0 = 1$, and recursively define

$$\mathfrak{m}_j = \frac{1}{2}\left((j+1)\mathfrak{s}_j - \sum_{i=1}^{j-1} \mathfrak{m}_i\mathfrak{m}_{j-i}\right), \qquad j \geq 1. \tag{3.9}$$

Then $\mathfrak{m}_j$ is the jth moment of the normalized root measure. Near infinity the normalized root Cauchy transform is recovered by

$$\mathscr{C}_P(z) = \sqrt{-C'_{\rho_P}(z)}, \qquad \mathscr{C}_P(z) = z^{-1} + O(z^{-2}), \tag{3.10}$$

where the asymptotic condition specifies the square-root branch.

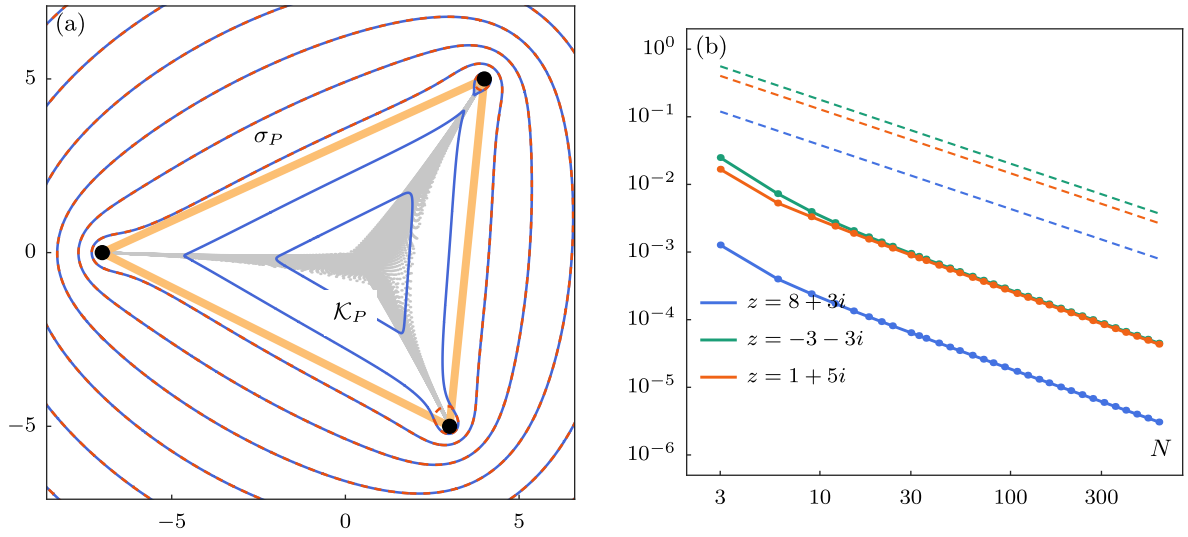


FIGURE 4. (a) Level curves of the logarithmic potentials of $\bar{\nu}_{P,n}$ (solid; $n = 110$) and of σ_P (dashed) for the cubic of Figure 1(a). They agree outside H_P, where $U_{\rho_P} = U_{\sigma_P}$, but not inside. Orange: the edges carrying σ_P; gray: zeros. (b) $|C_G^{\mathrm{pool}}(z) - C_{\sigma_G}(z)|$ for $G = P^n$, $N = 3n$, at three points $z \notin \mathsf{H}_P$, with the bounds $2/((N+1)\delta)$ of (3.3) (dashed).

Proof. The atomic formula (2.9) gives

$$\mu_P^{\mathrm{root}} = \sum_{a:\, \rho_P(\{a\})>0} \sqrt{\rho_P(\{a\})}\, \delta_a. \tag{3.11}$$

Thus (i) implies (ii) directly. The exterior reconstruction in (3.10) uses less information: Theorem 3.4 gives the square identity. The convergent Cauchy expansion of the compactly supported root probability is $\mathscr{C}_P(z) = z^{-1} + O(z^{-2})$. Any two square roots with this asymptotic have nonzero sum near infinity, so factoring the difference of their squares proves equality there. The rational continuation has the root locations as poles and the normalized multiplicities as residues. Finally, (3.9) follows by isolating the two terms containing $\mathfrak{m}_j$ in the moment identity of Theorem 3.5.

Write the distinct roots of P as a_j, their positive multiplicities as r_j, and $g = \gcd_j r_j$. Put $r_j^* = r_j/g$ and

$$P_*(z) = \prod_{j=1}^{s} (z - a_j)^{r_j^*}.$$

Then $P = {P_*}^g$ and the multiplicities of P_* have gcd one. If Q has the same normalized root measure, it has the same distinct roots. Writing its multiplicities as s_j, degree as e, and $h = \gcd_j s_j$, equality of root masses gives $er_j = ds_j$. Taking gcds gives $eg = dh$; cancellation therefore gives $r_j/g = s_j/h$ for every j. Hence $Q = {P_*}^h$. This also proves uniqueness of the primitive representative, and establishes (ii) implies (iii). Conversely, the complete pooled zero measures for $({P_*}^k)^n$ and ${P_*}^{kn}$ agree exactly, including their normalizations and all their derivative orders. Letting $n \to \infty$ proves $\rho_{P_*^k} = \rho_{P_*}$, so (iii) implies (i).

For the finite reconstruction at known degree, set $s_j = d\mathfrak{m}_j$ for $1 \le j \le d$. Newton's identities recover the elementary symmetric functions by $e_0 = 1$ and

$$je_j = \sum_{i=1}^{j} (-1)^{i-1} e_{j-i} s_i, \qquad 1 \le j \le d.$$

Then $P(z) = \sum_{j=0}^{d} (-1)^j e_j z^{d-j}$. The one-root case is included: its shadow measure is the point mass at that root. □

Corollary 3.7 (The discrete ambiguity in recovering the full shadow)**.** *If P_* is primitive, all polynomials in the fiber of ρ_{P_*} have the same core, while*

$$\mathcal{S}_{P_*^k} \subseteq \mathcal{S}_{P_*}, \qquad \bigcap_{k \ge 1} \mathcal{S}_{P_*^k} = \mathcal{K}_{P_*}, \qquad \mathcal{S}_{P_*^k} \longrightarrow \mathcal{K}_{P_*} \quad \textit{in Hausdorff distance.} \tag{3.12}$$

Thus the canonical measure determines both the common core and the maximal full shadow in its polynomial fiber. It need not determine which full shadow is present when the common multiplicity scale is unspecified.

Proof. Each descendant of $(P_*{}^k)^n$ is a descendant of $P_*{}^{kn}$, proving the inclusion. The intersection and convergence follow from the removal of transients under passage to powers [HS1, Proposition 5.9]. The measure determines P_* by Theorem 3.6, hence also determines the two sets so specified. The distinction between members of the fiber is genuine. The polynomial

$$Q_*(z) = z^4 + \tfrac{1}{2}z^3 - \tfrac{15}{32}z^2 + \tfrac{1}{2}z + 1$$

has a transient by [HS1, Theorem 6.3]. Let P_* be its primitive root, so that $Q_* = P_*{}^r$ for some positive integer r. Choose a transient x of Q_*. Power invariance of the core and the inclusion $\mathcal{S}_{P_*^r} \subseteq \mathcal{S}_{P_*}$ make x a transient of P_*. By [HS1, Proposition 5.9], x is absent from $\mathcal{S}_{P_*^k}$ for all sufficiently large k, although $\rho_{P_*^k} = \rho_{P_*}$. □

3.6. **Periods and winding transport.** Boundary integrals sum over finitely many disjoint regular simple C^1 loops. Positive orientation puts D on the left: in the local chart $w \mapsto \gamma(t + \operatorname{Re} w) + i \operatorname{Im} w\, \gamma'(t)$ it occupies $\operatorname{Im} w > 0$. The aggregate winding is consequently 1 on D and 0 outside its closure.

Proposition 3.8 (Moments inside a shadow-free boundary)**.** *Let D be a bounded open set with smooth positively oriented boundary and $\partial D \cap \mathcal{K}_P = \varnothing$. Then*

$$\oint_{\partial D} \mathscr{C}_P(z)^2 \, \mathrm{d}z = 0, \tag{3.13}$$

$$\int_D z^k \, \mathrm{d}\rho_P(z) = \frac{1}{2\pi i (k+1)} \oint_{\partial D} z^{k+1} \mathscr{C}_P(z)^2 \, \mathrm{d}z, \qquad k \ge 0. \tag{3.14}$$

In particular, $\rho_P(D) = 0$ if D contains no root, while $\rho_P(D) = 1$ if D contains every root.

Proof. Integrating $C'_{\rho_P} = -\mathscr{C}_P^2$ around every boundary component gives (3.13). Fubini's theorem in the Cauchy integral, followed by integration by parts, gives

$$2\pi i \int_D z^k \, \mathrm{d}\rho_P(z) = \oint_{\partial D} z^k C_{\rho_P}(z) \, \mathrm{d}z = \frac{1}{k+1} \oint_{\partial D} z^{k+1} \mathscr{C}_P(z)^2 \, \mathrm{d}z.$$

The core is connected by [HS1, Theorem 5.6] and contains all roots. Since it avoids ∂D, it lies wholly in D or wholly outside D. A root fixes the side, proving both mass assertions. □

Lemma 3.9 (Winding-number transport)**.** *Let γ be a closed piecewise smooth curve in $\mathbb{C} \setminus \mathcal{K}_P$. For each distinct root a_j,*

$$\operatorname{wind}(\gamma, a_j) = \operatorname{wind}(\gamma, \mathfrak{c}_P). \tag{3.15}$$

Proof. The endpoint limits in Proposition 2.3 complete the physical measures continuously by $\nu_{0,P} = \mu_P^{\mathrm{root}}$ and $\nu_{d,P} = \delta_{\mathfrak{c}_P}$. They are carried by $\mathcal{K}_P$, so their Cauchy transforms are jointly continuous on $[0,d] \times (\mathbb{C} \setminus \mathcal{K}_P)$ and are nonzero there. At the lower endpoint this uses the inclusion of all roots and critical points in the core; at the upper endpoint the transform is $(z - \mathfrak{c}_P)^{-1}$.

For a root a_j, define the rescaled displacement

$$H_\alpha(z) = \frac{d-\alpha}{d}(z - a_j) + \frac{\alpha}{d\, C_{\alpha,P}(z)}.$$

On the interior it equals $(d-\alpha)(w_\alpha(z) - a_j)/d$ and is nonzero by Proposition 2.2. Its endpoint values are $H_0(z) = z - a_j$ and $H_d(z) = z - \mathfrak{c}_P$, both nonzero outside the core. Thus $H_\alpha \circ \gamma$ is a continuous homotopy of nonvanishing closed curves between these two displacement curves. Their winding numbers about zero are equal, proving (3.15). □

The support continuity also gives paths to the centroid. Precisely, for each $z \in \mathcal{K}^{\mathrm{fix}}_{\alpha_0,P}$ there is a continuous definable path $\gamma : [\alpha_0, d] \to \mathcal{K}_P$ with

$$\gamma(\alpha_0) = z, \qquad \gamma(d) = \mathfrak{c}_P, \qquad \gamma(\alpha) \in \mathcal{K}^{\mathrm{fix}}_{\alpha,P} \quad (\alpha_0 \le \alpha < d)$$

[HS2, Corollary 4.8]. These pointwise paths are compatible with the presence of *holes*, that is, bounded complementary components. Winding transport supplies an additional constraint on each shadow-free loop: the residue condition alone can hold for a proper symmetric subset of the roots.

4. Switching sheets and the shadow measure

The nonatomic fixed-ratio measure is carried by interfaces at which the selected critical height changes. An equality between unselected heights contributes no mass. We compute the measure on these interfaces and then project it to the observation plane. The projection separates the planar and curve parts of the shadow measure and identifies the possible regular boundary points.

4.1. **Selected switching charts.**

Definition 4.1 (Regular switching chart)**.** A *regular switching chart* consists of an open set $V \subset \mathbb{C} \times (0,d)$ and two distinct functions w_1, w_2 on V, holomorphic in z and real-analytic in α, such that

$$F_{P,\alpha}(w_j(z,\alpha)) = z, \qquad F'_{P,\alpha}(w_j(z,\alpha)) \neq 0, \qquad P(w_j)P'(w_j)(w_j - z) \neq 0.$$

Here $F'_{P,\alpha}$ denotes differentiation with respect to w. Thus w_1 and w_2 are distinct simple ordinary-saddle branches. Choose compatible local logarithms and set

$$\Psi_j(z,\alpha) := \Phi_{P,z,\alpha}(w_j(z,\alpha)), \qquad \psi_j(z,\alpha) := \operatorname{Re}\Psi_j(z,\alpha).$$

The chart is *switching* if

$$\mathfrak{b}_P(z,\alpha) = \max\{\psi_1(z,\alpha), \psi_2(z,\alpha)\} \quad \text{on } V,$$

and both strict inequalities $\psi_1 > \psi_2$ and $\psi_2 > \psi_1$ occur. Define

$$g_{12} := \psi_1 - \psi_2, \qquad \Sigma_{12} := \{g_{12} = 0\} \cap V.$$

The set Σ_{12} is the local *switching sheet*. Each branch must be selected on its indicated side; equality of two unselected heights does not define a switching chart.

The maximum is a local representation involving only the branches that exchange selection. Continuity makes their heights equal on the interface; positivity of the Riesz measure determines the maximum, rather than the minimum.

Proposition 4.2 (Derivatives of a switching pair)**.** *On a regular switching chart,*

$$2\partial_z g_{12} = \alpha\left(\frac{1}{w_1 - z} - \frac{1}{w_2 - z}\right), \tag{4.1}$$

$$\partial_\alpha g_{12} = \log\frac{|w_2 - z|}{|w_1 - z|} = \log\frac{|h_P(w_2)|}{|h_P(w_1)|}. \tag{4.2}$$

Here ∇_z is the real gradient in the two coordinates of z. In particular, $\nabla_z g_{12} \neq 0$. Using the same local logarithm branches, define the auxiliary phase

$$\Theta_\alpha(w) := (1-\alpha)\log P(w) + \alpha \log P'(w).$$

At an ordinary saddle,

$$\Psi_j = \Theta_\alpha(w_j) - \alpha\log\alpha, \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\operatorname{Re}\Theta_\alpha(w_j(z,\alpha)) = 1 - \log|h_P(w_j)|. \tag{4.3}$$

The complex identity uses compatible logarithm branches; the real identities do not depend on those choices. Moreover,

$$\partial_\alpha^2 g_{12} = \operatorname{Re}\left(\frac{h_P'(w_2)}{1-\alpha h_P'(w_2)} - \frac{h_P'(w_1)}{1-\alpha h_P'(w_1)}\right). \tag{4.4}$$

Proof. At a saddle the derivative of $\Phi_{P,z,\alpha}$ with respect to w vanishes. Differentiation in the two base variables therefore gives

$$\partial_z \Psi_j = \frac{\alpha}{w_j - z}, \qquad \partial_\alpha \Psi_j = -\log(w_j - z).$$

Taking real parts and subtracting proves (4.1)–(4.2); the distinctness of w_1 and w_2 gives $\nabla_z g_{12} \neq 0$. Choose local logarithms $L = \log P$ and $M = \log P'$ and put $\log(w_j - z) = \log\alpha + L(w_j) - M(w_j)$. The saddle equation makes this a compatible displacement logarithm, and substitution gives (4.3). Finally, differentiating $z = w_j - \alpha h_P(w_j)$ at fixed z yields

$$\partial_\alpha w_j = \frac{h_P(w_j)}{1 - \alpha h_P'(w_j)}.$$

A second derivative of (4.2) now gives (4.4). □

For a fixed ratio in the chart, write

$$V_\alpha := \{z : (z,\alpha) \in V\}, \qquad \mathscr{T}_{12,\alpha} := \{z \in V_\alpha : g_{12}(z,\alpha) = 0\}.$$

By Proposition 4.2, every nonempty $\mathscr{T}_{12,\alpha}$ is a smooth real-analytic curve. We call it the local *tie curve*.

A point of the switching sheet is *stationary in the ratio* when $\partial_\alpha g_{12} = 0$. Two distinct ordinary saddles with equal values of both $|P|$ and $|P'|$ form a *balanced pair*. A *nondegenerate balance fold*, also called a *regular balance fold*, is a switching point with $\partial_\alpha g_{12} = 0$ and $\partial_\alpha^2 g_{12} \neq 0$. Its projection is *exposed* if it lies locally on $\partial\mathcal{K}_P$; it is an *internal density ridge* if other selected pieces fill both sides. The rank of the projection is always the rank of its real differential restricted to the relevant stratum. The *ordinary saddle-collision locus* consists of parameter values at which two inverse branches of $F_{P,\alpha}$ coalesce at a point where $P(w)P'(w) \neq 0$.

4.2. **Projection of the zero measure.** Hausdorff measures use the Euclidean metric and are normalized to agree with Lebesgue measure in the matching dimension. We use the following change-of-variables convention. If $F : M^k \to \mathbb{R}^\ell$ is a C^1 map from a smooth k-dimensional submanifold, with $k \geq \ell$, its *ℓ-dimensional Jacobian* $J_\ell F$ is the product of the ℓ singular values of its differential, including zeros when the rank is less than ℓ. The *coarea formula* reads

$$\int_M h\, J_\ell F \,\mathrm{d}\mathcal{H}^k = \int_{\mathbb{R}^\ell} \left(\int_{F^{-1}(y)} h \,\mathrm{d}\mathcal{H}^{k-\ell} \right) \mathrm{d}y$$

for nonnegative compactly supported Borel functions h. When $k = \ell$, the inner integral is a sum over preimages; this is the *area formula*. These formulas, used below only on smooth charts, convert densities on a surface into densities on its image or along its fibers; see [Fed].

Fix a regular switching chart and a nonnegative continuous cutoff χ compactly supported in it. Define the corresponding weighted contribution ρ_χ by

$$\int \varphi(z)\,\mathrm{d}\rho_\chi(z) := \frac{2}{d^2} \int_0^d (d-\alpha) \int \varphi(z)\chi(z,\alpha)\,\mathrm{d}\mu_{\alpha,P}(z)\,\mathrm{d}\alpha$$

for bounded continuous test functions φ. When several charts are used, a partition of unity prevents double counting.

Theorem 4.3 (Local density formula)**.** *On a regular switching chart, the nonatomic part of the fixed-ratio measure carried by the chart is*

$$\mu_{\alpha,P}^{\mathrm{na}} \restriction V_\alpha = \frac{|\nabla_z g_{12}|}{2\pi(d-\alpha)}\, \mathcal{H}^1_{\mathscr{T}_{12,\alpha}}. \tag{4.5}$$

Suppose that every contributing preimage of z in the chart satisfies $\partial_\alpha g_{12} \neq 0$; equivalently, the differential of the projection is surjective at every such preimage. Then the planar density of the weighted average is

$$\frac{\mathrm{d}\rho_\chi}{\mathrm{d}A}(z) = \frac{1}{\pi d^2} \sum_{\alpha:(z,\alpha)\in V,\ g_{12}(z,\alpha)=0} \frac{\chi(z,\alpha)|\nabla_z g_{12}(z,\alpha)|^2}{|\partial_\alpha g_{12}(z,\alpha)|}. \tag{4.6}$$

The sum is over solutions satisfying $\partial_\alpha g_{12} \neq 0$. Thus (4.6) *gives the rank-two contribution and makes no assertion about a stationary rank-one sheet.*

Proof. The distributional Laplacian of the maximum of two harmonic functions has, along a smooth interface where their gradient difference is nonzero, arclength density equal to the magnitude of that difference. Applied to the fixed-ratio potential, this jump formula gives (4.5). Equivalently, integrate by parts on the two sides of the tie curve; the outward normal derivatives differ by $|\nabla_z g_{12}|$.

The averaging factor $2(d-\alpha)/d^2$ cancels the denominator $d-\alpha$ in (4.5). Before projection, the resulting measure on Σ_{12} is

$$\frac{1}{\pi d^2} \frac{\chi|\nabla_z g_{12}|^2}{|\nabla_{z,\alpha} g_{12}|} \mathcal{H}^2 \restriction \Sigma_{12}. \tag{4.7}$$

Here $\nabla_{z,\alpha} g_{12}$ is the full real gradient on $\mathbb{C}\times\mathbb{R} \simeq \mathbb{R}^3$. The two-dimensional Jacobian of $\mathrm{pr}_z(z,\alpha) = z$, restricted to Σ_{12}, is $|\partial_\alpha g_{12}|/|\nabla_{z,\alpha} g_{12}|$. The area formula therefore gives (4.6). For fixed z, analyticity in α makes regular zeros locally finite, including on identically zero components, where $\partial_\alpha g_{12} = 0$. Compact support inside the chart therefore leaves only finitely many nonzero terms. □

Theorem 4.4 (Balance-fold singularity)**.** *Let (z_*, α_*) be a point of a regular switching chart satisfying*

$$g_{12} = \partial_\alpha g_{12} = 0, \qquad \kappa_* := \partial_\alpha^2 g_{12}(z_*, \alpha_*) \neq 0.$$

Then the projection of the switching sheet has a fold at (z_*, α_*): locally, points on one side of the fold image have two nearby preimages on the sheet, while points on the other side have none. Set

$$\mathfrak{n}_* := |\nabla_z g_{12}(z_*, \alpha_*)|, \qquad \mathbf{n} := \frac{\nabla_z g_{12}(z_*, \alpha_*)}{\mathfrak{n}_*},$$

and let $\tau_\perp$ be signed Euclidean distance from z_* in the direction $\mathbf{n}$. The side with two preimages is the occupied side*. For a cutoff equal to one near the fold, write D_χ^{fold} for the sum in* (4.6) *over these two nearby preimages. Using this canonical representative, along the normal line on the occupied side,*

$$D_\chi^{\mathrm{fold}}(z_* + \tau_\perp \mathbf{n}) = \frac{\sqrt{2}\,\mathfrak{n}_*^{3/2}}{\pi d^2 \sqrt{|\kappa_*|}} |\tau_\perp|^{-1/2}(1 + o(1)). \tag{4.8}$$

On the other side this local contribution vanishes. The fold image is an exposed boundary only if no other selected stratum fills the vacant side; otherwise it is an internal density ridge.

Proof. Because $\nabla_z g_{12} \neq 0$, choose local real coordinates in the observation plane and solve the switching equation as $y = \mathfrak{y}(x, \alpha)$. At the base point, $\partial_\alpha \mathfrak{y} = 0$ and $\partial_\alpha^2 \mathfrak{y} \neq 0$. The parameter-dependent Morse lemma gives a fold with two preimages on one side and none on the other. Put $\widehat{\alpha} := \alpha - \alpha_*$. Along the Euclidean normal at z_*, Taylor expansion gives

$$g_{12} = \mathfrak{n}_* \tau_\perp + \tfrac{1}{2}\kappa_* \widehat{\alpha}^2 + O(\tau_\perp^2 + |\tau_\perp \widehat{\alpha}| + |\widehat{\alpha}|^3).$$

On the occupied side the two solutions satisfy

$$|\widehat{\alpha}| = \left(\frac{2\mathfrak{n}_* |\tau_\perp|}{|\kappa_*|}\right)^{1/2} (1 + o(1)), \qquad |\partial_\alpha g_{12}| = \left(2\mathfrak{n}_* |\kappa_*| |\tau_\perp|\right)^{1/2} (1 + o(1)).$$

Their two contributions in (4.6), together with $|\nabla_z g_{12}| = \mathfrak{n}_* + o(1)$, give (4.8). Other incidence contributions are nonnegative. They cannot cancel the singularity, although they may fill the opposite side and make the fold internal. The critical incidence is a graph $\alpha = c(x)$ and is null for planar parameter measure. The curve area law, Fubini and the fixed-ratio Hausdorff law give zero averaged critical mass. The local preimage density is therefore the density of a finite cutoff measure, hence integrable. Its projected analytic critical curve is area-null and carries no mass. □

Remark 4.5 (Odd stationary order). If instead $\partial_\alpha g_{12} = \partial_\alpha^2 g_{12} = 0$ and $\partial_\alpha^3 g_{12} \neq 0$, the image of the switching sheet contains a full neighborhood of z_*. Indeed, in the representation $y = \mathfrak{y}(x, \alpha)$, the values at $\alpha_* \pm \varepsilon$ lie on opposite sides of $\mathfrak{y}(x_*, \alpha_*)$ for every sufficiently small fixed ε. This remains true for nearby x, and the intermediate-value theorem fills a common intervening interval. Thus an odd-order stationary point cannot by itself create an exposed boundary.

4.3. **The global measure decomposition.**

Theorem 4.6 (Planar, curve, and atomic parts)**.** *The decomposition* (1.5) *holds. Away from the prescribed atoms, the curve part is generated by two-dimensional selected incidence strata whose projection to the z-plane has rank one. Such strata are locally stationary families of support arcs. By contrast, an isolated fold curve inside a rank-two switching sheet creates an integrable singularity of the planar density, not an additional arclength measure.*

Proof. Use the positive incidence measure

$$\mathrm{d}\Omega_P(z, \alpha) = \frac{2}{d^2}(d - \alpha)\,\mathrm{d}\mu_{\alpha,P}(z)\,\mathrm{d}\alpha, \qquad 0 < \alpha < d.$$

Its spatial projection is ρ_P. The root lines give exactly the atoms in (2.9); write Ω_P° for the remaining measure.

First, $\mu_{\alpha,P} \restriction (\mathbb{C} \setminus Z(P)) \ll \mathcal{H}^1$. Indeed, the first-variation formula, valid also at multiple connecting saddles [HS2, Appendix], bounds the one-sided spatial derivatives of the barrier by the gate differentials. Nearby connecting saddles stay in a compact region away from the roots and the observation point, so these bounds are locally uniform. The one-sided mean-value inequality and (2.5) give local Lipschitz continuity of $U_{\mu_{\alpha,P}}$ off $Z(P)$, including at collisions. If this potential is L-Lipschitz on a disk of radius $2r$, the circle-mean formula gives

$$\mu_{\alpha,P}(\overline{B}(z,r)) \log 2 \le U^{(2r)}(z) - U^{(r)}(z) \le 3Lr,$$

where $U^{(r)}$ is the potential with $\log|z-a|$ replaced by $\log\max\{r,|z-a|\}$. Covering null sets by small disks proves the asserted arclength absolute continuity. Fubini therefore also makes the spatial marginal of Ω_P° absolutely continuous with respect to $\mathcal{H}^1$.

The completed support incidence is bounded and definable, of dimension at most two. Apply Proposition 2.4, retaining constant ranks of both the ratio and spatial projections. Strata of dimension at most one have finite fibers outside finitely many ratios and hence zero Ω_P°-mass. A connected stratum with zero ratio derivative has constant ratio and likewise contributes nothing.

On each remaining two-dimensional stratum, the inverse-function theorem gives local coordinates (t,α) retaining the actual ratio and one spatial coordinate, with spatial map $\gamma(t,\alpha)$ satisfying $\ell(\gamma(t,\alpha)) = t$. Its slices are regular injective arcs. Their arclength law and Fubini give $\Omega_P^\circ \ll \mathrm{d}t\,\mathrm{d}\alpha$ in these coordinates. If the spatial rank is two, the spatial map has smooth local inverses, so its projected contribution is absolutely continuous with respect to area. If the rank is one, $\partial_\alpha\gamma$ is a multiple of $\partial_t\gamma$. Applying ℓ shows that this multiple is zero. Thus γ is independent of α on a smaller product neighborhood: the original support arc is stationary.

The rank-one spatial images are bounded definable curves. Decompose them into finitely many regular C^1 coordinate-monotone arcs and finitely many points. Coordinate variation bounds their total length, and the points have zero nonatomic mass. Let Γ_P be the union of these arcs. It is area-null and has finite $\mathcal{H}^1$ measure. Countably many local charts suffice to patch the rank-two area laws. Consequently the marginal of Ω_P°, restricted off Γ_P, is area-absolutely-continuous; its restriction to Γ_P is arclength-absolutely-continuous. The Radon–Nikodym theorem supplies the nonnegative integrable densities in (1.5). Finally, Theorem 4.4 proves that an isolated fold's critical incidence has zero mass and its projected density is integrable. It adds no curve measure. Restoring the root atoms completes the decomposition. □

4.4. **Regular boundary points.**

Theorem 4.7 (The regular boundary alternative)**.** *Suppose (z_*,α_*) lies on the switching sheet Σ_{12} of a regular switching chart and $z_* \in \partial\mathcal{K}_P$. Then*

$$g_{12}(z_*,\alpha_*) = \partial_\alpha g_{12}(z_*,\alpha_*) = 0. \tag{4.9}$$

For two distinct ordinary saddles above the same (z,α), these two equations are equivalent to

$$|P(w_1)| = |P(w_2)|, \qquad |P'(w_1)| = |P'(w_2)|. \tag{4.10}$$

Consequently, the observation point lies on the perpendicular bisector of w_1 and w_2.

Proof. If $\partial_\alpha g_{12}(z_*,\alpha_*) \neq 0$, the implicit-function theorem solves the switching equation as $\alpha = \alpha(z)$ throughout a neighborhood of z_*. By the switching-chart hypothesis, the corresponding fixed-ratio interface belongs to the fixed-ratio support for every nearby z. Its projection therefore contains a neighborhood of z_* in $\mathcal{K}_P$, contrary to $z_* \in \partial\mathcal{K}_P$.

For the algebraic reduction, set

$$\mathfrak{d}_0 := \log\left|\frac{P(w_1)}{P(w_2)}\right|, \qquad \mathfrak{d}_1 := \log\left|\frac{P'(w_1)}{P'(w_2)}\right|.$$

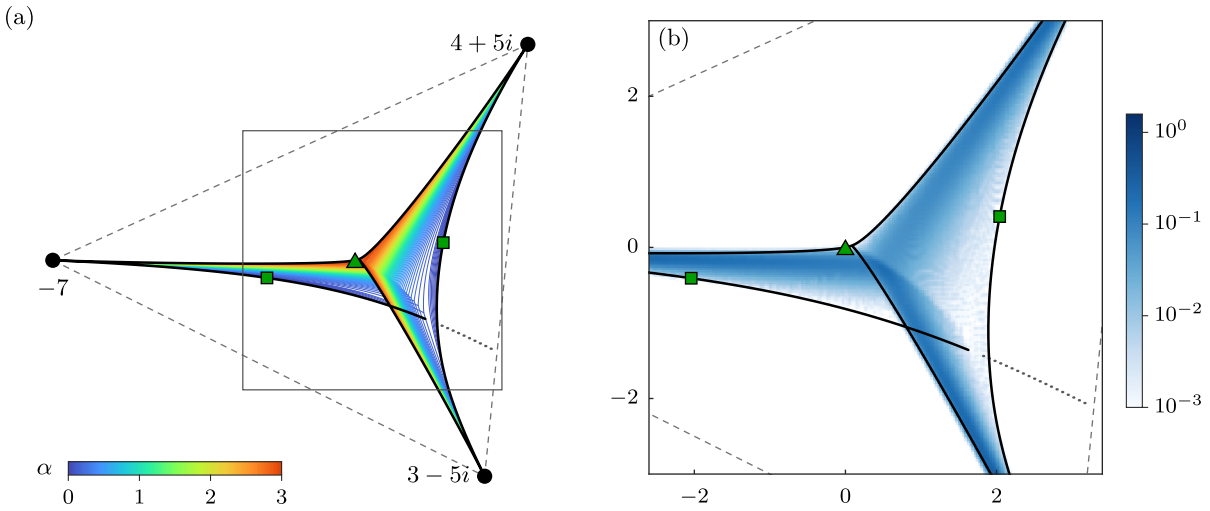


FIGURE 5. (a) The supports $\mathcal{K}^{\mathrm{fix}}_{\alpha,P}$, $0<\alpha<3$, for the cubic of Figure 1(a), colored by α. Black: critical values of $F_{P,\alpha}$ that are endpoints of $\mathcal{K}^{\mathrm{fix}}_{\alpha,P}$; dotted: the remaining caustic candidates; dashed: $\partial\mathsf{H}_P$. The boundary of $\mathcal{K}_P$ lies on the black curves, some of which also cross its interior. (b) The density of $\bar{\nu}_{P,n}$, $n=200$, in the box of (a), on a logarithmic scale; it changes abruptly across some caustic arcs inside the core.

By (4.3), the switching equation is $(1-\alpha)\mathfrak{d}_0+\alpha\mathfrak{d}_1=0$, while stationarity is $\mathfrak{d}_1-\mathfrak{d}_0=0$. The determinant of this linear system is one, so $\mathfrak{d}_0=\mathfrak{d}_1=0$. The converse is immediate. Equation (4.2) then gives $|w_1-z|=|w_2-z|$. □

This result concerns regular preimages with $0<\alpha<d$. Saddle collisions, endpoint ratios, and multiple-height transitions must also be included in a global boundary description.

4.5. **Caustics and balanced secants.** The spatial projection of the ordinary saddle-collision locus is called a *caustic*. Its incidence equations are

$$z=F_{P,\alpha}(w),\qquad 1-\alpha h_P'(w)=0,\qquad 0<\alpha<d. \tag{4.11}$$

It is parametrized by

$$\alpha=\frac{1}{h_P'(w)},\qquad z=w-\frac{h_P(w)}{h_P'(w)}. \tag{4.12}$$

Only values for which $1/h_P'(w)$ is real and belongs to $(0,d)$ are retained. The closure of the resulting z-image is the *caustic candidate set*. Its points are possible collision values, not automatically selected endpoints; see Figure 5(a). Since $h_P=\mathsf{Q}_P/\mathsf{N}_P$, the reduced critical equation is

$$\mathsf{N}_P(w)^2-\alpha\big(\mathsf{Q}_P'(w)\mathsf{N}_P(w)-\mathsf{Q}_P(w)\mathsf{N}_P'(w)\big)=0, \tag{4.13}$$

on $\mathsf{Q}_P(w)\mathsf{N}_P(w)\neq 0$.

For a monic polynomial with roots $b_1,\dots,b_D$, the discriminant $\prod_{i<j}(b_i-b_j)^2$ is a polynomial in its coefficients and vanishes exactly at repeated roots. The reduced rational map is essential when roots have multiplicity: an unreduced polynomial discriminant can acquire components unrelated to genuine branch collisions. Related geometry of critical values is developed in [HSS].

For distinct w_1,w_2 satisfying $h_P(w_1)\neq h_P(w_2)$, define the *secant ratio* and *secant image* by

$$\alpha_{\mathrm{sec}}(w_1,w_2):=\frac{w_1-w_2}{h_P(w_1)-h_P(w_2)},\qquad z_{\mathrm{sec}}(w_1,w_2):=\frac{w_2h_P(w_1)-w_1h_P(w_2)}{h_P(w_1)-h_P(w_2)}. \tag{4.14}$$

The equal-image condition $F_{P,\alpha}(w_1) = F_{P,\alpha}(w_2) = z$ is equivalent to $\alpha = \alpha_{\mathrm{sec}}(w_1, w_2)$ and $z = z_{\mathrm{sec}}(w_1, w_2)$. The *off-diagonal balance candidate set* is the z_{sec}-image of

$$\left\{ (w_1, w_2) : \begin{array}{l} w_1 \neq w_2, \quad P(w_j)P'(w_j) \neq 0, \quad \alpha_{\mathrm{sec}}(w_1, w_2) \in (0, d), \\ |P(w_1)| = |P(w_2)|, \qquad |P'(w_1)| = |P'(w_2)| \end{array} \right\}. \tag{4.15}$$

After denominators are cleared while retaining their nonvanishing conditions, every condition in (4.15) is semialgebraic. The candidate set is therefore semialgebraic. Selection and exposure nevertheless remain separate requirements: the candidate set is generally larger than the actual boundary.

For unconstrained secants, the diagonal limit $w_1, w_2 \to w$ gives

$$\alpha_{\mathrm{sec}}(w_1, w_2) \longrightarrow \frac{1}{h_P'(w)}, \qquad z_{\mathrm{sec}}(w_1, w_2) \longrightarrow w - \frac{h_P(w)}{h_P'(w)}.$$

These are the tangent limits of the secant formulas for the rational graph of h_P. The equal-modulus constraints may prevent the limit from being attained by balanced pairs. A caustic point therefore need not be a limit of off-diagonal balanced secants.

5. Semialgebraicity and the algebraic boundary

The regular boundary calculation gives algebraic candidates, but a candidate need not be selected. We first extend the calculation to all boundary points, including multiple-height transitions. The resulting finite algebraic graph contains the boundary. The conservation law then shows that the core includes either a whole root-free edge of this graph or none of it. Hausdorff continuity then makes this selection constant on the pieces of a compatible coefficient-space trivialization.

Theorem 5.1 (The global boundary alternative)**.** *Put*

$$\mathscr{D}_P := Z(P) \cup Z(P') \cup \{\mathfrak{c}_P\}.$$

For every $z \in \partial\mathcal{K}_P \setminus \mathscr{D}_P$ there is an $\alpha \in (0, d)$ for which at least one of the following holds:

(i) *An ordinary connecting saddle w satisfies*

$$z = w - \alpha h_P(w), \qquad 1 - \alpha h_P'(w) = 0.$$

(ii) *Two distinct simple ordinary connecting saddles w_1, w_2 have the same image z and satisfy*

$$|P(w_1)| = |P(w_2)|, \qquad |P'(w_1)| = |P'(w_2)|.$$

In the second case, all the connecting saddles have the same distance from z. Thus a multiple-height transition produces no additional, nonalgebraic boundary mechanism.

Proof. By the definition of $\mathcal{K}_P$, choose $z_j \in \mathcal{K}^{\mathrm{fix}}_{\alpha_j,P}$ with $z_j \to z$, and pass to a convergent subsequence of the ratios. The endpoint support theorem excludes limits 0 and d because $z \notin \mathscr{D}_P$. Lemma 2.6 gives $z \in \mathcal{K}^{\mathrm{fix}}_{\alpha,P}$ at the interior limiting ratio.

If a connecting saddle is multiple, (i) holds. Otherwise Proposition 2.1 writes the barrier near (z, α) as

$$B(\zeta, \beta) = \max_{1 \le j \le k} \psi_j(\zeta, \beta)$$

for the simple connecting saddles. There must be at least two, since a single branch would make the potential harmonic near z. At the base point all ψ_j have the same value. Set

$$b_j := \partial_\beta \psi_j(z, \alpha) = -\log|w_j - z|.$$

Suppose these numbers are not all equal. For a sufficiently small fixed $\varepsilon > 0$, the maximum at $(z, \alpha + \varepsilon)$ is attained only among the indices having largest b_j, and at $(z, \alpha - \varepsilon)$ only among those

having smallest b_j. The two index groups are disjoint. The strict inequalities excluding the other indices persist for all ζ in a small disk about z.

For each such ζ, as β crosses this interval, the maximizing index cannot remain unique throughout: a unique maximizing index is locally constant, whereas the endpoint groups are disjoint. At some ratio two distinct branches therefore attain the maximum. Their spatial gradients are distinct, since

$$2\partial_\zeta \psi_i - 2\partial_\zeta \psi_j = \beta \left(\frac{1}{w_i - \zeta} - \frac{1}{w_j - \zeta} \right) \neq 0.$$

The maximum is not harmonic at that spatial point. Indeed, if it were harmonic on a neighborhood, its difference from either touching harmonic branch would be a nonnegative harmonic function with an interior zero; both branches would then agree, contrary to the displayed gradient inequality. Thus ζ belongs to the fixed-ratio support at that ratio. The entire observation disk would be contained in $\mathcal{K}_P$, contradicting $z \in \partial \mathcal{K}_P$.

All b_j are consequently equal. All distances $|w_j - z|$ are equal, and equality of the critical heights gives equality of $|P(w_j)|$. The saddle equations then give equality of $|P'(w_j)|$. Any two distinct connecting saddles supply (ii). □

5.1. **A finite algebraic candidate graph.** Let $\mathscr{G}_P^{\mathrm{col}}$ be the image of the ordinary collision equations (4.11), with $P(w)P'(w) \neq 0$. Let $\mathscr{G}_P^{\mathrm{bal}}$ be the balanced secant image defined by (4.15). Define

$$\mathscr{G}_P := \partial \mathsf{H}_P \ \cup \ \mathscr{D}_P \ \cup \ \left(\mathsf{H}_P \cap \overline{\mathscr{G}_P^{\mathrm{col}} \cup \mathscr{G}_P^{\mathrm{bal}}} \right). \tag{5.1}$$

Closure here is taken in the observation plane for the fixed polynomial. The added hull boundary is convenient for a finite planar decomposition; it is not asserted to be part of the shadow boundary.

Proposition 5.2 (One-dimensional algebraic candidates)**.** *For fixed degree the family $\{(P, z) : z \in \mathscr{G}_P\}$ is semialgebraic. Each fiber is compact, has dimension at most one, and contains $\partial \mathcal{K}_P$. There is an integer D_d, depending only on d, such that each $\mathscr{G}_P$ is contained in the zero set of a nonzero real polynomial of degree at most D_d.*

Proof. The collision and balanced-pair equations are polynomial equalities and inequalities in the real and imaginary parts of their variables after nonzero denominators have been cleared. This assertion uses the original P, P' only at ordinary saddles, so collisions of roots introduce no spurious ordinary solutions. Projection, fiberwise closure, and intersection preserve semialgebraicity. The root convex hull and its boundary are semialgebraic families as well: one can quantify over a list of roots and their convex-combination coefficients. These facts prove the first assertion; compactness is immediate from (5.1). Theorem 5.1 proves the boundary inclusion.

For fixed α, the rational map $F_{P,\alpha}$ has finitely many critical values, so the collision incidence in (z, α) has dimension at most one. Its spatial image has dimension at most one. It remains to check that a symmetry cannot make the balanced image two-dimensional.

Away from collisions, choose two simple inverse branches locally. Their balanced incidence is contained in

$$g(z, \alpha) = 0, \qquad g_\alpha(z, \alpha) = 0, \qquad \nabla_z g \neq 0,$$

by Proposition 4.2 and the balance equivalence. The regular level $g = 0$ is a smooth surface. At each stationary point, the vertical ratio direction lies in its tangent plane, so the derivative of spatial projection has rank at most one. Sard's theorem in local surface charts makes this stationary image planar-null. Countably many branch and surface charts cover all simple balanced witnesses. Pairs involving a multiple saddle project into the collision image already considered. Thus the semialgebraic balanced image has empty interior and dimension at most one. Closure preserves semialgebraic dimension.

For the degree bound, quantifier elimination gives a fixed-degree quantifier-free description of the family (5.1) using finitely many real polynomials in the coefficient parameters and x, y. Fix P and discard the polynomials that specialize identically to zero. At any point of $\mathscr{G}_P$, at least one remaining polynomial vanishes; otherwise all signs in that description would be locally constant and the fiber would have interior. The product of the remaining nonconstant polynomials is nonzero and vanishes on the fiber. Their number and degrees are bounded in terms of the original degree d only. The fiber is nonempty and not the whole plane, so a nonconstant factor is present. This proves the assertion with their total degree as a uniform bound. Standard quantifier elimination and dimension facts are recalled in [BCR]. □

5.2. **Continuation of selection along an algebraic edge.** Here a *crossing arc* means that the neighborhood and its graph arc admit a homeomorphism to an open rectangle and its horizontal middle interval. This is a topological condition; no differentiability is required.

Lemma 5.3 (Continuation along a root-free candidate arc)**.** *Let e be a connected open semialgebraic arc of $\mathscr{G}_P$, disjoint from $Z(P)$. Suppose every point has a simply connected neighborhood meeting the graph in one embedded arc crossing that neighborhood. Then either $e \subseteq \mathcal{K}_P$ or $e \cap \mathcal{K}_P = \varnothing$. The arc need not be differentiable.*

Proof. Since $\mathcal{K}_P$ is closed, it suffices to prove that $e \cap \mathcal{K}_P$ is relatively open. Take $z \in e \cap \mathcal{K}_P$ and a small root-free Jordan neighborhood D as in the statement. Each of the two components of $D \setminus \mathscr{G}_P$ is wholly in the core or wholly outside it, since $\partial\mathcal{K}_P \subseteq \mathscr{G}_P$. An included component includes its boundary arc by closedness. Otherwise $D \cap \mathcal{K}_P$ is contained in the crossing arc. If any point of that arc is absent from the core, its gap joins the two sides, so $D \setminus \mathcal{K}_P$ is connected. On the simply connected root-free domain D, choose a holomorphic primitive J of $-\mathscr{C}_P^2$. The conservation law gives $C_{\rho_P} = J + c$ on this connected complement for one constant c. The exceptional arc has planar area zero, since it is one-dimensional and semialgebraic; no smoothness is needed. The equality thus holds almost everywhere in D, and distributional $\partial_{\bar z}$ gives $\rho_P(D) = 0$. This contradicts $z \in \operatorname{supp}\rho_P$. Hence the entire local arc is included, proving relative openness and the assertion by connectedness. □

Theorem 5.4 (Semialgebraicity of the limiting core)**.** *For every fixed degree d, the family*

$$\{(P, z) : P \text{ monic of degree } d,\ z \in \mathcal{K}_P\}$$

is semialgebraic. Each limiting boundary consists of finitely many real-algebraic arcs and points and lies on a real algebraic curve of degree bounded only by d. These conclusions concern $\mathcal{K}_P$, not the possibly infinite set of transient points in $\mathcal{S}_P$.

Proof. We begin with a fixed polynomial. Decompose the compact semialgebraic graph $\mathscr{G}_P$ into finitely many vertices and open embedded C^1 edges. Include $\mathscr{D}_P$ among the vertices, and split a closed loop by adding a vertex if necessary. Its complement has finitely many connected open semialgebraic components, which we call *faces*. Each face is either entirely included in $\mathcal{K}_P$ or entirely excluded, since the shadow boundary lies in the graph. By Lemma 5.3, the same is true of each open edge. Membership of the finitely many vertices completes a finite union of semialgebraic sets. Thus each core is semialgebraic, although the critical-height comparisons used to select it contain logarithms.

For uniformity in the coefficients, first compress each ambient plane into the open square $(-1, 1)^2$ by applying $x \mapsto x/(1 + |x|)$ to each coordinate. The candidate graphs and distinguished root sets remain compact semialgebraic families. Compatible compact triviality [BCR, Har], including the closed square and its boundary among the specified sets, gives a finite partition of coefficient space into connected semialgebraic pieces. On each piece its boundary-preserving product restricts to the open square; conjugating by the compression gives a semialgebraic ambient

homeomorphism over the parameter. Transport the finite cell decomposition from one sample fiber, including its marked roots among the vertices. Frontier containment and Lemma 5.3 make every transported cell wholly included or wholly excluded. Crossing charts are preserved by ambient homeomorphisms, so no smoothness of the transported edges is required.

Only finitely many unions of the sample cells are possible. The inverse image of each actual core under an ambient fiber homeomorphism is compact, so retain only the compact unions and identify any repeated models. Their transported images are continuous families of compact sets, pairwise distinct at every parameter because the spatial maps are injective. The core is itself Hausdorff-continuous by Proposition 2.3. For each compact model, equality with the core therefore defines a closed set of parameters. These equality sets form a finite disjoint cover; each complement is also a finite union of closed sets, so each equality set is open. Connectedness forces one fixed compact model throughout the parameter piece. That model is a finite union of semialgebraic cells, and its image under the semialgebraic ambient product is a semialgebraic family. Taking the finite union over the parameter pieces proves the family assertion.

Finally, Proposition 5.2 supplies the degree-bounded algebraic curve containing the boundary. A one-dimensional semialgebraic set admits a finite decomposition into real-analytic algebraic arcs and points. Applying this to the boundary proves the last geometric assertion. □

Corollary 5.5 (Finite topology in each degree)**.** *For fixed d, the cores have finitely many homeomorphism types and uniformly bounded Betti numbers. Each core is homeomorphic to a finite simplicial complex and has finitely many complementary components.*

Proof. Apply semialgebraic triviality to the coefficient family in Theorem 5.4, and triangulate its compact fibers [BCR, Har]. Apply the same finite decomposition to the complements. □

Corollary 5.6 (A degree-uniform boundary-length bound)**.** *Let D_d be as in Proposition 5.2. If H_P lies in a closed disk of radius R, then*

$$\mathcal{H}^1(\partial \mathcal{S}_P) = \mathcal{H}^1(\partial \mathcal{K}_P) \le \pi D_d R.$$

In particular, after a fixed bound on the root diameter, boundary length is bounded uniformly in each degree, including at degenerate polynomials.

Proof. The limiting boundary is rectifiable and contained in an algebraic curve of degree at most D_d. For almost every line it has at most D_d intersection points, by restricting the defining polynomial to that line; lines contained in the curve form a null exceptional family. The planar Cauchy–Crofton formula [Fed], integrating unoriented normal angle over $[0, \pi)$ and signed offset over $[-R, R]$, bounds its length by $\frac{1}{2}\pi(2R)D_d$. The transient set is countable and has zero $\mathcal{H}^1$ measure, so the boundary decomposition in Proposition 2.3 gives the same bound for the full shadow. A singleton hull is immediate. □

Corollary 5.7 (Local Hölder stability of the limiting shape)**.** *Write $P(z) = z^d + \sum_{k=0}^{d-1} c_k z^k$ and use the Euclidean norm of $(c_0, \dots, c_{d-1}) \in \mathbb{C}^d$ as the coefficient norm $\|P\|_{\mathrm{coef}}$. For each $R > 0$ there are $C_R > 0$ and $0 < \gamma_R \le 1$, depending also on d, such that*

$$d_{\mathrm{H}}(\mathcal{K}_P, \mathcal{K}_Q) \le C_R \|P - Q\|_{\mathrm{coef}}^{\gamma_R} \quad \text{if } \|P\|_{\mathrm{coef}}, \|Q\|_{\mathrm{coef}} \le R.$$

Proof. Hausdorff distance between compact semialgebraic fibers is a semialgebraic function: its defining distance inequalities involve only quantification over the two compact fibers. By Proposition 2.3, this function is continuous. On the compact semialgebraic product of two coefficient balls, apply the semialgebraic Łojasiewicz inequality [BCR] to this nonnegative function and $\|P - Q\|_{\mathrm{coef}}$. The latter vanishes only on the diagonal, where the former vanishes too. Thus a positive integer power of the Hausdorff distance is bounded by a constant times the coefficient distance, yielding the assertion. □

Remark 5.8 (Selection and degree bounds). The equations for the candidate graph have degree controlled by d, independently of the power level. The core is a union of whole faces, edges, and vertices of that graph. This description supplies neither a small explicit bound for D_d nor a list of the included cells; selection is still determined by the barrier. In particular, the union of all collision and balanced-pair candidates is not in general the boundary; in Figure 5(a), some caustic arcs cross the interior of the core.

5.3. **Consequences for the full shadow.**

Corollary 5.9 (Boundary size of the full shadow)**.** *For every nonconstant polynomial,*

$$\mathcal{H}^1(\partial \mathcal{S}_P) = \mathcal{H}^1(\partial \mathcal{K}_P) < \infty.$$

The limiting boundary is a finite union of points and bounded real-algebraic arcs. The full-shadow boundary adds at most countably many points, all isolated outside $\mathcal{K}_P$.

Proof. Combine Theorem 5.4 with Proposition 2.3; a countable set has zero $\mathcal{H}^1$ measure. Corollary 5.6 gives a degree-uniform quantitative bound. □

Corollary 5.10 (When the full shadow is connected or semialgebraic)**.** *The full shadow $\mathcal{S}_P$ is connected if and only if T_P is empty. It is semialgebraic if and only if T_P is finite. Equivalently, it is definable, with real parameters, in some o-minimal expansion of the real field if and only if T_P is finite. Its connected components are $\mathcal{K}_P$ and the singletons $\{z\}$ for $z \in T_P$.*

Proof. The limiting core is nonempty and connected, whereas every point of T_P is isolated in the full shadow. This gives the component statement. If T_P is finite, its union with the semialgebraic set $\mathcal{K}_P$ is semialgebraic. Conversely, if the full shadow is definable in any o-minimal expansion, its difference $T_P = \mathcal{S}_P \setminus \mathcal{K}_P$ is a definable discrete set. Such a set has finitely many connected components and is therefore finite. Conversely, a finite union with the semialgebraic core is semialgebraic, hence definable in every such expansion. □

6. Reflection rigidity and the planar body

We first use conservation to straighten an isolated support arc. The stronger statement comes from a support arc that remains stationary over an interval of differentiation ratios: it forces a reflection of the entire root multiset.

For $s \geq 2$, let $\mathscr{A}_P$ be the union of the reflection axes of P, with $\mathscr{A}_P = \varnothing$ when there are none. There are only finitely many axes, and all pass through $\mathfrak{c}_P$. Indeed, a reflection preserving the root measure fixes its centroid; on a finite set with two distinct points its action is one of finitely many permutations, and two such reflections with the same action coincide.

Definition 6.1 (Regular filament)**.** A *regular filament* is a set $F = D \cap \mathcal{K}_P$, where D is an open disk disjoint from $Z(P)$ and F is a single embedded regular C^1 arc crossing D and separating it into two components.

Embedded arcs are understood locally in ambient coordinates: at an interior point the carrier is a coordinate axis, and at a terminal point it is a closed half-axis.

6.1. **Straight filaments of constant density.**

Theorem 6.2 (Filament rigidity)**.** *Every regular filament is a straight segment and carries a constant positive density with respect to arclength. An isolated terminal arc cannot have a free endpoint away from the roots. Here a* free endpoint *is an endpoint with a disk disjoint from $Z(P)$ in which the limiting core consists only of that terminating arc.*

Proof. On the root-free disk, choose a holomorphic primitive $\mathscr{P}_D$ of $-\mathscr{C}_P^2$. On the two sides of the arc, Theorem 3.4 gives

$$C_{\rho_P} = \mathscr{P}_D + \kappa_+ \quad \text{and} \quad C_{\rho_P} = \mathscr{P}_D + \kappa_-$$

for constants κ_+, κ_-. They are distinct. Otherwise C_{ρ_P} would agree almost everywhere with one holomorphic function throughout D, and its distributional $\partial_{\bar z}$ would vanish there, contrary to $\partial_{\bar z} C_{\rho_P} = \pi \rho_P$ and the support identity.

Across an oriented arc, the distributional jump is the measure

$$\frac{\kappa_- - \kappa_+}{2\pi i}\, \mathrm{d}z,$$

up to reversal of orientation. Positivity forces the tangent direction to be constant. Hence the arc is straight and has arclength density $|\kappa_- - \kappa_+|/(2\pi)$. More explicitly, the local two-branch inversion gives the whole support line $\operatorname{Re}((\kappa_+ - \kappa_-)(z - z_0)) = 0$ near each $z_0 \in F$. Connectedness fixes one normal-coordinate level on F. The support is relatively closed and locally open in this line inside D, so F is the whole disk chord.

Suppose instead that the arc has a free endpoint, and choose a root-free disk meeting only the terminating arc. Its complement in the disk is connected, so there is only one additive constant. Then C_{ρ_P} would be holomorphic almost everywhere on the entire disk and its distributional $\partial_{\bar z}$ would vanish, again contradicting the support identity. □

6.2. **Stationary arcs and reflection.**

Theorem 6.3 (A stationary arc forces reflection)**.** *Suppose a regular switching chart contains a product $\Gamma \times I$ in its switching sheet, where Γ is an embedded nonconstant C^1 spatial arc and $I \subset (0, d)$ is an open interval. Then Γ lies on a reflection axis of the root multiset.*

Proof. Fix $z \in \Gamma$ and a unit tangent $\mathbf{v} \in \mathbb{C}$ at z. Write $\xi_j = w_j(z, \alpha) - z$ and $g = \psi_1 - \psi_2$. Since $g = 0$ on $\Gamma \times I$, its derivative in α vanishes there. Equation (4.2) gives $|\xi_1| = |\xi_2| =: r > 0$. Its tangential derivative also vanishes, and (4.1) gives

$$0 = \operatorname{Re}((2\partial_z g)\mathbf{v}) = \frac{\alpha}{r^2} \operatorname{Re}(\overline{w_1 - w_2}\, \mathbf{v}).$$

Thus $w_1 - w_2$ is perpendicular to the tangent line through z, while z is equidistant from the two saddles. Reflection in that line,

$$\mathfrak{r}(w) = z + \mathbf{v}^2 \overline{w - z},$$

therefore interchanges w_1 and w_2 for every $\alpha \in I$.

The saddle equations $\xi_j = \alpha h_P(w_j)$ imply

$$h_P(\mathfrak{r}(w_1(z, \alpha))) = \mathbf{v}^2 \overline{h_P(w_1(z, \alpha))}. \tag{6.1}$$

The branch $\alpha \mapsto w_1(z, \alpha)$ is injective: equality at two ratios, together with $w_1 - z = \alpha h_P(w_1)$ and $h_P(w_1) \neq 0$, forces the ratios to be equal. Its image is therefore infinite. Set $Q(w) := \overline{P(\mathfrak{r}(w))}$, a genuine complex polynomial. Conjugating (6.1) and clearing denominators gives $Q'P = QP'$ on this infinite image, hence identically as a polynomial identity. In particular,

$$h_P(\mathfrak{r}(w)) = \mathbf{v}^2 \overline{h_P(w)} \tag{6.2}$$

identically as a rational identity. Moreover, $(Q/P)' = 0$, so Q/P is a nonzero constant. Thus reflection preserves the entire root multiset, including multiplicities. The tangent line at z is a reflection axis.

The nonconstant arc lies in the root hull, so there are at least two distinct roots. An affine conjugate map $w \mapsto u\overline{w} + b$ is determined by its values on two such roots; preserving the finite root set leaves only finitely many possibilities. As z varies along the connected arc, the tangent

reflection parameters vary continuously among these possibilities and are constant. Hence the entire arc, including its endpoints by continuity, lies on one reflection axis. □

Theorem 6.4 (All curve mass lies on reflection axes)**.** *For any polynomial with at least two distinct roots, the curve part of ρ_P is supported on $\mathscr{A}_P$. In particular, if the root multiset has no reflection axis, then*

$$\rho_P = f_P \,\mathrm{d}A + \sum_{j=1}^{s} p_j^2 \delta_{a_j}, \qquad \int_{\mathbb{C}} f_P \,\mathrm{d}A = 1 - \sum_{j=1}^{s} p_j^2 > 0. \tag{6.3}$$

In this case $\mathcal{K}_P$ is the closure of its interior.

Proof. By Theorem 4.6, every curve contribution comes from a two-dimensional support-incidence stratum with nonzero ratio differential and spatial rank one. Ratio coordinates give a local product of a regular stationary support arc with an open ratio interval. Fix an interior ratio and discard the finitely many roots and branch values of its rational saddle map. Near each remaining observation the connecting gates continue as a finite analytic maximum. At every support point at least two gates are active: a unique maximizer would make the actual potential locally harmonic. The finitely many closed pairwise height-tie sets therefore cover the product, and their interiors are dense by the Baire theorem. On each open tie patch the calculation in the proof of Theorem 6.3, which uses the saddle equations and equality of heights, gives a tangent reflection preserving the entire root multiset. The finite union of axes is closed, so it contains the whole arc, including the discarded observations. The remaining lower-dimensional incidence strata have zero nonatomic incidence measure. This proves the curve-support assertion.

Without axes the curve contribution vanishes, giving (6.3); its planar mass is positive since $0 < p_j < 1$ and $\sum_j p_j^2 < \sum_j p_j = 1$. The semialgebraic core has area-null frontier. A non-root support point outside the closure of its interior would have a root-free disk meeting the core only in that frontier; absolute continuity would give the disk zero shadow mass, a contradiction. Connectedness and two distinct roots exclude isolated core points, so roots are limits of non-root support points as well. The reverse inclusion follows from closedness. □

Recall the planar body and axial set defined in the introduction. The next theorem proves (1.6) and identifies the geometry and measure of the portions of the core outside its body.

Theorem 6.5 (The planar body and all one-dimensional appendages)**.** *Suppose P has at least two distinct roots. Define the closed sets*

$$\mathcal{B}_P^{\mathrm{pl}} := \overline{\operatorname{int} \mathcal{K}_P} = \overline{\operatorname{int} \mathcal{S}_P}, \qquad \mathcal{L}_P := \overline{\mathcal{K}_P \setminus \mathcal{B}_P^{\mathrm{pl}}}.$$

The planar body is exactly the support of the absolutely continuous part:

$$\mathcal{B}_P^{\mathrm{pl}} = \operatorname{supp}(f_P \,\mathrm{d}A).$$

The set $\mathcal{L}_P$ is empty or a finite union of nondegenerate closed straight segments lying on reflection axes of the root multiset. The intersection $\mathcal{L}_P \cap \mathcal{B}_P^{\mathrm{pl}}$ is finite, and

$$\mathcal{K}_P = \mathcal{B}_P^{\mathrm{pl}} \cup \mathcal{L}_P, \qquad \mathcal{S}_P = \mathcal{B}_P^{\mathrm{pl}} \cup \mathcal{L}_P \cup T_P. \tag{6.4}$$

The segments can be subdivided so that their endpoints belong to $\mathcal{B}_P^{\mathrm{pl}} \cup Z(P) \cup \{\mathfrak{c}_P\}$. On every open segment away from this set the arclength density of ρ_P is constant and positive. Every root outside the union of reflection axes belongs to the planar body. If there is no reflection axis, $\mathcal{L}_P = \varnothing$.

Proof. Both $\mathcal{K}_P$ and $\mathcal{B}_P^{\mathrm{pl}}$ are semialgebraic. Every disk of positive $f_P \,\mathrm{d}A$-mass meets the semialgebraic set $\mathcal{K}_P$ in positive area, and hence meets its interior. Thus $\operatorname{supp}(f_P \,\mathrm{d}A) \subseteq \mathcal{B}_P^{\mathrm{pl}}$. Conversely, a disk meeting $\operatorname{int} \mathcal{K}_P$ contains a smaller disk in that interior avoiding the finitely many reflection axes and the roots. As a neighborhood of a support point it has positive ρ_P-mass, which is

entirely planar by Theorem 6.4. Taking closures proves the reverse inclusion. If a disk avoids the planar body, its intersection with $\mathcal{K}_P$ has empty interior and hence zero planar area. On such a disk Theorem 6.4 therefore places the measure on the reflection axes and the finitely many roots. A support point outside those axes could only be a root isolated in the support, which connectedness excludes. Consequently

$$\mathcal{K}_P \setminus \mathcal{B}_P^{\mathrm{pl}} \subseteq \mathscr{A}_P.$$

This difference is a bounded semialgebraic subset of finitely many lines. It has no isolated points: it is relatively open in $\mathcal{K}_P$, which has none. Its closure is thus a finite union of nondegenerate closed line segments, unless empty. This proves (6.4), using Proposition 2.3 for the full shadow.

Remove the centroid before taking line sections. Each body-exterior section has finite boundary by semialgebraicity, so its closure adds only finitely many points; distinct axes meet only at the centroid. Consequently $\mathcal{L}_P \cap \mathcal{B}_P^{\mathrm{pl}}$ is finite. Split at these points, at roots, and at the centroid. Near any remaining support point, a small root-free rectangle avoiding the body and centroid has support confined to one axis. The conservation law gives a common analytic primitive for its Cauchy-transform derivative. A gap on the middle line would connect the slit complement; analytic continuation would then exclude the support point, a contradiction. Thus the local support is the whole line section, excluding further endpoints. Theorem 6.2, or the same local Cauchy-inversion argument, gives constant positive arclength density. Constants agree on overlapping neighborhoods because their common support neighborhood has positive finite shadow mass, so the density is constant on each connected open segment. The statements about roots and the absence of axes follow from the displayed inclusion. □

Corollary 6.6 (Generic absolute continuity in every degree)**.** *For $d \geq 3$, the monic degree-d polynomials with no reflection axis form an open dense subset of coefficient space whose complement has Lebesgue measure zero. On that set* (6.3) *holds, and the limiting and full shadows have the same nonempty interior.*

Proof. Identify coefficient space with $\mathbb{R}^{2d}$. Write an axis as $e^{i\theta}(\mathbb{R} + it)$, with $\theta, t \in \mathbb{R}$. Reflection symmetry is equivalent to reality of the coefficients of the monic polynomial $q(w) = e^{-id\theta}P(e^{i\theta}(w + it))$. Conversely, $P(z) = e^{id\theta}q(e^{-i\theta}z - it)$ reconstructs every symmetric polynomial. Thus the symmetric locus is the image of a smooth map with $d + 2$ real parameters: the angle, the normal offset, and the d lower coefficients of q. On countably many compact parameter sets the derivatives are bounded, so the image has $2d$-dimensional Lebesgue measure zero because $d + 2 < 2d$ for $d \geq 3$. The locus is closed: every axis passes through the root centroid, centroids converge under coefficient convergence, and a subsequence of unit axis directions converges. Reality of the transformed coefficients passes to the limit. The complement is therefore open, dense, and of full measure. A polynomial without an axis has at least two distinct roots. By Theorem 6.4, its nonempty core is the closure of its interior, so that interior is nonempty; (2.8) gives the same interior for the full shadow. □

Corollary 6.7 (The hole examples have genuine planar bodies)**.** *For every sufficiently large N, the core of $P_N(z) = (z^2 - 1)^N(z - 2 - i)(z - 3 - i)$ is the closure of its nonempty interior and has a hole. The same conclusions hold on a sufficiently small open coefficient neighborhood consisting of polynomials with no reflection axis.*

Proof. For every sufficiently large N, the point $q_\diamond = \frac{2}{3} + \frac{i}{30}$ lies in a bounded component of $\mathbb{C} \setminus \mathcal{K}_{P_N}$, and this separation persists on an open coefficient neighborhood [HS2, Theorem 5.4]. These polynomials have no reflection axis. Theorem 6.4 gives $\mathcal{K}_{P_N} = \overline{\operatorname{int} \mathcal{K}_{P_N}}$. The core is nonempty, so its interior is nonempty. Intersect the hole-persistence neighborhood with the open no-reflection locus from Corollary 6.6. The intersection still contains the original polynomial, and every member has the same planar-body and marked-hole conclusions. □

7. Segments, radial stars, and zero-area classification

The chord law gives explicit measures for collinear and radially symmetric root configurations. In the first case, moments determine the measure on the root segment. In the second, differentiation preserves a finite radial star and a sampling formula determines the limiting mass on it. In both cases every finite-level zero already belongs to the limiting core.

7.1. **Collinear roots.**

Theorem 7.1 (The segment formula)**.** *Suppose all roots of P lie on one line, with distinct roots $a_1, \dots, a_s$ and weights $p_j = r_j/d$. Then*

$$\rho_P = \sum_{j=1}^{s} p_j^2 \delta_{a_j} + 2 \sum_{i<j} p_i p_j \operatorname{Unif}[a_i, a_j]. \tag{7.1}$$

Here $\operatorname{Unif}[a_i, a_j]$ *is the probability arclength measure on that segment. If $s \geq 2$, then*

$$\mathcal{S}_P = \mathcal{K}_P = [a_-, a_+], \qquad \mathcal{Z}_{P,n} \longrightarrow [a_-, a_+] \quad \textit{in Hausdorff distance},$$

where a_-, a_+ are the extreme roots along the line. If $s = 1$, both shadows equal the single root.

Proof. By affine covariance, reduce to real roots. Every descendant has real roots in $[a_-, a_+]$, by repeated Rolle's theorem, so the same is true of ρ_P. The chord measure σ_P is supported in that interval and has the same moments as ρ_P. Polynomials are uniformly dense in the continuous functions on a compact interval; hence the two measures coincide. Conditioning on the ordered pair of root endpoints in the chord law gives (7.1). The pair of extreme roots gives positive density on the whole interval, so this interval is the support. Every finite set $\mathcal{Z}_{P,n}$ is already contained in the interval, while Proposition 2.3 shows that every point of the interval is approached uniformly by $\mathcal{Z}_{P,n}$. This proves Hausdorff convergence and equality with the full shadow. □

The nonatomic density with respect to arclength on the root line is piecewise constant:

$$\frac{\mathrm{d}\rho_P^{\mathrm{na}}}{\mathrm{d}\mathcal{H}^1_{[a_-,a_+]}}(z) = 2 \sum_{i<j} \frac{p_i p_j}{|a_i - a_j|} \mathbf{1}_{[a_i,a_j]}(z),$$

up to an irrelevant choice of values at the finitely many roots. For $P = z^2 - 1$ this becomes

$$\rho_P = \tfrac{1}{4}\delta_{-1} + \tfrac{1}{4}\delta_1 + \tfrac{1}{4}\mathbf{1}_{[-1,1]}(x)\,\mathrm{d}x. \tag{7.2}$$

This constant density is an average over derivative ratios, not the zero-density limit at a single fixed ratio.

7.2. **Preservation of signed radial stars.**

Fix $q \geq 2$, set $\omega_q = e^{2\pi i/q}$ and $\zeta_q = e^{\pi i/q}$, and suppose

$$P(z) = z^{r_0} H_{\mathrm{rad}}(z^q), \qquad r_0 \geq 0, \tag{7.3}$$

where H_{rad} is monic and has only nonzero real roots, counted with multiplicity. Both signs are allowed. Zero roots can be absorbed into z^{r_0}; the constant case is a power of z and has singleton shadow. Assume henceforth that H_{rad} is nonconstant, and put

$$b_+ = \max\big(\{0\} \cup Z(H_{\mathrm{rad}})\big), \qquad b_- = \min\big(\{0\} \cup Z(H_{\mathrm{rad}})\big), \qquad R_+ = b_+^{1/q}, \quad R_- = (-b_-)^{1/q}.$$

Define the *signed radial star*

$$\mathscr{R}_{q,R_+,R_-} := \bigcup_{j=0}^{q-1} \omega_q^j [0, R_+] \;\cup\; \bigcup_{j=0}^{q-1} \zeta_q \omega_q^j [0, R_-]. \tag{7.4}$$

A zero radius contributes only the origin. Positive roots of H_{rad} produce one family of q spokes; negative roots produce the interlaced family, rotated by π/q.

Lemma 7.2 (Preservation of the signed radial star)**.** *Every zero of every nonconstant derivative of every power of* (7.3)*, including derivative order zero, lies in* $\mathscr{R}_{q,R_+,R_-}$.

Proof. Write a polynomial in the class as $z^\ell G(z^q)$, absorbing any zero roots of G into z^ℓ, so that $G(0) \neq 0$ unless G is constant. Its derivative is

$$\begin{cases} z^{\ell-1}(\ell G(y) + qyG'(y)), & \ell > 0, \\ qz^{q-1}G'(y), & \ell = 0, \end{cases} \qquad y = z^q.$$

The roots of G' remain real in the convex hull of the roots of G. For $\ell > 0$, apply Gauss–Lucas to $F(y) := y^\ell G(y)^q$. Its roots lie in $[\min(\{0\} \cup Z(G)), \max(\{0\} \cup Z(G))]$, and

$$F'(y) = y^{\ell-1}G(y)^{q-1}(\ell G(y) + qyG'(y)).$$

Every root of $\ell G + qyG'$ is therefore a root of F' and lies in the same real interval. Constant G causes no difficulty. Thus the real-rooted class and its outer bounds are preserved by differentiation, and they are also preserved by taking powers, proving the claim. □

Every descendant zero measure, and hence ρ_P, is invariant under multiplication by ω_q. The polynomial changes by a scalar under this rotation, and differentiation preserves that property.

7.3. **Sampling on a signed radial star.** Let

$$\lambda_q := (z \mapsto z^q)_* \mu_P^{\mathrm{root}},$$

a probability measure on $[b_-, b_+]$. Let Y_1, Y_2 be independent with law λ_q, and let U_q be independent and uniform on $[0, 1]$. Set

$$V_q = (1 - \mathsf{U}_q)Y_1 + \mathsf{U}_q Y_2.$$

Let R_q and J_q be mutually independent and independent of these variables, with

$$\mathrm{Law}(\mathsf{R}_q) = \frac{1}{q}\delta_1 + \frac{q-1}{q}\,\mathrm{Unif}[0,1], \qquad J_q \text{ uniform on } \{0, \dots, q-1\}. \tag{7.5}$$

Choose the following explicit qth root of a real number:

$$\chi_q(v) = \begin{cases} v^{1/q}, & v \geq 0, \\ e^{\pi i/q}|v|^{1/q}, & v < 0. \end{cases}$$

In particular, $\chi_q(v)^q = v$.

Theorem 7.3 (Signed radial sampling formula)**.** *For* (7.3) *with* H_{rad} *nonconstant,*

$$\rho_P = \mathrm{Law}\big(\omega_q^{J_q}\mathsf{R}_q\chi_q(V_q)\big). \tag{7.6}$$

Moreover,

$$\mathcal{S}_P = \mathcal{K}_P = \mathscr{R}_{q,R_+,R_-}, \qquad \mathcal{Z}_{P,n} \longrightarrow \mathscr{R}_{q,R_+,R_-} \quad \textit{in Hausdorff distance.}$$

All finite-level zeros are already contained in this limiting star.

Proof. Write $M_{q,k} = \int y^k \,\mathrm{d}\lambda_q(y)$. Rotational invariance gives $\mathfrak{m}_k(P) = 0$ unless q divides k, and $\mathfrak{m}_{qk}(P) = M_{q,k}$. The chord moment formula gives

$$\int z^{qk} \,\mathrm{d}\rho_P(z) = \frac{1}{qk+1}\sum_{j=0}^{k} M_{q,j}M_{q,k-j}, \tag{7.7}$$

while all other moments vanish. Direct integration of the real chord and of R_q gives

$$\mathbb{E}[V_q^k] = \frac{1}{k+1}\sum_{j=0}^{k} M_{q,j}M_{q,k-j}, \qquad \mathbb{E}[\mathsf{R}_q^{qk}] = \frac{k+1}{qk+1}.$$

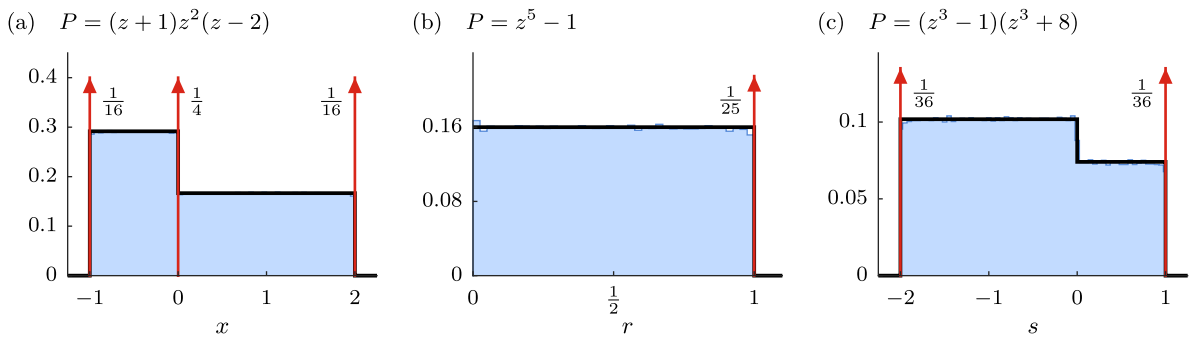


FIGURE 6. Histograms of $\bar{\nu}_{P,n}$ (shaded) and the limiting densities (lines); arrows: atoms, labeled by their masses. (a) Density on $\mathbb{R}$ for $P = (z+1)z^2(z-2)$, $n = 100$, from (7.1). (b) Arclength density on each spoke for $P = z^5 - 1$, $n = 80$, from (7.8). (c) Arclength density on each spoke for $P = (z^3-1)(z^3+8)$, $n = 60$, against the signed radius s, positive on the spokes through the cube roots of unity; (7.6) gives the values $\frac{2}{27}$ and $\frac{11}{108}$.

The law in (7.6) therefore has the same holomorphic moments and rotational invariance as ρ_P.

Both measures are supported on (7.4). Their pushforwards by $z \mapsto z^q$ are measures on the compact real interval $[b_-, b_+]$ with identical moments, hence coincide. A rotation-invariant measure on this star is determined by its pushforward: away from zero it divides mass equally among the q preimages, and zero has a single preimage. This proves the sampling formula.

Proposition 2.3 makes $\mathcal{K}_P$ connected and includes every nonzero extreme tip. A connected subset of the signed star containing these tips contains every spoke: the signed radius, positive on one spoke and negative on the others, is continuous on the star, so the intermediate value theorem forces the center and every intermediate radius. Thus $\mathcal{K}_P$ is the whole star. Lemma 7.2 and the core convergence in (2.8) give the full-shadow equality, finite-level inclusion, and Hausdorff convergence. □

For example, $(z^q - a^q)(z^q - b^q)$ with $0 < a < b$ has q spokes of length b; the inner ring changes the measure but not the support. In contrast, $(z^q - a^q)(z^q + b^q)$ with $a, b > 0$ has two interlaced families of spokes of lengths a and b. When $q = 2$ this gives the unequal cross with endpoints $\pm a, \pm ib$. Arbitrary multiplicities are allowed, and affine covariance gives the corresponding formulas after any affine change of coordinates.

Corollary 7.4 (Uniform spokes for a regular polygon)**.** *For $P(z) = z^q - 1$, $q \geq 2$,*

$$\rho_P = \frac{1}{q^2} \sum_{j=0}^{q-1} \delta_{\omega_q^j} + \frac{q-1}{q^2} \sum_{j=0}^{q-1} (r \mapsto \omega_q^j r)_* (\,\mathrm{d}r \restriction [0,1]). \tag{7.8}$$

Thus the nonatomic part has constant arclength density on each spoke, its total mass is $1 - 1/q$, and the mass at each root is $1/q^2$.

Proof. Here $\lambda_q = \delta_1$, so $V_q = 1$. Substitute (7.5) into (7.6) and distribute the mass equally over the rotations. Equivalently, the nonzero moments are $\int z^{qk}\,\mathrm{d}\rho_P = (k+1)/(qk+1)$, which are exactly the moments of (7.8). □

Figure 6 compares (7.1), (7.8) and (7.6) with the pooled zero measures at finite levels.

Example 7.5 (A spurious endpoint tie)**.** For $P(w) = w^3 - 1$, the formal saddle equation at $\alpha = 0$ is $3w^2(w - z) = 0$. Counting the double root $w = 0$ twice and taking the two largest limiting real

heights would declare a tie whenever $|z^3 - 1| < 1$. This includes an open set in the root triangle. Corollary 7.4, however, puts every descendant zero on the three spokes. At the endpoint, the formal equation no longer represents two distinct contributing saddles. The barrier is instead defined at interior ratios, and its endpoint behavior is determined by a limit.

Verification of the counterexample. The degree-normalized endpoint heights are $0, 0, \frac{1}{3}\log|z^3-1|$, with $-\infty$ at an original root. Off these roots, a saddle tending to $z \neq 0$ satisfies $w^2(w-z) = \frac{\alpha}{3}(w^3-1)$, so $\alpha \log|w-z| \to 0$; a saddle tending to zero has height limit zero by continuity. The point $z_0 = \frac{1}{4} + \frac{i}{8}$ lies strictly inside the root triangle, satisfies $|z_0^3 - 1| < 1$, and has $\Im(z_0^3) \neq 0$. These strict conditions persist on an open neighborhood, which therefore misses all three spokes while retaining the false maximal tie. □

7.4. **The classification.**

Theorem 7.6 (Complete zero-area classification)**.** *Let P be a nonconstant polynomial with at least two distinct roots. The following conditions are equivalent:*

(i) $\mathrm{Area}(\mathcal{K}_P) = 0$.
(ii) $\mathrm{Area}(\mathcal{S}_P) = 0$.
(iii) *Every root of P lies on a reflection axis of its root multiset.*
(iv) *There are $a, b, c \in \mathbb{C}$ with $ac \neq 0$, an integer $q \geq 1$, an integer $r_0 \geq 0$, and a nonconstant polynomial H with only nonzero real roots, such that*

$$cP(az+b) = z^{r_0} H(z^q). \tag{7.9}$$

In these cases $\mathcal{S}_P = \mathcal{K}_P$: the common set is a segment if $q = 1$, and the finite star described in Theorem 7.3 if $q \geq 2$, after the corresponding affine change of coordinates. In every other case both shadows have positive area and the same nonempty interior. A polynomial with a single distinct root has a singleton shadow.

Proof. The equivalence of (i) and (ii) is (2.8). If (i) holds, the planar part of ρ_P has zero mass. By Theorem 6.4, the support is contained in the union of $\mathscr{A}_P$ and the finite root set. A root outside $\mathscr{A}_P$ would be isolated in the support, contradicting Proposition 2.3. This proves (iii).

Suppose (iii). Center at the root centroid and choose one actual axis as the real line, using monic affine normalization to obtain a real polynomial Q of degree d. Let $q = \gcd\{d - n : n < d,\ [z^n]Q \neq 0\}$ and let r_0 be its least supported exponent. The gap set is nonempty, since Q has two distinct roots, so $q \geq 1$. For $\lambda \neq 0$, rotation by λ preserves the root multiset exactly when $Q(\lambda z) = \lambda^d Q(z)$; coefficient comparison makes this equivalent to $\lambda^q = 1$. Every reflection axis passes through zero. If a reflection fixes a root z, write $z = \lambda\overline{z}$. Composing with conjugation gives a rotation preserving the multiset, so $\lambda^q = 1$ and $z^q \in \mathbb{R}$. Every supported exponent is congruent to r_0 modulo q, hence $Q(z) = z^{r_0}H(z^q)$ for a monic nonconstant H with $H(0) \neq 0$. If $H(t) = 0$, choose z with $z^q = t$; then $Q(z) = 0$, so t is real and nonzero. This proves (iv), including $q = 1$.

Finally, for $q = 1$, repeated Rolle's theorem confines every descendant zero to the root segment. For $q \geq 2$, Lemma 7.2 confines every descendant zero to a finite union of radial segments; it allows both positive and negative roots of H. Thus (iv) implies (ii). Theorems 7.1 and 7.3 identify the support with the whole indicated segment or star, which also proves equality of the full and limiting cores. In the remaining cases the area is positive, and definability gives nonempty interior. The one-root assertion follows directly by differentiation. □

This criterion involves only the finite root multiset. It can be checked from the roots or from the normal form (7.9), without selecting saddles. Reflection symmetry alone is not sufficient: every root must lie on an axis. When some roots lie off the axes, planar and curve mass may coexist.

7.5. **Finite coefficient tests.**

Proposition 7.7 (Reflection and zero area from coefficients)**.** *Center P at its root centroid and write*

$$Q(w) := P(\mathfrak{c}_P + w) = w^d + \sum_{j=2}^{d} \beta_j w^{d-j}, \qquad J := \{j : \beta_j \neq 0\}.$$

If $J = \varnothing$, both shadows are the singleton $\{\mathfrak{c}_P\}$. Otherwise choose any $j_0 \in J$. All reflection axes are obtained from the at most $j_0 \leq d$ solutions of

$$\lambda^{j_0} = \frac{\beta_{j_0}}{\overline{\beta_{j_0}}} \quad \text{that also satisfy} \quad \beta_j = \overline{\beta_j}\lambda^j \quad (j \in J). \tag{7.10}$$

Every candidate has modulus one. A retained λ is the reflection $w \mapsto \lambda\overline{w}$ in centered coordinates, so its axis in the observation plane passes through $\mathfrak{c}_P$.

If no candidate is retained, the shadow has positive area. Otherwise put $q := \gcd J$, choose a retained $\lambda = e^{2i\theta}$, and set $r_0 := d - \max J$. Then

$$e^{-id\theta}Q(e^{i\theta}x) = x^{r_0}H(x^q), \tag{7.11}$$

where H is a monic real polynomial with $H(0) \neq 0$. Both shadows have zero area if and only if H is real-rooted. Thus the dimension classification requires only centering, finitely many coefficient identities, an integer greatest common divisor, and a real-rootedness test; no saddle-selection test is needed.

Proof. Reflection in $e^{i\theta}\mathbb{R}$ preserves the root multiset of Q if and only if the monic rotated polynomial $e^{-id\theta}Q(e^{i\theta}x)$ has real coefficients. Its coefficient indexed by j is $\beta_j e^{-ij\theta}$, whose reality is exactly $\beta_j = \overline{\beta_j}e^{2ij\theta}$. This proves (7.10), including completeness of the finite list.

Fix one solution λ_0. The other solutions are precisely $\lambda_0\omega_q$ with $\omega_q^j = 1$ for all $j \in J$, or equivalently $\omega_q^q = 1$. In coordinates rotated by θ, the axes are therefore the q lines of angles $\pi\ell/q$, $0 \leq \ell < q$. All exponents of the rotated polynomial are congruent to d modulo q, and their smallest is r_0. This proves (7.11) with H real and $H(0) \neq 0$. A nonzero point lies on one of the q axes exactly when its qth power is real. Hence all roots lie on the axes exactly when all roots of H are real. Theorem 7.6 completes the proof. □

For algebraic coefficients these tests can be performed exactly. To test real-rootedness even when H has multiple roots, take its square-free part $G := H/\gcd(H, H')$. Form $G_0 = G$, $G_1 = G'$, and successively $G_{k+1} = -\operatorname{rem}(G_{k-1}, G_k)$ stopping as soon as the last polynomial is constant, where rem is the remainder in polynomial division. Sturm's theorem states that the difference between the numbers of sign variations of this sequence at $-\infty$ and $+\infty$ counts the distinct real roots. It equals $\deg G$ exactly when H is real-rooted; see [BCR].

8. Analytic regularity of the planar density

The measure decomposition can be sharpened away from a finite exceptional set of arcs and points. The remaining incidence sheets project locally invertibly, so their densities are analytic and strictly positive.

Theorem 8.1 (Regularity of the planar measure)**.** *For every polynomial P there is a compact definable set $\mathscr{E}_P \subseteq \mathcal{K}_P$, consisting of finitely many points and bounded C^1 arcs of finite length, such that $\partial\mathcal{K}_P \subseteq \mathscr{E}_P$ and*

$$\rho_P \restriction (\mathbb{C} \setminus \mathscr{E}_P) = f_P \, \mathrm{d}A \restriction (\mathbb{C} \setminus \mathscr{E}_P).$$

On each component of $\operatorname{int}\mathcal{K}_P \setminus \mathscr{E}_P$, the density f_P has a strictly positive real-analytic representative. In particular

$$f_P(z) > 0 \quad \text{for almost every } z \in \operatorname{int}\mathcal{K}_P, \qquad f_P(z) = 0 \quad \text{for almost every } z \notin \operatorname{int}\mathcal{K}_P. \tag{8.1}$$

Thus the planar part of ρ_P and Lebesgue measure restricted to $\operatorname{int}\mathcal{K}_P$ *have exactly the same null sets.*

Proof. Use the compact definable incidence space $\mathcal{I}_P$ and a frontier-compatible C^2 stratification, refined by spatial projection rank. Away from the root and endpoint fibers and the ordinary saddle-collision image $\mathscr{G}_P^{\mathrm{col}}$, a two-dimensional rank-two stratum is locally a C^2 graph $\alpha = \alpha(z)$. The frontier condition isolates it from the whole incidence. A small tube about this graph has two connected complementary chambers. The unique selected simple critical height is fixed on each chamber; continuity makes their two-height maximum equal the original maximum throughout the tube. The support condition rules out the same winner on both sides. Their difference g satisfies $g(z, \alpha(z)) = 0$ and $\nabla_z g \neq 0$ by (4.1). Differentiating the graph identity gives $\partial_\alpha g \neq 0$, so the analytic implicit-function theorem makes the graph real-analytic. Additional heights may tie on it without changing this pair. Let $\mathscr{E}_P$ be the closure of the union of $\partial\mathcal{K}_P$, the distinguished finite set $Z(P) \cup Z(P') \cup \{\mathfrak{c}_P\}$, $\mathscr{G}_P^{\mathrm{col}} \cap \mathcal{K}_P$, and the spatial images of all lower-dimensional or rank-deficient strata. Include the root lines and endpoint fibers among the exceptional strata.

The image of each rank-deficient stratum has dimension at most one, as does the image of a stratum of dimension at most one. Definable closure preserves dimension, and the incidence space is compact. Consequently $\mathscr{E}_P$ is compact and definable of dimension at most one. The monotone-arc argument of Section 2.3 gives its stated decomposition and finite length. It contains every possible curve or atomic contribution to ρ_P.

Fix $z \in \mathcal{K}_P \setminus \mathscr{E}_P$. The incidence fiber over z is nonempty and compact, and all its points have projection rank two. Each such point is isolated in the fiber by the inverse-function theorem, so the fiber is finite: an infinite fiber would have an accumulation point in that same compact fiber, contradicting local invertibility. There is a disk D about z over which all these points continue as finitely many real-analytic graphs $\alpha = \alpha_j(\zeta)$. There are no additional preimages after shrinking D. Otherwise compactness would give a limiting preimage over z outside the chosen neighborhoods. In particular $D \subseteq \mathcal{K}_P$.

For each graph choose its actual switching pair and write g_j for the corresponding real height difference. Disjoint compact continuous cutoffs, equal to one near these preimages, partition every nearby incidence fiber. Apply (4.6) to each cutoff and add, counting each geometric preimage once. On D this gives

$$f_P(\zeta) = \frac{1}{\pi d^2} \sum_j \frac{|\nabla_z g_j(\zeta, \alpha_j(\zeta))|^2}{|\partial_\alpha g_j(\zeta, \alpha_j(\zeta))|}. \tag{8.2}$$

Every numerator is positive and every denominator is nonzero. The sign of each denominator is locally constant before taking its absolute value, so every summand is real-analytic. This proves local analyticity and strict positivity. The local representatives agree on overlaps, since they represent the same measure, proving the componentwise assertion. Finally $\mathscr{E}_P$ and the boundary have zero planar area, and the measure is supported on $\mathcal{K}_P$. This proves (8.1) and the null-set claim.

□

Figure 5(b) approximates the planar density of a cubic near its centroid. The caustic arcs inside the core lie in the set $\mathscr{E}_P$ constructed in the proof.

9. Finite atomic laws and vanishing root masses

9.1. **Weighted potentials and supports.** Let

$$\eta = \sum_{a \in A} p_a \delta_a, \qquad A \subset \mathbb{C} \text{ finite}, \quad p_a > 0, \quad \sum_{a \in A} p_a = 1. \tag{9.1}$$

The masses in this section are arbitrary real numbers. Put

$$C_\eta(w)=\sum_{a\in A}\frac{p_a}{w-a},\qquad h_\eta(w)=\frac{1}{C_\eta(w)},\qquad \mathfrak{c}_\eta=\sum_{a\in A}p_a a,$$
$$F_{t,\eta}(w)=w-th_\eta(w),\qquad 0<t<1.$$

For $z\notin A$, the weighted landscape and its local holomorphic phase are

$$L_{\eta,z,t}(w)=U_\eta(w)-t\log|w-z|,\qquad \Phi_{\eta,z,t}(w)=\sum_{a\in A}p_a\log(w-a)-t\log(w-z).$$

Define

$$E_\ell(\eta,z,t)=\{w\in\mathbb{C}\setminus(A\cup\{z\}):L_{\eta,z,t}(w)\geq\ell\}\cup\{z,\infty\},$$

and let $\mathfrak{b}_\eta(z,t)$ be the supremum of levels for which the two marked ends belong to the same connected component of this set. At the barrier level, a saddle is selected precisely when it belongs to the closures of both strict-superlevel components attached to the marked ends. The reduced saddle equation is

$$(w-z)\mathsf{N}_\eta(w)-t\mathsf{Q}_\eta(w)=0, \tag{9.2}$$

where

$$\mathsf{Q}_\eta(w)=\prod_{a\in A}(w-a),\qquad \mathsf{N}_\eta(w)=\sum_{a\in A}p_a\prod_{b\in A\setminus\{a\}}(w-b).$$

It has degree $|A|$ and leading coefficient $1-t$, and $C_\eta=\mathsf{N}_\eta/\mathsf{Q}_\eta$.

Proposition 9.1 (Weighted measures and selection)**.** [HS2, Propositions 2.1 and 2.5] *There is a unique probability measure $\mu_{t,\eta}$ with potential*

$$U_{\mu_{t,\eta}}(z)=\frac{\mathfrak{b}_\eta(z,t)-\mathfrak{h}_1(t)}{1-t},\qquad \mathfrak{h}_1(t)=-t\log t-(1-t)\log(1-t), \tag{9.3}$$

with subharmonic extension at the atoms. Set

$$S_{t,\eta}=\operatorname{supp}\mu_{t,\eta},\qquad \rho_\eta=2\int_0^1(1-t)\mu_{t,\eta}\,\mathrm{d}t,\qquad S_\eta=\operatorname{supp}\rho_\eta=\overline{\bigcup_{0<t<1}S_{t,\eta}}. \tag{9.4}$$

All these measures and sets are supported in $\operatorname{conv}A$*. Their atomic parts are*

$$\mu_{t,\eta}^{\mathrm{at}}=\sum_{a\in A}\frac{(p_a-t)_+}{1-t}\delta_a,\qquad \rho_\eta^{\mathrm{at}}=\sum_{a\in A}p_a^2\delta_a. \tag{9.5}$$

An atom location a belongs to $S_{t,\eta}$ whenever $0<t\leq p_a$ and $t<1$. The measures and supports are continuous in the interior ratio, and

$$\begin{aligned}&\mu_{t,\eta}\Longrightarrow\eta, && S_{t,\eta}\longrightarrow A\cup Z(C_\eta) && (t\downarrow 0),\\ &\mu_{t,\eta}\Longrightarrow\delta_{\mathfrak{c}_\eta}, && S_{t,\eta}\longrightarrow\{\mathfrak{c}_\eta\} && (t\uparrow 1),\end{aligned} \tag{9.6}$$

where $Z(C_\eta)$ is the set of finite zeros of the rational function C_η, and the set limits are in Hausdorff distance.

Off $S_{t,\eta}$, the Cauchy transform $C_{\mu_{t,\eta}}$ is nonzero and

$$\begin{aligned}W_{t,\eta}(z)&=z+\frac{t}{(1-t)C_{\mu_{t,\eta}}(z)},\\ F_{t,\eta}(W_{t,\eta}(z))&=z,\qquad C_\eta(W_{t,\eta}(z))=(1-t)C_{\mu_{t,\eta}}(z).\end{aligned} \tag{9.7}$$

At non-atom observation points this is the selected simple saddle; $F'_{t,\eta}(W_{t,\eta})\neq 0$. It is holomorphic in z and locally real-analytic in t. The separating-contour, local-maximum, and harmonic selection assertions of Proposition 2.1 hold with $L_{\eta,z,t}$ and $\mathfrak{b}_\eta$ in place of their polynomial counterparts, jointly under changes of the positive masses and distinct locations.

When $\eta = \mu_P^{\mathrm{root}}$, the normalizations agree through

$$\mu_{t,\eta} = \mu_{dt,P}, \quad S_{t,\eta} = \mathcal{K}^{\mathrm{fix}}_{dt,P}, \quad \rho_\eta = \rho_P, \quad S_\eta = \mathcal{K}_P, \quad \mathfrak{b}_\eta(z,t) = d^{-1}\mathfrak{b}_P(z,dt). \tag{9.8}$$

Proposition 9.2 (Definability, connectedness, and uniform complexity)**.** [HS2, Theorems 4.1–4.2 and Corollaries 4.4, 4.8–4.9] *For a fixed bound s on $|A|$, the weighted barriers, fixed-ratio supports, and cores form definable families in the atom locations and positive masses. The incidence set*

$$\mathcal{I}_\eta = \overline{\{(z,t) : 0 < t < 1,\ z \in S_{t,\eta}\}}$$

is compact and has dimension at most two, with spatial projection S_η. The set S_η is connected. Each point of a fixed-ratio support can be joined to $\mathfrak{c}_\eta$ by a continuous definable path through supports at increasing ratios.

There is an integer M_s, depending only on s, such that ∂S_η is a union of at most M_s points and at most M_s C^1 arcs on each of which both coordinates are monotone. In particular, if $A \subset \{|z| \le R\}$, then

$$\mathcal{H}^1(\partial S_\eta) \le 4M_s R.$$

The numbers of components and the Betti numbers of the fibers are uniformly bounded, and there are finitely many homeomorphism types.

9.2. **Finite-location limits and selected transport.** Let $B \supset A$ be another finite set. A point of $B \setminus A$ represents a root location whose limiting mass is zero. For such a point b with $C_\eta(b) \neq 0$, define its transport at ratio t by

$$\theta_b(t) = b - \frac{t}{C_\eta(b)}.$$

Only selected transports contribute to the complete zero-set limit. Set

$$\begin{aligned}\mathcal{O}_t(\eta, B) = \{\theta_b(t) :{}& b \in B \setminus A,\ C_\eta(b) \neq 0,\\ & \theta_b(t) \notin S_{t,\eta},\quad W_{t,\eta}(\theta_b(t)) = b\},\end{aligned} \tag{9.9}$$

and define the *finite-location limit set*

$$K_{\eta,B} := S_\eta \cup B \cup \overline{\bigcup_{0<t<1} \mathcal{O}_t(\eta, B)}. \tag{9.10}$$

This definition retains both the limiting masses and the locations at which mass disappears.

Proposition 9.3 (Complete finite-location limits)**.** [HS2, Theorem 3.1, Corollary 3.5, and Proposition 4.5] *Let G_j be monic of degree $N_j \to \infty$, with*

$$\eta_j = N_j^{-1} \operatorname{div}_0 G_j \Longrightarrow \eta, \qquad Z(G_j) \longrightarrow B \quad \text{in Hausdorff distance}, \tag{9.11}$$

where B is finite. If $m_j/N_j \to t \in (0,1)$, then

$$\frac{\operatorname{div}_0 G_j^{(m_j)}}{N_j - m_j} \Longrightarrow \mu_{t,\eta}, \qquad Z(G_j^{(m_j)}) \longrightarrow S_{t,\eta} \cup \mathcal{O}_t(\eta, B).$$

The pooled measures converge to ρ_η, and

$$\bigcup_{m=0}^{N_j-1} Z(G_j^{(m)}) \longrightarrow K_{\eta,B}.$$

More generally, if finite atomic laws $\eta_j \Longrightarrow \eta$ and finite sets $B_j \supset \operatorname{supp}\eta_j$ satisfy $B_j \to B$ with B finite, then $K_{\eta_j,B_j} \to K_{\eta,B}$ in Hausdorff distance.

The set $K_{\eta,B}$ is obtained from S_η by adjoining finitely many closed straight segments, each meeting S_η at an endpoint. It is contained in

$$S_\eta \cup B \cup \bigcup_{\substack{b\in B\setminus A\\ C_\eta(b)\neq 0}} \{\theta_b(t) : 0 \le t \le 1\}.$$

In particular, $K_{\eta,B}$ is connected and has the same interior and area as S_η. When $B = A$, it equals S_η.

Corollary 9.4 (Polynomial masses and finite-location limits)**.** *If $\eta = \mu_P^{\mathrm{root}}$ and $B \supset Z(P)$ is finite, then $K_{\eta,B}$ is semialgebraic.*

Proof. By Proposition 9.3, it is the union of $S_\eta = \mathcal{K}_P$ and finitely many closed straight segments. Apply Theorem 5.4. □

9.3. **Conservation and regularity for weighted laws.**

Proposition 9.5 (Weighted conservation)**.** *For every finite atomic law* (9.1)*,*

$$C'_{\rho_\eta} = -C_\eta^2 \qquad \text{on } \mathbb{C}\setminus S_\eta. \tag{9.12}$$

For every closed piecewise smooth curve γ in this complement,

$$\operatorname{wind}(\gamma, a) = \operatorname{wind}(\gamma, \mathfrak{c}_\eta) \qquad (a \in A).$$

Proof. On a disk disjoint from S_η, put $u_t = (1-t)C_{\mu_{t,\eta}}$. Differentiating (9.7) gives

$$u_t\,\partial_t u_t = \partial_z u_t, \qquad u_0 = C_\eta, \quad u_1 = 0.$$

Since $C_{\rho_\eta} = 2\int_0^1 u_t\,\mathrm{d}t$, integration yields (9.12). The bound $|\partial_z u_t| \le (1-t)/\operatorname{dist}(D, S_\eta)^2$ on a smaller disk justifies differentiation under the integral.

For $a \in A$, the curve $W_{t,\eta}\circ\gamma$ avoids a. The endpoint limits in (9.6) imply, uniformly on γ,

$$W_{t,\eta}(z) \longrightarrow z \quad (t\downarrow 0), \qquad \frac{1-t}{t}(W_{t,\eta}(z) - a) \longrightarrow z - \mathfrak{c}_\eta \quad (t\uparrow 1).$$

The winding number is constant in t, giving the assertion. □

Proposition 9.6 (Weighted measure decomposition and planar regularity)**.** *The measure ρ_η has a decomposition*

$$\rho_\eta = f_\eta\,\mathrm{d}A + \vartheta_\eta\,\mathcal{H}^1 \restriction \Gamma_\eta + \sum_{a\in A} p_a^2 \delta_a,$$

with nonnegative integrable densities and a finite union Γ_η of bounded C^1 arcs of finite length. There is a compact definable set $\mathscr{E}_\eta \subset S_\eta$ of dimension at most one, containing ∂S_η, such that off $\mathscr{E}_\eta$ the measure is $f_\eta\,\mathrm{d}A$. On each component of $\operatorname{int} S_\eta \setminus \mathscr{E}_\eta$, the density is strictly positive and real-analytic. In particular, $f_\eta\,\mathrm{d}A$ and Lebesgue measure on $\operatorname{int} S_\eta$ have the same null sets.

Proof. Local compact localization of all saddle roots, including multiple roots, and weighted first variation make the actual barrier locally Lipschitz in (z,t) off A, for $0 < t < 1$. Circle means of its potential and averaging therefore give $\rho_\eta \restriction A^c \ll \mathcal{H}^1$. This argument remains valid at saddle collisions.

Refine the compact weighted incidence by a frontier-compatible C^2 stratification and spatial projection rank, as in Theorem 8.1. Away from collisions, each two-dimensional rank-two stratum is a smooth ratio graph. Its two complementary chambers fix the two selected simple heights; continuity identifies their maximum with the actual barrier throughout a small tube. Their difference g has $\nabla_z g \neq 0$, since distinct saddles have distinct spatial derivatives $t/(w-z)$. Differentiating the graph identity gives $\partial_t g \neq 0$, so the analytic implicit-function theorem upgrades the graph to real-analytic. Additional heights may tie on it.

Let $\mathscr{E}_\eta$ be the closure, inside S_η, of the boundary, the sites, zeros of C_η, the centroid, the ordinary collision image, and the spatial images of lower-dimensional or rank-deficient strata; include root and endpoint fibers among the exceptional strata. Definable stratification and finite-length curve decomposition make $\mathscr{E}_\eta$ compact, of dimension at most one and zero area, and the union of finitely many points and bounded regular C^1 arcs of finite total length. Call the union of these arcs Γ_η.

Above any support point outside $\mathscr{E}_\eta$, compactness and local invertibility construct all preimages as finitely many analytic ratio graphs $t = t_j(z)$, with no further preimages on a smaller disk. Disjoint compact continuous cutoffs, equal to one near these preimages, partition every nearby incidence fiber. For each cutoff the actual fixed-ratio interface law is $|\nabla_z g_j|\,\mathcal{H}^1/(2\pi(1-t))$. Multiplication by $2(1-t)$ and the local area formula thus give

$$f_\eta(z) = \frac{1}{\pi}\sum_j \frac{|\nabla_z g_j(z, t_j(z))|^2}{|\partial_t g_j(z, t_j(z))|}. \tag{9.13}$$

The local measure laws agree on overlaps and give the global planar density, set to zero on $\mathscr{E}_\eta$ and outside S_η. Each summand is positive and real-analytic.

The remaining measure is carried by $\mathscr{E}_\eta$. After removing the sites, its arclength absolute continuity above and $\mathcal{H}^1(\mathscr{E}_\eta) < \infty$ give an integrable nonnegative Radon–Nikodym density. Finite points have zero arclength, so this is $\vartheta_\eta\,\mathcal{H}^1 \restriction \Gamma_\eta$. Formula (9.5) supplies precisely the remaining $p_a^2\delta_a$ terms. Finiteness of ρ_η makes both densities integrable. The exceptional set has zero area and contains the boundary; strict positivity of (9.13) on the remaining interior proves the asserted equality of null sets. □

9.4. **Weighted reflection and zero area.** The reflection argument also applies to arbitrary positive masses. A reflection axis of η is a line whose reflection preserves η, including the mass at each support point.

Theorem 9.7 (Reflection and dimension for every finite atomic law)**.** *Let $\eta = \sum_{a\in A} p_a\delta_a$ be a probability measure with finite support, $p_a > 0$, and at least two support points. The curve part of ρ_η is supported on reflection axes of η, and*

$$\operatorname{Area}(S_\eta) = 0 \quad\Longleftrightarrow\quad \textit{every point of } A \textit{ lies on a reflection axis of } \eta. \tag{9.14}$$

In this case, with $c = \int z\,\mathrm{d}\eta(z)$,

$$S_\eta = \bigcup_{a\in A} [c, a],$$

where $[c,a] = \{(1-t)c + ta : 0 \le t \le 1\}$; this is a segment or a finite radial star. If η has no reflection axis, then

$$\rho_\eta = f_\eta\,\mathrm{d}A + \sum_{a\in A} p_a^2\delta_a, \quad \int f_\eta\,\mathrm{d}A = 1 - \sum_{a\in A} p_a^2, \quad S_\eta = \overline{\operatorname{int} S_\eta}.$$

Proof. Proposition 9.6 supplies the planar, curve, and atomic decomposition. On a regular curve component, the rank-one incidence gives a stationary support arc over a ratio interval. Apply the argument of Theorem 6.3 with $h_\eta = 1/C_\eta$: equal selected heights force tangent reflection to exchange two distinct saddle branches. Their saddle equations give Cauchy covariance on the infinite image of either branch. Clearing the finite denominators and comparing poles therefore shows that this reflection preserves every original mass. There are only finitely many axes, so the curve measure is carried by their closed union. The no-axis conclusions follow as in Theorem 6.4, using connectedness from Proposition 9.2 and $\int f_\eta\,\mathrm{d}A = 1 - \sum_a p_a^2 > 0$.

If S_η has zero area, its planar measure vanishes. Outside the axes its support could then contain only the finitely many sites. A connected support with at least two points has no isolated

points, so removing these finite exceptions puts the whole support on the axes. Since $A \subset S_\eta$, this proves the forward implication in (9.14).

Conversely, suppose every site lies on an axis. Use the same integer approximants as in the finite atomic construction:

$$m_{a,j} = 1 + \lfloor j p_a \rfloor, \qquad P_j(w) = \prod_{a \in A} (w-a)^{m_{a,j}}, \qquad \eta_j = \frac{1}{\sum_a m_{a,j}} \sum_a m_{a,j} \delta_a.$$

Their normalized masses converge to p_a. Equal masses receive equal integers, so every original reflection preserves the entire root multiset of every P_j. All its roots thus lie on reflection axes. Theorem 7.6 gives exactly

$$\mathcal{K}_{P_j} = \bigcup_{a \in A} [c_j, a], \qquad c_j = \int z \, \mathrm{d}\eta_j(z).$$

For the segment case this uses $c_j \in \operatorname{conv} A$; for the signed star all nontrivial extreme tips are roots. Now $c_j \to c$, and moving the center changes the Hausdorff distance of these fans by at most $|c_j - c|$, using the same segment parameter. Proposition 9.3 also gives $\mathcal{K}_{P_j} = S_{\eta_j} \to S_\eta$. Uniqueness of the compact Hausdorff limit identifies S_η with $\bigcup_{a \in A}[c, a]$, a finite union of segments of zero area. □

10. Area stability and convergence

The uniform boundary complexity of finite atomic laws controls the area of small boundary neighborhoods. It yields quantitative area estimates when the number of atoms is bounded. For qualitative convergence, a persistence argument for regular interior points also allows the number of input locations to grow.

10.1. **Boundary neighborhoods and area stability.** For a compact set $E \subset \mathbb{C}$ and $\epsilon \geq 0$, define $E^{[\epsilon]} = \{z : \operatorname{dist}(z, E) \leq \epsilon\}$. We use the symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Lemma 10.1 (A uniform boundary-neighborhood bound)**.** *There is a constant $C_s > 0$, depending only on s, such that for any atomic probability measure η with at most s atoms in a disk of radius R, and every $\epsilon > 0$,*

$$\operatorname{Area}\big((\partial S_\eta)^{[\epsilon]}\big) \leq C_s(R\epsilon + \epsilon^2). \tag{10.1}$$

Proof. Use the at most M_s coordinate-monotone arcs and M_s points from Proposition 9.2. On each arc choose coordinate signs so that $u(z) = \pm\Re(z-c) \pm \Im(z-c)$ increases along it, where c is the disk center. Then $|z - w| \leq |u(z) - u(w)|$ for arc points and $-2R \leq u \leq 2R$. The occupied intervals of length ϵ in this scalar range give at most $4R/\epsilon + 1$ disks of radius ϵ covering the arc. Their closed versions also cover its closure, so the closed ϵ-neighborhood is covered by the same disks with radius 2ϵ. Its area is at most $16\pi R\epsilon + 4\pi\epsilon^2$. Each point piece contributes $\pi\epsilon^2$. Summing gives $16\pi M_s R\epsilon + 5\pi M_s \epsilon^2$, which proves the assertion with $C_s = 16\pi M_s$. Point pieces also cover a singleton hull. □

Theorem 10.2 (Quantitative stability in planar area)**.** *Let $\eta, \widetilde{\eta}$ be probability measures with at most s atoms each, all contained in one disk of radius R, and put $\delta = d_{\mathrm{H}}(S_\eta, S_{\widetilde{\eta}})$. For a constant A_s depending only on s,*

$$\operatorname{Area}(S_\eta \triangle S_{\widetilde{\eta}}) \leq A_s(R\delta + \delta^2). \tag{10.2}$$

In particular the area is continuous under weak convergence and Hausdorff convergence of finite atomic supports when the number of atoms is bounded.

For polynomials P, Q with at most s distinct roots each, the same bound holds with $\operatorname{Area}(\mathcal{S}_P \triangle \mathcal{S}_Q)$ *on the left and $\delta = d_{\mathrm{H}}(\mathcal{K}_P, \mathcal{K}_Q)$ on the right. The bound is independent of their degrees and multiplicities.*

Proof. For nonempty compact sets A, B at Hausdorff distance $\delta > 0$,

$$A \setminus B \subseteq (\partial B)^{[\delta]}, \qquad B \setminus A \subseteq (\partial A)^{[\delta]}.$$

Indeed a nearest point of B to a point outside B lies on ∂B. Apply Lemma 10.1 to both shadows and take $A_s = 2C_s$. If $\delta = 0$, the sets coincide and the assertion is immediate. For atomic-data continuity, assume that the supports themselves converge to $\operatorname{supp}\eta$, and apply Proposition 9.3 with $B_j = \operatorname{supp}\eta_j$ and $B = \operatorname{supp}\eta$. There are then no additional locations in the limiting data, so $K_{\eta,B} = S_\eta$. For polynomials, adding or removing the countable sets T_P, T_Q changes no planar area, by Proposition 2.3. This proves the last assertion. □

Corollary 10.3 (Hölder area stability in coefficients)**.** *For each fixed degree d and coefficient bound $B > 0$, there are constants $C_B > 0$ and $0 < \gamma_B \le 1$ such that*

$$\|\mathbf{1}_{\mathcal{S}_P} - \mathbf{1}_{\mathcal{S}_Q}\|_{L^1(\mathbb{C})} \le C_B \|P - Q\|_{\mathrm{coef}}^{\gamma_B}$$

whenever the monic degree-d polynomials P, Q have coefficient norms at most B. The same holds for the limiting cores. In particular, $P \mapsto \operatorname{Area}(\mathcal{S}_P) = \operatorname{Area}(\mathcal{K}_P)$ is locally Hölder continuous, and the positive-area locus is open. On every compact coefficient family containing only positive-area shadows, their areas have a common positive lower bound.

Proof. The Cauchy root bound puts all these roots in the disk of radius $1 + B$. Combine Corollary 5.7 with Theorem 10.2, using $s = d$ and $\delta \le 2(1 + B)$ to absorb the quadratic term into a constant times δ. The L^1 norm of the difference of two indicators is the area of their symmetric difference. The last conclusions follow from continuity and compactness. □

10.2. **Area convergence for varying root data.** The quantitative boundary estimate uses a bound on the number of distinct roots. Qualitative area convergence needs no such bound when the limiting location set is finite. The essential point is local: a regular interior point persists under perturbation of the logarithmic landscape, even when arbitrarily many new saddles appear near the root clusters.

Lemma 10.4 (Persistence of the regular interior)**.** *Let η_j be finitely supported probability measures with $\eta_j \Longrightarrow \eta$ and $\operatorname{supp}\eta_j \to B$ in Hausdorff distance, where B is finite. There is a set $E \subset \mathbb{C}$ of planar area zero such that every $z \in \operatorname{int} S_\eta \setminus E$ has a neighborhood contained in S_{η_j} for all sufficiently large j.*

Proof. Write $A = \operatorname{supp}\eta \subset B$. Use Proposition 9.6, and enlarge its exceptional set by B and the finitely many segments

$$\left\{ b - \frac{t}{C_\eta(b)} : 0 \le t \le 1 \right\}, \qquad b \in B, \quad C_\eta(b) \ne 0.$$

Their union E has area zero. At a remaining interior point z_0 there is a regular switching sheet whose real height difference g satisfies $\nabla_z g \ne 0$ and $\partial_t g \ne 0$ at (z_0, t_0). Its two actual simple saddle branches are distinct. Neither value at the center lies in B: a finite saddle $b \in B$ would give $z_0 = b - t_0/C_\eta(b) \in E$.

Choose two nearby ratios $t_- < t_+$ where opposite branches have strictly larger height. Such ratios exist in every neighborhood of t_0, since the nonzero derivative excludes a local extremum of g. On either strict side the barrier equals that branch's critical height locally. Its spatial differential, compared with (9.7), identifies the actual selected inverse with the winning branch. In particular $z_0 \notin S_{t_\pm,\eta}$. It also lies outside $\mathcal{O}_{t_\pm}(\eta, B)$, since all selected outliers belong to the removed transport segments.

The complete fixed-ratio continuity of [HS2, Corollary 3.5] therefore gives a disk D about z_0 disjoint from both perturbed endpoint supports for all sufficiently large j. We also have

$\mu_{t_\pm,\eta_j} \Longrightarrow \mu_{t_\pm,\eta}$. To justify this additional weak assertion, fix either endpoint and, for each j, choose a positive integer polynomial approximation of η_j with exactly its sites and degree at least j. Choose it so that both its input law and its rounded derivative law are within $1/j$ of η_j and $\mu_{t_\pm,\eta_j}$, respectively, in a metric for weak convergence. The integer approximation limits permit these simultaneous choices. The weak law in Proposition 9.3 identifies the diagonal derivative limit as $\mu_{t_\pm,\eta}$, proving the assertion.

On a smaller closed disk, these weak limits and the common support gaps give uniform convergence of the actual Cauchy transforms. Their limiting transforms do not vanish there. Formula (9.7) consequently gives uniform convergence of the selected endpoint inverses. They remain in fixed, disjoint neighborhoods of the two center saddles, uniformly for $z \in D$.

Choose a small disk about one center saddle whose larger concentric closed disk contains neither another limiting saddle nor a point of B. The intervening closed annulus contains no solution of $(w - z_0)C_\eta(w) = t_0$. Uniform Cauchy convergence on this compact annulus, which avoids B, excludes every solution of

$$(w - z)C_{\eta_j}(w) = t \tag{10.3}$$

there, uniformly for nearby z, t and sufficiently large j. Take $t_\pm$ close enough and shrink D so that this holds throughout $D \times [t_-, t_+]$, while the selected endpoints lie on opposite sides of the annulus. This allows arbitrarily many new saddles near B.

If some $z \in D$ were outside S_{η_j}, it would be outside every S_{t,η_j} on this interval. Its actual selected inverse would then vary continuously with t, by Proposition 9.1, and solve (10.3). A continuous path between the selected endpoints must cross the excluded annulus, by the intermediate value theorem applied to its distance from the center. This contradiction proves $D \subset S_{\eta_j}$ for all sufficiently large j. □

Theorem 10.5 (Qualitative area universality for finite root locations)**.** *Under the assumptions of Lemma 10.4,*

$$\|\mathbf{1}_{S_{\eta_j}} - \mathbf{1}_{S_\eta}\|_{L^1(\mathbb{C})} \longrightarrow 0. \tag{10.4}$$

In particular $\mathrm{Area}(S_{\eta_j}) \to \mathrm{Area}(S_\eta)$*, including when positive input masses tend to zero or the number of distinct input roots tends to infinity.*

For polynomial arrays G_j satisfying (9.11)*,*

$$\|\mathbf{1}_{\mathcal{K}_{G_j}} - \mathbf{1}_{S_\eta}\|_{L^1(\mathbb{C})} = \|\mathbf{1}_{\mathcal{S}_{G_j}} - \mathbf{1}_{S_\eta}\|_{L^1(\mathbb{C})} \longrightarrow 0. \tag{10.5}$$

Their Hausdorff limit can be the larger set $K_{\eta,B}$; its added segments do not change this area limit.

Proof. Since $K_{\eta_j,\operatorname{supp}\eta_j} = S_{\eta_j}$, Proposition 9.3 gives $S_{\eta_j} \to K_{\eta,B}$. Every point outside this last compact set is eventually excluded. The difference $K_{\eta,B} \setminus S_\eta$ has area zero by Proposition 9.3, as does ∂S_η by the weighted definability and finite-length result. Lemma 10.4 gives eventual inclusion at almost every point of the interior. Therefore $\mathbf{1}_{S_{\eta_j}} \to \mathbf{1}_{S_\eta}$ almost everywhere. All these sets lie in a common bounded disk, so dominated convergence proves (10.4). For polynomials, use $\eta_j = N_j^{-1} \operatorname{div}_0 G_j$ and $S_{\eta_j} = \mathcal{K}_{G_j}$. The full and limiting cores differ by at most countably many points, by Proposition 2.3; their indicators agree almost everywhere for each j. This proves the identity and the convergence in (10.5). □

Corollary 10.6 (Area continuity with disappearing atoms)**.** *Let $\eta_j \Longrightarrow \eta$ be probability measures, each supported on at most s points in one fixed compact set. Then $\mathbf{1}_{S_{\eta_j}} \to \mathbf{1}_{S_\eta}$ in $L^1(\mathbb{C})$.*

Proof. Every subsequence of the supports has a further Hausdorff-convergent subsequence with limit B of cardinality at most s. Weak convergence implies $\operatorname{supp}\eta \subset B$. Apply Theorem 10.5 to this further subsequence. Every subsequence thus has a further subsequence with the stated L^1 limit, which proves the assertion for the whole sequence. □

The qualitative conclusion permits unbounded numbers of distinct input roots; the constants in Theorem 10.2 depend on a fixed bound for that number.

Appendix A. Algebraic selection at rational ratios

For rational differentiation ratios, exponentiation makes the separation problem itself semialgebraic. Real quantifier elimination—elimination of quantified real variables from polynomial equalities and inequalities—then gives an exact terminating procedure on real algebraic data [BCR].

Proposition A.1 (Algebraic selection at rational ratios)**.** *Fix a rational ratio $\alpha = p_0/q_0 \in (0, d)$, where p_0, q_0 are positive integers. The support $\mathcal{K}^{\mathrm{fix}}_{\alpha,P}$ is semialgebraic in the observation plane, uniformly in the coefficients of P. For algebraic coefficients, a semialgebraic formula for the entire support is computable, including its singular points and root-atom thresholds. At an algebraic non-root observation point, a terminating exact real-algebraic connectivity procedure also determines the barrier height after exponentiation.*

Proof. Exponentiating the landscape by $2q_0$ gives the semialgebraic superlevel family

$$|P(w)|^{2q_0} \geq \lambda_{\mathrm{lev}}|w - z|^{2p_0}, \qquad \lambda_{\mathrm{lev}} > 0,$$

with the marked ends adjoined. For $z \notin Z(P)$, define

$$\mathfrak{B}_{P,\alpha}(z) := \exp\big(2q_0 \mathfrak{b}_P(z, \alpha)\big).$$

Finite preconnected covers compatible with the parameters, tests for separation by relative closures, and unboundedness tests give a first-order semialgebraic predicate $J(z, \lambda)$ for connectivity of the observation and infinity in this compactified superlevel. For $z \notin Z(P)$, the barrier characterization gives $J(z, \lambda) \iff \lambda \leq \mathfrak{B}_{P,\alpha}(z)$ for every $\lambda > 0$. Thus its graph is defined uniformly in the coefficients by

$$b > 0, \qquad \forall \lambda > 0 \quad \big(J(z, \lambda) \iff \lambda \leq b\big).$$

For algebraic coefficients and observation, quantifier elimination gives a polynomial formula for this graph. Complete real-algebraic sign sampling guarantees an algebraic witness, so exact algebraic search selects its unique positive value b. This is a terminating exact connectivity procedure; coincident critical heights require no separate genericity assumption.

Extend $\mathfrak{B}_{P,\alpha}$ by one at the roots to obtain a total positive semialgebraic graph. This leaves every non-root germ unchanged. On a root-free disk, $\mathfrak{b}_P = (2q_0)^{-1} \log \mathfrak{B}_{P,\alpha}$ is harmonic exactly when the positive function is C^2 there and

$$\mathfrak{B}_{P,\alpha} \Delta \mathfrak{B}_{P,\alpha} - |\nabla_z \mathfrak{B}_{P,\alpha}|^2 = 0$$

throughout that disk. Differentiability, derivatives and existence of such a disk have first-order real-algebraic limit formulas. By (2.5), their complement in the non-root plane is precisely the non-root support, including its singular points. Take its closure and adjoin exactly the roots with positive residual atom mass, namely $\alpha < \mathrm{ord}_a(P)$, by Proposition 2.2. A zero-mass root outside that closure would be an isolated support point of zero mass. These operations are uniformly semialgebraic and effective over the real algebraic numbers. □

Appendix B. Transient removability and capacity

For a probability measure ν carried by a compact set K, put

$$\mathcal{E}_{\log}(\nu) = \iint \log \frac{1}{|z - w|} \, \mathrm{d}\nu(z) \, \mathrm{d}\nu(w), \qquad \mathrm{cap}(K) = \exp\big(-\inf_{\nu} \mathcal{E}_{\log}(\nu)\big).$$

A minimizer of this energy is an *equilibrium measure*. A set is *polar* if it is contained in the set where a nonconstant subharmonic function equals $-\infty$. For these standard potential-theoretic notions, see [Ran, Chapters 3 and 5].

The analytic capacity of a compact set K is

$$\mathrm{cap}_{\mathrm{an}}(K) = \sup\{|f'(\infty)| : f \text{ holomorphic on } \widehat{\mathbb{C}} \setminus K, \quad |f| \le 1,\ f(\infty) = 0\},$$

where $f'(\infty) = \lim_{z\to\infty} zf(z)$. This extremal definition will be used only through bounded removability.

Theorem B.1 (Transient removability and unchanged capacity)**.** *For every polynomial with at least two distinct roots:*

(i) *Every bounded holomorphic function on $\mathbb{C} \setminus \mathcal{S}_P$ extends uniquely to a bounded holomorphic function on $\mathbb{C} \setminus \mathcal{K}_P$, with the same supremum norm. The same assertion holds for bounded harmonic functions.*

(ii) *The transient set is polar, and*

$$\mathrm{cap}(\mathcal{S}_P) = \mathrm{cap}(\mathcal{K}_P) > 0.$$

The two compact sets have the same equilibrium measure. They also have the same analytic capacity:

$$\mathrm{cap}_{\mathrm{an}}(\mathcal{S}_P) = \mathrm{cap}_{\mathrm{an}}(\mathcal{K}_P).$$

Proof. By Proposition 2.3, T_P is locally finite in $\mathbb{C} \setminus \mathcal{K}_P$. Each transient is an isolated puncture, across which a bounded holomorphic or bounded harmonic function extends with the same bound. Local finiteness makes the extensions compatible, and continuity gives uniqueness. This proves (i).

If T_P is nonempty, enumerate its points as t_j and choose $b_j > 0$ with $\sum_j b_j = 1$. Since T_P is bounded, the potential $\sum_j b_j \log|z - t_j|$ is subharmonic, finite outside a sufficiently large disk, and $-\infty$ at every t_j. Hence T_P is polar. A finite-energy probability measure on a compact set has no atoms: the logarithmic kernel is bounded below, while a positive atom gives infinite energy on the diagonal. It therefore assigns zero mass to the countable set T_P. Thus the finite-energy probability measures carried by $\mathcal{S}_P$ and by $\mathcal{K}_P$ coincide, as do their energy infima and minimizers. The core is a nondegenerate continuum and consequently has positive logarithmic capacity [Ran, Chapter 5]. Finally, the extensions in (i) identify the admissible functions in the two analytic-capacity problems, preserving their bounds and coefficients at infinity. Their analytic capacities are therefore equal. □

The equilibrium measure is different from ρ_P: the root atoms of ρ_P, of masses p_j^2, give it infinite logarithmic energy, whereas the equilibrium measure has finite energy. Thus transients affect neither energy minimization nor bounded analytic continuation.

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Department of Mathematics, Stockholm University, SE–106 91 Stockholm, Sweden
Email address: hagg@math.su.se

Department of Mathematics, Stockholm University, SE–106 91 Stockholm, Sweden
Email address: shapiro@math.su.se