On Involutive Nonassociative Lambek Calculus

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NL, Lambek (1961), On the calculus of syntactic types

connectives: $\otimes, \backslash, /$ (product, right and left implication)

Bunches (trees of formulas): elements of the free groupoid generated by the set of formulas

Atomic bunches: formulas; compound bunches: (Γ, Δ)

Sequents: $\Gamma \Rightarrow A$, where Γ is a bunch, A is a formula

$$\text{(NL-id)}\ A \Rightarrow A \quad \text{(NL-cut)}\ \frac{\Gamma[A] \Rightarrow B \quad \Delta \Rightarrow A}{\Gamma[\Delta] \Rightarrow B}$$

$$(\otimes \Rightarrow)\ \frac{\Gamma[(A, B)] \Rightarrow C}{\Gamma[A \otimes B] \Rightarrow C} \quad (\Rightarrow \otimes)\ \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{(\Gamma, \Delta) \Rightarrow A \otimes B}$$

$$(\backslash \Rightarrow)\ \frac{\Gamma[B] \Rightarrow C \quad \Delta \Rightarrow A}{\Gamma[(\Delta, A \backslash B)] \Rightarrow C} \quad (\Rightarrow \backslash)\ \frac{(A, \Gamma) \Rightarrow B}{\Gamma \Rightarrow A \backslash B}$$

$$(/ \Rightarrow)\ \frac{\Gamma[A] \Rightarrow C \quad \Delta \Rightarrow B}{\Gamma[(A/B, \Delta)] \Rightarrow C} \quad (\Rightarrow /)\ \frac{(\Gamma, B) \Rightarrow A}{\Gamma \Rightarrow A/B}$$

The algebraic models for NL are *residuated groupoids*, i.e. ordered algebras $(M, \otimes, \backslash, /, \leq)$ such that $(M, \leq)$ is a poset and $\otimes, \backslash, /$ are binary operations on M, satisfying the residuation laws:

$$a \otimes b \leq c \text{ iff } b \leq a \backslash c \text{ iff } a \leq c/b, \text{ for all } a, b, c \in M.$$

Language models are algebras of phrase structure languages.

Let Σ be an alphabet. By $\Sigma^{(+)}$ we denote the smallest set such that:

(i) $\Sigma \subseteq \Sigma^{(+)}$, (ii) if $X, Y \in \Sigma^{(+)}$ then $(XY) \in \Sigma^{(+)}$.

For $L_1, L_2 \subseteq \Sigma^{(+)}$, one defines:

$L_1 \otimes L_2 = \{(XY) : X \in L_1, Y \in L_2\}$

$L_1 \backslash L_2 = \{Z \in \Sigma^{(+)} : L_1 \otimes \{Z\} \subseteq L_2\}$, $L_1 / L_2 = \{Z \in \Sigma^{(+)} : \{Z\} \otimes L_2 \subseteq L_1\}$

$(\mathcal{P}(\Sigma^{(+)}), \otimes, \backslash, /, \subseteq)$ is a residuated groupoid.

One interprets each comma as $\otimes$ and $\Rightarrow$ as $\leq$.

$A, A\backslash B \Rightarrow B$; $A/B, B \Rightarrow A$ (application laws)

$A \Rightarrow (B/A)\backslash B$; $A \Rightarrow B/(A\backslash B)$ (type raising laws)

$A \Rightarrow B\backslash(B \otimes A)$; $A \Rightarrow (A \otimes B)/B$ (co-application laws)

Mary: pn, John: pn (proper noun)

knows: $(pn\backslash s)/pn$ (transitive verb)

(Mary (knows John)): s (sentence),

since $(pn, ((pn\backslash s)/pn, pn)) \Rightarrow s$ is provable in **NL**

$np = s/(pn\backslash s)$ noun phrase as subject, e.g. every student, she, he

$pn \Rightarrow np$ by type raising

$np' = (s/pn)\backslash s$ noun phrase as object, e.g. every student, her, him

$pn \Rightarrow np'$ as above

L, Lambek (1958), The mathematics of sentence structure

L adds to **NL**: $(A \otimes B) \otimes C \Leftrightarrow A \otimes (B \otimes C)$ (the associative law)

$\Leftrightarrow$ means: $\Rightarrow$ and $\Leftarrow$

In the sequent system one usually employs finite sequences of formulas instead of bunches.

$A \backslash B, B \backslash C \Rightarrow A \backslash C$; $A/B, B/C \Rightarrow A/C$ (composition laws)

$A \backslash B \Rightarrow (C \backslash A) \backslash (C \backslash B)$; $A/B \Rightarrow (A/C)/(B/C)$ (Geach laws)

$A \backslash (B/C) \Leftrightarrow (A \backslash B)/C$ (the associative law for \, /)

she knows him: s,

since $s/(pn \backslash s), (pn \backslash s)/pn, (s/pn) \backslash s \Rightarrow s$ is provable in **L**.

With **NL** one needs a new type, say, she: $(s/pn)/((pn \backslash s)/pn)$. In **L** it is derivable from $s/(pn \backslash s)$ by the second Geach law.

There were considered many extensions of **NL** and **L**.

NL1, **L1** admit constant 1 and axioms $1 \otimes A \Leftrightarrow A$, $A \otimes 1 \Leftrightarrow A$.

In sequent systems one employs the empty bunch ϵ, satisfying:

$(\epsilon, \Gamma) = (\Gamma, \epsilon) = \Gamma$.

One writes $\Rightarrow A$ for $\epsilon \Rightarrow A$. If this is provable, then A is *a theorem*. In **NL**, **L** no formula is a theorem.

Lattice connectives $\wedge, \vee$ and constants $\bot, \top$ (additives). This leads to Full **NL** (**FNL**), **FL**, etc.

Unary modalities $\Diamond, \Box^{\downarrow}$, connected by the residuation rule:

$$\frac{\Diamond A \Rightarrow B}{\overline{A \Rightarrow \Box^{\downarrow} B}}$$

The multi-modal framework, developed by Moortgat (1996), Morrill (1994) and others, employs many different modalities in order to make language processing more flexible.

Let $\mathcal{L}$ be a type logic, which yields sequents $\Gamma \Rightarrow A$.

A type grammar based on $\mathcal{L}$ can be defined as a triple $G = (\Sigma, I, A_0)$ such that:

- Σ is a finite lexicon (alphabet),

- I is a map which assigns finite sets of types (formulas of $\mathcal{L}$) to elements of Σ (*the type lexicon*),

- A_0 is a designated type.

One says that G *assigns* type B to a string $v_1 \ldots v_n$, where $v_i \in \Sigma$, if there exist types $A_i \in I(v_i)$, $i = 1, \ldots, n$, such that $A_1, \ldots, A_n \Rightarrow B$ is provable in $\mathcal{L}$. For nonassociative logics, one adds: under some bracketing of the sequence $A_1, \ldots, A_n$.

This bracketing induces a phrase structure on $v_1 \ldots v_n$.

The language of G, denoted by $L(G)$, consists of all $u \in \Sigma^+$ which are assigned A_0 by G.

CNL, de Groote and Lamarche (2002), Classical Non-Associative Lambek Calculus

It can be presented as the extension of **NL** by negation $^{\sim}$ with:

$A^{\sim\sim} \Leftrightarrow A, \quad A^{\sim}/B \Leftrightarrow A\backslash B^{\sim}$

$$(\mathrm{CON}^{\sim})\ \frac{A \Rightarrow B}{B^{\sim} \Rightarrow A^{\sim}}$$

de Groote and Lamarche (2002) defined and studied proof nets for **CNL**, presented as a one-sided sequent system.

A weaker system **InNL** can be presented as the extension of **NL** by two negations $^{\sim},^{-}$ with:

$A^{\sim-} \Leftrightarrow A, \quad A^{-\sim} \Leftrightarrow A, \quad A^{\sim}/B \Leftrightarrow A\backslash B^{-}$

and ($\mathrm{CON}^{\sim}$), (CON^{-}).

In a similar way one can extend **NL1**, **L**, **FL1** etc.

Involutive **FL1** is Noncommutative **MALL** of Abrusci (1991).

MALL: Multiplicative-Additive Linear Logic (Girard 1987). It assumes $A \otimes B \Leftrightarrow B \otimes A$.

Classical **FL1** is Cyclic Noncommutative **MALL** of Yetter (1990).

Involutive **L1** is Noncommutative **MLL**.

Classical **L1** is Cyclic Noncommutative **MLL**.

In these logics (also nonassociative): $1^{\sim} \Leftrightarrow 1^{-}$. One defines $0 = 1^{\sim}$.

Also: $(A^{\sim} \otimes B^{\sim})^{-} \Leftrightarrow (A^{-} \otimes B^{-})^{\sim}$.

One defines $A \oplus B = (B^{\sim} \otimes A^{\sim})^{-}$ (dual product, par).

$0 \oplus A \Leftrightarrow A$, $A \oplus 0 \Leftrightarrow A$, $A \backslash B \Leftrightarrow A^{\sim} \oplus B$, $A/B \Leftrightarrow A \oplus B^{-}$

$A^{\sim} \Leftrightarrow A \backslash 0$, $A^{-} \Leftrightarrow 0/A$, $(0/A)\backslash 0 \Leftrightarrow A \Leftrightarrow 0/(A\backslash 0)$ double negation

$(A\backslash 0)/B \Leftrightarrow A\backslash(0/B)$ contraposition

Some authors directly apply noncommutative linear logics in type grammars.

C. Casadio (2001), Non-Commutative Linear Logic in Linguistics

G. Morrill (1995), Higher-order linear logic programming of categorial deduction

J. Lambek (1999) employs a stronger logic, called by him Compact Bilinear Logic (**CBL**), which amounts to Noncommutative **MLL** with $\otimes = \oplus$ and $1 = 0$.

The algebraic models of **CBL** are called *pregroups*, hence the resulting grammars are called *pregroup grammars*.

Lambek (2008), From Word to Sentence, elaborated a detailed type lexicon for a fragment of English, using pregroup types. Several authors proposed partial type lexicons for other languages, e.g. French, German, Italian, Polish, Turkish, Chinese.

word	**NL**-type	**InNL**-type	pregroup type
works	$pn\backslash s$	$pn^{\sim} \oplus s$	$[pn]^r s$
likes	$(pn\backslash s)/pn$	$(pn^{\sim} \oplus s) \oplus pn^{-}$	$[pn]^r s[pn]^l$
whom	$(n\backslash n)/(s/pn)$	$(n^{\sim} \oplus n) \oplus (pn^{--} \otimes s^{-})$	$n^r n[pn]^{ll} s^l$

‘whom’ in contexts like ‘girl whom John likes’

n common noun

Lambek writes A^r for $A^{\sim}$ and A^l for A^{-}.

In **InNL** one sees more symmetries (dualities).

$A \otimes (A\backslash B) \Rightarrow B$ is equivalent to $B^{-} \Rightarrow (A \otimes (A\backslash B))^{-}$, which is translated into $B^{-} \Rightarrow (B^{-} \otimes A)/A$.

So the first application law is dual to the second co-application law, by contraposition.

In **InNL** one defines dual implications:

$A \searrow B = A^{-} \otimes B, \;\; A \swarrow B = A \otimes B^{\sim}$

We obtain dual residuation laws (here derivable rules)

$C \Rightarrow A \oplus B$ iff $A \searrow C \Rightarrow B$ iff $C \swarrow B \Rightarrow A$

Moortgat (2009), Symmetric Categorial Grammar, uses a logic which extends **NL** by $\oplus$, $\searrow$, $\swarrow$, satisfying the above and some additional axioms, going back to Grishin (1983).

Without Grishin's axioms this logic is a subsystem of **InNL** and with these axioms of (commutative) **MLL**. M. Moortgat shows several interesting linguistic interpretations of dual Lambek connectives.

A. Bastenhof (2013), Categorial Symmetry, PhD Thesis, Utrecht

Proof nets are interpreted as logical forms of expressions. R. Moot and C. Retoré (2012), The Logic of Categorial Grammars

The algebraic models of **InNL** are *involutive residuated groupoids*.

$(M, \otimes, \backslash, /, {}^{\sim}, {}^{-}, \leq)$ such that $(M, \otimes, \backslash, /, \leq)$ is a residuated groupoid and the following hold:

$a^{\sim -} = a = a^{-\sim}, \quad a^{\sim}/b = a \backslash b^{-}$

if $a \leq b$ then $b^{\sim} \leq a^{\sim}$ and $b^{-} \leq a^{-}$

The algebraic models of **CNL** are *cyclic* involutive residuated groupoids:

$a^{\sim} = a^{-}$ for any element a.

An algebra of this kind is said to be *unital*, if it contains an element 1, satisfying: $1 \otimes a = a = a \otimes 1$.

InNL1 - unital i.r.g.s, **CNL1** - unital cyclic i.r.g.s

InL - involutive residuated semigroups, **InL1** - involutive residuated monoids

Girard (1987), Yetter (1990), Abrusci (1991) for linear logics

$(M, \cdot, 1, O)$, where $(M, \cdot, 1)$ is a (commutative) monoid, $O \subseteq M$. For noncommutative logics, O must satisfy some conditions.

A phase space: $(M, \cdot, R)$ such that $(M, \cdot)$ is a groupoid and $R \subseteq M^2$

For $X \subseteq M$ define:

$X^{\sim} = \{b \in M : \forall_{a \in X} R(a, b)\}$, $X^{-} = \{a \in M : \forall_{b \in X} R(a, b)\}$.

Analogous to *polarities* $X^{\triangleright}, Y^{\triangleleft}$ in concept lattices.

The pair ${}^{\sim}, {}^{-}$ is a Galois connection: $X \subseteq Y^{\sim}$ iff $Y \subseteq X^{-}$.

Then: $X \subseteq Y$ implies $Y^{\sim} \subseteq X^{\sim}$ and $Y^{-} \subseteq X^{-}$,

$X^{\sim-\sim} = X^{\sim}$, $X^{-\sim-} = X^{-}$.

The operations $\phi_R(X) = X^{-\sim}$, $\psi_R(X) = X^{\sim-}$ are closure operations on $\mathcal{P}(M)$.

(C1) $X \subseteq C(X)$, (C2) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$,

(C3) $C(C(X)) = C(X)$

A set X is $C-closed$, if $C(X) = X$.

P1. The ϕ_R-closed sets are of the form $Y^{\sim}$. The ψ_R-closed sets are of the form Y^{-}.

Define M_R as the family of ϕ_R-closed subsets of M.

P2. If $\phi_R = \psi_R$, then M_R is closed under $^{\sim}$ and $^{-}$.

P3. R is symmetric iff for any $X \subseteq M$, $X^{\sim} = X^{-}$.

P4. If R is symmetric, then $\phi_R = \psi_R$.

One defines: $X \otimes Y = \{a \cdot b : a \in X, b \in Y\}$

$X \backslash Y = \{a \in M : X \otimes \{a\} \subseteq Y\}$, $\quad X/Y = \{a \in M : \{a\} \otimes Y \subseteq X\}$

Then $(\mathcal{P}(M), \otimes, \backslash, /, \subseteq)$ is a residuated groupoid.

(C4) $C(X) \otimes C(Y) \subseteq C(X \otimes Y)$

A closure operation C is *a nucleus*, if it satisfies (C4).

If C is a nucleus, then the family of $C-$closed sets is closed under $\backslash, /$.

One defines $X \otimes_C Y = C(X \otimes Y)$.

(Shift) for all $a, b, c \in M$, $R(a \cdot b, c)$ iff $R(a, b \cdot c)$.

P5. (Shift) is equivalent to: for all $X, Y \subseteq M$, $X^{\sim}/Y = X\backslash Y^{-}$.

We call $(M, \cdot, R)$ *a phase space for* **InNL** (resp. **CNL**), if it satifies (Shift) and $\phi_R = \psi_R$ (resp. R is symmetric).

Theorem. Let $(M, \cdot, R)$ be a phase space for **InNL** (resp. **CNL**). Then, ϕ_R is a nucleus, and $(M_R, \otimes_{\phi_R}, \backslash, /, ^{\sim}, ^{-}, \subseteq)$ is a (resp. cyclic) involutive residuated groupoid.

We refer to this algebra as *the complex algebra* of $(M, \cdot, R)$.

For $O \subseteq M$, one defines $R_O \subseteq M^2$:

$R_O(a, b)$ iff $a \cdot b \in O$.

If $(M, \cdot)$ is unital, then every $R \subseteq M^2$, satisfying (Shift), equals R_O for $O = \{a \in M : R(a, 1)\}$. We have: $X^{\sim} = X \backslash O$, $X^{-} = O/X$.

(Shift) is equivalent to: $(a \cdot b) \cdot c \in O$ iff $a \cdot (b \cdot c) \in O$.

If $(M, \cdot)$ is a free groupoid, then every $R \subseteq M^2$ equals R_O for $O = \{a \cdot b : R(a, b)\}$.

There exist non-unital phase spaces $(M, \cdot, R)$ such that $R \neq R_O$, for any $O \subseteq M$.

We have: $X^{\sim} = \{b \in M : \forall_{x \in M}(\neg R(x, b) \rightarrow x \notin X)\}$,
$X^{-} = \{a \in M : \forall_{x \in M}(\neg R(a, x) \rightarrow x \notin X)\}$.

So $X^{-} = \Box X^{c}$, $X^{\sim} = \Box^{\downarrow} X^{c}$, where $\Box$ corresponds to R^{c} and $\Box^{\downarrow}$ to the converse of R^{c}. $A^{-} \Leftrightarrow \Box \neg A$, $A^{\sim} \Leftrightarrow \Box^{\downarrow} \neg A$.

variables: $p, q, r, \ldots$, negated variables: $p^{\sim}, q^{\sim}, r^{\sim}, \ldots$

connectives: $\otimes, \oplus$

sequents: all bunches containing at least two formulas (we omit outer parentheses)

$$\text{(id) } p, p^{\sim} \quad \text{(cut) } \frac{A, \Gamma \quad A^{\sim}, \Delta}{\Delta, \Gamma}$$

$$\text{(r-}\otimes\text{) } \frac{(A, B), \Gamma}{A \otimes B, \Gamma} \quad \text{(r-}\oplus\text{) } \frac{A, \Gamma \quad B, \Delta}{A \oplus B, (\Delta, \Gamma)}$$

$$\text{(r-sym) } \frac{\Gamma, \Delta}{\Delta, \Gamma} \quad \text{(r-shift) } \frac{(\Gamma, \Delta), \Theta}{\Gamma, (\Delta, \Theta)}$$

This system is dual to that of de Groote and Lamarche (2002).

Metalanguage negation $A^{\sim}$

$(p^{\sim})^{\sim} = p, \ (A \otimes B)^{\sim} = B^{\sim} \oplus A^{\sim}, \ (A \oplus B)^{\sim} = B^{\sim} \otimes A^{\sim}$

A model: $(\mathbf{M}, \mu)$, $\mathbf{M}$ is a cyclic i.r.g., μ is a valuation of formulas.

$\mu((\Gamma, \Delta)) = \mu(\Gamma) \otimes \mu(\Delta)$

A sequent (Γ, Δ) is *true* in the model, if $\mu(\Gamma) \leq \mu(\Delta)^{\sim}$ (equivalently $\mu(\Delta) \leq \mu(\Gamma)^{\sim}$).

(de Groote and Lamarche 2002) is a purely proof-theoretic paper.

(1) a theory of proof nets,

(2) the cut-elimination theorem,

(3) **CNL** is a conservative extension of **NL**,

(4) the polynomial time complexity of **CNL**.

The paper, cited above, focuses on proof nets for **CNL**. (2), (3) are proved, using proof nets.

W. B., On Classical Nonassociative Lambek Calculus, *Logical Aspects of Computational Linguistics LACL 2016*, LNCS 10054.

(1) The weak completeness of the cut-free system (this implies cut elimination).

(2) **CNL** is a strongly conservative extension of **NL**.

$\Gamma \Rightarrow A$ an NL-sequent, Φ a set of NL-sequents.

$\Phi \vdash_{CNL} \Gamma \Rightarrow A$ iff $\Phi \vdash_{NL} \Gamma \Rightarrow A$

(3) The strong finite model property for **CNL**.

$\Phi \vdash_{CNL} \Gamma$ (Φ finite) iff Φ entails Γ in finite models.

(4) The polynomial time complexity of the finitary consequence relation.

(5) The context-freeness of the generated languages (by the type grammars based on **CNL**, possibly with nonlogical axioms).

We define: $\Gamma \sim \Delta$ iff Δ can be derived from Γ by (r-sym), (r-shift).

Extraction lemma. For any sequent Γ', containing one marked formula $\underline{A}$, there exists a unique bunch Δ' such that $\Gamma' \sim (\underline{A}, \Delta')$.

Every finite set of formulas can be extended to a finite set, closed under subformulas and $^{\sim}$.

By a $T-$sequent we mean a sequent whose formulas belong to T.

Every sequent Γ, Δ is deductively equivalent to a sequent of the form A, B.

We define $\Gamma \Rightarrow A$ as $\Gamma, A^{\sim}$ (equivalently: $A^{\sim}, \Gamma$).

Interpolation lemma. Let T be closed under subformulas and $^{\sim}$. Let Φ be a finite set of $T-$sequents of the form A, B. Let $\Phi \vdash_{CNL} \Gamma[\Delta]$, where $\Gamma[\Delta] \neq \Delta$ is a $T-$sequent. Then, there exists $D \in T$ such that $\Phi \vdash_{CNL} \Gamma[D]$ and $\Phi \vdash_{CNL} \Delta \Rightarrow D$.

We refer to D as *an interpolant* of Δ in $\Gamma[\Delta]$.

We define a phase space $(M, \cdot, R)$ such that

- $(M, \cdot)$ is the free groupoid of bunches,

- $R(\Gamma, \Delta)$ iff $\vdash \Gamma, \Delta$

Let $\vdash$ mean the provability in the cut-free **CNL**. R is symmetric and satisfies (Shift) due to (r-sym), (r-shift), hence this is a frame for **CNL**.

We define: $[A] = \{\Gamma \in M : \vdash \Gamma \Rightarrow A\}$

We have: $[A] = \{A^{\sim}\}^{\sim}$. So $[A]$ is ϕ_R−closed.

We define: $\mu(p) = [p]$, $\mu(p^{\sim}) = [p]^{\sim}$.

We prove: $A \in \mu(A) \subseteq [A]$, for any formula A.

Consequently $\Gamma \in \mu(\Gamma)$, for any bunch Γ.

If $\nvdash \Gamma, \Delta$, then $\neg R(\Gamma, \Delta)$. So $\Gamma \in \mu(\Gamma)$ but $\Gamma \notin \mu(\Delta)^{\sim}$. Consequently Γ, Δ is not true for μ in the complex algebra of the frame.

This proof is an adaptation of similar proofs for linear logics and other substructural logics (Lafont 1997, Galatos et al. 2007).

CNL with (cut) is *strongly complete* with respect to cyclic i.r.g.s: $\Phi \vdash_{CNL} \Gamma$ iff Γ is true in every cyclic i.r.g. and every valuation μ that satisfies all sequents from Φ.

The proof is similar. Due to (cut), we obtain: $\mu(A) = [A]$. This is needed to show that all sequents from Φ are true for μ.

In a similar way we prove the extended subformula property for the provability from assumptions in **CNL** and the strong finite model property for **CNL**.

Now $(M, \cdot)$ is the free groupoid generated by T (a finite set, closed under subformulas and $^{\sim}$).

Although the resulting frame is infinite, its complex algebra is finite. This follows from the interpolation lemma.

The strong conservativity of **CNL** over **NL** is obtained by a model-theoretic proof.

We consider a phase space $(M, \cdot, R)$ such that M is the free groupoid generated by all NL-formulas and their formal negations $A^{\sim}$, i.e. A with superscript $^{\sim}$.

R is the smallest relation containing all pairs $\langle A^{\sim}, \Gamma\rangle$ such that $\Phi \vdash_{NL} \Gamma \Rightarrow A$ and being closed under (r-sym), (r-shift). So $(M, \cdot, R)$ is a phase space for **CNL**.

Let A be an NL-formula. We define $[A] = \{\Gamma : \ \Phi \vdash_{NL} \Gamma \Rightarrow A\}$.

Let μ be defined as above. We prove $\mu(A) = [A]$, for any NL-formula A.

All sequents from Φ are true for μ. If $\Phi \not\vdash_{NL} \Gamma \Rightarrow A$, then $\Gamma \Rightarrow A$ is not true for μ, hence $\Phi \not\vdash_{CNL} \Gamma \Rightarrow A$.

This proof can easily be adapted for stronger logics (associative, commutative, with multiplicative constants, lattice connectives etc.).

From the interpolation lemma it follows that every type grammar $G = (\Sigma, I, A_0)$ based on **CNL** with Φ (finite) is equivalent to a context-free grammar such that:

- Σ is the terminal alphabet,

- the nonterminal alphabet is the closure of all types involved in G and Φ under subformulas and $^{\sim}$,

- A_0 is the start symbol,

- the lexical rules are $A \mapsto v$, whenever $A \in I(v)$,

- the non-lexical rules are $A \mapsto B$, whenever $\Phi \vdash_{CNL} B \Rightarrow A$, and $A \mapsto B, C$, whenever $\Phi \vdash_{NL} B, C \Rightarrow A$, where A, B, C are nonterminal symbols.

Theorem. For any finite set of sequents Φ, the type grammars based on **CNL** with Φ generate precisely the $\epsilon-$free context-free languages.

The sequents containing at most three formulas are said to be *restricted*.

Let T be a finite set of formulas, closed under subformulas and $\tilde{}$.

We prove that a restricted $T-$sequent is provable in **CNL** from Φ if and only if it is provable in this system limited to restricted $T-$sequents.

Accordingly the relation $\Phi \vdash_{CNL} A, B$, where Φ is finite, is P-TIME.

$O(n^7)$, where n is the size of T.

All results remain true for **CNL1**.

The finitary consequence relation for **CNL** with $\wedge, \vee$ (not satisfying the distributive laws) is undecidable.

This relation is undecidable for **FNL** (Chvalovsky 2015), and Full **CNL** is a strongly conservative extension of **FNL**.

Observe that **CNL** can be presented **InNL** with $A^{\sim} \Leftrightarrow A^{-}$.
Accordingly, **InNL** is intermediate between **NL** and **CNL**.

As a consequence, **InNL** is a strongly conservative extension of **NL**.

InNL does not possess the strong finite model property.

In finite i.r.g.s, $a^{\sim} \leq a^{-}$ entails $a^{\sim} = a^{-}$, since $a^{\sim} < a^{-}$ yields $a < a^{\sim\sim}$, which generates the infinite chain:

$a < a^{\sim\sim} < a^{\sim\sim\sim\sim} < \cdots$.

There exists an infinite i.r.g. with an element a such that $a^{\sim} < a^{-}$.

Not all tools employed for **CNL** can be adapted for **InNL**. In particular, a finite set T cannot be extended to a finite set closed under $^{\sim}$ and $^{-}$.

The complexity of the consequence relation for **InNL** remains an open problem.

atoms: $p^{(n)}$, where p is a variable, $n \in \mathbb{Z}$

sense: $p^{(0)}$ is p, $p^{(3)}$ is $p^{\sim\sim\sim}$, $p^{(-3)}$ is p^{---}

$$\text{(id) } p^{(n)}, p^{(n+1)}$$

$$\text{(r-}\otimes\text{) } \frac{\Gamma[(A, B)]}{\Gamma[A \otimes B]}$$

$$\text{(r-}\oplus\text{1) } \frac{\Gamma[B] \quad \Delta, A}{\Gamma[(\Delta, A \oplus B)]} \qquad \text{(r-}\oplus\text{2) } \frac{\Gamma[A] \quad B, \Delta}{\Gamma[(A \oplus B, \Delta)]}$$

$$\text{(r-shift) } \frac{(\Gamma, \Delta), \Theta}{\Gamma, (\Delta, \Theta)}$$

$$\text{(cut}^{-}\text{) } \frac{\Gamma[A] \quad A^{-}, \Delta}{\Gamma[\Delta]} \qquad \text{(cut}^{\sim}\text{) } \frac{\Gamma[A] \quad \Delta, A^{\sim}}{\Gamma[\Delta]}$$

$A^{\sim}$ and A^{-} are defined in metalanguage:

$(p^{(n)})^{\sim} = p^{(n+1)}, \quad (p^{(n)})^{-} = p^{(n-1)}$

$(A \otimes B)^{\sim} = B^{\sim} \oplus A^{\sim}$, $(A \oplus B)^{\sim} = B^{\sim} \otimes A^{\sim}$, and similarly for $^{-}$.

The following rules are admissible in the cut-free system.

$$(\text{r-}^{\sim\sim})\ \frac{A, \Gamma}{\Gamma, A^{\sim\sim}} \qquad (\text{r-}^{--})\ \frac{\Gamma, A}{A^{--}, \Gamma}$$

With the cut rules, they are derivable.

So A^{-}, Γ is equivalent to $\Gamma, A^{\sim}$. Both represent $\Gamma \Rightarrow A$.

A model: $(\mathbf{M}, \mu)$, $\mathbf{M}$ is an i.r.g., μ is a valuation, satisfying: $\mu(p^{(n+1)}) = \mu(p^{(n)})^{\sim}$. A sequent (Γ, Δ) is *true* in the model, if $\mu(\Gamma) \leq \mu(\Delta)^{-}$ (equivalently $\mu(\Delta) \leq \mu(\Gamma)^{\sim}$).

W. B., Involutive Nonassociative Lambek Calculus: Sequent Systems and Complexity. To appear in *Bulletin of The Section of Logic* (2017?).

This paper introduces the sequent system, presented above.

We prove its strong completeness with respect to i.r.g.s, using Lindenbaum-Tarski algebras. The cut elimination theorem is proved in a proof-theoretic way.

We also prove that the pure **InNL** is P-TIME.

W. B., Phase spaces for Involutive Nonassociative Lambek Calculus.

(1) phase spaces for **InNL**. (2) a model-theoretic proof of cut elimination,

(3) the P-TIME complexity of **InNL** by reduction to an auxiliary system **InNL**(k), which is P-TIME, (4) the equivalence of type grammars based on **InNL** and context-free grammars.

Let $k > 0$ be an even integer.

InNL(k) arises from **InNL** as follows.

One admits atoms $p^{(n)}$ for $0 \leq n < k$ only.

The axioms (id) are $p^{(n)}, p^{(n+1)}$, for $0 \leq n < k$, where $n + 1$ is computed modulo k.

CNL amounts to **InNL**(2).

Using phase spaces, one proves that **InNL**(k) is strongly complete with respect to i.r.g.s, satisfying $a^{(k)} = a$, for all a, i.e. k−cyclic i.r.g.s.

Our results for **CNL** can be extended for systems **InNL**(k) with quite similar proofs, e.g. interpolation, P-TIME complexity, context-freeness, strong finite model property.

One proves that a sequent Γ is provable in **InNL** if and only if it is provable in **InNL**(k), where k is computed from Γ in polynomial time. First, one must modify Γ to eliminate $p^{(n)}$ with $n < 0$.

Thank you!