Implementing Logical Grammar: CatLog3

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Abstract

CatLog3 is a Prolog parser/theorem-prover for (type) logical (categorial) grammar. In such logical grammar, grammar is *reduced* to logic: a string of words is grammatical if and only if an associated logical statement is a theorem. CalLog3 implements a logic extending displacement calculus, a sublinear fragment including as primitive connectives the continuous (Lambek) and discontinuous wrapping connectives of the displacement calculus, additives, 1st order quantifiers, normal modalities, bracket modalities and subexponentials. In this paper we survey how CatLog3 is implemented on the principles of Andreoli's focusing and a generalisation of van Benthem's count-invariance.

A view of the field

- Logical grammar dates back at least to Bar-Hillel (1953) and Ajdukiewicz (1935).
- It aspires to practice grammar as mathematical logic.
- For example, the seminal paper Lambek (1958) defines a (sublinear) syntactic calculus and proves Cut-elimination for it.
- Chomsky (1957) introduced transformational grammar as a formal system also, but such generative linguistics has relaxed the reigns of formalisation.
- Discrete formal grammar also flourished in computational linguistics in the 1980s: LFG, GPSG, HPSG, TAG, ... but such computational linguistics has given way to statistical NLP.
- Logical grammar remains as one of the oldest traditions of grammar.

Continuity and discontinuity

The Lambek calculus is a calculus of concatentation which is free of structural rules. The displacement calculus of Morrill et al. (2011) generalises Lambek calculus with intercalation, containing both continuous and discontinuous connective families, while remaining free of structural rules, and preserving Cut-elimination and its good corollaries: the subformula property, decidability, the finite reading property, and the focusing property.

CatLog

The CatLog program series comprises implementations in Prolog of type logical parser/theorem-provers starting from the basis of logic programming of displacement calculus theorem-proving (Morrill 2011):

- CatLog1 (Morrill 2012) was based on uniform proof (Miller et al. 1991), and count-invariance for multiplicatives (van Benthem 1991).
- CatLog2 was based on Andreoli's focusing (Andreoli 1992), and count-invariance for multiplicatives, additives and bracket modalities (Valentín et al. 2013).
- CatLog3 is based on focalisation and count-invariance for multiplicatives, additives, bracket modalities and (sub)exponentials (Kuznetov et al. 2017).

Outline

In this paper we survey the methods on which the implementation of CatLog3 is based.

- We describe the primitive connectives of the logical fragment for which parsing/theorem-proving is implemented.
- We discuss focusing.
- We discuss count-invariance.
- We illustrate in relation to the *Montague Test* (Morrill and Valentín 2016): the task of providing a computational grammar of Montague's (1973) fragment.

Displacement logic

The formalism used comprises the following connectives.

	cont. mult.	disc. mult.	add.	qu.	norm. mod.	brack. mod.	exp.	lim. contr. & weak.
primary	$/$ $\backslash$ $\bullet$ I	$\uparrow$ $\downarrow$ $\odot$ J	$\&$ $\oplus$	$\bigwedge$ $\bigvee$	$\Box$ $\Diamond$	$[\,]^{-1}$ $\langle\rangle$	$!$ $?$	$\mid$ W
sem. inactive variants	$\bullet\!\!-\ -\!\!\circ$ $\circ\!\!-\ -\!\!\bullet$ ◐ ◑	◓ ◒	$\sqcap$ $\sqcup$	$\forall$ $\exists$	■ ⧫			
det. synth.	$\triangleleft^{-1}$ $\triangleright^{-1}$ $\triangleleft$ $\triangleright$	ˇ ˆ						diff.
nondet. synth.	$\div$ $\times$	$\Uparrow$ $\Downarrow$ $\circledcirc$						$-$

- The heart of the logic is the displacement calculus of Morrill and Valentín (2010) and Morrill, Valentín and Fadda (2011) made up of twin continuous and discontinuous residuated families of connectives having a pure Gentzen sequent calculus —without labels and free of structural rules— and enjoying Cut-elimination (Valentín 2012).
- Other primary connectives making up $\mathbf{DA1S4b!_b?}$ include additives, 1st order quantifiers, normal (i.e. distributive) modalities, bracket (i.e. nondistributive) modalities, and (sub)exponentials.

We can draw a clear distinction between the primary connectives, the semantically inactive connectives, and the synthetic connectives; the latter two are abbreviatory and are there for convenience, and to simplify derivation.

There are semantically inactive variants of the continuous and discontinuous multiplicatives, and semantically inactive variants of the additives, 1st order quantifiers, and normal modalities. For example, the semantically inactive additive conjunction $A \sqcap B \colon \phi$ abbreviates $A \& B \colon (\phi, \phi)$.

Synthetic connectives (Girard 2011) divide into the continuous and discontinuous deterministic (unary) synthetic connectives, and the continuous and discontinuous nondeterministic (binary) synthetic connectives. For example, the nondeterministic continuous division $B \div A$ abbreviates $(A \backslash B) \sqcap (B / A)$.

Syntactic types

The syntactic types of displacement logic are sorted $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ according to the number of points of discontinuity $0, 1, 2, \ldots$ their expressions contain.

Each type predicate letter has a sort and an arity which are naturals, and a corresponding semantic type.

Assuming ordinary terms to be already given, where P is a type predicate letter of sort i and arity n and $t_1, \ldots, t_n$ are terms, $Pt_1 \ldots t_n$ is an (atomic) type of sort i of the corresponding semantic type.

Compound types of $\mathbf{DA1S4b!_b?}$ are formed as follows, and the structure preserving semantic type map T associates these with semantic types.

1.	$\mathcal{F}_i$	$::=$	$\mathcal{F}_{i+j}/\mathcal{F}_j$	$T(C/B)$	$=$	$T(B)\rightarrow T(C)$	over
2.	$\mathcal{F}_j$	$::=$	$\mathcal{F}_i\backslash\mathcal{F}_{i+j}$	$T(A\backslash C)$	$=$	$T(A)\rightarrow T(C)$	under
3.	$\mathcal{F}_{i+j}$	$::=$	$\mathcal{F}_i\bullet\mathcal{F}_j$	$T(A\bullet B)$	$=$	$T(A)\&T(B)$	continuous product
4.	$\mathcal{F}_0$	$::=$	I	$T(I)$	$=$	$\top$	continuous unit
5.	$\mathcal{F}_{i+1}$	$::=$	$\mathcal{F}_{i+j}\uparrow_k\mathcal{F}_j, 1 \leq k \leq i+j$	$T(C\uparrow_k B)$	$=$	$T(B)\rightarrow T(C)$	circumfix
6.	$\mathcal{F}_j$	$::=$	$\mathcal{F}_{i+1}\downarrow_k\mathcal{F}_{i+j}, 1 \leq k \leq i+1$	$T(A\downarrow_k C)$	$=$	$T(A)\rightarrow T(C)$	infix
7.	$\mathcal{F}_{i+j}$	$::=$	$\mathcal{F}_{i+1}\odot_k\mathcal{F}_j, 1 \leq k \leq i+1$	$T(A\odot_k B)$	$=$	$T(A)\&T(B)$	discontinuous product
8.	$\mathcal{F}_1$	$::=$	J	$T(J)$	$=$	$\top$	discontinuous unit
9.	$\mathcal{F}_i$	$::=$	$\mathcal{F}_i\&\mathcal{F}_i$	$T(A\&B)$	$=$	$T(A)\&T(B)$	additive conjunction
10.	$\mathcal{F}_i$	$::=$	$\mathcal{F}_i\oplus\mathcal{F}_i$	$T(A\oplus B)$	$=$	$T(A)+T(B)$	additive disjunction
11.	$\mathcal{F}_i$	$::=$	$\bigwedge V\mathcal{F}_i$	$T(\bigwedge vA)$	$=$	$F\rightarrow T(A)$	1st order univ. qu.
12.	$\mathcal{F}_i$	$::=$	$\bigvee V\mathcal{F}_i$	$T(\bigvee vA)$	$=$	$F\&T(A)$	1st order exist. qu.
13.	$\mathcal{F}_i$	$::=$	$\Box\mathcal{F}_i$	$T(\Box A)$	$=$	$\mathbf{L}T(A)$	universal modality
14.	$\mathcal{F}_i$	$::=$	$\Diamond\mathcal{F}_i$	$T(\Diamond A)$	$=$	$\mathbf{M}T(A)$	existential modality
15.	$\mathcal{F}_i$	$::=$	$[\,]^{-1}\mathcal{F}_i$	$T([\,]^{-1}A)$	$=$	$T(A)$	univ. bracket modality
16.	$\mathcal{F}_i$	$::=$	$\langle\rangle\mathcal{F}_i$	$T(\langle\rangle A)$	$=$	$T(A)$	exist. bracket modality
17.	$\mathcal{F}_0$	$::=$	$!\mathcal{F}_0$	$T(!A)$	$=$	$T(A)$	universal exponential
18.	$\mathcal{F}_0$	$::=$	$?\mathcal{F}_0$	$T(?A)$	$=$	$T(A)^+$	existential exponential

Gentzen sequent calculus

We use a Gentzen sequent presentation standard from Gentzen (1934) and Lambek (1958).
In Gentzen sequent antecedents for displacement logic with bracket modalities (structural inhibition) and exponentials (structural facilitation) there are also bracket constructors and 'stoups'.

Stoups (cf. the linear logic of Girard 2011[**?**]) (ζ) are stores read as multisets for re-usable (nonlinear) resources which appear at the left of a configuration marked off by a semicolon (when the stoup is empty the semicolon may be ommited). The stoup of linear logic is for resources which can be contracted (copied) or weakened (deleted). By contrast, our stoup is for a linguistically motivated variant of contraction, and does not allow weakening. Furthermore, whereas linear logic is commutative, our logic is in general noncommutative and here the stoup is used for resources which are also commutative.

A configuration together with a stoup is a *zone* (Ξ).
The bracket constructor applies not to a configuration alone but to a configuration with a stoup, i.e a zone: reusable resources are specific to their domain.
Stoups $\mathcal{S}$ and configurations $\mathcal{O}$ are defined by ($\emptyset$ is the empty stoup; Λ is the empty configuration; the *separator* 1 marks points of discontinuity) (note that only types of sort 0 can go into the stoup; reusable types of other sorts would not preserve the sequent antecedent-succedent sort equality under contraction or expansion: $0 + 0 = 0$, but $i + i \neq i$ for $i > 0$):

$$
(1) \quad \begin{array}{rcl}
\mathcal{S} & ::= & \emptyset \mid \mathcal{F}_0, \mathcal{S} \\
\mathcal{O} & ::= & \Lambda \mid \mathcal{T}, \mathcal{O} \\
\mathcal{T} & ::= & 1 \mid \mathcal{F}_0 \mid \mathcal{F}_{i>0}\{\underbrace{\mathcal{O} : \ldots : \mathcal{O}}_{i\,\mathcal{O}'\mathrm{s}}\} \mid [\mathcal{S}; \mathcal{O}]
\end{array}
$$

For a type A, its sort $s(A)$ is the i such that $A \in \mathcal{F}_i$.
For a configuration Γ, its sort $s(\Gamma)$ is $|\Gamma|_1$, i.e. the number of points of discontinuity 1 which it contains.
Sequents are of the form:

(2) $\mathcal{S}; \mathcal{O} \Rightarrow \mathcal{F}$ such that $s(\mathcal{O}) = s(\mathcal{F})$

The figure $\overrightarrow{A}$ of a type A is defined by:

(3) $\overrightarrow{A} = \begin{cases} A & \text{if } s(A) = 0 \\ A\{\underbrace{1 : \ldots : 1}_{s(A)\ 1\text{'s}}\} & \text{if } s(A) > 0 \end{cases}$

Where Γ is a configuration of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the *fold* $\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle$ is the result of replacing the successive 1's in Γ by $\Delta_1, \ldots, \Delta_i$ respectively.
Where Γ is of sort i, the hyperoccurrence notation $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0(\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle)$, i.e. a context configuration Δ (which is externally Δ_0 and internally $\Delta_1, \ldots, \Delta_i$) with a potentially discontinuous distinguished subconfiguration Γ (continuous if $i = 0$, discontinuous if $i > 0$).
Where Δ is a configuration of sort $i > 0$ and Γ is a configuration, the kth *metalinguistic intercalation* $\Delta\,|_k\,\Gamma$, $1 \leq k \leq i$, is given by:

(4) $$\Delta\,|_k\,\Gamma =_{df} \Delta \otimes \langle \underbrace{1 : \ldots : 1}_{k-1 \text{ 1's}} : \Gamma : \underbrace{1 : \ldots : 1}_{i-k \text{ 1's}} \rangle$$

i.e. $\Delta\,|_k\,\Gamma$ is the configuration resulting from replacing by Γ the kth separator in Δ.

Rules and linguistic applications

A semantically labelled sequent is a sequent in which the antecedent type occurrences $A_1, \ldots, A_n$ are labelled by distinct variables $x_1, \ldots, x_n$ of types $T(A_1), \ldots, T(A_n)$ respectively, and the succedent type A is labelled by a term of type $T(A)$ with free variables drawn from $x_1, \ldots, x_n$.
In this section we give the semantically labelled Gentzen sequent rules for the connectives of $\mathbf{DA1S4b!_b?}$, and indicate some linguistic applications.

1. $$\frac{\zeta_1; \Gamma \Rightarrow B\colon \psi \qquad \Xi(\zeta_2; \Delta\langle \overrightarrow{C}\colon z\rangle) \Rightarrow D\colon \omega}{\Xi(\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{C/B}\colon x, \Gamma\rangle) \Rightarrow D\colon \omega\{(x\ \psi)/z\}}\,/L \qquad \frac{\zeta; \Gamma, \overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\zeta; \Gamma \Rightarrow C/B\colon \lambda y \chi}\,/R$$

2. $$\frac{\zeta_1; \Gamma \Rightarrow A\colon \phi \qquad \Xi(\zeta_2; \Delta\langle \overrightarrow{C}\colon z\rangle) \Rightarrow D\colon \omega}{\Xi(\zeta_1 \uplus \zeta_2; \Delta\langle \Gamma, \overrightarrow{A\backslash C}\colon y\rangle) \Rightarrow D\colon \omega\{(y\ \phi)/z\}}\,\backslash L \qquad \frac{\zeta; \overrightarrow{A}\colon x, \Gamma \Rightarrow C\colon \chi}{\zeta; \Gamma \Rightarrow A\backslash C\colon \lambda x \chi}\,\backslash R$$

3. $$\frac{\Xi\langle \overrightarrow{A}\colon x, \overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Xi\langle \overrightarrow{A \bullet B}\colon z\rangle \Rightarrow D\colon \omega\{\pi_1 z/x, \pi_2 z/y\}}\,\bullet L \qquad \frac{\zeta_1; \Delta \Rightarrow A\colon \phi \qquad \zeta_2; \Gamma \Rightarrow B\colon \psi}{\zeta_1 \uplus \zeta_2; \Delta, \Gamma \Rightarrow A \bullet B\colon (\phi, \psi)}\,\bullet R$$

4. $$\frac{\Xi\langle \Lambda\rangle \Rightarrow A\colon \phi}{\Xi\langle \overrightarrow{I}\colon x\rangle \Rightarrow A\colon \phi}\,IL \qquad \frac{}{\emptyset; \Lambda \Rightarrow I\colon 0}\,IR$$

Figure: Lambek multiplicatives

The continuous multiplicatives, the Lambek connectives of Lambek (1958; 1988), defined in relation to concatenation/appending, are the basic means of categorial categorization and subcategorization.

Note that here and throughout the active types in antecedents are figures (vectorial) whereas those in succedents are not; intuitively this is because antecedents are structured but succedents are not.

The directional divisions over, $/$, and under, $\backslash$, are exemplified by assignments such as *the*: N/CN for *the man*: N and *sings*: $N\backslash S$ for *John sings*: S, and *loves*: $(N\backslash S)/N$ for *John loves Mary*: S.

5.
$$\frac{\zeta_1; \Gamma \Rightarrow B\colon \psi \qquad \Xi(\zeta_2; \Delta\langle \overrightarrow{C}\colon z\rangle) \Rightarrow D\colon \omega}{\Xi(\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{C{\uparrow_k}B}\colon x \,|_k\, \Gamma\rangle) \Rightarrow D\colon \omega\{(x\ \psi)/z\}} \uparrow_k L \qquad \frac{\zeta; \Gamma \,|_k\, \overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\zeta; \Gamma \Rightarrow C{\uparrow_k}B\colon \lambda y \chi} \uparrow_k R$$

6.
$$\frac{\zeta_1; \Gamma \Rightarrow A\colon \phi \qquad \Xi(\zeta_2; \Delta\langle \overrightarrow{C}\colon z\rangle) \Rightarrow D\colon \omega}{\Xi(\zeta_1 \uplus \zeta_2; \Delta\langle \Gamma \,|_k\, \overrightarrow{A{\downarrow_k}C}\colon y\rangle) \Rightarrow D\colon \omega\{(y\ \phi)/z\}} \downarrow_k L \qquad \frac{\zeta; \overrightarrow{A}\colon x \,|_k\, \Gamma \Rightarrow C\colon \chi}{\zeta; \Gamma \Rightarrow A{\downarrow_k}C\colon \lambda x \chi} \downarrow_k R$$

7.
$$\frac{\Xi\langle \overrightarrow{A}\colon x \,|_k\, \overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Xi\langle \overrightarrow{A{\odot_k}B}\colon z\rangle \Rightarrow D\colon \omega\{\pi_1 z/x, \pi_2 z/y\}} \odot_k L \qquad \frac{\zeta_1; \Delta \Rightarrow A\colon \phi \qquad \zeta_2; \Gamma \Rightarrow B\colon \psi}{\zeta_1 \uplus \zeta_2; \Delta \,|_k\, \Gamma \Rightarrow A{\odot_k}B\colon (\phi, \psi)} \odot_k R$$

8.
$$\frac{\Xi\langle 1\rangle \Rightarrow A\colon \phi}{\Xi\langle \overrightarrow{J}\colon x\rangle \Rightarrow A\colon \phi} JL \qquad \frac{}{\emptyset; 1 \Rightarrow J\colon 0} JR$$

Figure: Displacement multiplicatives

The discontinuous multiplicatives, the displacement connectives, Morrill and Valentín (2010), Morrill et al. (2011), are defined in relation to intercalation/plugging.

When the value of the k subindex indicates the first (leftmost) point of discontinuity it may be omitted, i.e. it defaults to 1.

Circumfixation, $\uparrow$, is exemplified by a discontinuous particle verb assignment $calls{+}1{+}up$: $(N\backslash S){\uparrow}N$ for *Mary calls John up*: S, and infixation, $\downarrow$, and circumfixation together are exemplified by a quantifier phrase assignment *everyone*: $(S{\uparrow}N){\downarrow}S$ simulating Montague's S14 treatment of quantifying in; see the demo at the end.

9.

$$\frac{\Xi\langle \overrightarrow{A} : x\rangle \Rightarrow C : \chi}{\Xi\langle \overrightarrow{A\&B} : z\rangle \Rightarrow C : \chi\{\pi_1 z/x\}}\&L_1 \qquad \frac{\Xi\langle \overrightarrow{B} : y\rangle \Rightarrow C : \chi}{\Xi\langle \overrightarrow{A\&B} : z\rangle \Rightarrow C : \chi\{\pi_2 z/y\}}\&L_2$$

$$\frac{\Xi \Rightarrow A : \phi \qquad \Xi \Rightarrow B : \psi}{\Xi \Rightarrow A\&B : (\phi, \psi)}\&R$$

10.

$$\frac{\Xi\langle \overrightarrow{A} : x\rangle \Rightarrow C : \chi_1 \qquad \Xi\langle \overrightarrow{B} : y\rangle \Rightarrow C : \chi_2}{\Xi\langle \overrightarrow{A\oplus B} : z\rangle \Rightarrow C : z \to x.\chi_1; y.\chi_2}\oplus L$$

$$\frac{\Xi \Rightarrow A : \phi}{\Xi \Rightarrow A\oplus B : \iota_1\phi}\oplus R_1 \qquad \frac{\Xi \Rightarrow B : \psi}{\Xi \Rightarrow A\oplus B : \iota_2\psi}\oplus R_2$$

Figure: Additives

The additives, Lambek (1961), Morrill (1990), Kanazawa (1992), have application to polymorphism.

For example the additive conjunction & can be used for *rice*: $N\&CN$ as in *rice grows*: S and *the rice grows*: S, and the additive disjunction $\oplus$ can be used for *is*: $(N\backslash S)/(N\oplus(CN/CN))$ as in *Tully is Cicero*: S and *Tully is humanist*: S.

The additive disjunction can be used together with the continuous unit to express the optionality of a complement as in *eats*: $(N\backslash S)/(N\oplus I)$ for *John eats fish*: S and *John eats*: S.

11. $$\frac{\Xi\langle\overrightarrow{A[t/v]:x}\rangle \Rightarrow B:\psi}{\Xi\langle\overrightarrow{\bigwedge vA:z}\rangle \Rightarrow B:\psi\{(z\ t)/x\}}\ \bigwedge L \qquad \frac{\Xi \Rightarrow A[a/v]:\phi}{\Xi \Rightarrow \bigwedge vA:\lambda v\phi}\ \bigwedge R^{\dagger}$$

12. $$\frac{\Xi\langle\overrightarrow{A[a/v]:x}\rangle \Rightarrow B:\psi}{\Xi\langle\overrightarrow{\bigvee vA:z}\rangle \Rightarrow B:\psi\{\pi_2 z/x\}}\ \bigvee L^{\dagger} \qquad \frac{\Xi \Rightarrow A[t/v]:\phi}{\Xi \Rightarrow \bigvee vA:(t,\phi)}\ \bigvee R$$

Figure: Quantifiers, where † indicates that there is no *a* in the conclusion

The quantifiers, Morrill (1994), have application to features.
For example, singular and plural number in *sheep*: $\bigwedge nCNn$ for *the sheep grazes*: S and *the sheep graze*: S.
And for a past, present or future tense finite sentence complement we can have *said*: $(N\backslash S)/\bigvee tSf(t)$ in *John said Mary walked*: S, *John said Mary walks*: S and *John said Mary will walk*: S.

$$13. \qquad \frac{\Xi\langle \overrightarrow{A}: x\rangle \Rightarrow B: \psi}{\Xi\langle \overrightarrow{\Box A}: z\rangle \Rightarrow B: \psi\{^\vee z/x\}}\Box L \qquad \frac{\boxtimes\Xi \Rightarrow A: \phi}{\boxtimes\Xi \Rightarrow \Box A: {}^\wedge\phi}\Box R$$

$$14. \qquad \frac{\boxtimes\Xi\langle \overrightarrow{A}: x\rangle \Rightarrow \diamondplus B: \psi}{\boxtimes\Xi\langle \overrightarrow{\Diamond A}: z\rangle \Rightarrow \diamondplus B: \psi\{^\cup z/x\}}\Diamond L \qquad \frac{\Xi \Rightarrow A: \phi}{\Xi \Rightarrow \Diamond A: {}^\cap\phi}\Diamond R$$

Figure: Normal modalities, where $\boxtimes/\diamondplus$ marks a structure all the types of which have main connective a box/diamond

With respect to the (**S4**) normal modalities, the universal (Morrill 1990) has application to intensionality.
For example, for a propositional attitude verb such as *believes* we can assign type $\Box((N\backslash S)/\Box S)$ with a modality outermost since the word has a sense, and a modality on the first argument but not the second, since the sentential complement is an intensional domain, but not the subject.
The modalities are in the categorial type, distinctly from, but in relation to, the logical interpretation of the propositional attitude verb.
The $\Box$ Right rule is semantically interpreted by intensionalisation $^\wedge$ and the $\Box$ Left rule is semantically interpreted by extensionalisation $^\vee$ in such a way that the Curry-Howard correspondence for the modality yields the law of down-up cancellation (Dowty et al. 1981): $^{\vee\wedge}\phi = \phi$.

15.
$$\frac{\Xi\langle\overrightarrow{A}:x\rangle \Rightarrow B:\psi}{\Xi\langle\overrightarrow{[[\,]^{-1}A}:x]\rangle \Rightarrow B:\psi}\,[\,]^{-1}L \qquad \frac{[\Xi] \Rightarrow A:\phi}{\Xi \Rightarrow [\,]^{-1}A:\phi}\,[\,]^{-1}R$$

16.
$$\frac{\Xi\langle[\overrightarrow{A}:x]\rangle \Rightarrow B:\psi}{\Xi\langle\overrightarrow{\langle\rangle A}:x\rangle \Rightarrow B:\psi}\,\langle\rangle L \qquad \frac{\Xi \Rightarrow A:\phi}{[\Xi] \Rightarrow \langle\rangle A:\phi}\,\langle\rangle R$$

Figure: Bracket modalities

The bracket modalities, Morrill (1992) and Moortgat 1995), have application to nonassociativity and syntactical domains such as prosodic phrases and extraction islands.
For example, single bracketing for weak islands: *walks*: $\langle\rangle N\backslash S$ for the subject condition, and *without*: $[\,]^{-1}(VP\backslash VP)/VP$ for the adverbial island constraint; and double bracketing for strong islands such as *and*: $(S\backslash[\,]^{-1}[\,]^{-1}S)/S$ for the coordinate structure constraint.

17.

$$\frac{\Xi(\zeta \uplus \{A\colon x\}; \Gamma_1, \Gamma_2) \Rightarrow B\colon \psi}{\Xi(\zeta; \Gamma_1, !A\colon x, \Gamma_2) \Rightarrow B\colon \psi}\ !L \qquad \frac{\zeta; \Lambda \Rightarrow A\colon \phi}{\zeta; \Lambda \Rightarrow !A\colon \phi}\ !R$$

$$\frac{\Xi(\zeta; \Gamma_1, A\colon x, \Gamma_2) \Rightarrow B\colon \psi}{\Xi(\zeta \uplus \{A\colon x\}; \Gamma_1, \Gamma_2) \Rightarrow B\colon \psi}\ !P$$

$$\frac{\Xi(\zeta_1 \uplus \zeta_2 \uplus \{A\colon x\}; \Gamma_1, [\zeta_2 \uplus \{A\colon y\}; \Gamma_2], \Gamma_3) \Rightarrow B\colon \psi}{\Xi(\zeta_1 \uplus \zeta_2 \uplus \{A\colon x\}; \Gamma_1, \Gamma_2, \Gamma_3) \Rightarrow B\colon \psi\{x/y\}}\ !C$$

18.

$$\frac{\Xi(A\colon x_1) \Rightarrow B\colon \psi([x_1]) \qquad \Xi(A\colon x_1, A\colon x_2) \Rightarrow B\colon \psi([x_1, x_2]) \qquad \ldots}{\Xi(?A\colon x) \Rightarrow B\colon \psi(x)}\ ?L$$

$$\frac{\Xi \Rightarrow A\colon \phi}{\Xi \Rightarrow ?A\colon [\phi]}\ ?R \qquad \frac{\zeta; \Gamma \Rightarrow A\colon \phi \qquad \zeta'; \Delta \Rightarrow ?A\colon \psi}{\zeta \uplus \zeta'; \Gamma, \Delta \Rightarrow ?A\colon [\phi|\psi]}\ ?M$$

Figure: Exponentials

Finally, there is nonlinearity. The universal exponential, Girard (1987), Barry, Hepple, Leslie and Morrill (1991), Morrill (1994), Morrill and Valentín (2015), and Morrill (2017), has application to extraction including parasitic extraction.

Using the universal exponential, !, for which contraction induces island brackets, we can assign a relative pronoun type *that*: $(CN\backslash CN)/(S/!N)$ allowing parasitic extraction such as *paper that John filed without reading*: CN, where parasitic gaps can appear only in (weak) islands, but can be iterated in subislands, for example, *man who the fact that the friends of admire without praising surprises*. Crucially, in the linguistic formulation ! does not have weakening, i.e. deletion, since, e.g., the body of a relative clause *must* contain a gap: **man who John loves Mary*.

In the formulation here $!L$ moves the operand of a universal exponential (e.g. the hypothetical subtype of relativisation) into the stoup, where it will percolate as indicated by the above rules. From there it can be copied into the stoup of a newly-created bracketed domain by the contraction rule $!C$ (producing a parasitic gap), and it can be moved into any position in the matrix configuration of its zone by $!P$ (producing a normal nonparasitic or host gap).

The existential exponential ? has application to iterated coordination (Morrill 1994; Morrill and Valentín 2015) and (unboundedly iterated) *respectively* (Morrill and Valentín 2016). Using the existential exponential, ?, we can assign a coordinator type *and*: $(?N\backslash N)/N$ allowing iterated coordination as in *John, Bill, Mary and Suzy*: N, or *and*: $(?(S/N)\backslash(S/N))/(S/N)$ for *John likes Mary dislikes, and Bill hates, London* (iterated right node raising), and so on.

Focusing

Spurious ambiguity is the phenomenon whereby distinct derivations in grammar may assign the same structural reading, resulting in redundancy in the parse search space and inefficiency in parsing.

Understanding the problem depends on identifying the essential mathematical structure of derivations.

This is trivial in the case of context free grammar, where the parse structures are ordered trees; in the case of type logical categorial grammar, the parse structures are proof nets.

However, with respect to multiplicatives intrinsic proof nets have not yet been given for displacement calculus (but see Morrill and Fadda (2008, Fadda 2010, and Moot 2014, 2016) In this context CatLog3 approaches spurious ambiguity by means of Andreoli's (1982) proof-theoretic technique of focalisation, which engenders a substantial reduction of spurious ambiguity.

In focalisation, *situated* (in the antecedent of a sequent, input, $^{\bullet}$/ in the succedent of a sequent, output, $^{\circ}$) non-atomic types are classified as of *reversible/negative* or *irreversible/positive polarity* according as their associated rule is reversible or not.
For example, $\backslash R$ is reversible, but $\& L$ is not reversible.

$$\frac{A, \Gamma \Rightarrow C}{\Gamma \Rightarrow A \backslash C}\backslash R \qquad\qquad \frac{\Delta(A) \Rightarrow C}{\Delta(A\&B) \Rightarrow C}\& L$$

There are alternating phases of don't-care nondeterministic negative rule application, and positive rule application locking on to *focalised* formulas.

Given a sequent with no occurrences of negative formulas, one chooses a positive formula as principal formula (which is boxed; we say it is focalised) and applies proof search to its subformulas while these remain positive.
When one finds a negative formula or a literal, reversible rules are applied in a don't care nondeterministic fashion until no longer possible, when another positive formula is chosen, and so on.
CatLog3 can be set to focus all atoms in the input (as in the example at the end) or in the output, i.e. it implements uniform *bias*.

A sequent is either unfocused and as before, or else focused and has exactly one type boxed. This is the focused type.
The focalised logical rules for displacement calculus are given below.
Sequents are accompanied by *judgements*: focalised or not focalised and reversible or not reversible.
The completeness of this focalisation, together with additives, is proved in Morrill and Valentín (2015).
The completeness of focalisation for other connectives of CatLog3 is a topic of ongoing research.

$$\frac{\overrightarrow{A}\colon x, \Gamma \Rightarrow C\colon \chi \quad \neg\mathbf{foc}}{\Gamma \Rightarrow A\backslash C\colon \lambda x\chi \quad \neg\mathbf{foc} \wedge \mathbf{rev}}\ \backslash R \qquad \frac{\Gamma, \overrightarrow{B}\colon y \Rightarrow C\colon \chi \quad \neg\mathbf{foc}}{\Gamma \Rightarrow C/B\colon \lambda y\chi \quad \neg\mathbf{foc} \wedge \mathbf{rev}}\ /R$$

$$\frac{\Delta\langle \overrightarrow{A}\colon x, \overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega \quad \neg\mathbf{foc}}{\Delta\langle \overrightarrow{A{\bullet}B}\colon z\rangle \Rightarrow D\colon \omega\{\pi_1 z/x, \pi_2 z/y\} \quad \neg\mathbf{foc} \wedge \mathbf{rev}}\ \bullet L$$

$$\frac{\Delta\langle\Lambda\rangle \Rightarrow A\colon \phi \quad \neg\mathbf{foc}}{\Delta\langle \overrightarrow{I}\colon x\rangle \Rightarrow A\colon \phi \quad \neg\mathbf{foc} \wedge \mathbf{rev}}\ IL$$

$$\frac{\overrightarrow{A}\colon x \,|_k\, \Gamma \Rightarrow C\colon \chi \quad \neg\mathbf{foc}}{\Gamma \Rightarrow A{\downarrow_k}C\colon \lambda x\chi \quad \neg\mathbf{foc} \wedge \mathbf{rev}}\ {\downarrow_k}R \qquad \frac{\Gamma \,|_k\, \overrightarrow{B}\colon y \Rightarrow C\colon \chi \quad \neg\mathbf{foc}}{\Gamma \Rightarrow C{\uparrow_k}B\colon \lambda y\chi \quad \neg\mathbf{foc} \wedge \mathbf{rev}}\ {\uparrow_k}R$$

$$\frac{\Delta\langle \overrightarrow{A}\colon x \,|_k\, \overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega \quad \neg\mathbf{foc}}{\Delta\langle \overrightarrow{A{\odot_k}B}\colon z\rangle \Rightarrow D\colon \omega\{\pi_1 z/x, \pi_2 z/y\} \quad \neg\mathbf{foc} \wedge \mathbf{rev}}\ \odot_k L$$

$$\frac{\Delta\langle 1\rangle \Rightarrow A\colon \phi \quad \neg\mathbf{foc}}{\Delta\langle \overrightarrow{J}\colon x\rangle \Rightarrow A\colon \phi \quad \neg\mathbf{foc} \wedge \mathbf{rev}}\ JL$$

Figure: Reversible multiplicative rules

$$\frac{\Gamma \Rightarrow \boxed{P}:\phi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D\colon \omega \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta\langle \Gamma, \overrightarrow{\boxed{P\backslash Q}}: y\rangle \Rightarrow D\colon \omega\{(y\ \phi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \backslash L$$

$$\frac{\Gamma \Rightarrow \boxed{P_1}:\phi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Delta\langle \overrightarrow{P_2}: z\rangle \Rightarrow D\colon \omega \quad \neg\mathbf{foc} \wedge ?P_2\mathbf{rev}}{\Delta\langle \Gamma, \overrightarrow{\boxed{P_1\backslash P_2}}: y\rangle \Rightarrow D\colon \omega\{(y\ \phi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \backslash L$$

$$\frac{\Gamma \Rightarrow Q_1:\phi \quad \neg\mathbf{foc} \wedge ?Q_1\mathbf{rev} \qquad \Delta\langle \overrightarrow{\boxed{Q_2}}: z\rangle \Rightarrow D\colon \omega \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta\langle \Gamma, \overrightarrow{\boxed{Q_1\backslash Q_2}}: y\rangle \Rightarrow D\colon \omega\{(y\ \phi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \backslash L$$

$$\frac{\Gamma \Rightarrow Q\colon\phi \quad \neg\mathbf{foc} \wedge ?Q\mathbf{rev} \qquad \Delta\langle \overrightarrow{P}: z\rangle \Rightarrow D\colon \omega \quad \neg\mathbf{foc} \wedge ?P\mathbf{rev}}{\Delta\langle \Gamma, \overrightarrow{\boxed{Q\backslash P}}: y\rangle \Rightarrow D\colon \omega\{(y\ \phi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \backslash L$$

$$\frac{\Gamma \Rightarrow \boxed{P} : \psi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Delta\langle \overrightarrow{\boxed{Q}} : z\rangle \Rightarrow D : \omega \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta\langle \overrightarrow{\boxed{Q/P}} : x, \Gamma\rangle \Rightarrow D : \omega\{(x\ \psi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} /L$$

$$\frac{\Gamma \Rightarrow Q_1 : \psi \quad \neg\mathbf{foc} \wedge ?Q_1\,\mathbf{rev} \qquad \Delta\langle \overrightarrow{\boxed{Q_2}} : z\rangle \Rightarrow D : \omega \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta\langle \overrightarrow{\boxed{Q_2/Q_1}} : x, \Gamma\rangle \Rightarrow D : \omega\{(x\ \psi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} /L$$

$$\frac{\Gamma \Rightarrow \boxed{P_1} : \psi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Delta\langle \overrightarrow{P_2} : z\rangle \Rightarrow D : \omega \quad \neg\mathbf{foc} \wedge ?P_2\,\mathbf{rev}}{\Delta\langle \overrightarrow{\boxed{P_2/P_1}} : x, \Gamma\rangle \Rightarrow D : \omega\{(x\ \psi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} /L$$

$$\frac{\Gamma \Rightarrow Q : \psi \quad \neg\mathbf{foc} \wedge ?Q\,\mathbf{rev} \qquad \Delta\langle \overrightarrow{P} : z\rangle \Rightarrow D : \omega \quad \neg\mathbf{foc} \wedge ?P\,\mathbf{rev}}{\Delta\langle \overrightarrow{\boxed{P/Q}} : x, \Gamma\rangle \Rightarrow D : \omega\{(x\ \psi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} /L$$

Figure: Left non-reversible continuous multiplicative rules

$$\frac{\Gamma \Rightarrow \boxed{P} : \phi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Delta\langle \overrightarrow{\boxed{Q}} : z\rangle \Rightarrow D : \omega \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta\langle \Gamma |_k \overrightarrow{\boxed{P {\downarrow_k} Q}} : y\rangle \Rightarrow D : \omega\{(y\ \phi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \downarrow_k L$$

$$\frac{\Gamma \Rightarrow \boxed{P_1} : \phi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Delta\langle \overrightarrow{P_2} : z\rangle \Rightarrow D : \omega \quad \neg\mathbf{foc} \wedge ?P_2\mathbf{rev}}{\Delta\langle \Gamma |_k \overrightarrow{\boxed{P_1 {\downarrow_k} P_2}} : y\rangle \Rightarrow D : \omega\{(y\ \phi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \downarrow_k L$$

$$\frac{\Gamma \Rightarrow Q_1 : \phi \quad \neg\mathbf{foc} \wedge ?Q_1\mathbf{rev} \qquad \Delta\langle \overrightarrow{\boxed{Q_2}} : z\rangle \Rightarrow D : \omega \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta\langle \Gamma |_k \overrightarrow{\boxed{Q_1 {\downarrow_k} Q_2}} : y\rangle \Rightarrow D : \omega\{(y\ \phi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \downarrow_k L$$

$$\frac{\Gamma \Rightarrow Q : \phi \quad \neg\mathbf{foc} \wedge ?Q\mathbf{rev} \qquad \Delta\langle \overrightarrow{P} : z\rangle \Rightarrow D : \omega \quad \neg\mathbf{foc} \wedge ?P\mathbf{rev}}{\Delta\langle \Gamma |_k \overrightarrow{\boxed{Q {\downarrow_k} P}} : y\rangle \Rightarrow D : \omega\{(y\ \phi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \downarrow_k L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta\langle \overrightarrow{\boxed{Q{\uparrow_k}P}}: x\,|_k\,\Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}}\ {\uparrow_k}L$$

$$\frac{\Gamma \Rightarrow Q_1: \psi \quad \neg\mathbf{foc} \wedge ?Q_1\mathbf{rev} \qquad \Delta\langle \overrightarrow{\boxed{Q_2}}: z\rangle \Rightarrow D: \omega \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta\langle \overrightarrow{\boxed{Q_2{\uparrow_k}Q_1}}: x\,|_k\,\Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}}\ {\uparrow_k}L$$

$$\frac{\Gamma \Rightarrow \boxed{P_1}: \psi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Delta\langle \overrightarrow{P_2}: z\rangle \Rightarrow D: \omega \quad \neg\mathbf{foc} \wedge ?P_2\mathbf{rev}}{\Delta\langle \overrightarrow{\boxed{P_2{\uparrow_k}P_1}}: x\,|_k\,\Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}}\ {\uparrow_k}L$$

$$\frac{\Gamma \Rightarrow Q: \psi \quad \neg\mathbf{foc} \wedge ?Q\mathbf{rev} \qquad \Delta\langle \overrightarrow{P}: z\rangle \Rightarrow D: \omega \quad \neg\mathbf{foc} \wedge ?P\mathbf{rev}}{\Delta\langle \overrightarrow{\boxed{P{\uparrow_k}Q}}: x\,|_k\,\Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\} \quad \mathbf{foc} \wedge \neg\mathbf{rev}}\ {\uparrow_k}L$$

Figure: Left non-reversible discontinuous multiplicative rules

$$\frac{\Delta \Rightarrow \boxed{P_1}: \phi \quad \textbf{foc} \wedge \neg\textbf{rev} \qquad \Gamma \Rightarrow \boxed{P_2}: \psi \quad \textbf{foc} \wedge \neg\textbf{rev}}{\Delta, \Gamma \Rightarrow \boxed{P_1 \bullet P_2}: (\phi, \psi) \quad \textbf{foc} \wedge \neg\textbf{rev}} \bullet R$$

$$\frac{\Delta \Rightarrow \boxed{P}: \phi \quad \textbf{foc} \wedge \neg\textbf{rev} \qquad \Gamma \Rightarrow Q: \psi \quad \neg\textbf{foc} \wedge ?Q\textbf{rev}}{\Delta, \Gamma \Rightarrow \boxed{P \bullet Q}: (\phi, \psi) \quad \textbf{foc} \wedge \neg\textbf{rev}} \bullet R$$

$$\frac{\Delta \Rightarrow N: \phi \quad \neg\textbf{foc} \wedge ?N\textbf{rev} \qquad \Gamma \Rightarrow \boxed{P}: \psi \quad \textbf{foc} \wedge \neg\textbf{rev}}{\Delta, \Gamma \Rightarrow \boxed{N \bullet P}: (\phi, \psi) \quad \textbf{foc} \wedge \neg\textbf{rev}} \bullet R$$

$$\frac{\Delta \Rightarrow N_1: \phi \quad \neg\textbf{foc} \wedge ?N_1\textbf{rev} \qquad \Gamma \Rightarrow N_2: \psi \quad \textbf{foc} \wedge ?N_2\textbf{rev}}{\Delta, \Gamma \Rightarrow \boxed{N_1 \bullet N_2}: (\phi, \psi) \quad \textbf{foc} \wedge \neg\textbf{rev}} \bullet R$$

$$\frac{}{\Lambda \Rightarrow \boxed{I}: 0 \quad \textbf{foc} \wedge \neg\textbf{rev}} IR$$

Figure: Right non-reversible continuous multiplicative rules

$$\frac{\Delta \Rightarrow \boxed{P_1} : \phi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Gamma \Rightarrow \boxed{P_2} : \psi \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta \,|_k\, \Gamma \Rightarrow \boxed{P_1 \odot_k P_2} : (\phi, \psi) \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \odot_k R$$

$$\frac{\Delta \Rightarrow \boxed{P} : \phi \quad \mathbf{foc} \wedge \neg\mathbf{rev} \qquad \Gamma \Rightarrow Q : \psi \quad \neg\mathbf{foc} \wedge ?N\mathbf{rev}}{\Delta \,|_k\, \Gamma \Rightarrow \boxed{P \odot_k Q} : (\phi, \psi) \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \odot_k R$$

$$\frac{\Delta \Rightarrow Q : \phi \quad \neg\mathbf{foc} \wedge ?N\mathbf{rev} \qquad \Gamma \Rightarrow \boxed{P} : \psi \quad \mathbf{foc} \wedge \neg\mathbf{rev}}{\Delta \,|_k\, \Gamma \Rightarrow \boxed{Q \odot_k P} : (\phi, \psi) \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \odot_k R$$

$$\frac{\Delta \Rightarrow Q_1 : \phi \quad \neg\mathbf{foc} \wedge ?Q_1\mathbf{rev} \qquad \Gamma \Rightarrow Q_2 : \psi \quad \neg\mathbf{foc} \wedge ?Q_2\mathbf{rev}}{\Delta \,|_k\, \Gamma \Rightarrow \boxed{Q_1 \odot_k Q_2} : (\phi, \psi) \quad \mathbf{foc} \wedge \neg\mathbf{rev}} \odot_k R$$

$$\frac{}{1 \Rightarrow \boxed{J} : 0 \quad \mathbf{foc} \wedge \neg\mathbf{rev}} JR$$

Figure: Right non-reversible discontinuous multiplicative rules

Count-invariance

We employ infinitary count invariance for categorial logic (Kuznetsov 2017), extending count invariance for multiplicatives (van Benthem 1991) and additives and bracket modalities (Valentín et al. 2013) to include exponentials.
This affords effective pruning of proof search in categorial parsing/theorem-proving.

Count invariance for multiplicatives in (sub)linear logic is introduced in van Benthem (1991).
This involves simply checking the number of positive and negative occurrences of each atom in a sequent.
Thus where $\#(\Sigma)$ is a count of the sequent Σ we have:

(5) $\vdash \Sigma \Longrightarrow \#(\Sigma) = 0$

I.e. the numbers of positive and negative occurrences of each atom must exactly balance.

This provides a necessary, but of course not sufficient, criterion for theoremhood, and it can be checked rapidly.
It can be used as a filter in proof search: if backward chaining proof search generates a goal which does not satisfy the count invariant, the goal can be safely made to fail immediately.
This notion of count for multiplicatives was included in the categorial parser/theorem-prover CatLog(1) (Morrill 2012).

In Valentín et al. (2013) the idea is extended to additives (and bracket modalities).
Instead of a single count for each atom of a sequent Σ we have a minimum count $\#_{\mathrm{min}}(\Sigma)$ and a maximum count $\#_{\mathrm{max}}(\Sigma)$ and for a sequent to be a theorem it must satisfy two inequations:

(6) $\vdash \Sigma \Longrightarrow \#_{\mathrm{min}}(\Sigma) \leq 0 \leq \#_{\mathrm{max}}(\Sigma)$

I.e. the count functions $\#_{\mathrm{min}}$ and $\#_{\mathrm{max}}$ define an interval which must include the point of balance 0; for the multiplicatives, $\#_{\mathrm{min}} = \#_{\mathrm{max}} = \#$ and (6) reduces to the special case (5).
This count-invariance is included in the categorial parser/theorem-prover CatLog2.

Here we describe the count-invariance of CatLog3 which includes further infinitary count functions for exponentials (Kuznetsov et al. 2017).

Infinitary Count Algebra

We consider terms built over constants 0, 1, $\bot$ ($-\infty$, minus infinity), and $\top$ ($+\infty$, plus infinity) by operations plus ($+$), minus ($-$), minimum (min) and maximum (max), and infinitary step functions X and Y thus; $i, j \in \mathcal{Z}$ and $n \in \mathcal{Z}^{+}$:

$+$	j	$\bot$	$\top$
i	$i+j$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$*$
$\top$	$\top$	$*$	$\top$

$-$	j	$\bot$	$\top$
i	$i-j$	$\top$	$\bot$
$\bot$	$\bot$	$*$	$\bot$
$\top$	$\top$	$\top$	$*$

min	j	$\bot$	$\top$
i	$\frac{\lvert i+j\rvert-\lvert i-j\rvert}{2}$	$\bot$	i
$\bot$	$\bot$	$\bot$	$\bot$
$\top$	j	$\bot$	$\top$

max	j	$\bot$	$\top$
i	$\frac{\lvert i+j\rvert+\lvert i-j\rvert}{2}$	i	$\top$
$\bot$	j	$\bot$	$\top$
$\top$	$\top$	$\top$	$\top$

$$X(i) = \begin{cases} \top & \text{if } i > 0 \\ i & \text{if } i \leq 0 \end{cases} \qquad Y(i) = \begin{cases} i & \text{if } i \geq 0 \\ \bot & \text{if } i < 0 \end{cases}$$

Where $\mathcal{P}$ is the set of primitive types, $P \in \mathcal{P}$, $Q \in \mathcal{P} \cup \{[]\}$, $p \in \{^\bullet, ^\circ\}$, and $\overline{^\bullet} = ^\circ$ and $\overline{^\circ} = ^\bullet$ we define the count functions for **DA1S4b!$_\mathbf{b}$?** as shown below.

$$
\begin{array}{rcccl}
\#^{p}_{m,Q}(P) & = & \begin{array}{ll} 1 & \text{if } Q = P \\ 0 & \text{if } Q \neq P \end{array} & & \\
\#^{p}_{m,Q}(A\backslash C) & = & \#^{p}_{m,Q}(A{\downarrow_k}C) & = & \#^{p}_{m,Q}(C) - \#^{\overline{p}}_{\overline{m},Q}(A) \\
\#^{p}_{m,Q}(C/B) & = & \#^{p}_{m,Q}(C{\uparrow_k}B) & = & \#^{p}_{m,Q}(C) - \#^{\overline{p}}_{\overline{m},Q}(B) \\
\#^{p}_{m,Q}(A\bullet B) & = & \#^{p}_{m,Q}(A\odot_k B) & = & \#^{p}_{m,Q}(A) + \#^{p}_{m,Q}(B) \\
\#^{p}_{m,Q}(I) & = & \#^{p}_{m,Q}(J) & = & 0 \\
\#^{\circ}_{m,Q}(A\&B) & = & \overline{m}(\#^{\circ}_{m,Q}(A), \#^{\circ}_{m,Q}(B)) & & \\
\#^{\bullet}_{m,Q}(A\&B) & = & m(\#^{\bullet}_{m,Q}(A), \#^{\bullet}_{m,Q}(B)) & & \\
\#^{\circ}_{m,Q}(A\oplus B) & = & m(\#^{\circ}_{m,Q}(A), \#^{\circ}_{m,Q}(B)) & & \\
\#^{\bullet}_{m,Q}(A\oplus B) & = & \overline{m}(\#^{\bullet}_{m,Q}(A), \#^{\bullet}_{m,Q}(B)) & & \\
\#^{p}_{m,Q}(\bigwedge xA) & = & \#^{p}_{m,Q}(\bigvee xA) & = & \#^{p}_{m,Q}(A) \\
\#^{p}_{m,Q}(\Box A) & = & \#^{p}_{m,Q}(\Diamond A) & = & \#^{p}_{m,Q}(A) \\
\#^{p}_{m,P}([]^{-1}A) & = & \#^{p}_{m,P}(A) & & \\
\#^{p}_{m,[]}([]^{-1}A) & = & \#^{p}_{m,[]}(A) - 1 & & \\
\#^{p}_{m,P}(\langle\rangle A) & = & \#^{p}_{m,P}(A) & & \\
\#^{p}_{m,[]}(\langle\rangle A) & = & \#^{p}_{m,[]}(A) + 1 & & \\
\#^{\bullet}_{\min,Q}(!A) & = & Y(\#^{\bullet}_{\min,Q}(A)) & & \\
\#^{\bullet}_{\max,P}(!A) & = & X(\#^{\bullet}_{\max,P}(A)) & & \\
\#^{\bullet}_{\max,[]}(!A) & = & \top & & \\
\#^{\circ}_{m,Q}(!A) & = & \#^{\circ}_{m,Q}(A) & & \\
\#^{\circ}_{\max,Q}(?A) & = & X(\#^{\circ}_{\max,Q}(A)) & & \\
\#^{\circ}_{\min,Q}(?A) & = & Y(\#^{\circ}_{\min,Q}(A)) & & \\
\#^{\bullet}_{m,Q}(?A) & = & \#^{\bullet}_{m,Q}(A) & &
\end{array}
$$

For zones, stoups, tree terms and configurations, counts are as follows:

$$
\begin{array}{rcl}
\#_{m,Q}(\mathcal{S};\mathcal{O}) & = & \#_{m,Q}(\mathcal{S}) + \#_{m,Q}(\mathcal{O}) \\
\#_{m,Q}(\emptyset) & = & 0 \\
\#_{m,Q}(\mathcal{F},\mathcal{S}) & = & \#_{m,Q}(\mathcal{F}) + \#_{m,Q}(\mathcal{S}) \\
\#_{m,Q}(\Lambda) & = & 0 \\
\#_{m,Q}(\mathcal{T},\mathcal{O}) & = & \#_{m,Q}(\mathcal{T}) + \#_{m,Q}(\mathcal{O}) \\
\#_{m,Q}(1) & = & 0 \\
\#_{m,Q}(\mathcal{F}) & = & \#^{\bullet}_{m,Q}(\mathcal{F}) \\
\#_{m,Q}(\mathcal{F}\{\mathcal{O}_1 : \ldots : \mathcal{O}_i\}) & = & \#^{\bullet}_{m,Q}(\mathcal{F}) + \Sigma_{n=1}^{i} \#_{m,Q}(\mathcal{O}_n) \\
\#_{m,[]}([\mathcal{Z}]) & = & \#_{m,[]}(\mathcal{Z}) + 1 \\
\#_{m,P}([\mathcal{Z}]) & = & \#_{m,P}(\mathcal{Z})
\end{array}
$$

The count-invariance theorem is:

(7) **Theorem**.

$\vdash \Xi \Rightarrow A \Longrightarrow \forall Q \in \mathcal{P} \cup \{[]\},$

$\#_{\mathrm{min},Q}(\Xi \Rightarrow A) \leq 0 \leq \#_{\mathrm{max},Q}(\Xi \Rightarrow A)$

where, $\#_{m,Q}(\Xi \Rightarrow A) = \overset{\circ}{\#}_{m,Q}(A) - \#_{\overline{m},Q}(\Xi)$.

Relativisation including medial and parasitic extraction is obtained by assigning a relative pronoun a type $(CN\backslash CN)/(!N\backslash S)$ whereby the body of a relative clause is analysed as $!N\backslash S$.

By way of example of count-invariance, we show how it discards $N, N\backslash S \Rightarrow !N\backslash S$ corresponding to the ungrammaticality of a relative clause without a gap: **paper that John walks*.

We have the max N-count:

$$\#_{\text{max},N}(N, N\backslash S \Rightarrow !N\backslash S) = \overset{\circ}{\#}_{\text{max},N}(!N\backslash S) - \overset{\bullet}{\#}_{\text{min},N}(N, N\backslash S) =$$
$$\overset{\circ}{\#}_{\text{max}.N}(S) - \overset{\bullet}{\#}_{\text{min},N}(!N) - \overset{\bullet}{\#}_{\text{min},N}(N) - \overset{\bullet}{\#}_{\text{min},N}(N\backslash S) =$$
$$0 - Y(\overset{\bullet}{\#}_{\text{min},N}(N)) - 1 - \overset{\bullet}{\#}_{\text{min},N}(S) + \overset{\circ}{\#}_{\text{min},N}(N) =$$
$$-Y(1) - 1 - 0 + 1 = -1 - 1 + 1 = -1 \not\geq 0$$

which means that the count-invariance is not satisfied.

Iterated sentential coordination is obtained by assigning a coordinator the type $(?S\backslash S)/S$.

By way of a second example we show how count-invariance discards $N, N, N\backslash S \Rightarrow ?S$ corresponding to the ungrammaticality of unequilibrated coordination: **John Mary walks and Suzy talks*.

Max N-count is:

$\#\mathrm{max},_N(N, N, N\backslash S \Rightarrow ?S) = \#^{\circ}{}_{\mathrm{max},N}(?S) - \#^{\bullet}{}_{\mathrm{min},N}(N, N, N\backslash S) = X(\#^{\circ}{}_{\mathrm{max},N}(S)) - \#^{\bullet}{}_{\mathrm{min},N}(N) - \#^{\bullet}{}_{\mathrm{min},N}(N) - \#^{\bullet}{}_{\mathrm{min},N}(N\backslash S) = X(0) - 1 - 1 - \#^{\bullet}{}_{\mathrm{min},N}(S) + \#^{\bullet}{}_{\mathrm{max},N}(N) = 0 - 2 - 0 + 1 = -1 \not\geq 0$

which means that the count-invariance is not satisfied.

Illustration

Morrill and Valentín (2016) proposes as the *Montague Test* the task of providing a computational grammar of the PTQ fragment of Montague (1973), and shows how CatLog meets this task. We are not aware of any other system which has passed the Montague Test. The example sentences of Chapter 7 of Dowty et al. (1981) and the CatLog lexicon for them are given below.

```
str(dwp(”7-7)’), [b([john]), walks], s(f)).
str(dwp(’(7-16)’), [b([every, man]), talks], s(f)).
str(dwp(’(7-19)’), [b([the, fish]), walks], s(f)).
str(dwp(’(7-32)’), [b([every, man]), b([b([walks, or, talks])])], s(f)).
str(dwp(’(7-34)’), [b([b([b([every, man]), walks, or, b([every, man]), talks])])], s(f)).
str(dwp(’(7-39)’), [b([b([b([a, woman]), walks, and, b([she]), talks])])], s(f)).
str(dwp(’(7-43, 45)’), [b([john]), believes, that, b([a, fish]), walks], s(f)).
str(dwp(’(7-48, 49, 52)’), [b([every, man]), believes, that, b([a, fish]), walks], s(f)).
str(dwp(’(7-57)’), [b([every, fish, such, that, b([it]), walks]), talks], s(f)).
str(dwp(’(7-60, 62)’), [b([john]), seeks, a, unicorn], s(f)).
str(dwp(’(7-73)’), [b([john]), is, bill], s(f)).
str(dwp(’(7-76)’), [b([john]), is, a, man], s(f)).
str(dwp(’(7-83)’), [necessarily, b([john]), walks], s(f)).
str(dwp(’(7-86)’), [b([john]), walks, slowly], s(f)).
str(dwp(’(7-91)’), [b([john]), tries, to, walk], s(f)).
str(dwp(’(7-94)’), [b([john]), tries, to, b([b([catch, a, fish, and, eat, it])])], s(f)).
str(dwp(’(7-98)’), [b([john]), finds, a, unicorn], s(f)).
str(dwp(’(7-105)’), [b([every, man, such, that, b([he]), loves, a, woman]), loses, her], s(f)).
str(dwp(’(7-110)’), [b([john]), walks, in, a, park], s(f)).
str(dwp(’(7-116, 118)’), [b([every, man]), doesnt, walk], s(f)).
```

a : $\blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\exists C[(A\ C)\wedge(B\ C)]$

and : $\blacksquare\forall f((\blacksquare ?Sf\backslash[]^{-1}[]^{-1}Sf)/\blacksquare Sf) : (\Phi^{n+}\ 0\ and)$

and : $\blacksquare\forall a\forall f((\blacksquare ?(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)) : (\Phi^{n+}\ (s\ 0)\ and)$

believes : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/(CPthat\sqcup\square Sf)) : \hat{}\lambda A\lambda B(Pres\ ((\check{}believe\ A)\ B))$ **bill** : $\blacksquare Nt(s(m)) : b$

catch : $\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa) : \hat{}\lambda A\lambda B((\check{}catch\ A)\ B)$

doesnt : $\blacksquare\forall g\forall a((Sg\uparrow((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)))\downarrow Sg) : \lambda A\neg(A\ \lambda B\lambda C(B\ C))$

eat : $\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa) : \hat{}\lambda A\lambda B((\check{}eat\ A)\ B)$

every : $\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C)\rightarrow(B\ C)]$

finds : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}find\ A)\ B))$ **fish** : $\square CNs(n) : fish$

he : $\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg)) : \lambda AA$

her : $\blacksquare\forall g\forall a(((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f)))) : \lambda AA$

in : $\square(\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/\exists aNa) : \hat{}\lambda A\lambda B\lambda C((\check{}in\ A)\ (B\ C))$

is : $\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I))) : \lambda A\lambda B(Pres\ (A\rightarrow C.[B=C]; D.((D\ \lambda E[E=B])\ B)))$ **it** : $\blacksquare\forall f\forall a(((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))) : \lambda AA$

it : $\blacksquare[]^{-1}\forall f((\blacksquare Sf|\blacksquare Nt(s(n)))/(\langle\rangle Nt(s(n))\backslash Sf)) : \lambda AA$ **john** : $\blacksquare Nt(s(m)) : j$

loses : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}lose\ A)\ B))$

loves : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}love\ A)\ B))$ **man** : $\square CNs(m) : man$

necessarily : $\blacksquare(SA/\square SA) : Nec$ **or** : $\blacksquare\forall f((\blacksquare ?Sf\backslash[]^{-1}[]^{-1}Sf)/\blacksquare Sf) : (\Phi^{n+}\ 0\ or)$

or : $\blacksquare\forall a\forall f((\blacksquare ?(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)) : (\Phi^{n+}\ (s\ 0)\ or)$

or : $\blacksquare\forall f((\blacksquare ?(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf))\backslash[]^{-1}[]^{-1}(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf)))/\blacksquare(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf))) : (\Phi^{n+}\ (s\ 0)\ or)$

park : $\square CNs(n) : park$

seeks : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\square\forall a\forall f(((Na\backslash Sf)/\exists bNb)\backslash(Na\backslash Sf))) : \hat{}\lambda A\lambda B((\check{}tries\ \hat{}((\check{}A\ \check{}find)\ B))\ B)$

she : $\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(f)))/(\langle\rangle Nt(s(f))\backslash Sg)) : \lambda AA$

slowly : $\square\forall a\forall f(\square(\langle\rangle Na\backslash Sf)\backslash(\langle\rangle\square Na\backslash Sf)) : \hat{}\lambda A\lambda B(\check{}slowly\ \hat{}(\check{}A\ \check{}B))$

such+that : $\blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))) : \lambda A\lambda B\lambda C[(B\ C)\wedge(A\ C)]$ **talks** : $\square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(Pres\ (\check{}talk\ A))$

that : $\blacksquare(CPthat/\square Sf) : \lambda AA$ **the** : $\blacksquare\forall n(Nt(n)/CNn) : \iota$ **to** : $\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))) : \lambda AA$

tries : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\square(\langle\rangle\exists gNt(s(g))\backslash Si)) : \hat{}\lambda A\lambda B((\check{}tries\ \hat{}(\check{}A\ B))\ B)$ **unicorn** : $\square CNs(n) : unicorn$

walk : $\square(\langle\rangle\exists aNa\backslash Sb) : \hat{}\lambda A(\check{}walk\ A)$ **walks** : $\square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(Pres\ (\check{}walk\ A))$ **woman** : $\square CNs(f) : woman$

The CatLog3 LaTeX output for the (ambiguous) last sentence is as follows:

(dwp((7-116, 118))) [**every**+**man**]+**doesnt**+**walk** : *Sf*

$[\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C) \rightarrow (B\ C)], \square CNs(m) : man], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg) : \lambda D\neg(D\ \lambda E\lambda F(E\ F)), \square(\langle\rangle(\exists aNa{-}\exists gNt(s(g)))\backslash Sf) : \hat{}\lambda G(Pres\ (\check{}walk\ G))\ \Rightarrow\ Sf$

$[\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C) \rightarrow (B\ C)], \square CNs(m) : man], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg) : \lambda D\neg(D\ \lambda E\lambda F(E\ F)), \square(\langle\rangle\exists aNa\backslash Sb) : \hat{}\lambda G(\check{}walk\ G)\ \Rightarrow\ Sf$

$$\dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{Nt(s(m)) \Rightarrow \boxed{\exists aNa}}\,\exists R}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists aNa}}\,\langle\rangle R$$

$$\dfrac{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists aNa} \qquad \boxed{Sb} \Rightarrow Sb}{[Nt(s(m))], \boxed{\langle\rangle\exists aNa\backslash Sb} \Rightarrow Sb}\,\backslash L$$

$$\dfrac{\dfrac{\dfrac{[Nt(s(m))], \boxed{\langle\rangle\exists aNa\backslash Sb} \Rightarrow Sb}{[Nt(s(m))], \boxed{\Box(\langle\rangle\exists aNa\backslash Sb)} \Rightarrow Sb}\,\Box L}{\langle\rangle Nt(s(m)), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sb}\,\langle\rangle L}{\Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow \langle\rangle Nt(s(m))\backslash Sb}\,\backslash R$$

$$\dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))}}\,\langle\rangle R \qquad \boxed{Sf} \Rightarrow Sf}{[Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}\,\backslash L$$

$$\dfrac{\Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow \langle\rangle Nt(s(m))\backslash Sb \qquad [Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}{[Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,/L$$

$$\dfrac{[Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}{[Nt(s(m))], 1, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf{\uparrow}((\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb))}\,\uparrow R$$

$$\dfrac{[Nt(s(m))], 1, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf{\uparrow}((\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb)) \qquad \boxed{Sf} \Rightarrow Sf}{[Nt(s(m))], \boxed{(Sf{\uparrow}((\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb))){\downarrow}Sf}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,\downarrow L$$

$$\dfrac{[Nt(s(m))], \boxed{(Sf{\uparrow}((\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb))){\downarrow}Sf}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}{[Nt(s(m))], \boxed{\forall a((Sf{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sf)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,\forall L$$

$$\dfrac{[Nt(s(m))], \boxed{\forall a((Sf{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sf)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}{[Nt(s(m))], \boxed{\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,\forall L$$

$$\dfrac{[Nt(s(m))], \boxed{\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}{[Nt(s(m))], \boxed{\blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,\blacksquare L$$

$$\dfrac{[Nt(s(m))], \boxed{\blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}{[1], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf{\uparrow}Nt(s(m))}\,\uparrow R$$

$$\dfrac{[1], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf{\uparrow}Nt(s(m)) \qquad \boxed{Sf} \Rightarrow Sf}{[\boxed{(Sf{\uparrow}Nt(s(m))){\downarrow}Sf}], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,\downarrow L$$

$$\dfrac{[\boxed{(Sf{\uparrow}Nt(s(m))){\downarrow}Sf}], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}{[\boxed{\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)}], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,\forall L$$

$$\dfrac{\boxed{CNs(m)} \Rightarrow CNs(m)}{\boxed{\Box CNs(m)} \Rightarrow CNs(m)}\,\Box L$$

$$\dfrac{\boxed{\Box CNs(m)} \Rightarrow CNs(m) \qquad [\boxed{\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)}], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}{[\boxed{\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)/CNs(m)}, \Box CNs(m)], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,/L$$

$$\dfrac{[\boxed{\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)/CNs(m)}, \Box CNs(m)], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}{[\boxed{\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g))}, \Box CNs(m)], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,\forall L$$

$$\dfrac{[\boxed{\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g))}, \Box CNs(m)], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}{[\boxed{\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g))}, \Box CNs(m)], \blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf}\,\blacksquare L$$

$$\forall C[(\check{}\,man\ C) \rightarrow \neg(\check{}\,walk\ C)]$$

$$Nt(s(m)) \Rightarrow Nt(s(m))$$

$$\exists R:\quad Nt(s(m)) \Rightarrow \boxed{\exists aNa}$$

$$\langle\rangle R:\quad [Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists aNa} \qquad \boxed{Sb} \Rightarrow Sb$$

$$\backslash L:\quad [Nt(s(m))], \boxed{\langle\rangle\exists aNa\backslash Sb} \Rightarrow Sb$$

$$\Box L:\quad [Nt(s(m))], \boxed{\Box(\langle\rangle\exists aNa\backslash Sb)} \Rightarrow Sb$$

$$\langle\rangle L:\quad \langle\rangle Nt(s(m)), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sb$$

$$\backslash R:\quad \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow \langle\rangle Nt(s(m))\backslash Sb$$

$$Nt(s(m)) \Rightarrow Nt(s(m))$$

$$\langle\rangle R:\quad [Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))} \qquad \boxed{Sf} \Rightarrow Sf$$

$$\backslash L:\quad [Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf$$

$$/L:\quad [Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\uparrow R:\quad [1], (\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf{\uparrow}Nt(s(m)) \qquad \boxed{Sf} \Rightarrow Sf$$

$$\downarrow L:\quad [\boxed{(Sf{\uparrow}Nt(s(m))){\downarrow}Sf}], (\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\forall L:\quad [\boxed{\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)}], (\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\boxed{CNs(m)} \Rightarrow CNs(m)$$

$$\Box L:\quad \boxed{\Box CNs(m)} \Rightarrow CNs(m)$$

$$/L:\quad [\boxed{\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)/CNs(m)}, \Box CNs(m)], (\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\forall L:\quad [\boxed{\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g))}, \Box CNs(m)], (\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\blacksquare L:\quad [\boxed{\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g))}, \Box CNs(m)], (\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb), \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\uparrow R:\quad [\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \Box CNs(m)], 1, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf{\uparrow}((\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb)) \qquad \boxed{Sf} \Rightarrow Sf$$

$$\downarrow L:\quad [\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \Box CNs(m)], \boxed{(Sf{\uparrow}((\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb))){\downarrow}Sf}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\forall L:\quad [\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \Box CNs(m)], \boxed{\forall a((Sf{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sf)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\forall L:\quad [\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \Box CNs(m)], \boxed{\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\blacksquare L:\quad [\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \Box CNs(m)], \boxed{\blacksquare\forall g\forall a((Sg{\uparrow}((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))){\downarrow}Sg)}, \Box(\langle\rangle\exists aNa\backslash Sb) \Rightarrow Sf$$

$$\neg\forall G[(\check{}man\ G) \to (\check{}walk\ G)]$$

Conclusions

- We have implemented a substantial fragment of categorial logic in CatLog3, a type logical parser/theorem-prover comprising 6000 lines of Prolog code.
- CatLog3 uses focalisation to deal with spurious ambiguity, and count-invariance for the full categorial fragment to improve efficiency.
- CatLog3 passes the Montague test which, to our knowledge, no other system has passed.

THANK YOU!