Formally representable functions from $\mathbf{N}^{\mathbf{N}}$ to $\mathbf{N}$

Tatsuji Kawai

School of Information Science
Japan Advanced Institute of Science and Technology

This work is supported by Core-to-Core Program A. Advanced Research Networks by Japan Society for the Promotion of Science.

Continuity on the Baire space $\mathbb{N}^{\mathbb{N}}$

There are two reasonably constructive notions of continuous function from the Baire space to $\mathbb{N}$.

- A function $f\colon\ \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that

$$\begin{tikzcd} \mathbb{N}^{\mathbb{N}} \arrow[r, "i_{\mathcal{B}}", "\cong"'] \arrow[d, "f"'] & \mathrm{Pt}(\mathcal{B}) \arrow[d, "\mathrm{Pt}(r)"] \\ \mathbb{N} \arrow[r, "i_{\mathcal{N}}", "\cong"'] & \mathrm{Pr}(\mathcal{N}) \end{tikzcd}$$

  for some formal topology map $r\colon \mathcal{B} \to \mathcal{N}$, viz **formally representable** function or **FT-continuous** function.
- A function **realized** by some inductively generated neighbourhood function $\alpha\colon\ \mathbb{N}^* \to \mathbb{N}$.

Continuity on the Baire space $\mathbb{N}^\mathbb{N}$

There are two reasonably constructive notions of continuous function from the Baire space to $\mathbb{N}$.

- A function $f\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}$ such that

$$\begin{tikzcd} \mathbb{N}^\mathbb{N} \arrow[r, "i_\mathcal{B}", "\cong"'] \arrow[d, "f"'] & \mathrm{Pt}(\mathcal{B}) \arrow[d, "\mathrm{Pt}(r)"] \\ \mathbb{N} \arrow[r, "i_\mathcal{N}", "\cong"'] & \mathrm{Pr}(\mathcal{N}) \end{tikzcd}$$

  for some formal topology map $r\colon \mathcal{B} \to \mathcal{N}$, viz **formally representable** function or **FT-continuous** function.

- A function **realized** by some inductively generated neighbourhood function $\alpha\colon \mathbb{N}^* \to \mathbb{N}$.

Remark. We work constructively (intuitionistically and predicatively), allowing some constructive choice principles (i.e. countable choice).

Neighbourhood functions $\alpha\colon \mathrm{N}^* \to \mathrm{N}$

The class $K \subseteq \mathbb{N}^* \to \mathbb{N}$ of **neighbourhood functions** is inductively generated by the following clauses:

$$\frac{n \in \mathbb{N}}{\lambda a.n+1 \in K}, \qquad \frac{\alpha(\langle\rangle) = 0 \quad (\forall n \in \mathbb{N})\lambda a.\alpha(\langle n\rangle * a) \in K}{\alpha \in K}.$$

Remark. A neighbourhood function $\alpha \in K$ can be identified with a well-founded tree labelled by $\mathbb{N}$.

1. $\lambda a.n+1$ corresponds to a single node tree $\{(\langle\rangle, n+1)\}$ labelled by $n+1$.
2. if $\alpha(\langle\rangle) = 0$ and for each $n \in \mathbb{N}$, $\lambda a.\alpha(\langle n\rangle * a)$ corresponds to a labelled tree T_n, then α corresponds to a tree $T = \{(\langle\rangle, 0)\} \cup \{(\langle n\rangle * a, L) \mid n \in \mathbb{N}, (a, L) \in T_n\}$.

The leaves of tree corresponding to $\alpha \in K$ determines a bar

$$P_\alpha = \left\{a \in \mathbb{N}^* \mid \alpha(a) > 0 \mathbin{\&} \left(\forall a' \prec a\right) \alpha(a') = 0\right\},$$

so that $\left(\forall \beta \in \mathbb{N}^{\mathbb{N}}\right) (\exists k \in \mathbb{N})\, \overline{\beta}k \in P_\alpha$.

Neighbourhood functions $\alpha\colon \mathrm{N}^* \to \mathrm{N}$

A neighbourhood function $\alpha \in K$ determines a (unique) continuous function $f_\alpha\colon\ \mathbb{N}^\mathbb{N} \to \mathbb{N}$ such that for each $\beta \in \mathbb{N}^\mathbb{N}$

$$f_\alpha(\beta) = \alpha(\overline{\beta}k) - 1$$

where $k \in \mathbb{N}$ is such that $\overline{\beta}k \in P_\alpha$.

A function $f\colon\ \mathbb{N}^\mathbb{N} \to \mathbb{N}$ is **realizable** if $f = f_\alpha$ for some $\alpha \in K$. In this case, we say that α realizes f.

Neighbourhood functions $\alpha\colon \mathrm{N}^* \to \mathrm{N}$

A neighbourhood function $\alpha \in K$ determines a (unique) continuous function $f_\alpha\colon\ \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that for each $\beta \in \mathbb{N}^{\mathbb{N}}$

$$f_\alpha(\beta) = \alpha(\overline{\beta}k) - 1$$

where $k \in \mathbb{N}$ is such that $\overline{\beta}k \in P_\alpha$.

A function $f\colon\ \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is **realizable** if $f = f_\alpha$ for some $\alpha \in K$. In this case, we say that α realizes f.

Proposition

A function $f\colon\ \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is realizable iff there exists $\alpha \in K$ such that for all $a \in P_\alpha$ the composition $f \circ \mathrm{cons_a}$ is constant, where $\mathrm{cons_a}\colon\ \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a mapping $\beta \mapsto a * \beta$.

Proof. ($\Rightarrow$) Obvious.
($\Leftarrow$) If $\alpha \in K$ satisfies the condition, replace all the labels of the leaf nodes $(a, \alpha(a))$ of the tree determined by α with values $f(a * 0^\omega) + 1$. The resulting tree represents $\alpha' \in K$, which realizes f.

Formal topologies

A **formal topology** is a triple $\mathcal{S} = (S, \lhd, \leq)$ where $(S, \leq)$ is a preorder and $\lhd$ is a relation $\lhd \subseteq S \times \mathrm{Pow}(S)$ such that

$$\frac{a \leq b}{a \lhd b}, \quad \frac{a \in U}{a \lhd U}, \quad \frac{a \lhd U \quad U \lhd V}{a \lhd V}, \quad \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V}.$$

$$U \lhd V \overset{\text{def}}{\iff} (\forall a \in U)\, a \lhd V, \qquad U \downarrow V \overset{\text{def}}{=} {\downarrow}U \cap {\downarrow}V$$

A morphism (**formal topology map**) between formal topologies $\mathcal{S}$ to $\mathcal{S}'$ is a relation $r \subseteq S \times S'$ such that

1. $S \lhd r^{-}S'$,
2. $r^{-}\{a\} \downarrow r^{-}\{b\} \lhd r^{-}(a \downarrow b)$,
3. $a \lhd' U \implies r^{-}\{a\} \lhd r^{-}U$

for all $a, b \in S'$ and $U \subseteq S'$. Two formal topology maps $r_1, r_2 : \mathcal{S} \to \mathcal{S}'$ are defined to be equal if

$$\mathcal{A}\, r_1^{-}\{a\} = \mathcal{A}\, r_2^{-}\{a\}$$

for all $a \in S'$, where $\mathcal{A}\, U = \{a \in S \mid a \lhd U\}$.

A **point** of a formal topology $\mathcal{S}$ is a subset $\alpha \subseteq S$ such that

1. $(\exists a \in S)\, a \in \alpha$,
2. $a, b \in \alpha \implies (\exists c \in a \downarrow b)\, c \in \alpha$,
3. $a \lhd' U \;\&\; a \in \alpha \implies (\exists a \in U)\, a \in \alpha$.

$\mathrm{Pt}(\mathcal{S})$ denotes the collection of points of $\mathcal{S}$.

A formal topology map $r : \mathcal{S} \to \mathcal{S}'$ determines a point map $\mathrm{Pt}(r) : \mathrm{Pt}(\mathcal{S}) \to \mathrm{Pt}(\mathcal{S}')$ given by

$$\mathrm{Pt}(r)(\alpha) = r\alpha$$

for all $\alpha \in \mathrm{Pt}(\mathcal{S})$.

FT-continuous functions

Formal Baire space $\mathcal{B} = (\mathbb{N}^*, \lhd_{\mathcal{B}}, \leq)$ is defined by

$$a \leq b \overset{\text{def}}{\iff} b \preceq a$$

and $\lhd_{\mathcal{B}}$ is the smallest covering relation satisfying

$$a \lhd_{\mathcal{B}} \{a * \langle n \rangle \mid n \in \mathbb{N}\}$$

for all $a \in \mathbb{N}^*$.
Formal natural numbers $\mathcal{N}$ is a structure $(\mathbb{N}, \in, =)$.
There are homeomorphisms:

$$i_{\mathcal{B}} : \mathbb{N}^{\mathbb{N}} \to \mathrm{Pt}(\mathcal{B}), \quad \beta \mapsto \left\{\overline{\beta}k \mid k \in \mathbb{N}\right\},$$
$$i_{\mathcal{N}} : \mathbb{N} \to \mathrm{Pt}(\mathcal{N}), \quad n \mapsto \{n\}.$$

A function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is **FT-continuous** if there exists $r : \mathcal{B} \to \mathcal{N}$ such that $f = i_{\mathcal{N}}^{-1} \circ \mathrm{Pt}(r) \circ i_{\mathcal{B}}$.

FT-continuous functions

Proposition

There exists a bijective correspondence between the FT-continuous functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ *and the formal topology maps* $r : \mathcal{B} \to \mathcal{N}$.

Remark. This is an instance of a more general result about complete metric spaces using the technique of localic completion due to S. Vickers.

FT-continuous functions

Define a (new) covering relation $\mathrm{Cov} \subseteq \mathbb{N}^* \times \text{Pow}(\mathbb{N}^*)$ inductively by

$$\frac{}{\{a\} \in \mathrm{Cov}(a)}, \qquad \frac{(\forall n \in \mathbb{N})\, U_n \in \mathrm{Cov}(a * \langle n \rangle)}{\bigcup_{n \in \mathbb{N}} U_n \in \mathrm{Cov}(a)},$$

where $U \in \mathrm{Cov}(a) \overset{\text{def}}{\iff} (a, U) \in \mathrm{Cov}$.

Remark. Cov is a presentation of formal Baire space $\mathcal{B}$, i.e.

$$a \lhd_{\mathcal{B}} U \iff (\exists V \in \mathrm{Cov}(a))\, V \subseteq\downarrow U.$$

Define a (new) covering relation $\mathrm{Cov} \subseteq \mathbb{N}^* \times \mathrm{Pow}(\mathbb{N}^*)$ inductively by

$$\frac{}{\{a\} \in \mathrm{Cov}(a)}, \qquad \frac{(\forall n \in \mathbb{N})\, U_n \in \mathrm{Cov}(a * \langle n \rangle)}{\bigcup_{n \in \mathbb{N}} U_n \in \mathrm{Cov}(a)},$$

where $U \in \mathrm{Cov}(a) \overset{\text{def}}{\Longleftrightarrow} (a, U) \in \mathrm{Cov}$.

Remark. Cov is a presentation of formal Baire space $\mathcal{B}$, i.e.

$$a \lhd_{\mathcal{B}} U \iff (\exists V \in \mathrm{Cov}(a))\, V \subseteq\downarrow U.$$

Proposition

A function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is FT-continuous iff there exists $U \in \mathrm{Cov}(\langle\rangle)$ such that $f \circ \mathrm{cons}_a$ is constant for each $a \in U$.

Proof.

($\Rightarrow$) Use the fact that Cov is a presentation of $\mathcal{B}$.

($\Leftarrow$) Define $r : \mathcal{B} \to \mathcal{N}$ by $a \; r \; n \overset{\text{def}}{\Longleftrightarrow} a \in U \;\&\; f(a * 0^\omega) = n$. $\square$

Lemma

For any $U \in \mathsf{Pow}(\mathbb{N}^*)$, we have

$$U \in \mathrm{Cov}(\langle\rangle) \iff (\exists \alpha \in K)\, P_\alpha = U.$$

Proof. Induction on Cov and K. □

Theorem

A function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is realizable iff f is FT-continuous.

Corollary

There exists a bijective correspondence between the formal topology maps $r : \mathcal{B} \to \mathcal{N}$ and the realizable functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$.

A function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is **realizable** if $\pi_n \circ f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is realizable for each $n \in \mathbb{N}$.

Proposition

There exists a bijective correspondence between the formal topology maps $r : \mathcal{B} \to \mathcal{B}$ and the realizable functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.

Proposition

A function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is realizable iff f is uniformly continuous with respect to the covering uniformity $\mathrm{Cov}(\langle\rangle)$, i.e. for any $V \in \mathrm{Cov}(\langle\rangle)$ there exists $U \in \mathrm{Cov}(\langle\rangle)$ such that

$$(\forall a \in U)\,(\exists b \in V)\left(\forall \beta \in \mathbb{N}^{\mathbb{N}}\right) \beta \in a \implies f(\beta) \in b.$$

References

C. Xu and M. Escardó.
A constructive model of uniform continuity.
volume 7941 of *Lecture Notes in Computer Science*, pages 236–249. Springer, 2013.

G. van der Hoeven and I. Moerdijk.
Sheaf models for choice sequences.
Annals of Pure and Applied Logic, 27(1):63–107, 1984.

E. Palmgren.
A constructive and functorial embedding of locally compact metric spaces into locales.
Topology and its Applications., 154:1854–1880, 2007.