Abstracts of talks at the

Fifth Workshop on Formal Topology: Spreads and Choice Sequences

Institute Mittag-Leffler, Djursholm,
June 8-10, 2015

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INVITED TALKS

The Canonicity of Point-free Topology

Andrej Bauer
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Abstract. Properties of topological spaces are not absolute, as they may depend on the ambient mathematical universe. For instance, the answers to some questions in classical point-set topology hinge on non-standard set-theoretic axioms, while in intuitionistic mathematics even such basic properties as the Heine-Borel compactness of the closed interval depend on the ambient constructive variety.

We show that, in contrast, there is a robust and natural formulation of point-free topology: the category of countably presented topologies (or locales, or formal spaces) is determined up to equivalence by the number-theoretic functions of the ambient framework. As most varieties of constructive mathematics agree that such functions are the Turing computable ones, they all have the same notion of countably-presented point-free spaces.

(This talk presents joint work with Alex Simpson, University of Ljubljana.)

A spatiality-like property for pointfree topologies with a positivity relation

Francesco Ciraulo - University of Padova

It is an unchangeable fact of history that pointfree topology appeared after pointwise one. For this reason the former is "condemned" to compare itself with the latter and, in particular, to consider the case of spatial topologies as a notable case. I will talk about one of the constructive manifestations of the classical notion of spatiality. This has recently been isolated by Sambin [4], who gave it the name of *reducibility*. It was then studied in [1], of which I will present some results. Here are two examples: reducibility for the pointfree version of Cantor space amounts to the Weak König's Lemma; reducibility for the pointfree Baire space states that every element of a (not necessarily decidable) spread belongs to a choice sequence contained in that spread [3]. In general, reducibility is a point existence property. Reducibility was born in the more general context of formal topologies with a positivity relation, also called positive topologies, where it emerged for structural motivations and reasons of symmetry. In [2] we managed to give a "localic" description of the category of positive topologies; in view of that work, reducibility can be now understood as statement about the weakly closed sublocales of the given locale. Classically, reducibility is equivalent to spatiality, at least in the particular case of locales; and assuming either spatiality implies reducibility or the converse yields the full law of excluded middle. In the more general case of positive topologies, instead, even if classical logic is assumed, reducibility remains distinct from, actually weaker than, spatiality. This makes the notion of reducibility potentially interesting also to a classical mathematician.

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The inconsistency of a Brouwerian continuity principle with the Curry-Howard interpretation

Martin Escardó

Abstract. If all functions $(\mathrm{N} \to \mathrm{N}) \to \mathrm{N}$ are continuous then $0 = 1$. We establish this in intensional (and hence also in extensional) intuitionistic dependent-type theories, with existence in the formulation of continuity expressed as a Σ type via the Curry-Howard interpretation. But with an intuitionistic notion of anonymous existence, defined as the propositional truncation of Σ, it is consistent that all such functions are continuous. A model is Johnstone's topological topos. On the other hand, any of these two intuitionistic conceptions of existence give the same, consistent, notion of uniform continuity for functions $(\mathrm{N} \to 2) \to \mathrm{N}$, again valid in the topological topos. It is open whether the consistency of (uniform) continuity extends to homotopy type theory. The theorems of type theory informally proved here are also formally proved in Agda, but the development presented here is self-contained and doesn't show Agda code.

Sheaf models for Introspection

Michael Fourman
Edinburgh University

Abstract. Our aim is to present a formal representation of Brouwer's development of intuitionistic analysis, "exceeding the frontiers of classical mathematics".

We argue that simple sheaf models provide a faithful representation of salient aspects of Brouwer's arguments. Examples include Brouwer's use of fleeing properties to provide counter-examples to classical truths, and Brouwer's use of choice sequences to derive the Fan Theorem and Bar Induction.

Universal fibrations and univalence

Ieke Moerdijk
Radboud University, Nijmegen

Abstract. I will discuss various notions of being "universal" for fibrations in the category of simplicial sets, and show how to get the univalence property for free.

Choice Sequences and Their Uses

Joan R. Moschovakis
Occidental College

Abstract. Brouwer's (destructive) "First Act of Intuitionism" questioned the universal applicability of the classical laws of double negation and excluded third. The resulting limitation to intuitionistic (constructive) reasoning made possible – and was justified by – Brouwer's "Second Act of Intuitionism" which accepted arbitrary choice sequences of natural numbers as legitimate mathematical objects, and required every function defined on all choice sequences to be continuous in the initial segment topology.

In the 20th century Heyting, Kleene, Vesley, Kreisel, Troelstra, and others clarified Brouwer's intuitionistic logic and mathematics by means of formal axiomatic systems; finally choice sequences could be compared with classical number-theoretic functions, and Brouwer's universal spread with classical Baire space. We explain this development, with the advantages of considering Brouwer's choice sequences as individual objects in the process of generation, spreads as structured sets, and species as extensional properties.

Choice Sequences vs Formal Topology

Thomas Streicher
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Abstract. It is known from the beginning of Formal Topology that in presence of Bar Induction (BI) sufficiently many formal spaces have enough points. Classically BI is equivalent to Dependent Choices. Thus BI introduces the (not not) existence of noncomputable elements in Baire space aka choice sequences. We discuss models of FIM = BI + Continuity, in particular function realizability. By a recent result of Escardo and Xu Continuity is incompatible even with ITT. Finally, we recall a theorem of Fourman which allows one to construct a gros topos over a small category of formal spaces which validates FIM. One may reason in this model and then use Kripke-Joyal to turn the results into Formal Topology terms. Thus, the illusion of points can be preserved even when working in Type Theory.

Kripke's Schema, transfinite proofs, and Troelstra's Paradox

Mark van Atten

Abstract. According to Brouwer, truth is experienced truth. Assumptions then have epistemic import: To assume that p is true is to assume that the subject has experienced its truth. This makes a difference for the way in which principles that involve an assumption can be justified. In this talk, I will discuss Kripke's Schema (KS) in this Brouwerian setting.

In the first part, I will give some examples of applications of KS in analysis, and go through an argument in its favour.

In the second part, I will try to meet some objections to KS that have been voiced by Kreisel, Veldman, and Vesley; in particular the objection that there is an incompatibility between (1) the idea, embodied in KS, that mathematical evidence comes in an ω-ordering, and (2) Brouwer's acceptance of transfinite (fully analysed) proofs.

In the third part, I argue that even though KS does not entail the theory of the Creating Subject (CS), in a Brouwerian setting objections to CS also undermine KS. The most prominent problem for CS is Troelstra's Paradox. As is well known, the construction of that paradox depends on the acceptability of a certain impredicativity; I argue that it moreover depends on Markov's Rule, and is therefore less threatening than it may seem.

The Almost-Fan Theorem

Wim Veldman

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Let A be a decidable subset of the set $\mathbb{N}$ of the natural numbers. A is *finite* if and only if there exists n such that, for all m in A, $m \leq n$. A is *almost-finite* if and only if, for every infinite strictly increasing sequence ζ of natural numbers there exists n such that $\zeta(n) \notin A$.

Every s in $\mathbb{N}$ codes a finite sequence of natural numbers. A function β from $\mathbb{N}$ to $\{0, 1\}$ is a *spread-law* if and only if, for each s, $\beta(s) = 0$ if and only if $\exists n[\beta(s * \langle n \rangle) = 0]$. Let $\mathcal{F}$ be a subset of the set $\mathcal{N}$ of all infinite sequences of natural numbers. $\mathcal{F}$ is a *spread* if and only if there exists a spread-law β such that, for all α in $\mathcal{N}$, $\alpha \in \mathcal{F}$ if and only if $\forall n[\beta(\overline{\alpha}n) = 0]$. $\mathcal{F}$ is a *finitary spread* or a *fan* if the spread-law β satisfies the additional condition: for each s, the set $\{n | \beta(s * \langle n \rangle) = 0\}$ is finite. $\mathcal{F}$ is an *almost-fan* if and only if the spread-law σ satisfies the additional condition: for each s, the set $\{n | \beta(s * \langle n \rangle) = 0\}$ is almost-finite.

Let $\mathcal{F}$ be a subset of $\mathcal{N}$ and let B be a subset of $\mathbb{N}$. B is a *bar in* $\mathcal{F}$ if and only if $\forall \alpha \in \mathcal{F} \exists n[\overline{\alpha}n \in B]$. ($\overline{\alpha}n$ denotes the initial part $\big(\alpha(0), \alpha(1), \ldots, \alpha(n-1)\big)$ of α.) B is *thin* if and only if, for all s, t in B, if $s \neq t$, then $s \perp t$, that is: s is, as a finite sequence, not an initial part of t and neither is t an initial part of s.

The *Fan Theorem* **FT** is the statement:

> Let $\mathcal{F}$ be a fan. Every decidable subset of $\mathbb{N}$ that is thin and a bar in $\mathcal{F}$ is a finite subset of $\mathbb{N}$.

The *Almost-Fan Theorem* **AFT** is the statement:

> Let $\mathcal{F}$ be an almost-fan. Every decidable subset of $\mathbb{N}$ that is thin and a bar in $\mathcal{F}$ is an almost-finite subset of $\mathbb{N}$.

Both theorems follow from Brouwer's Thesis on Bars in $\mathcal{N}$. The Almost-Fan Theorem implies the Fan Theorem. The relation between **FT** and **AFT** may be compared to the relation between *Weak König's Lemma* **WKL** and *König's Lemma* **KL** in classical Reverse Mathematics. The Principle of Open Induction on Cantor space follows from **AFT** and implies **FT**.

We consider some consequences and equivalents both of the Almost-Fan Theorem and of the Principle of Open Induction on Cantor space.

A coherent account of geometricity

Steve Vickers
University of Birmingham

Abstract. I shall present a coherence issue that arises on the trail of fibrewise topology, or a dependent type theory of spaces (always point-free). The results can be applied to space constructions such as powerlocales (hyperspaces) and valuation locales. At a certain point the treatment is impredicative, but I conjecture that this can be circumvented.

In topos theory, internal spaces and reindexing along continuous maps (geometric morphisms) are equivalent to bundles and pullback; and if a map is viewed as a generalized point then the bundle pullback is a generalized fibre. From the point of view of dependent types, we are therefore interested in "geometric" constructions of spaces, which are preserved - up to isomorphism - by reindexing and hence work fibrewise. The present work provides sufficient conditions for the coherence of those isomorphisms.

The first part of the argument is predicative, and depends on a careful analysis of the known techniques of working geometrically on presentations of spaces (or formal topologies). It uses the essentially algebraic theory of arithmetic universes as framework for presenting each space construction in a uniform, generic way, instantiated over any base by substitution.

The second part is still impredicative, and requires a condition that the construction should, on morphisms between presentations, preserve the property of inducing a homeomorphism between the spaces.

CONTRIBUTED TALKS

Formally representable functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

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A function $F : \mathcal{P}t(\mathcal{S}) \to \mathcal{P}t(\mathcal{S}')$ between the formal spaces of formal topologies $\mathcal{S}$ and $\mathcal{S}'$ is said to be *formally representable* if there exists a morphism $r : \mathcal{S} \to \mathcal{S}'$ such that $F = \mathcal{P}t(r)$, where $\mathcal{P}t \colon \mathbf{FTop} \to \mathbf{Top}$ is the right adjoint of the standard adjunction between the category of topological spaces **Top** and that of formal topologies **FTop**.

Here, we are interested in a particular case where $\mathcal{S}$ is the formal Baire space $\mathcal{B}$ and $\mathcal{S}'$ is the formal discrete space $\mathcal{N}$ of natural numbers. Identifying $\mathcal{P}t(\mathcal{B})$ with $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{P}t(\mathcal{N})$ with $\mathbb{N}$ by the axiom of unique choice, we ask what kind of function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is formally representable. Classically, the monotone bar induction, which is equivalent to the spatiality of $\mathcal{B}$, implies that the formally representable functions are exactly the pointwise continuous functions. Constructively, this need not be the case since the statement that every pointwise continuous function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is formally representable is equivalent to a version of bar induction stronger than the bar induction for decidable bars [1].

In this talk, we show that a function $F \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is formally representable if and only if F is realised by some neighbourhood function $\alpha \colon \mathbb{N}^* \to \mathbb{N}$, i.e.

$$(\forall \gamma \in \mathbb{N}^{\mathbb{N}})\,(\exists n \in \mathbb{N})\,\alpha(\langle \gamma(0), \ldots, \gamma(n-1) \rangle) = F(\gamma) + 1.$$

By a *neighbourhood function*, we mean an element of the class $K \subseteq \mathbb{N}^{\mathbb{N}^*}$ of inductively generated neighbourhood functions [2, Chapter 4, Section 8.4].

We work in Bishop constructive mathematics with the axiom of countable choice and generalised inductive definitions which have rules with countable premises.

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Duality and Monoidal Structures on Sambin's Basic Pairs

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We aim at examining duality and monoidal structures on basic pairs, which were introduced by Sambin to lay down a foundation for formal topology, a predicative form of point-free topology or locale theory.

Basic pairs have internal dualities inherent in them, and in this talk, we externalise them in the form of duality between categories (as opposed to duality in categories), thereby deriving different dualities between (topological, convex, and measurable) point-set spaces and point-free spaces from the internal dualities of basic pairs.

Moreover, the duality structure on basic pairs leads us to taking into account a monoidal structure on the category of basic pairs. The monoidal category of basic pairs, then, turns out to form a model of what is called categorical quantum mechanics (which was introduced by Abramsky and Coecke to account for the substructural logic of quantum mechanics).

A Marriage of Brouwer's Intuitionism and Hilbert's Finitism

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In this talk, we consider the consistency strength of various Brouwerian axioms. We show that the system consists of the following axioms over intuitionistic logic still has the same proof theoretic strength as PRA, which is considered as a formalization of Hilbert's finitism (cf. [4]).

1. Basic arithmetic
2. Fan theorem for (not necessarily binary) decidable fan with arbitrary bar
3. Bar induction for Π^0_1 property
4. Generalized continuity principle
5. Axiom of choice
6. Σ^0_2 induction (not only Σ^0_1-induction)

We show that the above system together with Markov's Principle can be interpreted into a system of second order arithmetic with the same consistency strength as PRA via realizability interpretation based on continuous function application. We also consider which combination of axioms lifts up the consistency strength. We show

- $1+2+3+5+6+$ "4 with uniqueness condition" $+$ LLPO has the same consistency strength as PRA (via Lifschitz functional realizability interpretation introduced by [2]), while $1+4+$ LLPO is inconsistent (cf. [2]);
- $1+$ "2 restricted to Δ^0_0 bar" $+$ LPO has the same consistency strength as PA (via negative translation and the method used in [3, Theorem III.7.2]);
- $1+$ "2 restricted to binary fan with Π^0_1 bar" $+$ LPO has the same consistency strength as PA (by extending the forcing method in [?] to second order forcing relation of higher complexity);
- $1+\Pi^0_2$ induction over intuitionistic has the same consistency strength as the one over classical logic (via forcing interpretation used in [?]).

This is a joint work with Kentaro Sato.

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WEAK FACTORIZATION SYSTEMS FOR INTENSIONAL TYPE THEORY

PAIGE NORTH

In this talk, I will characterize the weak factorization systems which can serve to interpret intensional type theory.

In 1984, Per Martin-Löf introduced his intensional type theory [ML84], in which the notion of equality was bifurcated into *definitional* equality and *propositional* equality. Only a decade later, with Hofmann and Streicher's groupoid interpretation of type theory [HS96], did it become clear that these two notions of equality do not coincide. While definitional equality is a judgement as to whether two things are equal, as equality usually is, propositional equality is represented by a type, the *identity type*, which Hofmann and Streicher showed can possess much more information. In his thesis [War08], Michael Warren showed how the rules governing this identity type produce a weak factorization system and that this weak factorization system bears much similarity to the classical Hurewicz weak factorization system on the category of topological spaces. Thus the connection between intensional type theory and homotopy theory became clear.

In their paper, van den Berg and Garner [BG12] described algebraic conditions on an endofunctor of a category which enable it to serve as an interpretation of the identity type. In this talk, I will describe the weak factorization systems that can give rise to such an endofunctor, thus characterizing the weak factorization systems that can interpret intensional type theory. In fact, they are exactly those in which (1) every object is fibrant and (2) the left class of maps is stable under pullback along the right class. I will also talk about internal categories and presheaves in such a category, and under which conditions they themselves form a category that can interpret intensional type theory.

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Date: 15 March 2015.

The Cantor space as a Bishop space

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Roughly speaking, in topology with points there are two ways to determine the continuous functions between sets X and Y. The first is to start from some common space-structure on X and Y that determines a posteriori which functions $X \to Y$ are continuous with respect to it, while the second is to start from a given class C of "continuous" functions $X \to Y$ that determines a posteriori a common space-structure on X and Y that corresponds to C. One could say that Brouwer used the second way for $X = 2^{\mathbb{N}}$ and $Y = \mathbb{N}$, and the first for $X = Y = \mathbb{R}$. The theory of Bishop spaces (TBS) is an approach to constructive topology with points which is based on the second way to determine continuity. In [1] Bishop introduced function spaces, here called Bishop spaces, without really exploring them. In [2] Bridges revived the subject, in [3] Ishihara studied a subcategory of the category of Bishop spaces, while in [4]-[7] we try to develop TBS.

A *Bishop space* is a structure $\mathcal{F} = (X, F)$, where the *topology* F of functions on X is a subset of the real-valued functions $\mathbb{F}(X)$ on X, which includes the constant functions and it is closed under addition, uniform limits and composition with the *Bishop-continuous* functions $\mathrm{Bic}(\mathbb{R})$ i.e., the functions $\mathbb{R} \to \mathbb{R}$ which are uniformly continuous on every bounded subset of the constructive reals $\mathbb{R}$. Since these definitional clauses can be seen as inductive rules, one can talk about the least Bishop space $\mathcal{F}(F_0)$ including a given set $F_0 \subseteq \mathbb{F}(X)$. As a special case of the product Bishop topology, the *Cantor topology* $\bigvee_{n\in\mathbb{N}} \pi_n = \mathcal{F}(\{\pi_n \mid n \in \mathbb{N}\})$ is the least topology on $2^{\mathbb{N}}$ including the projections.

Our aim is to present results on the Cantor space, seen as the Bishop space $(2^{\mathbb{N}}, \bigvee_{n\in\mathbb{N}} \pi_n)$, that are related to intuitionistic analysis. If ρ is the standard metric on $2^{\mathbb{N}}$ and $C_u(2^{\mathbb{N}})$ denotes the set of uniformly continuous real-valued functions on $2^{\mathbb{N}}$, we show that $\bigvee_{n\in\mathbb{N}} \pi_n \subseteq C_u(2^{\mathbb{N}})$, therefore the Fan theorem holds for the elements of $\bigvee_{n\in\mathbb{N}} \pi_n$. Moreover, $\bigvee_{n\in\mathbb{N}} \pi_n$ includes the set $\{\rho_\alpha \mid \alpha \in 2^{\mathbb{N}}\}$, where $\rho_\alpha(\beta) = \rho(\alpha, \beta)$, for every $\beta \in 2^{\mathbb{N}}$. Consequently, $\bigvee_{n\in\mathbb{N}} \pi_n = C_u(2^{\mathbb{N}})$ and, most importantly, a compact metric space endowed with the topology of the uniformly continuous functions is a 2-compact Bishop space. We call a Bishop space $\mathcal{F}$ *2-compact*, if there is an epimorphism from some Bishop product 2^I to $\mathcal{F}$. In this way 2-compactness generalizes the notion of a compact metric space and seems to be an appropriate function-theoretic notion of compactness for TBS. All our proofs are within Bishop's informal system of constructive mathematics BISH.

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Formal Topology, Domains and Finite Density

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Abstract. The connections between formal topology and domains have been thoroughly investigated in the literature. In particular, unary formal topologies represent a natural way to describe constructively algebraic cpo's. In the most recent formulation of formal topology, convergence is modelled by means of a multi-valued operation. By tuning the properties of such operation, we first obtain natural constructive presentation of Scott and bifinite domains. This setting allows us, by means of an appropriate hierarchy of hereditarily total functionals together with an abstract notion of totality, to do a proof of the Kleene-Kreisel-Berger Density Theorem. In the context of non-flat domains, moreover, this proof can be done with finite methods and without points.

Minimal Criminals and Minimal Logic

Peter M. Schuster
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Abstract. In many a proof by contradiction of a universal statement the existence of a minimal counterexample is proved by invoking Zorn's Lemma. Whenever minimal logic suffices for establishing the hypotheses, we can reread the proof in a systematic way as a direct and inductive proof with Raoult's principle of Open Induction. As the extremal elements disappear, we thus eliminate the corresponding ideal objects or points of spaces. Our result further opens up the road to extracting the computational content from the classical proofs we have started with. We go beyond earlier work also inasmuch as our new method does not require the presence of a binary operation.

Cubical sets as a classifying topos*

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Abstract

Coquand's cubical set model for homotopy type theory provides the basis for a computational interpretation of the univalence axiom and some higher inductive types, as implementated in the *cubical* proof assistant. We show that the underlying cube category is the opposite of the Lawvere theory of De Morgan algebras. The topos of cubical sets itself classifies of the theory of 'free De Morgan algebras'. This provides us with a topos with an internal 'interval'. Using this we construct a model of type theory following van den Berg and Garner. However, it is possible that the interval can also be used to construct other models.

Introduction

Kreisel and Troelstra's elimination translation [KT70] provides an explanation of choice sequences. A semantic, topos theoretic, presentation of this technique was developed in the eighties [Fou84, vdHM84] and has found renewed interest [Fou13, XE13] recently. One goal of homotopy type theory [Uni13] is to serve as an internal language for elementary higher toposes, a theory that this currently being developed by a number of researchers. Elementary higher toposes should be a natural home for the development of spread models of type theory, as it is a natural place to study sheaf models of homotopy type theory. In this abstract we use tools from topos theory to present the cubical set model.

The topos of cubical sets

Simplicial sets from a standard framework for homotopy theory. The topos of simplicial sets is the classifying topos of the theory of strict linear orders with endpoints. Cubical sets turn out to be more amenable to a constructive treatment of homotopy type theory. Grandis and Mauri [GM03] describe the classifying theories for several cubical sets without diagonals. We consider the most recent cubical set model [Coq15]. This consists of symmetric cubical sets with connections $(\wedge, \vee)$, reversions $(\neg)$ and diagonals. Let $\mathbb{F}$ be the category of finite sets with all maps. Consider the monad DM on $\mathbb{F}$ which assigns to each finite set F the *finite* set of the free DM-algebra on F. That this set is finite can be seen using the disjunctive normal form. The *cube category* in [Coq15] is the Kleisli category for the monad DM.

Lawvere theory Recall that for each algebraic (=finite product) theory T, the Lawvere theory $C_{fp}[T]$ is the opposite of the category of free finitely generated models. This is the classifying category for T: models of T in any finite product category category E correspond to product-preserving functors $C_{fp}[T] \to E$. The Kleisli category KL_{DM} is precisely the *opposite* of the Lawvere theory for DM-algebras: maps $I \to DM(J)$ are equivalent to DM-maps $DM(I) \to DM(J)$ since each such DM-map is completely determined by its behavior on the atoms, as $DM(I)$ is free.

*I gratefully acknowledge the support of the Air Force Office of Scientific Research through MURI grant FA9550-15-1-0053. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the AFOSR.

Classifying topos To obtain the classifying topos for an algebraic theory, we first need to complete with finite limits, i.e. to consider the category C_{fl} as the *opposite* of finitely *presented* DM-algebras. Then $C_{fl}^{op} \to Set$, i.e. functors on finitely presented T-algebras, is the classifying topos. This topos contains a generic T-algebra M. T-algebras in any topos $\mathcal{F}$ correspond to *left exact left adjoint* functors from the classifying topos to $\mathcal{F}$.

Let FG be the category of *free finitely generated* DM-algebras and let FP the category of *finitely presented* ones. We have a fully faithful functor $f : FG \to FP$. This gives a geometric morphism ϕ between the functor toposes. Since f is fully faithful, ϕ is an embedding.

The subtopos Set^{FG} of the classifying topos for DM-algebras is given by a quotient theory, the theory of the model $\phi^* M$. This model is given by pullback and thus is equivalent to the canonical DM-algebra $\mathbb{I}(m) := m$ for each $m \in FG$. So cubical sets are the classifying topos for 'free DM-algebras'. Each finitely generate DM-algebra has the disjunction property and is strict, $0 \neq 1$. These properties are geometric and hence also hold for $\mathbb{I}$. This disjunction property is important in the implementation [Coq15, 3.1].

This result can be generalized to related algebraic structures, e.g. Kleene algebras. A Kleene algebra is a DM-algebra with the property for all x, y, $x \wedge \neg x \leq y \vee \neg y$. With Coquand we checked that free finitely generated Kleene algebras also have the disjunction property.

Model of type theory

Coquand's presentation of the cubical model does not depend on a general categorical framework for constructing models of type theory. Docherty [Doc14] presents a model on cubical sets with connections using the general theory of path object categories [vdBG12]. The precise relation with the model in [BCH14] is left open. We present a slightly different construction using similar tools, but combined with internal reasoning, starting from the observation [Coq15] that $\mathbb{I}$ represents the interval. To obtain a model of type theory on a category C it suffices to provide an involutive 'Moore path' category object on C with certain properties. Now, category objects on cubical sets are categories in that topos. The Moore path category MX consists of lists of composable paths $\mathbb{I} \to X$ with the zero-length paths e_x as left and right identity. To obtain a *nice* path object category, we quotient by the relation which identifies constant paths of any length. The reversion $\neg$ on $\mathbb{I}$ allows us to reverse paths of length 1. This reversion extends to paths of any length. We obtain an involutive category: Moore paths provide strictly associative composition, but non-strict inverses.

A path contraction is a map $MX \to MMX$ which maps a path p to a path from p to the constant path on tp (t for target). Like Docherty, we use connections to first define the map from $X^{\mathbb{I}}$ to $X^{\mathbb{I}\times\mathbb{I}}$ by $\lambda p.\lambda ij.p(i \vee j)$ and then extended this to a contraction. All these constructions are algebraic and hence work functorially. We obtain a nice path object category.

We have obtained a model of type theory [vdBG12, Doc14] starting from the interval $\mathbb{I}$ in the cubical model, we plan to compare this more carefully with the one in [Coq15]. Finally, like in Voevodsky's HTS [Voe13], we define intensional identity types inside the extensional type theory of a topos. One wonders whether HTS can be interpreted in the present model. Like the presentation above, the topos simplicial sets, on which HTS is based, carries an generic interval object with reversion and a sup operation. It appears that much of the models construction above carries over.

Acknowledgements I would like to thank Awodey and Coquand for discussions on the topic of this paper. Independently, Awodey showed that Cartesian cubical sets (without connections or reversions) classify strictly bipointed objects.

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OTHER TALKS

TBA

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A minimalist predicative generalization of elementary toposes

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Abstract. We propose a generalization of the notion of Lawvere-Tierney's elementary topos where the axiom of unique choice is not necessarily valid.

The outcome is a variant of Moerdijk-Palmgren-Van Den Berg's notion of predicative topos and it has an elementary topos as an example.

We name "minimalist elementary topos" such a notion since its definition is inspired by the design of the two-level Minimalist Foundation (for short MF), ideated with G. Sambin in [5] and completed in [1]. An example is provided by the quotient model used to interpret one level of MF into the other and analyzed categorically in [2],[3],[4].

As a consequence it shares the same properties of the Minimalist Foundation reported in [6] in allowing

1. a definition of "choice sequence" distinct from that of "lawlike sequence";

2. a definition of boolean minimalist topos which is predicative;

3. examples where both Cauchy reals and also Dedekind reals do not form a set but only proper collections of choice sequences whose topology must the be defined in point-free term by using formal topology;

4. examples whose internal logic validates both Bar Induction for choice sequences and formal Church thesis for lawlike sequences.

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Two Pictures of a Continuum

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Abstract. The usual picture of a topological space, based on Venn diagrams and utilized in set-theoretic topology, will be compared with the picture of a spread, which is Brouwer's substitute for the notion of a topological space, and a type-theoretic formalization of the notion of a spread will then be given.

Spreads and choice sequences in the minimalist foundation. Spatial intuition and computational interpretation reconciled

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Abstract. In a dynamic, evolutionary approach to philosophy of mathematics, which I proposed to call dynamic constructivism, one is not so much interested in what mathematical entities are, as in why and how we construct them, how we communicate them, to which domains we can fruitfully apply them.

The formal counterpart of dynamic constructivism is a foundational system, developed in collaboration with Milly Maietti and called minimalist foundation MF, which acts as a framework in which all notions and all foundational conceptions can be formulated and can interact dynamically. In particular, MF allows us to introduce the two notions of function (total singlevalued relation) and operation (hypothetical construction of elements, or indexed family of elements, or its λ-abstraction) and keep them distinct. This means that the so called axiom of unique choice $\mathtt{AC!}$ is not valid in MF. In turn, this means conceiving $\exists x \varphi(x)$ true when we have a guarantee that a witness c can eventually be found, also when no operation providing it is available.

My claim is that a choice sequence is nothing but a function from $\mathbb{N}$ to $\mathbb{N}$, that is a relation φ such that $\forall n \exists! m\, \varphi(n, m)$, as opposed to a sequence $a_n \in \mathbb{N}$ ($n \in \mathbb{N}$) given by an explicit law. Assuming $\mathtt{AC!}$ to be valid means, conversely, that the distinction between operations and functions is essentially lost, either because one considers only functions (as in the classical approach) or because one restricts application of mathematics to operations and their graphs.

The pointfree approach to topology seems essential to give a general mathematical status to these ideas. Pointfree topology represents the real, computational side of mathematics. In particular, spreads are surprisingly well expressed as inhabited formal closed subsets, in the definition of positive topology (a positive topology is, roughly speaking, a formal topology in which the positivity predicate is replaced by a positivity relation).

Ideal aspects are then expressed through the notion of ideal point of a positive topology. Ideal points of Baire positive topology on $List(\mathbb{N})$ turn out to coincide with functions from $\mathbb{N}$ to $\mathbb{N}$, i.e. choice sequences. So an ideal point belonging to a given formal closed subset in Baire positive topology becomes a precise, definite expression for a choice sequence to stay in a given spread.

In this set up, Bar Induction $\mathtt{BI}$ is just an equivalent formulation of spatiality of Baire positive topology. So $\mathtt{BI}$ is a specific example of a general property of positive topologies. The very nature of choice sequences says that also the constructive dual to spatiality, called reducibility, should be valid in Baire positive topology. In fact, it amounts to Spread Habitation $\mathtt{SH}$, which says that every spread is inhabited by a choice sequence.

Both BI and SH are perfectly precise and clear mathematical statements. Both are intuitively obvious, but from the perspective of MF and positive topology one would not expect them to be provable. This situation suggests that a simple way out is to prove meta-mathematically that such ideal principles are conservative over real, pointfree topology; intuitively, whatever one can prove on pointfree topology passing through choice sequences and using BI or SH, one can also prove without. Trying to prove conservativity of BI and SH is work in progress.

This reconstruction of the notions of spread and choice sequence is in my opinion faithful to Brouwer's ideas. It is mathematically well-defined and technically elementary; the price is a little change in the foundational attitude of constructivism. The benefit is that it opens the possibility of reconciling Brouwer's still debated assumptions on the continuum with the successful computational approaches by Bishop and Martin-Löf, without any modification of the latter. Suggestively, our spatial intuitions as human beings are totally reliable, as long as they do not destroy computational content.