Research in Mathematical Logic at the Department of Mathematics, Stockholm University

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Research in the Logic Group

- Systems for foundations of constructive mathematics:
    - Type theory, especially Martin-Löf type theory (MLTT) and Univalent type theory
    - Constructive set theory
    - Computer supported formalization (Coq, Agda)
- Model theory and proof theory of constructive systems
    - homotopy-theoretic models of MLTT
    - internal models of MLTT
    - categorical logic and model theory
- Constructive Mathematics
    - pointfree methods in topology

The Logic Group

Professors

- Per Martin-Löf, professor emeritus
- Erik Palmgren, professor

Assistant professors and postdocs

- Guillaume Brunerie, Postdoc
- Peter LeFanu Lumsdaine, assistant professor
- Anders Mörtberg, assistant professor

PhD students

- Daniel Ahlsén
- Menno de Boer
- Johan Lindberg
- Anna Giulia Montaruli

What is Constructive Mathematics?

Mathematical existence

Students starting to study mathematics at a more advanced level are often perplexed to learn that proofs of existence of a solution to some problem may not actually construct or present an explicit solution.

A well worn (and slightly contrived) example is the following:

Theorem: there are two irrational numbers a and b such that a^b is rational.

Proof: $\sqrt{2}$ is irrational. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we are done with $a = b = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is irrational, then

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\sqrt{2})} = \sqrt{2}^2 = 2$$

is rational. In this case, we may take $a = \sqrt{2}$, and $b = \sqrt{2}^{\sqrt{2}}$.

But which one is our b?

Programming and constructive mathematics

In programming tasks (e.g. numerical calculations) we demand more of our solutions, they should not only be explicit but also (effectively) computable.

Is it possible to sharpen the demands on mathematical existence so that that existence proofs actually provide computable or constructive solutions?

Yes! (L.E.J.Brouwer 1908) Proofs in mathematics can often avoid assuming the principle of excluded middle

$$A \text{ or not } A$$

which was the source of the confusion about b above. (Namely we took A to be $\sqrt{2}^{\sqrt{2}}$ is rational).

The Brouwer-Heyting-Kolmogorov (BHK) interpretation of logical statements

- p is a proof of $A \vee B$ precisely when either $p = \mathrm{left}(a)$ and a is a proof of A or $p = \mathrm{right}(b)$ and b is a proof of B,
- p is a proof of $A \wedge B$ precisely when $p = (a, b)$ and a is a proof of A and b is a proof of B,
- p is a proof of $\exists x : S, A(x)$ precisely when $p = (t, a)$, $t \in S$ and a is a proof of $A(t)$,
- p is a proof of $A \longrightarrow B$ precisely when p is a method (algorithm) for transforming a proof a of A, to a proof $p(a)$ of B,
- p is a proof of $\forall x : S, A(x)$ precisely when p is a method (algorithm) for assigning each element $t \in S$, a proof $p(t)$ of $A(t)$.

  The BHK-interpretation guarantees honest existence proof:. What exists, can be constructed (in principle). But: The Law of Excluded Middle is not true under BHK.

All laws of intuitionistic ("constructive") logic are however true under the BHK-interpretation.

Constructive mathematics is loosely speaking mathematics carried out with intuitionistic logic and constructive set axioms.

Some methods of classical mathematics involving strong uses of the axiom of choice and Zorn's lemma does not work at all. These have to be systematically replaced.

Point-free methods in:

- Topology: focus development on the lattice of open sets rather than points. Generate open covers constructively to obtain useful notion of compactness.
- Commutative algebra: Avoid prime ideals, try to reformulate using sheaf-theoretic methods.
- Measure theory: focus on measure algebras of rather than sets of points.

The BHK-interpretation is the basis for further connections between constructive proofs and computation.

H.B. Curry (1958), W.A. Howard (1968), P. Martin-Löf (1972) showed that proofs in logic and set theory could be viewed as computer programs (more precisely λ-terms of type theory).

Martin-Löf type theory is now part of the foundations for proof assistants such as Agda and Coq.

Study of formal systems for constructive mathematics

Dependent type theory (aka Martin-Löf type theory):

- proof-theoretic questions: decidability of typing, normalization of terms
- models and independence of axioms
- nonstandard extensions: univalence axiom and higher inductive types. Connections to homotopy theory.
- Finding the internal logic of ∞-toposes.

Constructive set theory: set theory based on intuitionistic logic and predicative set existence axioms (roughly: function spaces instead of power sets, restricted separation, and only countable axiom of choice). Similar questions as for type theory.

Algorithm extraction from (constructive) proofs

Given a proof (in a suitable system, e.g. Peano Arithmetic) that there are infinitely many primes

$$\frac{\ldots \quad \ldots}{\forall x \exists y (y > x \,\&\, \mathrm{Prime}(y))}$$

A program or algorithm f may be extracted from the proof, which is such that it on input x, finds a prime number $f(x)$ greater than x:

$$\frac{\ldots \quad \ldots}{p : \forall x (f(x) > x \,\&\, \mathrm{Prime}(f(x)))}$$

Remark: The standard proof of Euclid can easily be made constructive.

Applications of type theories in formalization of mathematics

The type theory of Martin-Löf (MLTT) is currently used in proof assistant systems (Agda, Coq) for formalizing huge proofs in mathematics, and in verifications of computer systems. Commonly cited examples are:

- The Appel–Haken four-color theorem (Gonthier, Werner 2005)
- The Feit–Thompson theorem (Gonthier et al. 2012): every finite group of odd order is solvable.
- verified C compiler to specific hardware (CompCert project)

More modest formalization projects are currently done in the Logic Group, revolving around models of type theory and set theory.

Category theory and categorical logic

A category is an abstraction on the notion of a class of mathematical structures: it is given by its objects and its morphisms (= structure preserving notion of map) between such.

Numerous close connection between category theory, logic and type theory are known since long, and further are waiting to be discovered.

- Categories and deductive systems.
- Cartesian closed categories and typed λ-calculus (can be made into an equivalence) — Lambek.
- Elementary toposes and higher order logic (Lawvere and Tierney).
- Locally cartesian closed categories and Martin-Löf type theory (more problematic, needs fibered categories to make sense)

Logic Group Activities

The Logic Group is currently investigating:

- Erik Palmgren: models of MLTT and their internalization, constructive mathematics, categorical logic.
- Peter LeFanu Lumsdaine: model theory of MLTT, homotopy-theoretic models, higher categories and type theory
- Anders Mörtberg: MLTT and cubical type theory. Synthetic homotopy theory.
- Guillaume Brunerie: Synthetic homotopy theory. Homotopy theory and proof theory formalized in type theory.

Stockholm Logic Seminar: logic.math.su.se/seminar/

Education in Mathematical Logic

The logic group offers yearly one basic undergraduate course and at least two advanced level courses in mathematical logic, and usually also some PhD-level course. Advanced courses are given on a rotating scheme

- Logic II (every year)
- Theory for Computation and Formal Languages
- Set Theory and Forcing
- Proof theory
- Type theory
- Model theory

PhD-level courses (examples): Homotopy Type Theory, Category Theory, Realizability, Logics for Linguistics, Advanced topics in proof theory and foundations

Education in Logic at Stockholm University

In cooperation with the Department of Philosophy we offer a joint BSc program in Logic, Philosophy and Mathematics.
The program includes, in addition to basic courses in mathematics, philosophy, and mathematical logic, also more advanced courses in philosophical logic and logic for computer science and AI.

Master level studies: combining advanced level courses from both departments it is possible to obtain a broad master level education in logic. The master exam can be either in mathematics or philosophy, depending of the composition.

PhD-level studies: Dept. of Mathematics (Prof. Erik Palmgren), Dept. of Philosophy (Prof. Valentin Goranko, Prof. Dag Westerståhl).