Ottawa talk (Oct 31, 2015)

New title for changed contents:

Sketch-theoretic reformulations of Michael Batanin's concept of weak higher category and its variants

1. Sketches. Charles Ehresmann's concept of sketch deserves more attention than has been accorded it. The big bang in the theory of sketches was Christian Lair's 1981 theorem that says: “a category is sketchable iff it is accessible”: any accessible category (one that is equivalent, for some cardinal kappa, to the category of Set-valued kappa-filtered colimit-preserving functors on a small category) is equivalent to the category of models of a small (Ehresmann) small-colimit-small-limit sketch –-- and conversely. With Bob Pare, we rediscovered Lair theorem in 1985, and used it, among others, to prove the limit theorem for accessible categories: any small pseudo-limit of a diagram of accessible categories and accessible functors calculated in the standard 2-category of (possibly large) categories is itself accessible. An essential element of our proof of the limit theorem was the use of the *category* of sketches.

In the middle 1990's, I developed a theory of something I called generalized sketches – which I am going to call simply sketches, with various qualifiers when needed –-, and used them, among others, for a framework of categorical completeness theorems, in which the *category* of sketches of a specific kind is the main ingredient.

Sketch theory is a categorical approach to logical syntax. The sketches (the objects of the category) correspond (quite closely, in fact) to the formulas of logic. The morphisms stand for logical axioms and theorems, and also figure in certain infinite colimits which are the *theory-categories* of a particular “doctrine”, say that of regular category. This last use of the morphisms of sketches is the important one for my present goals. I introduce a new kind of sketch and the *category* of them, to obtain a tool to talk about equational *theories* of weak higher-dimensional categories, or more specifically, monads over the category of globular sets (omega graphs) whose algebras are such weak higher categories.

I make some further introductory remarks. Take the most well-known kind of sketch, finite-limit sketch, and, on the other hand, the concept of finite-limit category (category having all finite limits). In particular, the notion of *category* is a finite-limit notion: there is a small finite-limit category whose models (finite-limit-preserving functors to Set) are exactly the (small) categories. But how do we know this? I submit that we only know this because (1) we have a finite-limit sketch *S-sub-cat* whose models are categories, and (2) we know that for any (small) finite-limit sketch *S*, there is a finite-limit category <*S*> that has the same (equivalent) category of models as *S*. Category theorists usually will not take the sketch *S-sub-cat* seriously, and will be satisfied with talking about the *finite-limit theory T-sub-cat* = <*S-sub-cat*> of categories being “presented” in an informal sense by some prescription. This prescription is indeed necessary to write out in detail when the category theorist has to define a category object in a (finite-limit) category – but they will probably still not use the word “sketch” to name the prescription when doing so.

Even if the category theorist admits that there is a sketch involved, they will still (correctly) argue that, whereas the theory T-sub-cat is essentially uniquely determined (by Gabriel-Ulmer duality), the sketch *S-sub-cat* is far from being unique – it is a mere "presentation", and the essential entity *T-sub-cat has*, of course, many different presentations; the presentation is a second-class entity, and it is best to avoid it when possible. This last point is where I disagree with the category theorist. In fact, I am bound to say that I want the sketch, and don't want the theory, since the sketch already tells me what the models (algebras) are, and that's enough.

But, suppose we agree that there are sketches and we should use them. How do we know fact (2) above about the completion *<S>* of *S*? An obvious attempt at the answer is that *<S>* is the value of the left adjoint of the inclusion of the category *Lex* of finite-limit categories into the category *LexSketch* of finite-limit sketches. Thus, the *category* of finite-limit sketches appears on the scene. However, there are problems with the argument: said left adjoint does not exist, basically because Lex is not a good category; it does not have limits, for instance. We have to go to 2-categories and pseudo-limits and the like to make the argument stick. But there is another, more elementary, solution, which I will now describe, since the method involved in it is the one that I need for my talk.

Some general definitions. Let **A** be a locally presentable category, and *SIGMA* a (small) set of morphisms in **A**. An object A is *SIGMA-injective* if

[1] (Oct 17, p.2).

We also write A |= SIGMA, and say "A satisfies SIGMA", for A SIGMA-injective.

An arrow *a:A---->B* is *SIGMA-cellular* if it belongs to the least class *X* of arrows of **A** (the *cellular closure* of *SIGMA,* denoted *cell(SIGMA)*) which

(i) contains *SIGMA*,
(ii) is closed under pushout

[2] (Oct 17. p. 2),

and (iii) is closed under transfinite composition

[3] (Oct 17. p. 2),

or equivalently (having already condition (ii)), closed under *good* composition

[4] (Oct 17. p. 2)

(Jacob Lurie).

Let us say that a class of arrows is *cellular,* or *cellularly saturated*, if it is closed under pushout and transfinite composition. Thus, the cellular closure of SIGMA is the least cellular class containing SIGMA.

(Remark: although this will not (yet?) play a role in what I am to say here, I note that we have the classical *Gabriel-Zisman saturation*, or *cellular-retract-closure*, of SIGMA, denoted cell-rtr(SIGMA), which is obtained by adding the condition of being closed under retracts of arrows (in the arrow category). The full small-object argument gives that the left-orthogonal closure of the right-orthogonal closure of

SIGMA is cell-rtr(SIGMA): we have a combinatorial (weak) factorization system with cell-rtr(SIGMA) being the left class of the system. This structure is a part of a combinatorial Quillen model structure, where both the class TrCof of the trivial cofibrations, and the class Cof of the cofibrations are of the form

cell-rtr(SIGMA-sub-anod),

resp.,

cell-rtr(SIGMA-sub-cof) .)

Call *a:A--->B* a *SIGMA-completion* of A if *a* is *SIGMA* cellular, and the codomain *B* is *SIGMA*-injective. A simple special case of the so-called small-object argument yields

Proposition (**A** locally presentable, SIGMA small) For any object *A*, a *SIGMA* completion

gamma-sub-A:A---> A<SIGMA>

exists.

The completion is not unique in general, not even up to isomorphism. However, if f:A---->B and g:A--->C are SIGMA-completions, then there are (non-unique) morphisms h:B--->C and k:C---->B such that hf=g, kg=f . In significant special cases, the pair (h,k) turns out to be an *equivalence* in a good sense of the two completions (B,f), (C,g).

Now, take **A** to be the category *LexSketch*, very “simple” category, locally finitely presentable, in fact, a presheaf category. There is a finite set *SIGMA* of *finite* morphisms in **A ,** the set of the “finite-limit axioms”, such that, for any sketch *S* in **A**, any *SIGMA*-

completion

g=gamma-sub-S:S--->S[SIGMA] (=<S>)

with domain *S* has the usual universal property of the finite-limit-category completion; in particular, *<S>* is the underlying sketch of a finite-limit category, and it has the same (equivalent) category of models as *S*.

2. Linton sketches

In 2009 (see my web-site), I formulated a notion I called “Linton sketch”, to get a precise tool to present and specify algebraic theories, or monads over essentially arbitrary categories. Just like sketches before, a Linton sketch has models, or algebras, via a direct definition that makes it unnecessary to pass to the monad whose presentation the Linton sketch is. The name comes from the fact that every monad gives rise to a canonical large Linton sketch (just like a finite-limit category gives rise to a finite-limit sketch, without any loss of information), which is Linton's algebraic theory associated with the monad. In the same year, John Power gave a talk in Montreal about generalized Lawvere algebraic theories. He did not mention sketches; his algebraic theories are, however, the *theory*-Linton sketches, the completions of the Linton sketches.

Let **G** be any category – soon we will make some mild assumptions on **G**. The first example is, of course, Set; the second one is *Graph*, the category of graphs (in the category-theoretic sense). The important example is *omega-Graph*, or *Glob*, the category of omega-graphs, or globular sets.

A *Linton sketch* S over **G** , simply a “*sketch*” from now on, has an

(i) underlying graph (always ordinary graph!), denoted |S| ;

(ii) a subgraph P (=P[S], “P-of-S”) of |S|, the “scaffolding” of S, with the *same* objects as |S|;

(iii) a sketch-for-categories structure on the graph |S|, i.e., commutative diagram specifications, including equality spec's of parallel arrows.

It is required that P also be a subgraph of the underlying graph of the category **G** (it would be enough to have a diagram P--->**G** as additional structure, but it is simpler to handle the present definition). (End of definition of “Linton sketch”.)

A *morphism* phi:S--->T of sketches is a map phi:|S|--->|T| of graphs that acts as the identity on objects (phi(U)=U for all U in Ob(S)) and all arrows of P[S], and which, furthermore, is a morphism of category sketches (takes commutative diagrams to commutative diagrams). We have the *category* Linton[**G**] of small Linton sketches over the category **G**.

Sometimes it is convenient to assume that S is not just a sketch-for-categories, but also a category itself (the underlying sketch-for-categories of a category). Such a sketch we call *c-closed*. It is clear what the *c-closure*, cS, of any S is; it comes with a morphism

S--->cS

which is the identity on objects. cS is determined up to isomorphism in Linton[**G**] under S.

Examples: the sketch for the notion of category over **G** = Graph, and other related examples

[5] (slides)

A model, or *algebra*, M, of the sketch S consists of an object |M| of **G**, and further structure:

[6] (slides)

Returning to the examples:

[7] (slides)

A morphism (*homomorphism*) of algebras of S : h:M---->N is, by definition, an arrow h:|M|------>|N| such that

[8] (slides)

We have the *category* Alg(S) of algebras of S, with the faithful, isomorphism-reflecting forgetful functor

U=U-sub-S : Alg (S) ----> **G**

U creates limits (from those in **G**), in the precise sense Mac Lane meant; thus, if **G** has small limits, so does Alg (S), and U preserves them.

Now assume that

> **G** is locally presentable (accessible, complete, cocomplete), and the Linton sketch S is small.

Then Alg(S) is also locally presentable, since it can be seen to be the same as (equivalent to) the category of **G**-valued models of a small finite-limits sketch. Moreover,

Proposition. For **G** and S as above, U:Alg(S)---->**G** has a left adjoint, and U is monadic.

The proof (outlined in my 2009 posting “Lintonism”) is via Beck monadicity theorem, and copies Mac Lane's proof (“Categories for ...”) of the special case for the ordinary algebraic case.

Let mon(S) be the name for the monad associated with the monadic functor U:Alg(S)---->**G .**

Taking the category of models is obviously functorial:

Alg : (Linton[G[)^op --------> CAT ;

for phi:S----T, a morphism of sketches,

(-)|phi = Alg (phi) : Alg (T)---> Alg (S)

takes a model M to its phi-reduct M|phi, with the same underlying object as M, and operations

M|phi(f) : |M|^V ---> |M|^U

for f:U--->V in S as M|phi(f) = M(phi(f)) (remember that phi(U)=U, phi(V)=V ; thus phi(f):U--->V still!).

Moreover, U-sub-(-): Alg(-) ---> '**G**' (constant-**G** functor) is a natural transformation: for phi:S--->T, the diagrams commutes.

Let us call a sketch-morphism phi:S---->T *tautological* (thinking of sketch-morphisms as “axioms”) if the associated functor

Alg (phi) : Alg(T)--->Alg(S)

is an *isomorphism* of categories. One notes easily that the class of tautological morphisms of Linton[**G**] is cellular(ly closed) (see above); in fact, even also retract closed. Next, we specify certain specific tautological sketch-morphisms.

First, there is the set

SIGMA-sub-cat (1)

of those axioms (arrows of Linton[**G**]) that are designed so that the SIGMA-sub-cat completion of S is cS; I will skip the details.

I then have the axioms of the form

empty-sketch -----> subgraph X of **G,**

that, when applied (injectivity …), simply attaches X as new parts of the scaffolding P. They are obviously tautological. Call said sketch-arrow

SIGMA-sub-scaf [X] (2)

(X denotes a (small) graph of **G**)

Then, more interestingly, we have the set

SIGMA-sub-colimit. (3)

Before giving details, I will give you the effect of the set

SIGMA-sub-taut **=def** union of (1) , (2) and (3).

(I suppressed the dependence on X.)

The SIGMA-sub-taut-completion S[SIGMA] (see above) of a sketch S is now determined up to isomorphism, and it is the *theory-completion* of S, denoted th(S) for short. It is obtained by extending the scaffolding "executing" the prescriptions (commutativities) of S together with the (commutativities and) the colimits consisting of objects and arrows of P[S], colimits meant in the category **G** ; remember that P[S] is a sub-graph of **G**. Since the arrows in SIGMA-sub-taut are all tautological (as we will see), the canonical morphisms

gamma : S -------->S[SIGMA-sub-taut]

is, by being SIGMA-sub-taut- cellular, a tautological extension itself.
We call a Linton sketch of the form th(S) (equivalently, S such that S=th(S)) a *Linton theory.*

The extensions S1--->S2, S2--->S4 under [5] are tautological, because they are SIGMA-sub-taut cellular (some commutativities are omitted in the notations).

For any colimit diagram D in G, we define PHI[D]^ex, and PHI[D]^un, arrows in Linton[G], to be the elements of SIGMA-sub-colimit . For the first, abbreviated PHI:A--->B, we have

[10]1

It is tautological because:

[10]2

The Linton theory th(S) associated with the sketch S contains all the equational consequences of the axioms of S, together with their proofs. Consider, for instance, the sketch S3, the axiom-system for “category”. From the associative law h(gf)=(hg)f, for composable triples (f,g,h), one derives the equalities

i(h(gf)) = (ih)(gf) = ((ih)g)f , and
i(h(gf)) = i((hg)f) = (i(hg))f = ((ih)g)f ,

thus i(h(gf)) = ((ih)g)f “in two different ways”.

All this is displayed in the structure of th(S3). It will use the additional graph <4>, ceratin colimits in Graph, and the equalities will become equalities of (already parallel) arrows of the form

<4>----><1>

3. Batanin sketches

From now on, the category **G** is Glob, the category of globular sets. The decisive discovery of Michael Batanin's is his description of the monad T on Glob whose algebras are the strict omega-categories. For any omega-graph G, T(G), the underlying omega-graph of the free omega-category <G> on G is described by Batanin by the use (countable) set of particular omega-graphs, called here *Batanin cells*, or B-cells for shor*t*.

For an omega-graph H, a *composite* of H (if any) is any non-identity element x of <H> such that for any *proper* sub-omega-graph K of H, x is *not* in <K>, a sub-omega category of <H> : “x uses *all* of G”. x is a *unique composite* if it is the only composite of H. The abstract definition says that the omega-graph B is a B-cell if it has a unique composite, which we denote as mu-sub-B. It is obvious that B-cells are finite omega graphs. We select a representative of each isomorphism class, and by a “B-cell” we always mean the fixed representative. The dimension of a B-cell B is the dimension of its unique composite, also equal to the maximum dimension of the elements of the omega-graph B.

The description of <G> says that every non-unit element (n-cell, for n any integer) x of <G> is associated *uniquely* (!) by a pair (B,xi), where B is a B-cell, an xi is a graph-map xi;B---->G , the association given by the equality <xi>(mu-sub-B) = x, where

<xi> : <B> –----> <G>

is the canonical map. B is the “type” (“shape”) of x; x is the result of realizing B by the omega-graph map xi.

A unit-element is given a B-cell B, and a number k=1,2,...; the pair (B,k) will give the k-fold unit (identity) element on the non-unit element x (x is also regarded as the element given by the pair (B,0)). We have that the set of n-cells of <G> is in a bijective correspondence with the disjoint union

DU of hom(B,G) over all B-cells with dim(B)<=n

The reason why we are concerned here with the Batanin cells is that, in Batanin's “operadic” definition of *weak* omega category, they are the “arities” of the operations. In our terminology here this means that the operations, in an algebra M, are all of the form

|M|^B   -------> |M|-sub-(k) ,

with k>=dim(B) , where |M|-sub-(k) is the set of k-cells of the omega-graph |M|. The restriction to such Batanin-type operations seems enough to prevent the algebras from becoming strict higher dimensional categories unless one makes explicit identifications to force strictness.

For my present versions of Batanin's definition, the Batanin cells are the objects of the scaffolding P[S] of G-sketches S, objects of Linton[G] .

Let **B** denote the full subcategory of Glob with objects the B-cells. I note that all morphisms of **B** are monomorphisms.

Let us say that a *Batanin-sketch* (B-sketch) is any Linton sketch S over **G** = Glob such that the “scaffolding” of S, P[S], is a sub-graph of **B**. All the examples shown before are B-sketches. In other words: the category of B-sketches is Linton[**B**], with B the category of B-cells. However, note that models of a sketch in Linton[**B**] will still have arbitrary omega-graphs, not just objects of **B.**

I introduce two B-sketches, S-sub-magma, and S-sub-strict. Both are defined to have scaffolding P = **B** : all Batanin cells and their morphisms in Glob. S-sub-magma consists of the primitive operations of strict omega-categories, with the domain-and-codomain laws, but without the other laws. S-sub-strict is the extension of S-sub-magma obtained by adding the rest of the laws of omega-categories. Of the examples above, S-sub-2, together with the domain-codomain laws shown for S-sub-3, is the first fragment (sub-sketch) of S-sub-magma, the part that concerns ordinary categories. S-sub-3 is a fragment of S-sub-strict. The details of these sketches can be chosen in different ways. (In my paper "The word problem for computads" (see my website) I introduced primitive operations and laws for strict omega-categories, different from the usual ones.) Only some of the simplest Batanin-cells are necessary for the formulations; I added all of them to make the discussion simpler

By definition, the category of algebras, Alg(S-sub-strict), is the category of (small) strict omega-categories. Let T-sub-strict denote cc(S-sub-strict), the category-colimit closure (completion) of S-sub-strict. T-sub-strict is the *Linton-Batanin theory* of strict omega categories.

To formulate a fact about T-sub-strict, I use some notation.

The *globes* are particular B-cells. The globe **g**k (k=0,1,2,...) is the shape of the individual k-cell, k=0,1,2,.... . Glob is the presheaf category Set^((**gl**)^op), with **gl** having objects (k) = (0), (1), (2),...and morphisms d,c:(k) –--> (k+1) satisfying the globular equalities dd=cd, dc=cc . **g**k is the representable functor

gk = hom(-,(k)) : **gl**^op ----> Set .

We write d,c:**g**(k+1) ----->**g**k for the arrows hom(-,d), hom(-,c).

Proposition Let B be any B-cell, k>=dim(B). Suppose that in the category T-sub-strict, we have the diagram

[11] 1 (slides)

The hypothesis means that, in any model (algebra) of T-sub-strict, and any B-tuple x:B--->|M|, the k-cells delta(x), gamma(x) are parallel k-cells:

[11]2

"In omega-categories, all generic diagrams that can commute, will commute."

The proof of the proposition uses, of course, Michael Batanin's work " ...", and also our subsequent analysis (MM and Marek Zawadiwski: …), and it goes through an explicit description of the structure of the category T-sub-strict. We may safely say that Batanin's (and Tom Leister's) operadic construction of the "theory" of weak omega-categories, as well as of related theories such as T-su-strict, differs from my present one in the fact that I add to their operations the coprojections, arrows of the scaffolding P[S] – with sticking to B-cells just as MB and TL do. Indeed, the normal form one can prove for arrows in T-sub-strict is that any arrow of the form **g**k ----> B in T-sub-strict is uniquely factored as

gk ----------> A --------->B

with k>=n=dim(A), where the first factor is mu[A,k] "corresponding" to the (n-k)-fold identity on the unique composite mu-sub-A, an n-cell in <A>. For more general arrows

C ------->B

in T-sub-strict, one uses a “canonical” colimit representation of the domain C interms of globes.

Next, I define my version of Batanin's definition of “weak omega-category”, in the form of a Linton-theory over B, the category of B-cells, denoted T-sub-B. For a particular set, SIGMA-sub-B, of sketch-axioms, T-sub-B is “the” (SIGMA-sub-B)-*completion* of the sketch S-sub-magma

T-sub-B = (S-sub-magma) < SIGMA-sub-B>
def

For each B in **B**, and each integer k>dim(B), I define an arrow

PHI[B,k] : S1[B,k] ------> S2[B,k] ,

abbreviated PHI : S1 ---> S2. S1 and S2 are as follows; PHI is the inclusion:

[12] (slides)

To see the effect of the “axiom” PHI[B,k], assume that the sketch S is PHI[B,k]-injective (S |= PHI[B,k]). This means that *whenever* we have a pair of co-operations

delta, gamma: gk ----->B

(with arity B, and vale type “k-cell”) in S which are “parallel” –-- in particular, in a model M of S, and for any B-tuple x:B---> |M|, delta(x), gamma(x) are parallel k-cells in the usual sense –-- , we also have a co-operation alpha "contracting" delta and gamma; thus, in the model M,

alpha(x) : delta(x) ----> gamma(x).

SIGMA-sub-contr is defined to be the set of all PHI[B,k] where B is in **B**, and k>dim(B).

Let SIGMA-sub-B denote the union of SIGMA-sub-taut (defined above with an arbitrary base-category **G** in mind, which is now **B**) and SIGMA-sub-contr. Example S5 above is a fragment of SIGMA-sub-B; it introduces the associativity constraint alpha.

SIGMA-sub-B = SIGMA-sub-taut (union) SIGMA-sub-contr

Alg(SIGMA-sub-B) is the category of *Batanin-weak omega categories*, T-sub-B, "the" completion of SIGMA-sub-B, is the Batanin-Linton-theory of the same.

Proposition. There is a unique morphism of sketches

T-sub-B –-----> T-sub-strict

that extends the identity on S-sub-magma.

The proof uses the previous proposition, and the well-ordered/ recursive nature of the definition of T-sub-B as a cellular arrow.

Remember that T-sub-B is not unique, even up to isomorphism – just like for Batanin, who has a weakly initial object in a category of operads. However, as we said above in a general context, we have

[13] (slide)