Reflexive polytopes, \textbf{Gorenstein polytopes}, and combinatorial mirror symmetry

Benjamin Nill

U Kentucky 10/04/10
Goals of this talk:

Convince you that (reflexive &) Gorenstein polytopes

1. turn up naturally
2. consist of interesting examples
3. have fascinating and not yet understood properties
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I. Reflexive polytopes
Combinatorial polytopes and duality

*Combinatorial types* of polytopes

*Isomorphisms*: combinatorially isomorphic face posets
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Realized polytopes and duality

*Embedded* polytopes: $P \subset \mathbb{R}^d$

*Isomorphisms*: affine isomorphisms
Realized polytopes and duality

*Embedded* polytopes: \( P \subset \mathbb{R}^d \)

*Isomorphisms*: affine isomorphisms

\( P \subset \mathbb{R}^d \) \( d \)-polytope with interior point 0 \( \implies \)

\[
P^* := \{ y \in (\mathbb{R}^d)^* : \langle y, x \rangle \geq -1 \ \forall \ x \in P \}
\]
Lattice polytopes and duality

*Lattice polytopes*: \( P = \text{conv}(m_1, \ldots, m_k) \) for \( m_i \in \mathbb{Z}^d \)

*isomorphisms*: affine lattice isomorphisms of \( \mathbb{Z}^d \) (unimodular equivalence)

---

Definition (Batyrev '94)

A reflexive polytope is a lattice polytope \( P \) with \( 0 \in \text{int}(P) \) such that \( P^* \) is also a lattice polytope.
Lattice polytopes and duality

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Isomorphisms: affine lattice isomorphisms of $\mathbb{Z}^d$ (unimodular equivalence)

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A **reflexive polytope** is a lattice polytope \( P \) with \( 0 \in \text{int}(P) \) such that \( P^* \) is also a lattice polytope.

\( \leadsto \) origin only interior lattice point.
Reflexive polytopes

Facts

1. [Lagarias/Ziegler ’91]: In each dimension only finitely many reflexive polytopes up to lattice isomorphisms.

2. [Haase/Melnikov ’06]: Any lattice polytope is a face of a (higher-dimensional) reflexive polytope.

3. [Kreuzer/Skarke ’98-00]: Tons of them: $d_2^3 = 16, 473,800,776$

4. Even basic questions are open: maximal number of vertices?

$d_2^3 = 6, 14, 36$
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Let $P$ be a lattice polytope with 0 in its interior.
Reflexive polytopes

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Definition

$P$ is reflexive if and only if

- each facet $F$ has lattice distance 1 from the origin,
- each vertex is a primitive lattice point.
Reflexive polytopes

Let $P$ be a lattice polytope with 0 in its interior.

**Definition**

$P$ is reflexive of **Gorenstein index** 1 if and only if

- each facet $F$ has *lattice distance* 1 from the origin,
- each vertex is a primitive lattice point.
Reflexive polytopes of higher index! (Joint work with A. Kasprzyk)

Let $P$ be a lattice polytope with 0 in its interior.

**Definition (Kasprzyk/N. ’10)**

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$\Rightarrow \ell P^* \text{ $\ell$-reflexive}$
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$\rightsquigarrow$

$\ell P^*$ is $\ell$-reflexive and $P = \ell(\ell P^*)^*$. 

Duality of $\ell$-reflexive polytopes!
Examples of $\ell$-reflexive polygons!

$l=2$ : No!

$l=3$ :
Classification of \( \ell \)-reflexive polygons (Joint work with A. Kasprzyk)

**Theorem**

\( P \) \( \ell \)-reflexive polygon; \( \Lambda \) := \( \langle \partial P \cap \mathbb{Z}^2 \rangle_\mathbb{Z} \) \( \implies \) \( P \) is 1-reflexive w.r.t. \( \Lambda \).
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**Applications**

- No $\ell$-reflexive polygons for $\ell$ odd.

Yields fast classification algorithm: $\ell$

\begin{align*}
&1, 3, 5, 7, 9, 11, 13, 15, 17, \cdots \\
&1, 13, 29, 1, 61, 81, 1, 113, \cdots
\end{align*}
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The “number 12”

\[
\# \mathcal{P} \cap \mathbb{Z}^2 = 7 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \# \mathcal{P} \cap \mathbb{Z}^2 = 5
\]
The "number 12"

\[ \# \partial P \cap \mathbb{Z}^2 = 9 \]

\[ \# \partial P \cap \mathbb{Z}^2 = 3 \]
The “number 12”

12-Property

$P$ reflexive polygon $\implies$

$$|\partial P \cap \mathbb{Z}^2| + |\partial P^* \cap \mathbb{Z}^2| = 12.$$
The “number 12” generalizes! (Joint work with A. Kasprzyk)

12-Property

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12-Property

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What else can be generalized?
What about higher dimensions?
What about algebro-geometric implications?
II. Gorenstein polytopes
Definition and duality

**Def.** [Batyrev/Borisov ’97] A **Gorenstein polytope of codegree** \( r \) is a lattice polytope \( P \) such that \( rP \) is a reflexive polytope (up to lattice translation).
Definition and duality

Let \( C_P := \mathbb{R}_{\geq 0}(P \times 1) \).

**Proposition (Batyrev/Borisov '97)**

\( P \) is a Gorenstein polytope if and only if

\[(C_P)^* \cong C_Q\]

for some lattice polytope \( Q \).

\( \Rightarrow \) Natural duality of Gorenstein polytopes!
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$$\text{codeg}(P) = \text{codeg}(P^*).$$
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Characterization via Commutative Algebra

Facts (see Bruns & Gubeladze, Miller & Sturmfels)

If $P$ is a lattice $d$-polytope, then $S_P := \mathbb{C}[C_P \cap \mathbb{Z}^{d+1}]$ is a positively graded normal Cohen-Macaulay ring,

$$\omega_{S_P} = \langle x^m : m \in \text{int}(C_P) \cap \mathbb{Z}^{d+1} \rangle.$$

T.f.a.e. $P$ Gorenstein polytope there exists $x \in \text{int}(C_P) \cap \mathbb{Z}^{d+1}$ s.t. $x + C_P \cap \mathbb{Z}^{d+1} = \text{int}(C_P) \cap \mathbb{Z}^{d+1}$.

$S_P$ Gorenstein ring the Hilbert series $H_{S_P}(t)$ satisfies

$$H_{S_P}(t) = (-1)^{d+1} H_{S_P}(t-1).$$
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If $P$ is a lattice $d$-polytope, then $S_P := \mathbb{C}[C_P \cap \mathbb{Z}^{d+1}]$ is a positively graded normal Cohen-Macaulay ring, and $R$ has a canonical module $\omega_R$:

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Characterization via Lattice-Point-Enumeration

$P$ lattice $d$-polytope.

\[
\sum_{k \geq 0} |C_P \cap (\mathbb{Z}^d \times k)| \ t^k = \frac{h^*(t)}{(1 - t)^{d+1}},
\]

where $h^*(t)$ is a polynomial with nonnegative integer coefficients of degree $\leq d$. 
Characterization via Lattice-Point-Enumeration

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**Ehrhart theory:** $k \mapsto kP \cap \mathbb{Z}^d$ is a polynomial of degree $d$. 
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**Def.:** The **degree** of $P$ is defined as the degree of its $h^*$-polynomial.
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\[ \implies \deg(P) = d + 1 - \text{codeg}(P). \]
Characterization via Lattice-Point-Enumeration

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$$\sum_{k \geq 0} |kP \cap \mathbb{Z}^d| t^k = \frac{h^*(t)}{(1 - t)^{d+1}},$$

where $h^*(t)$ is a polynomial with nonnegative integer coefficients of degree $\leq d$.

Proposition (Stanley)

T.f.a.e.

- $P$ Gorenstein polytope (of codegree $\text{codeg}(P)$)
- $h^*$-polynomial of $P$ is symmetric (of degree $\text{deg}(P)$)
Finiteness of Gorenstein polytopes

**Observation:** Lattice pyramids don’t change the $h^*$-polynomial.
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**Theorem (Batyrev/N. ’08; Haase/N./Payne ’09; Batyrev/Juny ’09)**

There exist only *finitely* many Gorenstein polytopes of degree $s$ that are not lattice pyramids.

$$S, h^*_s = h^*_0 = 1$$
Finiteness of Gorenstein polytopes

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Gorenstein polytopes in algebraic and polyhedral combinatorics

- Toric ideals of Gorenstein polytopes are “classical”; “Nice initial complexes on some classical ideals” (Conca/Hosten/Thomas ’06).

Order polytope of a pure poset is Gorenstein (Hibi, Stanley)

Tropically & ordinarily convex polytopes are associated to Gorenstein products of simplices (Joswig/Kulas ’08)
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'The’ example of a Gorenstein polytope: the Birkhoff polytope

Def.: An $n \times n$ matrix is called **doubly stochastic**, if any entry is $\geq 0$ and the row and column sums equal 1.
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**Birkhoff-von Neumann theorem**

The set of $n \times n$ matrix of *doubly stochastic matrices* is the convex hull of the $n!$ permutation matrices:
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The example of a Gorenstein polytope: the Birkhoff polytope

**Facts**

\[ B_n \subseteq \mathbb{R}^{n^2} \] is a lattice polytope

- dimension: \((n - 1)^2\)
'The' example of a Gorenstein polytope: the Birkhoff polytope

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A point in \(kB_n \cap \mathbb{Z}^{n^2}\) is \(n \times n\)-matrix with entries in \(\{0, \ldots, k\}\) and row and column sum \(k\): **semi-magic square with magic number** \(k\).

\[ \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
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\end{bmatrix} \]

Lattice distance of \(x\) equals 1 from any facet \(\Rightarrow B_n\) is reflexive.
'The' example of a Gorenstein polytope: the Birkhoff polytope

**Facts**

\[ B_n \subseteq \mathbb{R}^{n^2} \] is a lattice polytope

- **Dimension:** \((n - 1)^2\)
- **Number of facets:** \(n^2\) (inequalities: \(x_{i,j} \geq 0\))

A point in \(kB_n \cap \mathbb{Z}^{n^2}\) is \(n \times n\)-matrix with entries in \(\{0, \ldots, n\}\) and row and column sum \(k\): **semi-magic square with magic number** \(k\).

A semi-magic square is in the interior of \(kB_n\) if and only if any entry is \(\neq 0\).
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Athanasiadis’ proof of Stanley’s conjecture

Def.: The coefficient vector \((h_0^*, \ldots, h_s^*)\) is unimodal, if

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Proof relies on the notion of special simplices.
Special simplices

Let $P$ be a Gorenstein $d$-polytope of codegree $r$.

**Proposition (Batyrev/N. '07)**

$S$ is a *special* $(r - 1)$-simplex, if the vertices of $S$ are $r$ affinely independent lattice points of $P$ such that

- any facet of $P$ contains precisely $r - 1$ vertices of $S$,
- $S$ is not contained in the boundary of $P$,
- the sum of the vertices of $S$ sum up to unique interior lattice point $x$ of $rP$.

Then $S$ is unimodular.

**Example:** $B_n$ contains special $(n - 1)$-simplex: permutation matrices corresponding to elements in cyclic subgroup generated by $(1 2 \cdots n)$. 
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Proposition (Bruns/Roemer ’05; Batyrev/N.’07)

Projecting $P$ along a special $(r - 1)$-simplex yields a reflexive polytope with the same $h^*$-polynomial.
Main open question
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**Theorem (Bruns-Roemer ’05)**

The $h^*$-vector of a Gorenstein polytope $P$ is unimodal, if $P$ admits a regular unimodular triangulation.

**Theorem (Mustata/Payne ’05)**

There exist reflexive 6-polytopes with non-unimodal $h^*$-vector.

**Def.:** $P$ is normal, if $C_P \cap \mathbb{Z}^d_{+}$ is generated by lattice points in $P$.

**Question:** $P$ normal Gorenstein polytope $\Rightarrow h^*P$ unimodal ?
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III. Combinatorial mirror symmetry
Philosophy

Gorenstein polytopes are combinatorial models of Calabi-Yau varieties.
Mirror symmetry

Calabi-Yau $n$-fold, if its canonical divisor is trivial.
Mirror symmetry

A Calabi-Yau $n$-fold, if its canonical divisor is trivial.

Example: Let $P$ be reflexive polygon. For generic coefficients $c_{(a,b)} \in \mathbb{C}^*$

$$Y := \{(x, y) \in (\mathbb{C}^*)^2 : \sum_{(a,b) \in P \cap \mathbb{Z}^2} c_{(a,b)} x^a y^b = 0\}$$
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is an elliptic curve (Calabi-Yau 1-fold).
Mirror symmetry

String Theory proposes mirror pairs of CY-$n$-folds $Y$, $Y^*$!

Topological mirror symmetry test

$$h^{p,q}(Y) = h^{p,n-q}(Y^*)$$

for Hodge numbers $h^{p,q} = h^q(Y, \Omega_Y^p)$. 

$n=3$: 

[Diagram of a complex network or graph]
Batyrev’s construction

**Theorem (Batyrev ’94)**

$P, P^*$ dual reflexive polytopes $\leadsto$ Calabi-Yau hypersurfaces $Y_P, Y_{P^*}$ in Gorenstein toric Fano varieties whose stringy Hodge numbers satisfy the topological mirror symmetry test.
Batyrev-Borisov-construction

Theorem (Batyrev/Borisov ’96)

Dual *nef-partitions* $\leadsto$ Calabi-Yau *complete intersections* in Gorenstein toric Fano varieties whose stringy Hodge numbers satisfy the topological mirror symmetry test.
Nef-partitions

Families of lattice polytopes $\leadsto$ complete intersections $Y$. 
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\( Y \) is Calabi-Yau, if \( Q_1 + \cdots + Q_r \) is reflexive.
Nef-partitions

Families of lattice polytopes $\leadsto$ complete intersections $Y$.

$Y$ is Calabi-Yau, if $Q_1 + \cdots + Q_r$ is reflexive.

$Q_1, \ldots, Q_r$ nef-partition, if $0 \in Q_1, \ldots, 0 \in Q_r$. 

\begin{center}
\begin{tikzpicture}
  \fill[gray!20] (0,0) -- (1,1) -- (2,0) -- cycle;
  \fill[gray!20] (3,0) -- (4,1) -- (5,0) -- cycle;
  \fill[gray!20] (0,2) -- (1,3) -- (2,2) -- cycle;
  \fill[gray!20] (3,2) -- (4,3) -- (5,2) -- cycle;
\end{tikzpicture}
\end{center}
Gorenstein polytopes enter the picture

\[ Q_1 + \cdots + Q_r \text{ reflexive} \]

\[ \leadsto \]

Cayley-polytope is Gorenstein of codegree \( r \)!
Gorenstein polytopes enter the picture

Prop. (Batyrev/N. ’08)

$P$ Gorenstein polytope of codegree $r$:

Cayley structures of length $r$ on $P$ $\iff$ Special $(r-1)$-simplices of $P^*$
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The stringy $E$-polynomial of $Y$

**Def.:** Stringy $E$-polynomial:

$$E_{st}(Y; u, v) := \sum_{p, q} (-1)^{p+q} h_{st}^{p,q}(Y) \ u^p \ v^q.$$
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Given Gorenstein polytope $P$ as Cayley polytope of length $r$ and CY complete intersection $Y$:

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Let us call $n := d + 1 - 2r$ the Calabi-Yau dimension of $P$.

Beautiful facts (Batyrev/Borisov ’96; Borisov/Mavlyutov ’03)

- **”Hodge duality”:** $E_{st}(P; u, v) = E_{st}(P; v, u)$.
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Benjamin Nill (U Georgia) 

Gorenstein polytopes
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Proof relies on

Open: Is the degree of $E_{st}(P; u, v)$ \(\neq 2\)?
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**Open:** Is the degree of $E_{st}(P; u, v) \neq 0$ equal to $2n$?
Finally, the main challenge

Conjecture (Batyrev/N. ’08)
There exist only *finitely* many stringy $E$-polynomials of Gorenstein polytopes with fixed Calabi-Yau dimension $n$ and fixed constant coefficient.

Would imply the finiteness of Hodge numbers of irreducible CY-manifolds constructed via the Batyrev-Borisov-procedure.

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