Polyhedral Adjunction Theory

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I. Classical Adjunction Theory
Polarized varieties

Polarized variety

\((X, L)\) where

- \(X\) is a normal projective variety of dimension \(n\)
- \(L\) ample line bundle on \(X\)
Polarized varieties

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Adjunction theory = study of adjoint bundles \(tL + K_X\)
Polarized varieties

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Adjunction theory = study of adjoint bundles $L + c \ K_X$
Polarized varieties

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Adjunction theory = study of adjoint bundles $L + c K_X$

Minimal assumption

$X$ is $\mathbb{Q}$-Gorenstein, i.e., $K_X$ is $\mathbb{Q}$-Cartier.
Two algebro-geometric invariants

The unnormalized spectral value $\mu$

$$\mu = \sup\{ c \in \mathbb{R} : L + c \ K_X \ \text{big} \}^{-1}$$

The nef-value $\tau$

$$\tau = \sup\{ c \in \mathbb{R} : L + c \ K_X \ \text{nef} \}^{-1}$$
Two algebro-geometric invariants

**The unnormalized spectral value $\mu$**

$$\mu = \sup \{ c \in \mathbb{R} : L + c \ K_X \big \}^{-1}$$

$-\mu$ is also called *Kodaira energy*.

**The nef-value $\tau$**

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Two algebro-geometric invariants

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$$\mu \leq \tau$$
Two algebro-geometric invariants
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\[ L + \frac{1}{t} K_X \]
Results and conjectures

Most work on polarized manifolds:

\[ \tau \leq n + 1, \]

with equality only for \((\mathbb{P}^n, O(1))\).
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**Fujita, Beltrametti/Sommese, et. al:** Classification for \(\tau > n - 3\).
Results and conjectures

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Conjectures on polarized manifolds

- **Q-normality conjecture:**

\[ \mu > \frac{n + 1}{2} \implies \mu = \tau \]
Results and conjectures

Most work on polarized manifolds:

\[ \tau \leq n + 1, \]

with equality only for \( (\mathbb{P}^n, O(1)) \).

Fujita, Beltrametti/Sommese, et. al: Classification for \( \tau > n - 3 \).

Conjectures on polarized manifolds

\begin{itemize}
  \item \textbf{\( \mathbb{Q} \)-normality conjecture:}
    \[ \mu > \frac{n + 1}{2} \implies \mu = \tau \]
  \item \textbf{Spectrum conjecture:}
    For \( \varepsilon > 0 \), there are only finitely many \( \mu > \varepsilon \).
\end{itemize}
II. Polyhedral Adjunction Theory
The adjoint polytope

Study initiated by [Dickenstein, Di Rocco, Piene ’09].
The adjoint polytope

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Let $P \subseteq \mathbb{R}^n$ be $n$-dimensional lattice polytope

Polyhedral adjunction: “Move facets simultaneously inwards”
The adjoint polytope

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Let \( P \subseteq \mathbb{R}^n \) be \( n \)-dimensional lattice polytope

**Adjunct polytope**

\( P^{(c)} \) is the set of points in \( P \) having lattice distance \( \geq c \) from each facet.
The adjoint polytope

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**Adjoint polytope**

\( P^{(c)} \) is the set of points in \( P \) having lattice distance \( \geq c \) from each facet.

If \( P \) is given by \( m \) facet-inequalities

\[
P = \{ x \in \mathbb{R}^n : A_i x \geq b_i \text{ for } i = 1, \ldots, m \}
\]

where \( A_i \in \mathbb{Z}^n \) primitive and \( b_i \in \mathbb{Z} \)
The adjoint polytope

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Let \( P \subseteq \mathbb{R}^n \) be \( n \)-dimensional lattice polytope

**Adjoint polytope**

\( P(c) \) is the set of points in \( P \) having lattice distance \( \geq c \) from each facet.

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\[
P(c) = \{ x \in \mathbb{R}^n : A_i x \geq b_i + c \text{ for } i = 1, \ldots, m \}.
\]
The adjoint polytope

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*Polyhedral adjunction: ”Move facets simultaneously inwards”*
The adjoint polytope

\[ P = P^{(0)} \]
The adjoint polytope

\[ P(0.1) \]
The adjoint polytope

\( P(0.2) \)
The adjoint polytope

$P(0.3)$
The adjoint polytope

\[ P(0.4) \]
The adjoint polytope

\[ P(0.5) \]
The adjoint polytope

$P(0.6)$
The adjoint polytope

\[ P(0.7) \]
The adjoint polytope

\[ P(0.8) \]
The adjoint polytope

$P(0.9)$
The adjoint polytope

\[ P^{(1)} \text{ point} \]
The adjoint polytope

\[ P(c) = \emptyset \text{ for } c > 1 \]
The adjoint polytope

\[ P = P^{(0)} \]
The adjoint polytope

\[ P(0.2) \]
The adjoint polytope

\[ P^{(0.4)} \]
The adjoint polytope

\[ P(0.6) \]
The adjoint polytope

\[ P(0.8) \]
The adjoint polytope

$P^{(1)}$ combinatorics changes!
The adjoint polytope

\[ P(1.2) \]
The adjoint polytope

$P(1.4)$
The adjoint polytope

\[ P(1.6) \]
The adjoint polytope

\[ \mathcal{P}(1.8) \]
The adjoint polytope

$P^{(2)}$ interval
The adjoint polytope

\[ P(c) = \emptyset \text{ for } c > 2 \]
μ, τ for polarized toric varieties

(X_P, L_P) polarized toric variety.
\( \mu, \tau \) for polarized toric varieties

\((X_P, L_P)\) polarized toric variety. Assume \(X_P\) \(\mathbb{Q}\)-Gorenstein. Then

\[ P^{(c)} \cap \mathbb{Z}^n \iff \text{global sections of } \quad L_P + cK_{X_P} \]
\( \mu, \tau \) for polarized toric varieties

\((X_P, L_P)\) polarized toric variety. Assume \(X_P\) \(\mathbb{Q}\)-Gorenstein. Then

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\[(\text{multiples of}) \ P^{(c)} \cap \mathbb{Z}^n \iff \text{global sections of (multiples of)} \ L_P + cK_{X_P}\]

\[\mu = (\sup\{c \in \mathbb{R} : L_P + cK_{X_P} \text{ big}\})^{-1}\]
\( \mu, \tau \) for polarized toric varieties

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\[(\text{multiples of})\ P^{(c)} \cap \mathbb{Z}^n \iff \text{global sections of (multiples of)} \ L_P + cK_{X_P}\]

**\( \mu \)**

\[
\mu = \left( \sup \{ c > 0 : P^{(c)} \text{ full-dimensional} \} \right)^{-1}
\]
$\mu, \tau$ for polarized toric varieties

$(X_P, L_P)$ polarized toric variety. Assume $X_P \mathbb{Q}$-Gorenstein. Then

(multiples of) $P^{(c)} \cap \mathbb{Z}^n \iff$ global sections of (multiples of) $L_P + cK_{X_P}$

$$\mu = \left( \sup \{ c > 0 : P^{(c)} \neq \emptyset \} \right)^{-1}$$
\( \mu, \tau \) for polarized toric varieties

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(multiples of) \( P^{(c)} \cap \mathbb{Z}^n \leftrightarrow \) global sections of (multiples of) \( L_P + cK_{X_P} \)

\(\mu\)

\[ \mu = \left( \sup \{ c > 0 : P^{(c)} \neq \emptyset \} \right)^{-1} \]

\(\nu\)

\[ \tau = \left( \sup \{ c \in \mathbb{R} : L_P + c K_{X_P} \text{ nef} \} \right)^{-1} \]
μ, τ for polarized toric varieties

\((X_P, L_P)\) polarized toric variety. Assume \(X_P \mathbb{Q}\)-Gorenstein. Then

\((\text{multiples of})\ P^{(c)} \cap \mathbb{Z}^n \iff \text{global sections of } (\text{multiples of})\ L_P + cK_{X_P}

\[\mu = \left(\sup\{c > 0 : P^{(c)} \neq \emptyset\}\right)^{-1}\]

\[\tau = \left(\sup\{c > 0 : P^{(c)} \text{ combinatorially equal to } P\}\right)^{-1}\]
Two polyhedral invariants

Definition makes sense for **general** lattice polytopes!

**Definition**

- $\mu_P := \left( \sup\{ c > 0 : P^{(c)} \neq \emptyset \} \right)^{-1}$
- $\tau_P := \left( \sup\{ c > 0 : P^{(c)} \text{ combinatorially equal to } P \} \right)^{-1}$
Two polyhedral invariants

Definition makes sense for **general** lattice polytopes!

**Definition**

- \( \mu_P := (\sup\{ c > 0 : P^{(c)} \neq \emptyset \})^{-1} \)
- \( \tau_P := (\sup\{ c > 0 : P^{(c)} \text{ combinatorially equal to } P \})^{-1} \), with \( (\sup\{\})^{-1} := \infty \).
Two polyhedral invariants

\[ P = P^{(0)} \]
Two polyhedral invariants

$p(0.2)$
Two polyhedral invariants

\[ p(0.4) \]
Two polyhedral invariants

$P(0.6)$
Two polyhedral invariants

\[ p(0.8) \]
Two polyhedral invariants

\[ P^{(1)} \text{ combinatorics changes} \implies \tau_P = 1^{-1} = 1 \]
Two polyhedral invariants

$p^{(1.2)}$
Two polyhedral invariants

$p(1.4)$
Two polyhedral invariants

\( p(1.6) \)
Two polyhedral invariants

$p(1.8)$
Two polyhedral invariants

\[ p^{(2)} \text{ point} \implies \mu_P = 2^{-1} = \frac{1}{2} \]
Two polyhedral invariants

\[ P = P^{(0)} \text{ combinatorics changes immediately} \implies \tau = \infty \]
Two polyhedral invariants

$p(0.05)$
Two polyhedral invariants

\( P(0.1) \)
Two polyhedral invariants

$p(0.15)$
Two polyhedral invariants

\[ p(0.2) \]
Two polyhedral invariants

\[ p(0.25) \]
Two polyhedral invariants

\( p(0.3) \)
Two polyhedral invariants

\( p(0.35) \)
Two polyhedral invariants

\[ p(0.4) \]
Two polyhedral invariants

\( p(0.45) \)
Two polyhedral invariants

\[ P^{(0.5)} \text{ polygon } \implies \mu_P = 0.5^{-1} = 2 \]
Two polyhedral invariants

Criterion

\[ \tau_P < \infty \]
Two polyhedral invariants

Criterion

\[ \tau_P < \infty \iff X_P \text{ is } \mathbb{Q}\text{-Gorenstein} \]

(i.e., generators of each maximal cone lie in affine hyperplane)
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Polyhedral approach allows to deal with \( \mu_P \) even if \( \tau_P = \infty \).
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Polyhedral approach allows to deal with \( \mu_P \) even if \( \tau_P = \infty \).

Polyhedral adjunction theory

\[ \supset \]

Adjunction theory of polarized toric varieties
III. The Main Theorem
Large $\mu_P$ implies $P$ flat

Theorem [Di Rocco, Haase, N., Paffenholz ’11]

$$\mu_P \geq \frac{n + 2}{2} \implies P \text{ has lattice width one.}$$
Large $\mu_P$ implies $P$ flat

Theorem [Di Rocco, Haase, N., Paffenholz ’11]

$\mu_P \geq \frac{n + 2}{2} \iff P$ has lattice width one.

“If you cannot move the facets of $P$ very far, then $P$ has to be flat.”
Large \( \mu_P \) implies \( P \) flat

**Theorem** [Di Rocco, Haase, N., Paffenholz ’11]

\[
\mu_P \geq \frac{n + 2}{2} \implies P \text{ has lattice width one.}
\]

“If you cannot move the facets of \( P \) very far, then \( P \) has to be flat.”

**Theorem is sharp:** \((\mathbb{P}^n, O(2)), \mu = \frac{n+1}{2}, \text{ lattice width} > 1\)
Large $\mu_P$ implies $P$ flat

Theorem [Di Rocco, Haase, N., Paffenholz ’11]

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“If you cannot move the facets of $P$ very far, then $P$ has to be flat.”

Theorem is sharp: $(\mathbb{P}^n, O(2)), \mu = \frac{n+1}{2}$, lattice width $> 1$
Relation to Ehrhart theory

[Dickenstein, Di Rocco, Piene ’09]: $\mu_P$ is called $\mathbb{Q}$-codegree of $P$. 

$\text{codeg}(P) := \min\{k \in \mathbb{N} : \text{int}(kP) \cap \mathbb{Z}^n \neq \emptyset\}$

$\text{codeg}(P) \leq n + 1$, with equality only for unimodular $n$-simplex.

Proof follows from $\text{int}(kP) \cap \mathbb{Z}^n \subset (kP)_{\frac{1}{k}}$. 
Relation to Ehrhart theory

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**Codegree**

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with equality only for unimodular \( n \)-simplex.

**Relation to \( \mathbb{Q} \)-codegree**

\[
\mu_P \leq \text{codeg}(P)
\]
Relation to Ehrhart theory

[Dickenstein, Di Rocco, Piene ’09]: \( \mu_P \) is called \textbf{Q-codegree} of \( P \).

\begin{itemize}
  \item \textbf{Codegree}

  \[
  \text{codeg}(P) := \min \{ k \in \mathbb{N} : \text{int}(kP) \cap \mathbb{Z}^n \neq \emptyset \}
  \]

  \[
  \text{codeg}(P) \leq n + 1,
  \]

  with equality only for unimodular \( n \)-simplex.

  \item \textbf{Relation to Q-codegree}

  \[
  \mu_P \leq \text{codeg}(P)
  \]

  Proof follows from

  \[
  \text{int}(kP) \cap \mathbb{Z}^n \subset (kP)^{(1)} = kP^{(\frac{1}{k})}.
  \]
\end{itemize}
Relation to Ehrhart theory

Cayley conjecture [Batyrev, N. ’07]

\[ \text{codeg}(P) > \frac{n+2}{2} \implies P \text{ lattice width one.} \]
Relation to Ehrhart theory

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Proofs for

- [Haase, N., Payne ’09] general \( P \), but weaker bound
Relation to Ehrhart theory

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- [Dickenstein, Di Rocco, Piene ’09] \( X_P \) smooth with \( \mu = \tau \)
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- [Dickenstein, N. ’10] \( X_P \) smooth
- [Main theorem] \( X_P \) Gorenstein and \( \mu = \tau \)
Relation to Ehrhart theory

Cayley conjecture [Batyrev, N. ’07]
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\text{codeg}(P) > \frac{n+2}{2} \implies P \text{ lattice width one.}
\]

Proofs for
- [Haase, N., Payne ’09] general $P$, but weaker bound
- [Dickenstein, Di Rocco, Piene ’09] $X_P$ smooth with $\mu = \tau$
- [Dickenstein, N. ’10] $X_P$ smooth
- [Main theorem] $X_P$ Gorenstein and $\mu = \tau$

Philosophy: $\mathbb{Q}$-codegree is more tractable than codegree!
Relation to dual defective polarized manifolds

**Dual defectivity**

$(X, L)$ is **dual defective**, if $X^*$ is not a hypersurface.

---

[Beltrametti, Fania, Sommese '92] $\Rightarrow \mu = \tau > n + 2/2$.

[>Dickenstein, N. '10

Let $X_P$ be smooth. $\mu_P > n + 2/2 \iff X_P$ is dual defective.

$\mu_P > n + 2/2 \Rightarrow \mu_P = \tau_P$.

This is (nearly) the $Q$-normality conjecture!
Relation to dual defective polarized manifolds

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Let $X_P$ be smooth.

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$(X, L)$ is **dual defective**, if $X^*$ is not a hypersurface.

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This is (nearly) the $\mathbb{Q}$-normality conjecture!
Relation to dual defective polarized manifolds

What about the singular situation?

\[ \mu_P > n + 2 \implies X_P \text{ dual defective?} \]

Main theorem shows that this may be true!

\[ \text{[Curran/Cattani'07, Esterov'08]} \]

\[ X_P \text{ dual defective} = \implies P \text{ lattice width one.} \]

Main conjecture

\[ \text{codeg}(P) > n + 2 \implies X_P \text{ dual defective.} \]
Relation to dual defective polarized manifolds

What about the singular situation?

**Question**

\[ \mu_P > \frac{n + 2}{2} \implies X_P \text{ dual defective?} \]
Relation to dual defective polarized manifolds

What about the singular situation?

Question

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Relation to dual defective polarized manifolds

What about the singular situation?

**Question**

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[Curran/Cattani’07, Esterov’08]

$$X_P \text{ dual defective} \implies P \text{ lattice width one.}$$

**Main conjecture**

$$\text{codeg}(P) > \frac{n+2}{2} \implies X_P \text{ dual defective.}$$
Proof sketch

Let $\mu_P \geq \frac{n+2}{2}$.
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1. The core of $P$: $P\left(\frac{1}{\mu}\right)$ is lower-dimensional.
Proof sketch

Let $\mu_P \geq \frac{n+2}{2}$.

1. The core of $P$: $P(\frac{1}{\mu})$ is lower-dimensional. Projecting along the core non-decreases $\mu$. 

2. Look at big facets of $P$ that define the core.

Let $C \subset (\mathbb{R}^{n+1})^*$ be cone spanned by the big primitive normals. 

3. Tricky part: in $C$ the point $(0, 1)$ is a non-trivial sum of lattice points. 

[Batyrev, N. '07] $P$ has lattice width one.

Do methods also help to attack the Spectrum Conjecture?

Benjamin Nill (U Georgia)
Proof sketch

Let \( \mu_P \geq \frac{n+2}{2} \).

1. The core of \( P \): \( P(\frac{1}{\mu}) \) is lower-dimensional. Projecting along the core non-decreases \( \mu \).

\( \leadsto \) may assume core is a point.
Proof sketch

Let $\mu_P \geq \frac{n+2}{2}$.

1. **The core of $P$: $P(\frac{1}{\mu})$ is lower-dimensional.** Projecting along the core **non-decreases** $\mu$.
   $\Rightarrow$ may assume core is a point.

2. **Look at big facets of $P$ that define the core.**
Proof sketch

Let $\mu_P \geq \frac{n+2}{2}$.

1. The core of $P$: $P(\frac{1}{\mu})$ is lower-dimensional. Projecting along the core non-decreases $\mu$. $\rightsquigarrow$ may assume core is a point.

2. Look at big facets of $P$ that define the core.
   Let $C \subset (\mathbb{R}^{n+1})^*$ be cone spanned by the big primitive normals.

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Proof sketch

Let $\mu_P \geq \frac{n+2}{2}$.

1. The core of $P$: $P(\frac{1}{\mu})$ is lower-dimensional. Projecting along the core non-decreases $\mu$. $\Rightarrow$ we may assume core is a point.

2. Look at big facets of $P$ that define the core. Let $C \subset (\mathbb{R}^{n+1})^*$ be cone spanned by the big primitive normals. *Tricky part:* in $C$ the point $(0, 1)$ is a non-trivial sum of lattice points.

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2. Look at big facets of $P$ that define the core. Let $C \subset (\mathbb{R}^{n+1})^*$ be cone spanned by the big primitive normals. *Tricky part*: in $C$ the point $(0,1)$ is a non-trivial sum of lattice points.

3. [Batyrev, N. ’07] $P$ has lattice width one.
Proof sketch

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Do methods also help to attack the Spectrum Conjecture?