Chicken McNuggets

Square magic

Polyhedra

The art of bookkeeping

The magic unravelled

Barvinok's algorithm

Closing the circle

Fast-food, square magic and polyhedra.

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(& Christian Haase)

Research Group Lattice Polytopes - FU Berlin

IMA Minneapolis, October 2, 2007

$=$

^{11} $\cdot$

^{6} $\cdot$

^{4}

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1. Chicken McNuggets

How many ways are there to order 200 Chicken McNuggets?
(Available: 6, 9, 20.)

$$
\begin{array}{rllll}
= & t^{10}, & n^{20}t, & sn^{6}t^{7}, & s^{2}n^{12}t^{4} \\
 & s^{3}n^{18}t, & s^{4}n^{4}t^{7}, & s^{5}n^{10}t^{4}, & s^{6}n^{16}t \\
 & s^{7}n^{2}t^{7}, & s^{8}n^{8}t^{4}, & s^{9}n^{14}t, & s^{10}t^{7} \\
 & s^{11}n^{6}t^{4}, & s^{12}n^{12}t, & s^{14}n^{4}t^{4}, & s^{15}n^{10}t \\
 & s^{17}n^{2}t^{4}, & s^{18}n^{8}t, & s^{20}t^{4}, & s^{21}n^{6}t \\
 & s^{24}n^{4}t, & s^{27}n^{2}t, & s^{30}t &
\end{array}
$$

…twenty three possibilities.

Exercise. Determine the largest impossible order (Frobenius number).

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On planet Qkargogg they have Value Menus with
12'223, 12'224, 36'674, 61'119, and 85'569
Chicken McNuggets each.

How many ways are there to order 89'643'482 Chicken McNuggets?

$$12223x_1+12224x_2+36674x_3+61119x_4+85569x_5 = 89643482$$

How many NON-NEGATIVE, INTEGRAL solutions?

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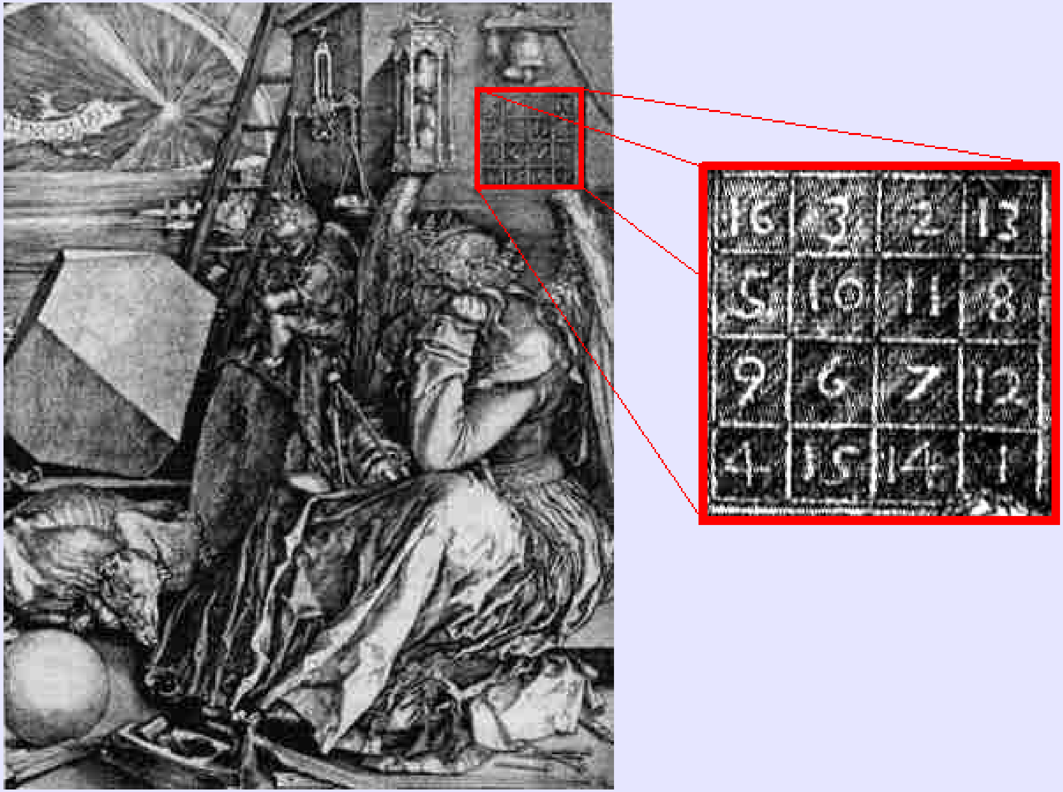
2. Square magic



A magic square is a matrix with all row sums, column sums and diagonal sums equal to the magic constant.

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 30 & 41 & 54 & 23 & 12 & 63 & 36 \\ 47 & 52 & 7 & 28 & 57 & 38 & 17 & 14 \\ 21 & 10 & 61 & 34 & 3 & 32 & 43 & 56 \\ 59 & 40 & 19 & 16 & 45 & 50 & 5 & 26 \\ 42 & 53 & 2 & 29 & 64 & 35 & 24 & 11 \\ 8 & 27 & 48 & 51 & 18 & 13 & 58 & 37 \\ 62 & 33 & 22 & 9 & 44 & 55 & 4 & 31 \\ 20 & 15 & 60 & 39 & 6 & 25 & 46 & 49 \end{bmatrix}$$

Exercise. Determine the magic constant, if all entries are different.

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = \mathrm{mc} \quad x_4 + x_5 + x_6 = \mathrm{mc} \quad x_7 + x_8 + x_9 = \mathrm{mc}$$
$$x_1 + x_4 + x_7 = \mathrm{mc} \quad x_2 + x_5 + x_8 = \mathrm{mc} \quad x_3 + x_6 + x_9 = \mathrm{mc}$$
$$x_1 + x_5 + x_9 = \mathrm{mc} \quad x_3 + x_5 + x_7 = \mathrm{mc}$$

How many NON-NEGATIVE, INTEGRAL solutions?

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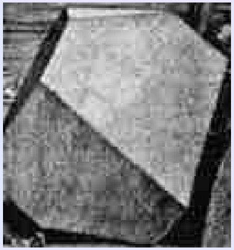
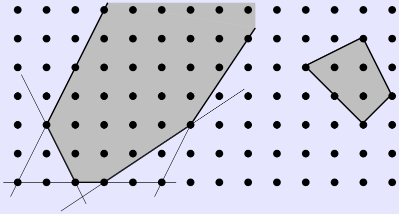
3. Polyhedra

3.1. Definition

The set of (real) solutions to finitely many linear (in)equalities is a **polyhedron**. The convex hull of finitely many points is a **polytope**.





Theorem. bounded polyhedron = polytope

Lattice points in $\mathbb{R}^n$ are elements in the lattice $\mathbb{Z}^n$.

3.2. Polytope of Sudokus

5	8	4	1	7	3	2	9	6
6	9	3	5	4	2	7	1	8
7	1	2	8	6	9	5	3	4
4	5	8	7	2	1	9	6	3
1	3	9	6	5	4	8	2	7
2	7	6	9	3	8	4	5	1
8	2	5	3	1	7	6	4	9
3	6	7	4	9	5	1	8	2
9	4	1	2	8	6	3	7	5

Counting number of Sudokus
= counting lattice points in a polytope?!

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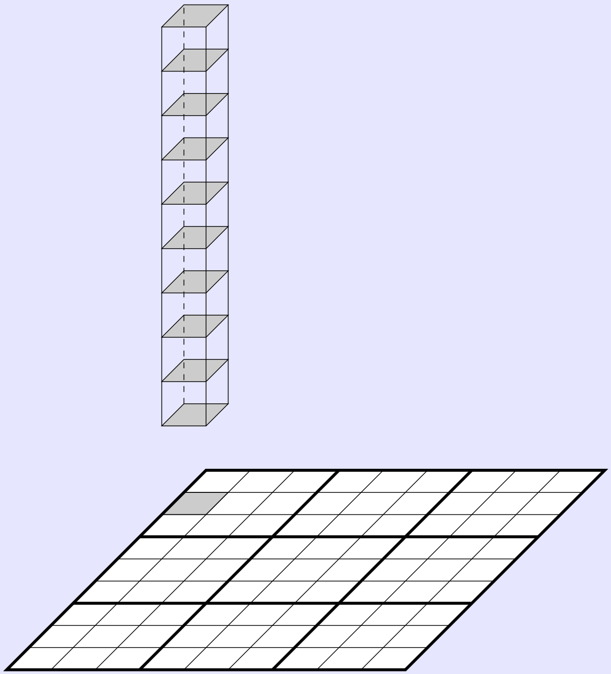

Polytope in $\mathbb{R}^{9\times 9\times 9}$ given by :

- Any entry between 0 and 1.
- Sum over each tower equals 1.
- Sum over each floor of row/column/square equals 1.

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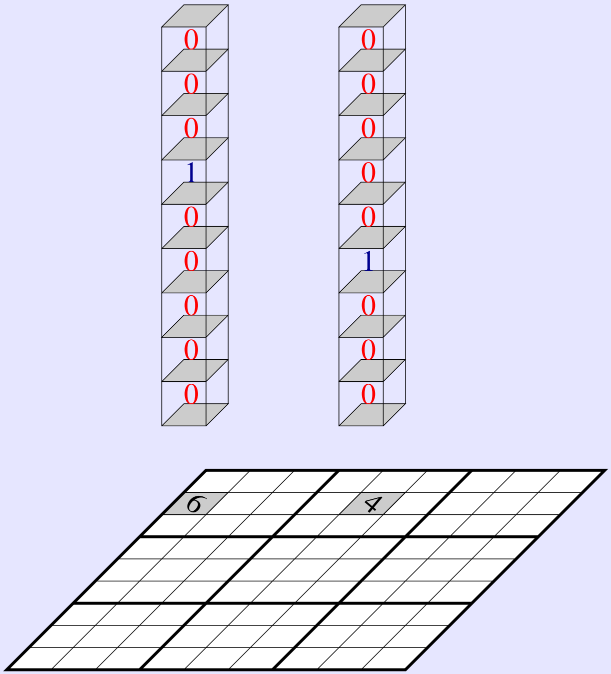

Polytope in $\mathbb{R}^{9\times 9\times 9}$ given by :

- Any entry between 0 and 1.
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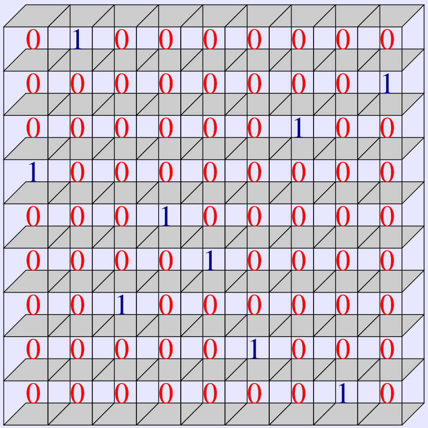

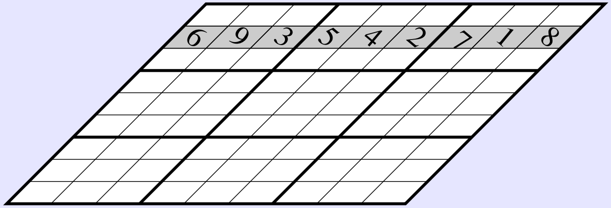

Polytope in $\mathbb{R}^{9\times 9\times 9}$ given by :

- Any entry between 0 and 1.
- Sum over each tower equals 1.
- Sum over each floor of row/column/square equals 1.

3.3. Ehrhart polynomials

Theorem. [Ehrhart 1967]
Let P be a polytope whose vertices have rational coordinates. Define for a natural number k

$$L(k) := \text{ number of lattice points in } kP.$$

Then $k \mapsto L(k)$ is a **quasi-polynomial** (of period N), i.e., it becomes polynomial on the set of numbers with the same remainder modulo N.

Example:
The number of 4×4 magic squares with magic constant c equals

$$\begin{cases} \frac{1}{480}c^7 + \frac{7}{240}c^6 + \frac{89}{480}c^5 + \frac{11}{16}c^4 + \frac{49}{30}c^3 + \frac{38}{15}c^2 + \frac{71}{30}c + 1 & \text{if } c \text{ is even,} \\ \frac{1}{480}c^7 + \frac{7}{240}c^6 + \frac{89}{480}c^5 + \frac{11}{16}c^4 + \frac{779}{4800}c^3 + \frac{593}{240}c^2 + \frac{1051}{480}c + \frac{13}{16} & \text{if } c \text{ is odd.} \end{cases}$$

3.4. Other applications

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Counting lattice points in polyhedra

turns up in

- graph theory/integer linear programming (colorings and flows)
- statistics (contingency tables)
- representation theory (Kostka and Littlewood-Richardson coefficients, saturation conjecture)
- algebraic geometry (global sections, Todd classes)
- string theory (stringy Hodge numbers)

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4. The art of bookkeeping

Throughout all polyhedra are given by *rational* inequalities.

4.1. Why rational functions are nice

List all lattice points in the polyhedron $[0,3]$.

- 0,1,2,3
- $g_{[0,3]} = 1 + x + x^2 + x^3$
- $g_{[0,3]} = \dfrac{1 - x^4}{1 - x}$

List all lattice points in the polyhedron $[0, 10000]$.

- 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,2
- $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} +$
- $g_{[0,10000]} = \dfrac{1 - x^{10001}}{1 - x}$

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List all lattice points in the polyhedron $[0, 10000]$.

- 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,2
- $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} +$
- $g_{[0,10000]} = \dfrac{1 - x^{10001}}{1 - x}$

...in the polyhedron $[0, \infty)$

-
-
- $g_{[0,\infty)} = \dfrac{1}{1 - x}$

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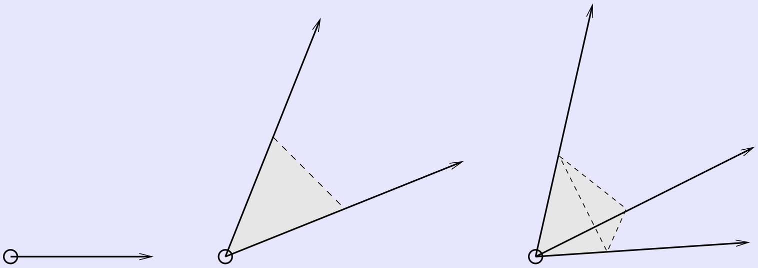
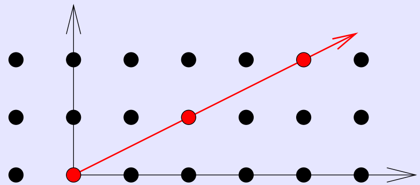
4.2. Why simple cones are *simple*

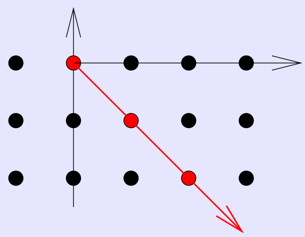
Simple cones in dimension 1,2,3:



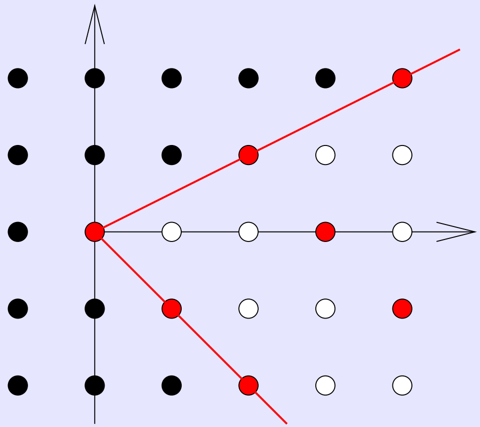
How to enumerate lattice points?



$$1 + x^2y + x^4y^2 + \ldots = \sum_{k \geq 0} (x^2y)^k = \frac{1}{1 - x^2y}$$



$$1 + x/y + x^2/y^2 + \ldots = \sum_{k \geq 0} (x/y)^k = \frac{1}{1 - x/y}$$



$$(x^2y)^2 \cdot 1$$

$$x^2y \cdot 1$$

$$\frac{1}{1-x^2y} \cdot \frac{1}{1-x/y} = 1 \cdot 1 \qquad x^2y \cdot x/y$$

$$1 \cdot x/y \qquad x^2y \cdot (x/y)^2$$

$$1 \cdot (x/y)^2$$

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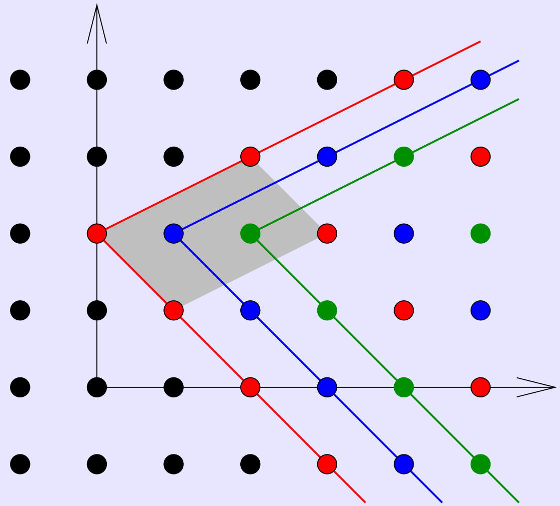

$$\frac{y^2 + xy^2 + x^2y^2}{(1 - x/y)(1 - x^2y)}$$

C a simple cone in $\mathbb{R}^n$ $\quad\Rightarrow$

$$g_C = \sum_{x \in C \cap \mathbb{Z}^n} x$$

rational function of this form.

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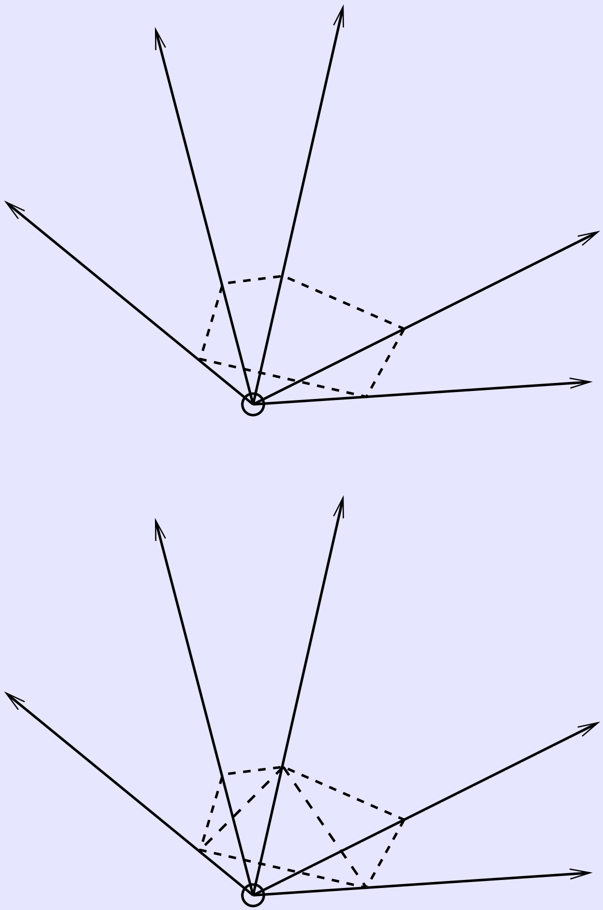
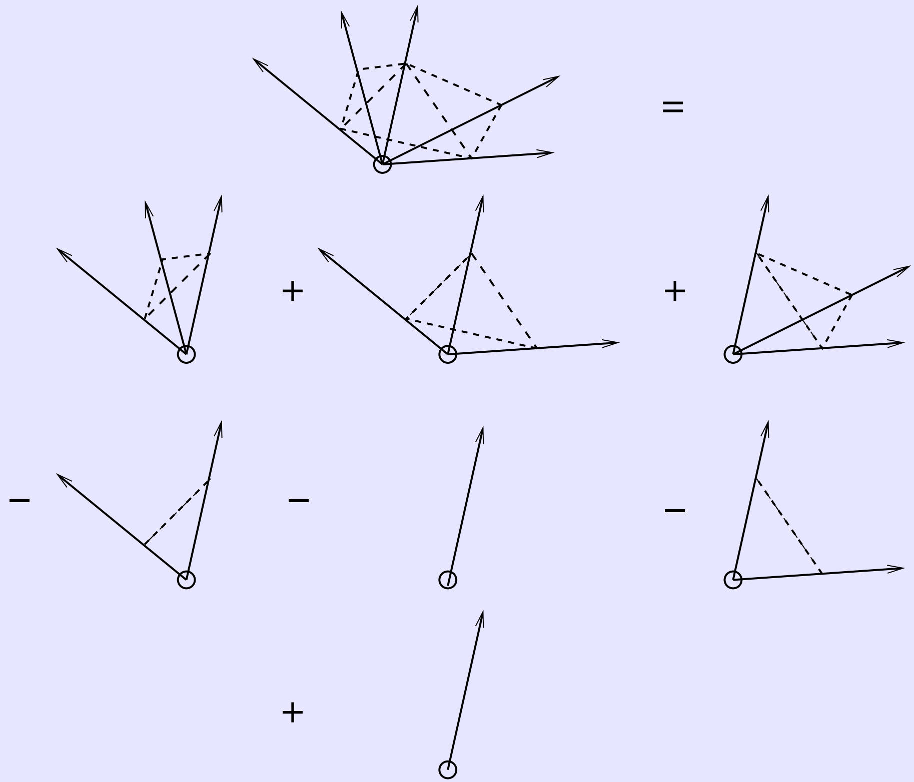
4.3. Cones triangulate into simple ones



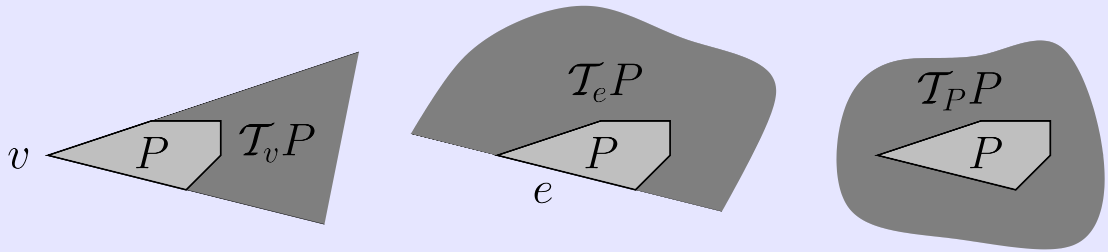


C cone $\Rightarrow$ g_C rational function!

4.4. So, what about polytopes?

For a face F of a polytope P, define the tangent cone

$$\mathcal{T}_F P = \{f + x \in \mathbb{R}^d \;:\; f \in F \text{ and } f + \epsilon x \in P \text{ for some } \epsilon > 0\}$$



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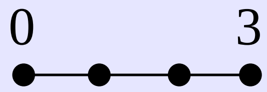
Our favourite example:

$$P = [0, 3]$$



$$g_P = x^0 + x^1 + x^2 + x^3 = \frac{1 - x^4}{1 - x}$$

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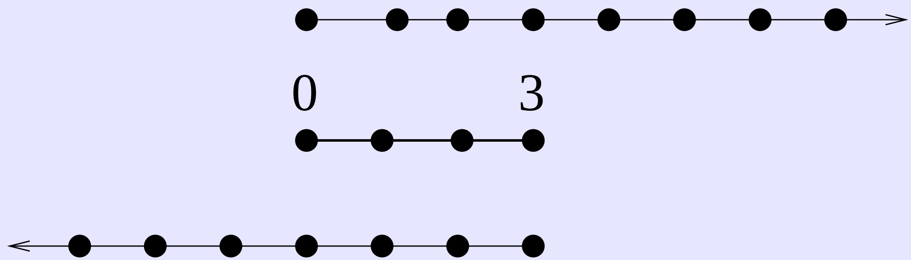
Look at the tangent cones:



$$g_{\mathcal{T}_0 P} \;=\; x^0 + x^1 + x^2 + x^3 + x^4 + x^5 + \cdots \;=\; \frac{1}{1-x}$$

$$g_{\mathcal{T}_3 P} \;=\; \cdots + x^{-2} + x^{-1} + x^0 + x^1 + x^2 + x^3 \;=\; \frac{x^3}{1-1/x}$$

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A magic sum:

$$
\begin{array}{rcc}
g_{\mathcal{T}_0 P} & = & \dfrac{1}{1-x} \\
+ \quad g_{\mathcal{T}_3 P} & = & \dfrac{x^3}{1-1/x} \\
\hline
g_P & = & \dfrac{1-x^4}{1-x}
\end{array}
$$

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Theorem. [Brion 1988]
P polytope with rational vertex coordinates

$$\Rightarrow \quad g_P = \sum_{v \text{ vertex of } P} g_{\mathcal{T}_v P}.$$

In particular, g_P is a rational function.

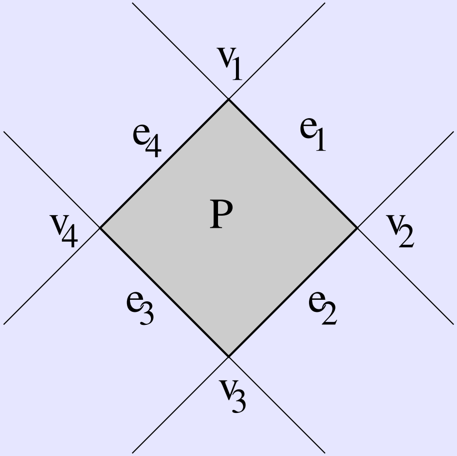
5. The magic unravelled

Theorem. [Brianchon 1837, Gram 1874]

$$g_P = \sum_v g_{\mathcal{T}_v P} + \sum_F (-1)^{\dim F} g_{\mathcal{T}_F P}$$

where F are faces of P with $\dim F > 0$.

Illustration:



$$g_P = \sum_{i=1}^4 g_{\mathcal{T}_{v_i} P} - \sum_{j=1}^4 g_{\mathcal{T}_{e_j} P} + g_{\mathcal{T}_P P}.$$

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The magic trick:

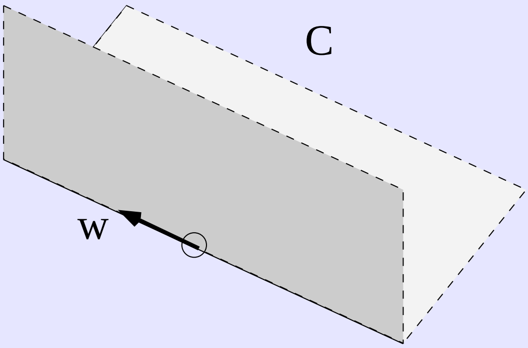
Claim:

C cone containing a line $\Rightarrow$ $g_C = 0$ (as a rational function).

Corollary:

F is a face of positive dimension $\Rightarrow$ $g_{\mathcal{T}_F P} = 0.$

Thus, Brion's theorem follows from Brianchon-Gram.



Proof of claim:

Exists lattice point w:

$$w + C = C.$$

$$\Rightarrow \quad x^w g_C = g_C.$$

$$\Rightarrow \quad (1 - x^w) g_C = 0.$$

$$\Rightarrow \quad g_C = 0,$$

since rational function $f \neq 0$ times polynomial $p \neq 0$ is rational function $f \cdot p \neq 0$.

$\square$

6. Barvinok's algorithm

6.1. Our algorithm so far

Given a linear (in-)equality description of polytope P in $\mathbb{R}^n$.

1. Calculate vertices v.
2. Calculate $\mathcal{T}_v P$.
3. Triangulate $\mathcal{T}_v P$ into simple cones C_i.
4. Calculate rational function g_{C_i}.
5. Calculate rational function $g_{\mathcal{T}_v P}$.
6. Calculate rational function g_P by Brion's theorem.
7. Evaluate rational function

$$g_P(1, \ldots, 1) = \sum_{x \in P \cap \mathbb{Z}^n} 1$$

to get number of lattice points in P.

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Fix the dimension n. Given P with input size α.

Bit input size: $\log(\alpha)$.

Goal: Enumeration complexity polynomial in $\log(\alpha)$.

6.2. Analyzing the algorithm

1. Calculate vertices v - polynomial
2. Calculate $\mathcal{T}_v P$ - polynomial
3. Triangulate $\mathcal{T}_v P$ into simple cones C_i - polynomial
4. Calculate rational function g_{C_i} - unclear

Possibly MANY lattice points in parallelepiped!

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5. Calculate rational function $g_{\mathcal{T}_v P}$ - polynomial

   This can be done via inclusion-exclusion.

   Nowadays, via *irrational decomposition* even disjoint union of lattice points in **full-dimensional** cones.

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6. Calculate rational function g_P by Brion's theorem - polynomial

7. Evaluate $g_P(1,\ldots,1)$ to get number of lattice points - polynomial

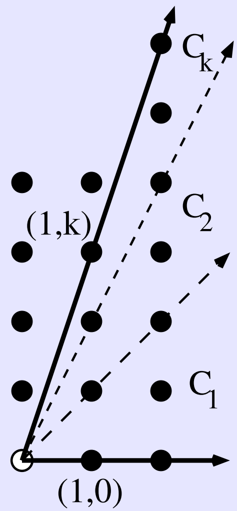
$(1,\ldots,1)$ is a pole of rational function g_P.
Evaluation via complex methods.

Only problem left is:
4. Calculate rational function g_{C_i} - unclear

Approach: Triangulate simple cone as far as possible, into *unimodular* cones.

UNIMODULAR CONE $=$
simple cone & parallepiped contains only apex.

Example:
C cone given by directions $(1,0)$ and $(1,k)$.
Decomposition into k unimodular cones:



complexity not polynomial $(k = e^{\log(k)})$!

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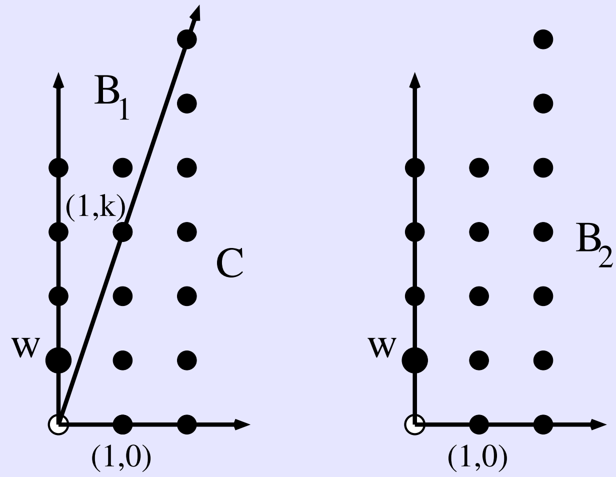
6.3. Barvinok's trick

Write a **signed decomposition** "$C = B_2 - B_1$" into unimodular cones B_1, B_2:



$$\boxed{g_C = g_{B_2} - g_{B_1}} \quad \pm \sum_F g_F \quad (F \text{ lower dimensional}).$$

Only 2 unimodular cones needed!

This idea works always:

Minkowski's lattice point theorem $\Rightarrow$
exists lattice point w $\rightsquigarrow$ signed decomposition into "smaller" simple cones.

Theorem. [Barvinok 1994] Let n be fixed.
There exists a polynomial-time algorithm for computing the rational generating function g_P of a polyhedron $P \subseteq \mathbb{R}^n$ given by rational inequalities.

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7. Closing the circle

Let N be a natural number.

Theorem. [Jacobi 1829] The number of representations of N as sums of four squares equals 8 times the sum of all divisors of N that are not divisible by 4.

Example: Let $N = p \cdot q$ for different primes p, q.
Number of representations of N as four squares $=$

$$8(1 + p + q + N).$$

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Application:
Let $N = p \cdot q$ for different primes p, q.

Define

$$B(N) := \{x \in \mathbb{Z}^4 \,:\, x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq N\},$$

set of lattice points in 4-dim. ball of radius $\sqrt{N}$.

$$\begin{aligned} & |B(N)| - |B(N-1)| \\ = \;& |\{x \in \mathbb{Z}^4 \,:\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = N\}| \\ = \;& 8(1 + p + q + N). \end{aligned}$$

In other words:
N, $|B(N)| - |B(N-1)|$ known $\quad\Leftrightarrow$
$p \cdot q, p + q$ known $\quad\Leftrightarrow$
p, q known.

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RSA cryptosystem attack. [De Loera 2005]

IF we could count lattice points in 4-dim. balls *fast*,
THEN we could factorize $N = p \cdot q$ *fast*.

References

The many aspects of counting lattice points in polytopes

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