Combinatorial aspects of mirror symmetry

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1. Reflexive polytopes

$M, N$ dual lattice of rank $d$.

Let $P \subseteq M_\mathbb{R} := M \otimes \mathbb{Z} \mathbb{R}$ be a $d$-dimensional lattice polytope containing $0$ in its interior.

**Definition:**

$$P^* := \{ y \in N_\mathbb{R} : \langle x, y \rangle \geq -1 \ \forall \ x \in P \}.$$  

$$(P^*)^* = P.$$  

**Definition:** $P$ reflexive $: \iff P^*$ lattice polytope.

$P$ reflexive $\iff P^*$ reflexive

Reflexive polytopes turn up in dual pairs!
Reflexive polytopes

$P \subseteq M_{\mathbb{R}}$ a lattice polytope $\Rightarrow$

$$\sigma_P := \text{pos}(P \times \{1\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{R},$$

$$S_P := \sigma_P \cap (M \oplus \mathbb{Z}),$$

$$X_P := \text{Proj} \mathbb{C}[S_P].$$

**Bijection** (of isomorphism classes):

Reflexive polytopes $P, P^*$

$\leftrightarrow$

Gorenstein toric Fano varieties $X_P, X_{P^*}$

(i.e., projective toric varieties with ample anticanonical Cartier divisor)
Let $P \subseteq M_\mathbb{R}$ be reflexive.

**Definition:** Let $Y_P$ denote generic anticanonical hyper-surface in (a crepant resolution of) $X_P$.

**Theorem.** [Batyrev 1994] Let $P$ be reflexive. $Y_P, Y_P^*$ are mirror-symmetric CY’s on the level of stringy Hodge numbers:

$$h_{\text{str}}^{p,q} = h_{\text{str}}^{(d-1)-p,q} \quad \forall \ 0 \leq p, q \leq d - 1.$$ 

There are explicit formulas for computing stringy Hodge numbers in terms of the reflexive polytopes $P, P^*$!
2. Nef-partitions

Let $X$ be a Gorenstein toric Fano variety.

**Definition:** A *nef-partition* is a partition of the torus-invariant prime divisors of $X$ into effective nef Cartier divisors $D_1, \ldots, D_r$, i.e.,

$$-K_X = D_1 + \cdots + D_r.$$

Then the associated generic anticanonical complete intersection in (crepant resolution of) $X$ is a (singular) CY.

**Theorem.** [Borisov 93] Exists duality of nef-partitions.

**Theorem.** [Batyrev, Borisov 96] The associated complete intersections are mirror-symmetric CY’s on the level of stringy Hodge numbers.
Combinatorial characterization

- Nef-partitions correspond to Minkowski sums

\[ P_1 + \cdots + P_r \text{ reflexive,} \]

where \( P_i \) are lattice polytopes containing 0.

- Duality of nef-partitions can be characterized as follows:

1. \( \Delta := P_1 + \cdots + P_r \subseteq M_\mathbb{R} \text{ reflexive,} \)
2. \( \nabla := Q_1 + \cdots + Q_r \subseteq N_\mathbb{R} \text{ reflexive,} \)
3. \( \langle P_i, Q_j \rangle \geq -\delta_{i,j} \).
3. Gorenstein polytopes

Let $P \subseteq M_\mathbb{R}$ be $d$-dimensional lattice polytope.

Definition:
- **Gorenstein cones** are cones of the form $\sigma_P$.
- $P$ is called **Gorenstein polytope** of index $r$ :
  $\iff rP$ reflexive.

**Proposition.** [Batyrev, Borisov 97]

\[
P \text{ Gorenstein polytope } \iff \sigma_P^\vee \text{ Gorenstein cone}
\]

Then the **dual Gorenstein polytope** $P^*$ defined via

\[
\sigma_{P^*} \simeq \sigma^\vee.
\]

The index of $P^*$ equals the index of $P$. 
**Example:** \( d = 3 \) (index \( r = 2 \))

The lattice point \( q = n_\sigma \) in \( \sigma^\vee \) with

\[
\langle q, P \times \{1\} \rangle = 1
\]

is the unique interior lattice point of \( rP^* \).
Back to complete intersections

Let $P_1, \ldots, P_r \subseteq M_{\mathbb{R}}$ be lattice polytopes.

**Definition:** The **Cayley polytope** $P_1 \ast \cdots \ast P_r$ is defined as

$$\text{conv}(P_1 \times \{e_1\}, \ldots, P_r \times \{e_r\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{Z}^r,$$

where $e_1, \ldots, e_r$ is a lattice basis of $\mathbb{Z}^r$.

**Proposition.** [Batyrev, Borisov 97]

$$P := P_1 * \cdots * P_r \text{ Gorenstein polytope of index } r \iff P_1 + \cdots + P_r \text{ reflexive.}$$

Then the Gorenstein cone $\sigma_P$ is called **completely split**.
Stringy $E$-functions

**Definition:** Let $P$ be a Gorenstein polytope of index $r$.

$$E_{str}(P; u, v) := (uv)^{-r} \sum_{\emptyset \leq F \leq P} (-u)^{\dim(F)+1} \tilde{S}(F, u^{-1}v) \tilde{S}(F^*, uv),$$

where

$$\tilde{S}(P, t) := \sum_{\emptyset \leq F \leq P} (-1)^{\dim(P) - \dim(F)} h_F^*(t) g[F,P](t).$$

**Theorem.** [Batyrev, Borisov 96; Borisov, Mavlyutov 03]

Let $P = P_1 \ast \cdots \ast P_r$ be Gorenstein polytope of index $r$. Then

$$E_{str}(P; u, v) = E_{str}(Y; u, v) = \sum_{p,q} (-1)^{p+q} h_{str}^{p,q}(Y) u^p v^q$$

for associated generic complete intersection $Y \subset X_{P_1+\cdots+P_r}$.

**Reciprocity (Mirror symmetry):**

$$E_{str}(P^*; u, v) = (-u)^{d+1-2r} E_{str}(P; u^{-1}, v).$$
Conjectures:

- The stringy $E$-function of any Gorenstein polytope is a **polynomial** of degree $2(d + 1 - 2r)$ with non-negative integers.

- Up to scalar there are only **finitely** many polynomials of given degree occuring as $E_{\text{str}}$-functions of Gorenstein polytopes.
4. Nef-partitions - 2.0

Let $P \subseteq M_\mathbb{R}$ be a $d$-dimensional lattice polytope.

**Proposition.** [Batyrev, N. 07]

Cayley polytope structures of $P = P_1 \ast \cdots \ast P_r$

$q = q_1 + \cdots + q_r$, where $q_i$ are non-zero lattice points in $\sigma_P$.

Given $q_1, \ldots, q_r$ we define

$P_i := \{ p \in P : \langle (p, 1), q_i \rangle = 1 \} \quad \text{for} \quad i = 1, \ldots, r.$
Corollary. [Batyrev, N. 07] Let $P$ be a Gorenstein polytope of index $r$. T.f.a.e.:

- $P$ is a Cayley polytope of length $r$
- $\sigma_P$ is completely split
- $q = q_1 + \cdots + q_r$ for lattice points $q_i$ in $P^*$
- $P^*$ has an $(r - 1)$-dimensional lattice simplex $R$ not in the boundary of $P^*$
- $P^*$ has a special lattice simplex $R$
  (i.e., has $r$ vertices, any facet of $P^*$ contains $r - 1$ vertices of $R$)
**Proposition.** [Batyrev, Borisov 97]

Cayley polytopes of dual nef-partitions are dual to each other Gorenstein polytopes.

**Definition:** A **Gorenstein-nef-partition** is a dual pair of Gorenstein polytopes $P$, $P^*$ together with choices of special lattice simplices $R \subseteq P$ and $R' \subseteq P^*$. Exist if and only if $\sigma_P$ and $\sigma_P^*$ are completely split.

Cayley polytopes of dual nef-partitions form Gorenstein-nef-partitions.
Back to (dual) nef-partitions

Let dual Gorenstein polytopes $P, P^*$ of index $r$ be given, admitting Gorenstein-nef-partitions.

The choice of a special lattice simplex $R' \subseteq P^*$ yields lattice polytopes $P_1, \ldots, P_r$ such that $P = P_1 \ast \cdots \ast P_r$, i.e.,

$$P_1 + \cdots + P_r \text{ is reflexive.}$$

There is a correspondence between

- special lattice simplices $R \subseteq P$
- lattice points $p_i \in P_i$ such that $p_1 + \cdots + p_r = 0$

Then $P_1 - p_1, \ldots, P_r - p_r$ is a nef-partition.

The associated Cayley polytope structure of $P^*$

$$P^* = Q_1 \ast \cdots \ast Q_r,$$

yields the dual nef-partition $Q_1, \ldots, Q_r$. 
**The choice matters**

Different choices of the special lattice simplex $R \subseteq P$ may yield extremely different dual nef-partitions!

Still, all corresponding dual CYs have the same stringy Hodge numbers!

**Questions:** Are they birationally isomorphic? Are their derived categories of coherent sheaves related?