Combinatorial aspects of mirror symmetry

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AMS Meeting, Rutgers University, October 7, 2007

V.V. Batyrev, B. Nill: *Combinatorial aspects of mirror symmetry*, `math/0703456`
(Proceedings of AMS-conference on integer points at Snowbird 2006)

1. Reflexive polytopes

M, N dual lattice of rank d.

Let $P \subseteq M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ be a d-dimensional lattice polytope containing 0 in its interior.

Definition:

$$P^* := \{y \in N_{\mathbb{R}} \ : \ \langle x, y \rangle \geq -1 \ \forall \, x \in P\}.$$

$$(P^*)^* = P.$$

Definition: P **reflexive** $:\Longleftrightarrow$ P^* lattice polytope.

$$P \text{ reflexive} \Longleftrightarrow P^* \text{ reflexive}$$

Reflexive polytopes turn up in dual pairs!

$P \subseteq M_{\mathbb{R}}$ a lattice polytope $\Rightarrow$

$$\sigma_P := \mathrm{pos}(P \times \{1\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{R},$$
$$S_P := \sigma_P \cap (M \oplus \mathbb{Z}),$$
$$X_P := \mathrm{Proj}\ \mathbb{C}[S_P].$$

Bijection (of isomorphism classes):

Reflexive polytopes P, P^*

$$\longleftrightarrow$$

Gorenstein toric Fano varieties X_P, X_{P^*}

(i.e., projective toric varieties with ample anticanonical Cartier divisor)

Let $P \subseteq M_{\mathbb{R}}$ be reflexive.

Definition: Let Y_P denote generic anticanonical *hypersurface* in (a crepant resolution of) X_P.

Theorem. **[Batyrev 1994]** Let P be reflexive.
Y_P, Y_{P^*} are mirror-symmetric CY's on the level of stringy Hodge numbers:

$$h_{\mathrm{str}}{}^{p,q} = h_{\mathrm{str}}{}^{(d-1)-p,q} \quad \forall\, 0 \leq p, q \leq d-1.$$

There are explicit formulas for computing stringy Hodge numbers in terms of the reflexive polytopes P, P^*!

2. Nef-partitions

Let X be a Gorenstein toric Fano variety.

Definition: A **nef-partition** is a partition of the torus-invariant prime divisors of X into effective nef Cartier divisors $D_1, \ldots, D_r$, i.e.,

$$-K_X = D_1 + \cdots + D_r.$$

Then the associated generic anticanonical complete intersection in (crepant resolution of) X is a (singular) CY.

Theorem. **[Borisov 93]** Exists duality of nef-partitions.

Theorem. **[Batyrev, Borisov 96]** The associated complete intersections are mirror-symmetric CY's on the level of stringy Hodge numbers.

Combinatorial characterization

- Nef-partitions correspond to Minkowski sums

$$P_1 + \cdots + P_r \text{ reflexive,}$$

  where P_i are lattice polytopes containing 0.

- Duality of nef-partitions can be characterized as follows:

  1. $\Delta := P_1 + \cdots + P_r \subseteq M_{\mathbb{R}}$ reflexive,
     $\nabla := Q_1 + \cdots + Q_r \subseteq N_{\mathbb{R}}$ reflexive,
  2. $\nabla^* = \mathrm{conv}(P_1, \ldots, P_r)$,
     $\Delta^* = \mathrm{conv}(Q_1, \ldots, Q_r)$,
  3. $\langle P_i, Q_j \rangle \geq -\delta_{i,j}$.

3. Gorenstein polytopes

Let $P \subseteq M_{\mathbb{R}}$ be d-dimensional lattice polytope.

Definition:

- **Gorenstein cones** are cones of the form σ_P.
- P is called **Gorenstein polytope** of index r $:\Longleftrightarrow$ rP reflexive.

Proposition. **[Batyrev, Borisov 97]**

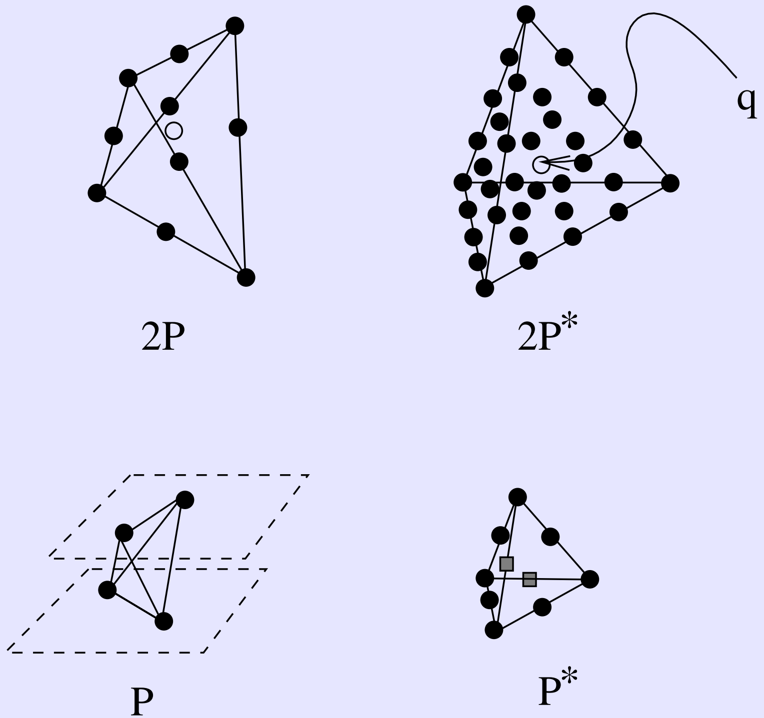
$$P \text{ Gorenstein polytope } \Leftrightarrow \sigma_P^{\vee} \text{ Gorenstein cone}$$

Then the **dual Gorenstein polytope** P^* defined via

$$\sigma_{P^*} \cong \sigma^{\vee}.$$

The index of P^* equals the index of P.

Example: $d = 3$ (index $r = 2$)



The lattice point $q = n_\sigma$ in $\sigma^\vee$ with

$$\langle q, P \times \{1\} \rangle = 1$$

is the unique interior lattice point of rP^*.

Back to complete intersections

Let $P_1, \ldots, P_r \subseteq M_{\mathbb{R}}$ be lattice polytopes.

Definition: The **Cayley polytope** $P_1 * \cdots * P_r$ is defined as

$$\operatorname{conv}(P_1 \times \{e_1\}, \ldots, P_r \times \{e_r\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{Z}^r,$$

where $e_1, \ldots, e_r$ is a lattice basis of $\mathbb{Z}^r$.

Proposition. **[Batyrev, Borisov 97]**

$$P := P_1 * \cdots * P_r \text{ Gorenstein polytope of index } r \iff P_1 + \cdots + P_r \text{ reflexive.}$$

Then the Gorenstein cone σ_P is called **completely split**.

Stringy E-functions

Definition: Let P be a Gorenstein polytope of index r.

$$E_{\mathrm{str}}(P; u, v) := (uv)^{-r} \sum_{\emptyset \leq F \leq P} (-u)^{\dim(F)+1}\, \widetilde{S}(F, u^{-1}v)\, \widetilde{S}(F^*, uv),$$

where $$\widetilde{S}(P, t) := \sum_{\emptyset \leq F \leq P} (-1)^{\dim(P)-\dim(F)}\, h^*_F(t)\, g_{[F,P]}(t).$$

Theorem. **[Batyrev, Borisov 96; Borisov, Mavlyutov 03]**
Let $P = P_1 * \cdots * P_r$ be Gorenstein polytope of index r. Then

$$E_{\mathrm{str}}(P; u, v) = E_{\mathrm{str}}(Y; u, v) = \sum_{p,q} (-1)^{p+q}\, {h_{\mathrm{str}}}^{p,q}(Y)\, u^p v^q$$

for associated generic complete intersection $Y \subset X_{P_1+\cdots+P_r}$.

Reciprocity (Mirror symmetry):

$$E_{\mathrm{str}}(P^*; u, v) = (-u)^{d+1-2r} E_{\mathrm{str}}(P; u^{-1}, v).$$

Conjectures:

- The stringy E-function of any Gorenstein polytope is a **polynomial** of degree $2(d+1-2r)$ with non-negative integers.
- Up to scalar there are only **finitely** many polynomials of given degree occuring as E_{str}-functions of Gorenstein polytopes.

4. Nef-partitions - 2.0

Let $P \subseteq M_{\mathbb{R}}$ be a d-dimensional lattice polytope.

Proposition. **[Batyrev, N. 07]**

$$\text{Cayley polytope structures of } P = P_1 * \cdots * P_r$$
$$\longleftrightarrow$$
$$q = q_1 + \cdots + q_r, \text{ where } q_i \text{ are non-zero lattice points in } \sigma_P^\vee.$$

Given $q_1, \ldots, q_r$ we define

$$P_i := \{p \in P \,:\, \langle (p,1), q_i \rangle = 1\} \quad \text{for} \quad i = 1, \ldots, r.$$

Corollary. **[Batyrev, N. 07]** Let P be a Gorenstein polytope of index r. T.f.a.e.:

- P is a Cayley polytope of length r
- σ_P is completely split
- $q = q_1 + \cdots + q_r$ for lattice points q_i in P^*
- P^* has an $(r-1)$-dimensional lattice simplex R not in the boundary of P^*
- P^* has a **special lattice simplex** R

  (i.e., has r vertices, any facet of P^* contains $r-1$ vertices of R

Reflexive polytopes
Nef-partitions
Gorenstein polytopes
Nef-partitions - 2.0

Proposition. **[Batyrev, Borisov 97]**
Cayley polytopes of dual nef-partitions are dual to each other Gorenstein polytopes.

Definition: A **Gorenstein-nef-partition** is a dual pair of Gorenstein polytopes P, P^* together with choices of special lattice simplices $R \subseteq P$ and $R' \subseteq P^*$.

Exist if and only if σ_P and $\sigma_P^{\vee}$ are completely split.

Cayley polytopes of dual nef-partitions form Gorenstein-nef-partitions.

Back to (dual) nef-partitions

Let dual Gorenstein polytopes P, P^* of index r be given, admitting Gorenstein-nef-partitions.

The choice of a special lattice simplex $R' \subseteq P^*$ yields lattice polytopes $P_1, \ldots, P_r$ such that $P = P_1 * \cdots * P_r$, i.e.,

$$P_1 + \cdots + P_r \text{ is reflexive.}$$

There is a correspondence between

- special lattice simplices $R \subseteq P$
- lattice points $p_i \in P_i$ such that $p_1 + \cdots + p_r = 0$

Then $P_1 - p_1, \ldots, P_r - p_r$ is a nef-partition.
The associated Cayley polytope structure of P^*

$$P^* = Q_1 * \cdots * Q_r,$$

yields the dual nef-partition $Q_1, \ldots, Q_r$.

The choice matters

Different choices of the special lattice simplex $R \subseteq P$ may yield extremely different dual nef-partitions!

Still, all corresponding dual CYs have the same stringy Hodge numbers!

Questions: Are they birationally isomorphic? Are their derived categories of coherent sheaves related?